The Arithmetic of Consecutive Polynomial Sequences over Finite Fields

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Abstract. Motivated by a question of van der Poorten about the existence of infinite chain of prime numbers (with respect to some base), in this paper we advance the study of sequences of consecutive polynomials whose coefficients are chosen consecutively from a sequence in a finite field of odd prime characteristic. We study the arithmetic of such sequences, including bounds for the largest degree of irreducible factors, the number of irreducible factors, as well as for the number of such sequences of fixed length in which all the polynomials are irreducible.

1. Introduction

1.1. Motivation. In [24], van der Poorten observed that the numbers 19, 197, 1979, 19793, 197933, 1979339, 19793393, 197933933, 1979339339 are all prime numbers and raised a question that whether there is such an infinite chain of prime numbers (with respect to some base $b$). One related question is whether there exists the largest truncatable prime in a given base $b$ (such a prime can yield a sequence of primes when digits are removed away from the right). Note that the above integer 1979339339 is not a truncatable prime. The authors in [1] have given heuristic arguments for the length of the largest truncatable prime in base $b$ (roughly, the length is $be/\log b$, where $e$ is the base of the natural logarithm) and computed the largest truncatable primes in base $b$ for $3 \leq b \leq 15$. Both questions might be very hard.

Mullen and Shparlinski [22, Problem 31] asked an analogous question about polynomials over finite fields. More precisely, let $p$ be an odd prime number and $q = p^s$ for some positive integer $s$. We denote by $\mathbb{F}_q$ the finite field of $q$ elements, and use $\mathbb{F}_q[X]$ to denote the ring of polynomials with coefficients in $\mathbb{F}_q$.

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For a (finite or infinite) sequence \( \{u_n\}, n \geq 0 \), of non-zero elements in \( \mathbb{F}_q \), we define a consecutive polynomial sequence \( \{f_n\}, n \geq 1 \), associated to the sequence \( \{u_n\} \), in \( \mathbb{F}_q[X] \) as follows:

\[
\tag{1.1} f_n = u_nX^n + \ldots + u_1X + u_0, \quad n \geq 1.
\]

If all the polynomials \( f_n, n \geq 1 \), are irreducible, then the sequence \( \{f_n\} \) is called a consecutive irreducible polynomial sequence, and \( \{u_n\} \) is called a consecutive irreducible sequence. Given a sequence \( \{u_n\} \), let \( L(\{u_n\}) \) be either \( \infty \) if \( \{u_n\} \) is infinite, or a non-negative integer such that \( L(\{u_n\}) + 1 \) is the length of \( \{u_n\} \). That is, \( L(\{u_n\}) \) is the length of the associated polynomial sequence \( \{f_n\} \).

Mullen and Shparlinski [22, Problem 31] asked for lower and upper bounds for the maximum length \( L(q) = \max \{L(\{u_n\})\} \) (possibly infinite), where \( \{u_n\} \) runs through all consecutive irreducible sequences over \( \mathbb{F}_q \). The only known result is a lower bound due to Chow and Cohen [5, Theorem 1.2],

\[
\tag{1.2} L(q) > \frac{\log q}{2\log \log q},
\]

whenever \( q \neq 3 \); they also observed that for \( q = 3 \), \( L(3) = 3 \).

The work on irreducible polynomials with prescribed coefficients might reflect that such an upper bound of \( L(q) \) indeed exists. Twenty years ago, Hansen and Mullen [13, Conjecture B] conjectured that for any \( n \geq 3 \), there exists a monic irreducible polynomial of degree \( n \) over \( \mathbb{F}_q \) with a prescribed coefficient. This conjecture has been proved by Wan [35] and Ham and Mullen [12]. Most recently, Pollack [23] has showed that for any \( \varepsilon > 0 \), and all large enough \( n \) depending on \( \varepsilon \), one can find a monic irreducible polynomial of degree \( n \) over \( \mathbb{F}_q \) with any \( [(1 - \varepsilon)\sqrt{n}] \) coefficients preassigned. However, to search for consecutive irreducible sequences, we need to fix \( n \) values \( u_0, u_1, \ldots, u_{n-1} \in \mathbb{F}_q^* \) and find \( u_n \in \mathbb{F}_q^* \) such that the polynomial \( u_nX^n + \cdots + u_1X + u_0 \) is irreducible. Thus, the difficulty of the above work suggests that searching for consecutive irreducible sequences of infinite length might be inaccessible. Moreover, at the end of Section 5.2 we give a heuristic argument to predict that \( L(q) < 3q \).

We also want to remark that it is easy to construct an infinite chain of consecutive irreducible polynomials over the rational integers \( \mathbb{Z} \). For example, given a prime number \( \ell \), all the polynomials \( 1 + \ell X, 1 + \ell X + \ell X^2, 1 + iX + \ell X^2 + iX^3, \ldots \) are irreducible, which can be obtained by using Eisenstein’s criterion to their reciprocal polynomials and with respect to the prime number \( \ell \).
Throughout the paper, we use the Landau symbols $O$ and $o$ and the Vinogradov symbol $\ll$. We recall that the assertions $U = O(V)$ and $U \ll V$ (sometimes we write this also as $V \gg U$) are both equivalent to the inequality $|U| \leq cV$ with some constant $c > 0$, while $U = o(V)$ means that $U/V \to 0$. In this paper, the constants implied in the symbols $O$, $\ll$ are absolute and independent of any parameters. If the implied constant is not absolute and depends on some parameter $\rho$, then we write $O_\rho$ and $\ll_\rho$.

1.2. Our results and methods. In this paper, we study the arithmetic of consecutive polynomial sequences defined by (1.1), such as, the growth of the largest degree of irreducible factors, the number of irreducible factors, as well as giving upper and lower bounds for the number of consecutive irreducible sequences of fixed length. We describe below our results and the techniques we use in more details.

In Section 2 we introduce the main tools that we use to prove our results. In Section 3, we use a method introduced in [11], relying on the polynomial $ABC$ theorem (proved first by Stothers [33], and then independently by Mason [20, 21] and Silverman [30], see also [31]), to give a lower bound on the largest degree $D(f_n)$ of the irreducible factors of $f_n$, $n \geq 1$, defined by (1.1). In particular, we prove that given an arbitrary sequence \{f_n\} of infinite length defined by (1.1), for almost all integers $n \geq 1$,

$$D(f_n) \gg \frac{\log n}{\log q}.$$ 

In Section 4, using similar ideas as in [25], we prove that for any consecutive polynomial sequence \{f_n\} defined by (1.1), for any integers $m \geq 0$ and $H \geq 2$, the number $\omega(f_{m+1}f_{m+2}\cdots f_{m+H})$ of irreducible factors of the product of $H$ consecutive polynomials $f_{m+1}, f_{m+2}, \ldots, f_{m+H}$, satisfies

$$\omega(f_{m+1}f_{m+2}\cdots f_{m+H}) \gg \frac{(m+H)H}{m+H \log(m+H)}.$$ 

Given a finite set $S$ of irreducible polynomials in $\mathbb{F}_q[X]$, we also give an upper bound for the number of $S$-polynomials among $H$ consecutive polynomials $f_{m+1}, f_{m+2}, \ldots, f_{m+H}$.

We conclude this section by showing that there exists a consecutive polynomial sequence \{f_n\} of length at least $\lfloor (3q)^{1/3} \rfloor$ defined by (1.1) such that all the polynomials are pairwise coprime. In this setting, the bound is much better than that in (1.2).

In Section 5, we give upper and lower bounds for the number $I_N$ of consecutive irreducible polynomial sequences of length $N$. This is also the most technical part of the paper. To give such bounds, we use a
sieve for large values of $N$ and also the Weil bound for multiplicative character sums, together with Stickelberger’s Theorem [32, 34] (which gives the parity of irreducible factors of a polynomial) for $N$ that are not too large compared to $q$. We prove that for any integer $N \geq 2$, we have

$$I_N < 2^{-N/5}q^{N+1}$$

and

$$I_N < q^{N+1}/2^{N-1} + 2 \left( N^2 q^{N+1/2} + N^4 q^N \right)/3. \tag{1.4}$$

Note that (1.4) is better than (1.3) when $q$ is much larger than $N$. The rest of the section is dedicated to obtaining a formula for $I_2$ and explicit lower bounds for $I_3$ and $I_4$, which are better than those implied in [5].

2. Preliminaries

In this section we gather some tools which are used in the proofs for the convenience of the reader.

We start by recalling a few properties of discriminants and resultants of polynomials. A detailed exposition on this subject can be found in [4, Part III, Chapter 15]. For two polynomials $f, g \in \mathbb{F}_q[X]$, we denote by $\text{Disc}(f)$ and $\text{Res}(f, g)$ the discriminant of $f$ and the resultant of $f$ and $g$, respectively. The following well-known formula for the discriminant of the product $fg$ can be found in [4, Part III, Chapter 15, Proposition 2] (see also [15, Theorem 3.10]),

$$\text{Disc}(fg) = \text{Disc}(f) \text{Disc}(g) \text{Res}(f, g)^2. \tag{2.1}$$

The discriminant of a polynomial $f$ can be viewed as a polynomial function in the coefficients of $f$. This point of view gives the following simple formula, which can be regarded as a relation between discriminants of polynomials of consecutive degrees; see [15, Theorem 3.11]. If $f \in \mathbb{F}_q[X]$ is of degree at most $d$ (not necessarily fixed $d$),

$$f = a_d X^d + g, \quad g = a_{d-1} X^{d-1} + \cdots + a_1 X + a_0,$$

then if we set $a_d = 0$, we can get the following relation

$$\text{Disc}(f) = a_{d-1}^2 \text{Disc}(g). \tag{2.2}$$

One of the main tools used for our results is Stickelberger’s Theorem (see [32] or [34, Corollary 1]), which gives the parity of the number of distinct irreducible factors of a square-free polynomial over a finite field of odd characteristic. This provides a powerful tool to study the number of irreducible factors of polynomials.
Lemma 2.1. Suppose that $f \in \mathbb{F}_q[X]$, where $q$ is odd, is a polynomial of degree $d \geq 2$ and is the product of $r$ pairwise distinct irreducible polynomials over $\mathbb{F}_q$. Then $r \equiv d \pmod{2}$ if and only if $\text{Disc}(f)$ is a square element in $\mathbb{F}_q$.

For proving our results, we treat the discriminant of a general polynomial $f$ as a multivariate polynomial in the coefficients of $f$ and study for which substitutions of the variables the discriminant is a square. This technical result has been given in [10, Lemma 5.2], which in fact implies an explicit result. Here, we reproduce the proof briefly.

Lemma 2.2. Let $G \in \mathbb{F}_q[Y_0, Y_1, \ldots, Y_d]$ be a polynomial of degree $D$, which is not a square polynomial in the algebraic closure of $\mathbb{F}_q$. Then there exists $i \in \{0, 1, \ldots, d\}$ such that $G(a_0, \ldots, a_{i-1}, Y_i, a_{i+1}, \ldots, a_d)$ is not a square polynomial in $Y_i$ up to a multiplicative constant for all but at most $D^2q^{d-1}$ values of $a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d \in \mathbb{F}_q$.

Proof. As the proof in [10, Lemma 5.2], let

$$G(Y_0, \ldots, Y_d) = a G_1(Y_0, \ldots, Y_d)d_1 \cdots G_h(Y_0, \ldots, Y_d)^{d_h}$$

be the decomposition of the polynomial in a product of a non-zero constant and monic irreducible polynomials, and assume that $d_1$ is an odd integer and $G_1(Y_0, \ldots, Y_d)$ depends on some variable $Y_i$. The result in [10, Lemma 5.2] comes from the sum of three upper bounds $Dq^{d-1}$, $\deg G_1(\deg G_1 - 1)q^{d-1}$ and $\deg G_1 \deg G_jq^{d-1}$, where $j$ is some integer between 2 and $h$ (it may not exist).

In fact, if the polynomial $G(a_0, \ldots, a_{i-1}, Y_i, a_{i+1}, \ldots, a_d)$ is a constant polynomial under the specialisation $a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d \in \mathbb{F}_q$, then for some $k$, $G_k(a_0, \ldots, a_{i-1}, Y_i, a_{i+1}, \ldots, a_d)$ is also a constant. So, the bound $Dq^{d-1}$ can be replaced by $q^{d-1}\max_{1 \leq k \leq h} \deg G_k$. Noticing

$$D^2 \geq \max_{1 \leq k \leq h} \deg G_k + \deg G_1(\deg G_1 - 1) + \deg G_1 \deg G_j,$$

we get the desired result. \qed

To estimate the number of consecutive irreducible sequences, we need the Weil bound for character sums with polynomial arguments (see [19, Theorem 5.41]).

Lemma 2.3. Let $\chi$ be a multiplicative character of $\mathbb{F}_q$ of order $m > 1$, and let $f \in \mathbb{F}_q[X]$ be a polynomial of positive degree that is not, up to a multiplicative constant, an $m$-th power of a polynomial. Let $d$ be the number of distinct roots of $f$ in its splitting field over $\mathbb{F}_q$. Under these
conditions, the following inequality holds:

$$\left| \sum_{x \in \mathbb{F}_q} \chi(f(x)) \right| \leq (d - 1)q^{1/2}.$$ 

Some of our results are also based on the polynomial $ABC$ theorem [20, 21, 30, 31, 33].

For a polynomial $f \in \mathbb{F}_q[X]$ we use $\text{rad}(f)$ to denote the product of all monic irreducible factors of $f$.

**Lemma 2.4.** Let $A$, $B$, $C$ be non-zero polynomials over $\mathbb{F}_q$ with $A + B + C = 0$ and $\gcd(A, B, C) = 1$. If $\deg A \geq \deg \text{rad}(ABC)$, then for their derivatives, we have $A' = B' = C' = 0$.

To obtain an upper bound for the number of consecutive irreducible sequences of fixed length, we need the following result due to Johsen [17, Corollary 2] on the number of irreducible polynomials over $\mathbb{F}_q$ in an arithmetic progression.

**Lemma 2.5.** Let $n$ and $r$ be positive integers such that $1 \leq r < n$, and let $f \in \mathbb{F}_q[X]$. Denote by $\pi(f; n, r)$ the number of irreducible polynomials of degree $n$ which are congruent to $f$ modulo $X^r$. Then

$$\pi(f; n, r) < \frac{2q^{n-r+1}}{n-r}.$$ 

Finally, we recall a classical result on using the cubic resolvent to solve quartic equations, which is due to Euler [7, §5]. Here, we reproduce a form from [16, Theorem 3.2].

**Lemma 2.6.** Let $K$ be an arbitrary field of characteristic not equal to 2 or 3. Given a quartic polynomial $f(X) = X^4 + aX^2 + bX + c \in K[X]$, define its cubic resolvent by $R(X) = X^3 + 2aX^2 + (a^2 - 4c)X - b^2$. Let $u, v, w$ be the roots of $R$, and put $\gamma_1 = \sqrt{u}, \gamma_2 = \sqrt{v}, \gamma_3 = \sqrt{w}$, where we choose the signs so that $\gamma_1\gamma_2\gamma_3 = -b$. Then, the roots of $f$ are given by

$$\begin{cases} 
\beta_1 = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3), \\
\beta_2 = \frac{1}{2}(\gamma_1 - \gamma_2 - \gamma_3), \\
\beta_3 = \frac{1}{2}(-\gamma_1 + \gamma_2 - \gamma_3), \\
\beta_4 = \frac{1}{2}(-\gamma_1 - \gamma_2 + \gamma_3). 
\end{cases}$$

3. The Largest Degree of Irreducible Factors

We recall that for a consecutive polynomial sequence $\{f_n\}$ defined by (1.1), we use $D(f_n)$ to denote the largest degree of irreducible factors of $f_n$ for each $n \geq 1$. 

The following is our main result of this section. We use the same technique as in the proof of \([11, \text{Theorem 10}]\). Recall that \(p\) is an odd prime and the characteristic of \(\mathbb{F}_q\).

**Theorem 3.1.** Let \(\{f_n\}, n \geq 1\) be any consecutive polynomial sequence defined by (1.1) of infinite length. For any integers \(n \geq 2q\) and \(d\) satisfying \(0 < d \leq \frac{\log(n/2)}{\log q}\), we have

\[
\max\{D(f_n), D(f_{n+d})\} > \frac{\log((n+1)/2) + \log \log q - \log \log(n/2)}{\log q}.
\]

Moreover, if \(p \nmid n + 1\) or \(p \nmid d\), then

\[
\max\{D(f_n), D(f_{n+d})\} > \frac{\log((n+1)/2)}{\log q}.
\]

**Proof.** Fix an integer \(n \geq 2q\) and fix an integer \(d\) such that

\[
0 < d \leq \frac{\log(n/2)}{\log q}.
\]

By construction we have that

\[
f_{n+d} - f_n = X^{n+1}\left(\sum_{i=1}^{d} u_{i+n}X^{i-1}\right).
\]

Let \(g = \gcd(f_n, f_{n+d})\). Then we must have that \(g\) divides \(\sum_{i=1}^{d} u_{i+n}X^{i-1}\), and so \(\deg g \leq d - 1\). Put \(A = f_{n+d}/g, B = -f_n/g\) and

\[
C = -X^{n+1}\left(\sum_{i=1}^{d} u_{i+n}X^{i-1}\right)/g.
\]

Then,

\[A + B + C = 0\quad \text{and}\quad \gcd(A, B, C) = 1.\]

Let \(m\) be the largest non-negative integer such that \(A = A_1 p^m, B = B_1 p^m, C = C_1 p^m\) for some polynomials \(A_1, B_1, C_1\) such that the identity about derivatives \(A'_1 = B'_1 = C'_1 = 0\) is not true. Note that \(m = 0\) if and only if the identity \(A' = B' = C' = 0\) does not hold. Then, we have

\[A_1 + B_1 + C_1 = 0\quad \text{and}\quad \gcd(A_1, B_1, C_1) = 1.\]

By the form of \(C\), we can write \(C_1\) as

\[C_1 = X^{(n+1)/p^m}h(X)\quad \text{with}\quad \deg h \leq (d - 1)/p^m\]

for some polynomial \(h(X)\) (note that we indeed have \(p^m \mid n + 1\)).
Since both $\deg A$ and $\deg B$ are divisible by $p^m$, we get $p^m \mid d$. So the choice of $d$ implies that

\begin{equation}
 p^m \leq \frac{\log(n/2)}{\log q}.
\end{equation}

We define $N$ as the largest integer satisfying

\begin{equation}
 2q^N \leq (n + 1)/p^m.
\end{equation}

So, we have

\begin{equation}
 N + 1 > \frac{\log((n + 1)/2) - m \log p}{\log q}.
\end{equation}

If $N = 0$, then we obtain

\[ n + 1 < 2q^m \leq \frac{2q \log(n/2)}{\log q}, \]

which implies that the right side of (3.1) is less than 1, and thus (3.1) is true automatically.

In the following we assume that $N \geq 1$. Now, we prove the desired result by contradiction. Suppose that

\[ \max\{D(f_n), D(f_{n+d})\} \leq N. \]

This means that the polynomial $f_n f_{n+d}$ can be factorized by irreducible polynomials of degree at most $N$. So, any root of $f_n$ or $f_{n+d}$ belongs to $\mathbb{F}_{q^j}$ with $j \leq N$. Then, the product $f_n f_{n+d}$ has at most

\begin{equation}
 \sum_{j=1}^{N} q^j < 2q^N
\end{equation}

distinct roots.

Then, applying Lemma 2.4 to $A_1$, $B_1$ and $C_1$, we obtain

\[ \frac{n + 1}{p^m} \leq \deg A_1 < \deg \text{rad}\ (A_1 B_1 C_1) \]
\[ < \deg \text{rad}\ (f_{n+d} f_n X h(X)) \leq 2q^N, \]

where the last inequality comes from (3.6) and the fact $\deg h < N$ (which can be straightforward proved by collecting (3.3) and (3.5) and noticing the choices of $h(X)$ and $d$). Hence, we get $(n + 1)/p^m < 2q^N$, which contradicts (3.4). So, we must have

\begin{equation}
 \max\{D(f_n), D(f_{n+d})\} \geq \frac{\log((n + 1)/2) - m \log p}{\log q},
\end{equation}

which, together with (3.3), concludes the proof of (3.1).
Now, it remains to prove (3.2). If the derivatives $A' = B' = 0$, then we get that both $n + d - \deg g$ and $n - \deg g$ are divisible by $p$, and thus $p \mid d$. Since $C$ can be written as $C = X^{n+1} r(X)$, where $r(X)$ is some polynomial with $r(0) \neq 0$, if $C' = 0$, then we must have $p \mid n + 1$.

Thus, under the condition $p \nmid n + 1$ or $p \nmid d$, the identity $A' = B' = C' = 0$ is not true, and then the integer $m = 0$. So, the desired result follows from (3.7) directly.

We want to point out that the conclusions in Theorem 3.1 also hold for consecutive polynomial sequences of finite but sufficiently large length. One can understand other relevant results in this paper from the same point of view.

Now, we want to give an example to show that without the condition $p \nmid n + 1$ or $p \nmid d$, the case $A' = B' = C' = 0$ can happen in the proof of Theorem 3.1.

**Example 3.2.** Choose $q = 3$, and $u_n = 1$ for all integers $n \geq 0$, and use the notation in the proof of Theorem 3.1. Fix $n = 56$ and pick $d = 3$, then we have
\[
\begin{align*}
f_n &= (X^{54} + X^{51} + \cdots + X^3 + 1)(X^2 + X + 1), \\
f_{n+d} &= (X^{57} + X^{54} + \cdots + X^3 + 1)(X^2 + X + 1).
\end{align*}
\]
So, we can get that $m = 1$, $N = 3$ and $\gcd(f_n, f_{n+d}) = X^2 + X + 1$. It is easy to see that $A' = B' = C' = 0$.

Moreover, we can get the following asymptotic result.

**Corollary 3.3.** Let $\{f_n\}, n \geq 1$, be a consecutive polynomial sequence of infinite length defined by (1.1). For almost all integers $n \geq 1$, we have
\[
D(f_n) \gg \frac{\log n}{\log q}.
\]

**Proof.** By (3.1), there exists an absolute constant $c$ such that
\[
\max\{D(f_n), D(f_{n+d})\} \geq \frac{c \log(n + d)}{\log q}
\]
for any integer $n \geq 2q$ and any $0 < d \leq \frac{\log(n/2)}{\log q}$ (note that the choice of $c$ is independent of $q$).

Now, for any sufficiently large $n$, if $D(f_n) < \frac{c \log n}{\log q}$, then by (3.8), for any $0 < d \leq \frac{\log(n/2)}{\log q}$, we have
\[
D(f_{n+d}) \geq \frac{c \log(n + d)}{\log q}.
\]
This implies that
\[
\lim_{N \to \infty} \frac{|\{1 \leq n \leq N : D(f_n) < \frac{c \log n}{\log q}\}|}{N} = 0,
\]
which completes the proof. \(\square\)

Theorem 3.1 tells us that there exist irreducible factors with arbitrary large degree in such a given sequence \(\{f_n\}\). However, it is generally false that \(D(f_n)\) grows with \(n\) or even that \(D(f_n) > 1\) for all sufficiently large \(n\). As an example, taking \(u_n = 1\) for all \(n \geq 0\), it is easy to check that
\[
f_n(X)(X - 1) = X^{n+1} - 1, \quad n \geq 1.
\]

Fix an integer \(n \geq 1\) and write \(n + 1 = p^k m\) with \(\gcd(m, p) = 1\), then according to [19, Theorem 2.47], \(D(f_n)\) is exactly the multiplicative order of \(q\) modulo \(m\). Especially, when \(n + 1 = p^k\) for some integer \(k\), then \(f_n(X)(X - 1) = (X - 1)^{p^k}\), and thus \(D(f_n) = 1\).

In addition, given two non-zero coprime integers \(g, m\) with \(m \geq 1\), denote by \(\ell_g(m)\) the multiplicative order of \(g\) modulo \(m\). In [18, Theorem 1] (see [29, Theorem 3.4] for previous work), the authors have showed that if the Generalized Riemann Hypothesis is true, then for the average multiplicative order, we have
\[
\frac{1}{x} \sum_{\substack{m \leq x \\gcd(m, g) = 1}} \ell_g(m) = \frac{x}{\log x} \exp \left( \frac{B \log \log x}{\log \log \log x} (1 + o(1)) \right)
\]
as \(x \to \infty\), uniformly in \(g\) with \(1 < |g| \leq \log x\), where \(B\) is an explicit absolute constant. This can give a conditional asymptotic formula of the average value of \(D(f_n)\) for the above sequence \(\{f_n\}\).

**Theorem 3.4.** Let \(\{f_n\}\) be the consecutive polynomial sequence defined by (1.1) such that all the coefficients of \(f_n\) for any \(n \geq 1\) are equal to 1. Under the Generalized Riemann Hypothesis, we have
\[
\frac{1}{x} \sum_{n \leq x} D(f_n) = \frac{p^2 x}{(p^2 - 1) \log x} \exp \left( \frac{B \log \log x}{\log \log \log x} (1 + o(1)) \right),
\]
as \(x \to \infty\), where \(B\) is the explicit constant in (3.9).

**Proof.** From the above discussions, for any \(n \geq 1\), \(D(f_n) = \ell_q(m)\) for some integer \(m\), where \(n + 1 = p^k m\) with \(\gcd(m, p) = 1\). So using (3.9),
for sufficiently large $x$ we have
\[
\frac{1}{x} \sum_{n \leq x} D(f_n) = \frac{1}{x} \sum_{\substack{m \leq n+1 \atop \gcd(m,q)=1}} \ell_q(m) + \frac{1}{x} \sum_{\substack{m \leq (n+1)/p \atop \gcd(m,q)=1}} \ell_q(m) + \frac{1}{x} \sum_{\substack{m \leq (n+1)/p^2 \atop \gcd(m,q)=1}} \ell_q(m) + \cdots
\]
\[
= \left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots\right) \frac{x}{\log x} \exp\left(\frac{B \log \log x}{\log \log \log x} (1 + o(1))\right)
\]
\[
= \frac{p^2 x}{(p^2 - 1) \log x} \exp\left(\frac{B \log \log x}{\log \log \log x} (1 + o(1))\right),
\]
as $x \to \infty$, which completes the proof.

Theorem 3.4 suggests that the bound in Corollary 3.3 might be not tight for the sequence $\{f_n\}$ in Theorem 3.4 and thus might be not optimal in general.

Furthermore, we can say more about the above sequence $\{f_n\}$. One can see that $f_n$ is irreducible if and only if $n+1$ is a prime number coprime to $q$ and $q$ is a primitive root modulo $n+1$. Recall that $q = p^s$. If $s$ is even, for any $n \geq 2$, if $n+1$ is prime, then $q$ is not a primitive root modulo $n+1$; thus, for any $n \geq 2$, $f_n$ is reducible. Otherwise, if $s$ is odd, under Artin’s conjecture on primitive roots, there are infinitely many integers $n$ such that $f_n$ is irreducible.

4. The Number of Irreducible Factors

From now on, for a polynomial $f \in \mathbb{F}_q[X]$ we use $\omega(f)$ to denote the number of distinct irreducible factors of $f$. In this section, we study irreducible factors of consecutive polynomial sequences. First we need a lemma based on similar ideas as in [25, Lemma 1].

**Lemma 4.1.** Let $\{f_n\}$ be any consecutive polynomial sequence defined by (1.1). Given a non-constant polynomial $g \in \mathbb{F}_q[X]$, $g(0) \neq 0$, and integers $m \geq 0$ and $H \geq 2$, denote by $T(m, H; g)$ the number of positive integers $n$ with $m+1 \leq n \leq m+H$ such that $g \mid f_n$, and let $e(m, H; g)$ be the power of $g$ in the product $f_{m+1}f_{m+2}\cdots f_{m+H}$. Then, we have
\[ T(m, H; g) \leq 1 + \frac{H}{\deg g}, \]
and
\[ e(m, H; g) \ll \frac{m + H \log(m + H)}{\deg g}. \]
In particular, if \( H \geq 3 \), we have
\[
T(0, H; g) \leq \frac{H}{\deg g} \quad \text{and} \quad e(0, H; g) \leq \frac{2H \log H}{\deg g}.
\]

**Proof.** For any integers \( n, d \geq 1 \), we have
\[
f_{n+d} = f_n + X^{n+1} \sum_{i=1}^{d} u_{n+i} X^{i-1}.
\]
If \( g \mid f_n \), then we can see that \( g \mid f_{n+d} \) if and only if \( g \mid \sum_{i=1}^{d} u_{n+i} X^{i-1} \). Thus, if \( g \mid f_n \) and \( d \leq \deg g \), then we must have \( g \nmid f_{n+d} \). This implies that
\[
T(m, H; g) \leq 1 + \frac{H}{\deg g}.
\]
Now, let \( \theta(m, H; g) \) be the maximal power of \( g \) in a factorization of any one polynomial \( f_{m+1}, f_{m+2}, \ldots, f_{m+H} \). Then, we deduce that
\[
e(m, H; g) = \sum_{k=1}^{\theta(m, H; g)} T(m, H; g^k) \leq \sum_{k=1}^{\left\lceil \frac{(m+H)/\deg g}{1+H/(k \deg g)} \right\rceil} (1 + H/(k \deg g)) \lesssim \frac{m + H \log(m + H)}{\deg g}.
\]
For the case \( m = 0 \), one can apply the same arguments to get the desired explicit estimates without using the symbol \( \ll \). Here, in order to bound \( e(0, H; g) \), one should use the assumption \( H \geq 3 \) and also the trivial upper bound for the partial sum of the harmonic series:
\[
\sum_{k=1}^{n} \frac{1}{k} \leq 1 + \log n, \quad n \geq 1.
\]
\[\square\]

Now, we are ready to estimate the number of distinct irreducible factors of the product of consecutive terms in a consecutive polynomial sequence \( \{f_n\} \), similarly as in [25, Theorem 2].

**Theorem 4.2.** Let \( \{f_n\} \) be any consecutive polynomial sequence defined by (1.1). For any integers \( m \geq 0 \) and \( H \geq 2 \), we have
\[
\omega(f_{m+1} f_{m+2} \cdots f_{m+H}) \gg \frac{(m + H)H}{m + H \log(m + H)}.
\]
In particular, if \( H \geq 3 \), we have
\[
\omega(f_1 f_2 \cdots f_H) \geq \frac{H}{4 \log H}.
\]
Proof. It follows from Lemma 4.1 that for any irreducible polynomial \( g \in \mathbb{F}_q[X] \) we get
\[
\deg(g^{e(m,H,g)}) \ll m + H \log(m + H).
\]
On the other hand, since \( \deg f_n = n \) for any \( n \geq 1 \), we have
\[
\deg(f_{m+1}f_{m+2} \cdots f_{m+H}) \gg mH + H^2.
\]
Thus, the above two bounds yield the first desired result.

The second desired lower bound can be obtained by applying the same arguments and using the explicit estimates in Lemma 4.1. \( \square \)

In the following, we give a direct consequence of Theorem 4.2.

Corollary 4.3. Let \( \{f_n\} \) be any consecutive polynomial sequence defined by (1.1) such that all the polynomials split completely over \( \mathbb{F}_{q^k} \) for a fixed integer \( k \geq 1 \). Then the length of the sequence \( \{f_n\} \) is \( O(qk^k \log q) \). Moreover, if \( k = 1 \), the the length of \( \{f_n\} \) is at most \( 2q + 1 \).

Proof. It follows directly from Theorem 4.2 as for any \( H \geq 3 \) (\( H \) is not greater than the length of \( \{f_n\} \)), we have
\[
q^k \geq \omega(f_1f_2 \cdots f_H) \geq H/(4\log H),
\]
which concludes the proof of the first part.

Notice that for \( \{f_n\} \) the largest degree of irreducible factors is equal to 1, then the second part follows from (3.1) (choosing \( n = 2q \) there). \( \square \)

Let \( S \) be a finite set of irreducible polynomials in \( \mathbb{F}_q[X] \). We call a polynomial \( f \in \mathbb{F}_q[X] \) an \( S \)-polynomial if all its irreducible factors are contained in \( S \).

Theorem 4.4. Let \( \{f_n\} \) be a consecutive polynomial sequence defined by (1.1). For any integers \( m \geq 0 \) and \( H \geq 2 \), denote by \( Q(m,H;S) \) the number of \( S \)-polynomials amongst \( f_{m+1}, f_{m+2}, \ldots, f_{m+H} \). Then, we have
\[
Q(m,H;S) \ll |S| \log H \log(m + H).
\]

Proof. We follow that same approach as in [25, Theorem 3].

Set \( L_0 = 1 \). Split the interval \([1, H]\) into \( k = O(\log H) \) intervals \([L_{i-1}, L_i]\), where \( L_i = \min\{2^i, H\} \), \( i = 1, 2, \ldots, k \). For any \( 1 \leq i \leq k \), let \( M_i \) be the number of \( S \)-polynomials among \( f_n, n \in [m + L_{i-1}, m + L_i] \). Since \( \deg f_n = n \) for each \( n \geq 1 \) and \( L_i \leq 2L_{i-1} \) for any \( 1 \leq i \leq k \),
combining with Lemma 4.1, we obtain

\[
(m + L_{i-1})M_i \leq \sum_{g \in S} \deg(g^{e(m+L_{i-1}-1,L_{i-1}+1;g)})
\ll |S|(m + L_{i-1}) \log(m + L_i).
\]

So, we get \(M_i \ll |S| \log(m + L_i)\). Thus,

\[
Q(m, H; S) = \sum_{i=1}^{k} M_i \ll |S| \sum_{i=1}^{k} \log(m + L_i) \ll |S| \log H \log(m + H).
\]

□

The equation (1.2) says that when \(q\) is large enough, there exists a consecutive irreducible polynomial sequence whose length is greater than \(\frac{\log q}{2 \log \log q}\). We can improve this lower bound if we want to search for a consecutive polynomial sequence whose terms are pairwise coprime.

**Theorem 4.5.** There exists a consecutive polynomial sequence \(\{f_n\}_{n=1}^{H}\) defined by (1.1) of length

\[
H \geq \left\lfloor (3q)^{1/3} \right\rfloor,
\]

such that all the terms are pairwise coprime.

**Proof.** First, we note that two polynomials \(f\) and \(g\) are coprime if and only if their resultant \(\text{Res}(f, g) \neq 0\). So, given a consecutive polynomial sequence \(\{f_n\}\) of length \(H\), the polynomials \(f_1, f_2, \ldots, f_H\) are pairwise coprime if and only if

\[
\prod_{1 \leq i < j \leq H} \text{Res}(f_i, f_j) \neq 0.
\]

Without loss of generality, we fix \(u_0 = 1\). Since

\[
\prod_{1 \leq i < j \leq H} \text{Res}(f_i, f_j) = \prod_{1 \leq i < j \leq H} \text{Res}(f_i, f_j - f_i)
= \pm \prod_{1 \leq i < j \leq H} \text{Res}(f_i, (f_j - f_i)/X^{i+1}),
\]

as a polynomial in variables \(u_1, \ldots, u_H\), we have

\[
\deg \prod_{1 \leq i < j \leq H} \text{Res}(f_i, (f_j - f_i)/X^{i+1})
= \sum_{i=1}^{H-1} \sum_{j=i+1}^{H} (i + (j - i - 1))
= \frac{1}{6} H(H - 1)(2H - 1) < \frac{1}{3} H^3.
\]
Thus, the multivariate polynomial \( \prod_{1 \leq i < j \leq H} \text{Res} \left( f_i, (f_j - f_i)/X^{i+1} \right) \) has less than \( H^3q^{H-1}/3 \) zeros. However, the vector \( (u_1, \ldots, u_H) \) can take \( q^H \) choices. So, when \( q \geq H^3/3 \), we can choose \( u_0, u_1, \ldots, u_H \) such that the inequality (4.1) holds; that is, we get a consecutive polynomial sequence whose terms are pairwise coprime. This completes the proof. \( \square \)

5. The Number of Consecutive Irreducible Sequences

For any integer \( N \geq 2 \), we denote by \( I_N \) the number of consecutive irreducible sequences \( \{u_n\} \), \( n \geq 0 \), of length \( N + 1 \). That is, \( I_N \) is the number of consecutive irreducible polynomial sequences of length \( N \). In this section, we give some upper and lower bounds for \( I_N \), as well as an asymptotic formula.

5.1. Trivial bound. Recall that for integer \( n \geq 1 \), \( \pi_q(n) \) is the number of monic irreducible polynomials of degree \( n \) over \( \mathbb{F}_q \). By [23, Lemma 4], we have

\[
\frac{q^n}{2n} \leq \pi_q(n) \leq \frac{q^n}{n}.
\]

Trivially, we have that \( I_N \) is not greater than the number of irreducible polynomials of degree \( N \) over \( \mathbb{F}_q \). So for \( N \geq 2 \), we have

\[
I_N \leq (q - 1)\frac{q^N}{N} < \frac{q^{N+1}}{N}.
\]

5.2. Upper bounds. Here, under some circumstances we establish two upper bounds for \( I_N \) better than the trivial one in (5.2). We recall that \( q \) is an odd prime power.

**Theorem 5.1.** For any integer \( N \geq 2 \), the number \( I_N \) of consecutive irreducible polynomial sequences of length \( N \) satisfies

\[
I_N < 2^{-N/5}q^{N+1} < 0.871^N q^{N+1}.
\]

Moreover, for \( N \geq 7 \) we have

\[
I_N < 3^{-N/7}q^{N+1} < 0.855^N q^{N+1}.
\]

**Proof.** Later on in Theorem 5.5 we will show that \( I_2 = \frac{1}{2}(q-1)^3 \). Then, we have \( I_3 \leq \frac{1}{2}(q - 1)^4 \). It is easy to check that both \( I_2 \) and \( I_3 \) satisfy the first desired inequality. Now, assume that \( N \geq 4 \).

Note that for each consecutive irreducible polynomial sequence \( \{f_n\} \) of length \( N \) defined by \( \{u_n\} \) and for any positive integer \( 2 \leq m \leq N \), we have

\[
u_NX^N + \cdots + u_1X + u_0 \equiv u_{N-m}X^{N-m} + \cdots + u_1X + u_0 \mod X^{N-m+1},
\]
which, together with Lemma 2.5, implies that

\[(5.3) \quad I_N < I_{N-m} \cdot \frac{2q^m}{m-1}.\]

Write \(N = km + r\) with \(0 \leq r < m\). Using (5.3) repeatedly, we obtain

\[I_N < I_{r} \left( \frac{2q^m}{m-1} \right)^k,
\]

where one should note that \(I_0 = q - 1\) and \(I_1 = (q - 1)^2\).

Applying the trivial estimate \(I_{r} < q^{r+1}\), we have

\[I_N < q^{N+1} \left( \frac{2}{m-1} \right)^k \leq q^{N+1} g(m)^{N},\]

where \(g(m) = \left( \frac{2}{m-1} \right)^{1/m}\). It is easy to see that when \(m = 7\) the term \(g(m)\) attains its minimum value, which requires that \(N \geq 7\). Since \(I_4 \leq \frac{1}{2}(q - 1)^5\), we choose the value \(g(5)\) for the general bound, and pick \(g(7)\) for \(N \geq 7\). This completes the proof.

In the following, we want to improve the upper bound in Theorem 5.1 when \(q\) is much larger than \(N\). To give such an improvement on bounding \(I_N\), we use the same technique as in [10, Theorem 5.5]. For this we need the following lemma.

**Lemma 5.2.** Let \(\{f_n\}, n \geq 1\), be a consecutive irreducible polynomial sequence defined by \(\{u_n\}, n \geq 0\), in (1.1). Then, for any \(\nu \geq 1\),

\[D_{n_1, \ldots, n_\nu} = \prod_{j=1}^{\nu} \text{Disc} (f_{n_j}), \quad 2 \leq n_1 < \ldots < n_\nu,
\]

is not a square polynomial in \(u_0, u_1, \ldots, u_{n_\nu}\) (as a multivariate polynomial).

**Proof.** The proof follows by induction on \(\nu \geq 1\). Although the sequence \(\{u_n\}\) is given, we sometimes view \(u_0, u_1, \ldots\) as variables when considering discriminants without specific indication.

For the induction argument we need to prove that \(D_{n_1}\) and \(D_{n_1, n_2}\) is not a square polynomial.

We prove first that \(D_{n_1}\) is not a square polynomial. If \(D_{n_1} = \text{Disc} (f_{n_1})\) would be a square polynomial as a multivariate polynomial in \(u_0, \ldots, u_{n_1}\), then for any specialisation of the variables \(u_0, \ldots, u_{n_1}\), we would get that \(\text{Disc} (f_{n_1})\) is a square element in \(\mathbb{F}_q\). From Lemma 2.1, this implies that for any choice of \(u_0, \ldots, u_{n_1} \in \mathbb{F}_q, u_{n_1} \neq 0\), the number of irreducible factors of \(f_{n_1}\) is congruent to \(n_1\) modulo 2 when
$f_{n_1}$ is square-free, which is obviously not true in general. Thus, $D_{n_1}$ is not a square multivariate polynomial.

We prove now that $D_{n_1,n_2}$ is not a square polynomial in $u_0, \ldots, u_{n_2}$. If $D_{n_1,n_2}$ is a square polynomial, then it is also a square polynomial for the specialisation $u_{n_2} = 0$. Using now (2.2) with $u_{n_2} = 0$, we get

$$\text{Disc} \left( f_{n_2} \right) = u_{n_2-1}^2 \text{Disc} \left( f_{n_2-1} \right),$$

which implies that

$$D_{n_1,n_2} = u_{n_2-1}^2 \text{Disc} \left( f_{n_2-1} \right) D_{n_1} = u_{n_2-1}^2 \text{Disc} \left( f_{n_2-1} \right) \text{Disc} \left( f_{n_1} \right),$$

which is a square if and only if $\text{Disc} \left( f_{n_1} \right) \text{Disc} \left( f_{n_2-1} \right)$ is a square.

If $n_2 - 1 > n_1$ we continue the same process as above, that is, if $\text{Disc} \left( f_{n_1} \right) \text{Disc} \left( f_{n_2-1} \right)$ is a square then it is a square for the specialisation $u_{n_2-1} = 0$. From (2.2), we get

$$\text{Disc} \left( f_{n_2-1} \right) = u_{n_2-2}^2 \text{Disc} \left( f_{n_2-2} \right).$$

We apply this reduction until we obtain $n_2 - k = n_1 + 1$, that is for $k = n_2 - n_1 - 1$ times. Putting everything together we get that if $D_{n_1,n_2}$ is a square polynomial, then so is

$$\left( \prod_{k=1}^{n_2-n_1-1} u_{n_2-k} \right)^2 \text{Disc} \left( f_{n_1} \right) \text{Disc} \left( f_{n_2} \right),$$

and thus $\text{Disc} \left( f_{n_1} \right) \text{Disc} \left( f_{n_2} \right)$ is also a square polynomial. Using (2.1), this is equivalent with that $\text{Disc} \left( f_{n_1} f_{n_2} \right)$ is a square polynomial. Suppose that $\text{Disc} \left( f_{n_1} f_{n_2} \right)$ is a square polynomial in $u_0, \ldots, u_{n_1+1}$, then by Lemma 2.1, the number of irreducible factors of $f_{n_1} f_{n_2}$, which is exactly 2 (as $f_{n_1}$ and $f_{n_2}$ are irreducible), is congruent to 1 modulo 2 (as $2n_1 + 1$ is the degree of $f_{n_1} f_{n_2}$); this is not true. We finally conclude that $D_{n_1,n_2}$ is not a square polynomial.

We now assume that $\nu \geq 3$ and the statement true for $D_{n_1,\ldots,n_j}$ for any $j \leq \nu - 1$. If $D_{n_1,\ldots,n_\nu}$ is a square polynomial, then using exactly the same reductions as the above (using (2.2)), but $n_\nu - n_{\nu-1}$ times, we obtain that

$$\left( \prod_{k=1}^{n_\nu-n_{\nu-1}} u_{n_\nu-k} \right)^2 \text{Disc} \left( f_{n_{\nu-1}} \right) D_{n_1,\ldots,n_{\nu-1}}$$

$$= \left( \prod_{k=1}^{n_\nu-n_{\nu-1}} u_{n_\nu-k} \text{Disc} \left( f_{n_{\nu-1}} \right) \right)^2 D_{n_1,\ldots,n_{\nu-2}}$$

is also a square polynomial. Thus, $D_{n_1,\ldots,n_{\nu-2}}$ is a square polynomial, which contradicts the induction hypothesis. Now, we conclude the proof. \[\square\]
Remark 5.3. In the second paragraph of the above proof, we actually prove that for any polynomial $f \in \mathbb{F}_q[X]$ of degree greater than 1, its discriminant is not a square polynomial as a multivariate polynomial in the coefficients of $f$ (treated as variables).

Now, we are ready to get a better upper bound of $I_N$ when $q$ is very large compared to $N$.

Theorem 5.4. For any integer $N \geq 2$, the number $I_N$ of consecutive irreducible polynomial sequences of length $N$ satisfies

$$I_N < q^{N+1}/2^{N-1} + 2 \left( N^2 q^{N+1/2} + N^4 q^N \right)/3.$$

Proof. Let $\{f_n\}$ be a consecutive polynomial sequence of length $N$ defined by a sequence $\{u_n\}$ in (1.1). If $f_2, \ldots, f_N$ are irreducible polynomials, by Lemma 2.1, we know that

$$\chi(\text{Disc}(f_n)) = (-1)^{n+1}, \quad n = 2, 3, \ldots, N,$$

where $\chi$ is the multiplicative quadratic character of $\mathbb{F}_q$.

Thus, we have

$$I_N \leq \sum_{u_0, \ldots, u_N \in \mathbb{F}_q} \frac{1}{2^{N-1}} \prod_{n=2}^{N} \left( 1 - (-1)^n \chi(\text{Disc}(f_n)) \right)$$

$$= \frac{1}{2^{N-1}} \sum_{u_0, \ldots, u_N \in \mathbb{F}_q} \prod_{n=2}^{N} \left( 1 - (-1)^n \chi(\text{Disc}(f_n)) \right).$$

(5.4)

Just expanding the product in (5.4), we obtain $2^{N-1} - 1$ character sums of the shape

$$(-1)^{n_1 + \cdots + n_\nu} \sum_{u_0, \ldots, u_{n_\nu} \in \mathbb{F}_q} \chi \left( \prod_{j=1}^{\nu} \text{Disc}(f_{n_j}) \right), \quad 2 \leq n_1 < \cdots < n_\nu \leq N,$$

with $\nu \geq 1$ and one trivial sum that equals $q^{N+1}$ (corresponding to the terms 1 in the product of (5.4)).

So, the trivial summand of the right-hand side in (5.4) is equal to $q^{N+1}/2^{N-1}$. We view each $\prod_{j=1}^{\nu} \text{Disc}(f_{n_j})$ as a multivariate polynomial in $u_0, u_1, \ldots, u_{n_\nu}$, whose degree is equal to

$$\sum_{j=1}^{\nu} (2n_j - 2) = 2(n_1 + \cdots + n_\nu) - 2\nu \leq N^2.$$

Note that if we associate values to $n_\nu$ variables among $u_0, u_1, \ldots, u_{n_\nu}$, the resulted polynomial might be a square polynomial in the remaining variable up to a multiplicative constant. By Lemma 5.2, we know that
\( \prod_{j=1}^{\nu} \text{Disc} (f_{n_j}) \) is not a square polynomial in \( u_0, u_1, \ldots, u_{n_{\nu}} \), and thus by Lemma 2.2 we obtain that there exists \( i \in \{0, 1, \ldots, n_{\nu}\} \) such that \( \prod_{j=1}^{\nu} \text{Disc} (f_{n_j}) \) is not a square polynomial in \( u_i \) up to a multiplicative constant for all but at most \( N^4q^{n_{\nu}-1} \) values of \( u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{n_{\nu}} \in \mathbb{F}_q \).

We use Lemma 2.3 for those specialisations for which \( \prod_{j=1}^{\nu} \text{Disc} (f_{n_j}) \) is not a square polynomial in \( u_i \) up to a constant; and for the rest, we use the trivial bound. Thus, we deduce that

\[
\left| \sum_{u_0, \ldots, u_{n_{\nu}} \in \mathbb{F}_q} \chi \left( \prod_{j=1}^{\nu} \text{Disc} (f_{n_j}) \right) \right| \leq N^2q^{n_{\nu}+1/2} + N^4q^{n_{\nu}}.
\]

Note that the sum of the terms \( q^{n_{\nu}} \) through all the choices \( 2 \leq n_1 < \cdots < n_{\nu} \leq N \) is exactly equal to

\[
\sum_{n=2}^{N} 2^{n-2}q^n < \frac{q}{2q-1} \cdot 2^{N-1}q^N < \frac{1}{3} \cdot 2^Nq^N.
\]

So, regarding (5.4) and putting everything together, we obtain

\[
I_N < q^{N+1}/2^{N-1} + 2 \left( N^2q^{N+1/2} + N^4q^N \right) /3,
\]

which completes the proof. \( \square \)

We remark that when \( N \geq 4 \) and \( q \geq 4^{1+N/5}N^4 \), for the three summation terms in the bound of Theorem 5.4 each of them is not greater than one third of the bound \( 2^{-N/5}q^{N+1} \) in Theorem 5.1, so Theorem 5.4 is better than the first bound in Theorem 5.1.

5.3. **Heuristic upper bound.** Here, we present a heuristic estimate for \( I_N, N \geq 2 \), which is compatible with numerical data and implies an upper bound for \( L(q) \) (defined in Section 1.1).

For any integer \( n \geq 1 \), let \( \pi_q^*(n) \) be the number of monic irreducible polynomials of degree \( n \) over \( \mathbb{F}_q \) whose coefficients are all non-zero. Then for \( n \geq 2 \), given a polynomial of degree \( n \) and without zero coefficients, the probability that it is irreducible is

\[
\frac{(q-1)\pi_q^*(n)}{(q-1)^n+1} = \frac{\pi_q^*(n)}{(q-1)^n}.
\]

So, when fixing non-zero coefficients \( u_0, u_1, \ldots, u_{n-1} \) and varying the leading coefficient \( u_n \neq 0 \), we seemingly can get

\[
\frac{(q-1)\pi_q^*(n)}{(q-1)^n} = \frac{\pi_q^*(n)}{(q-1)^{n-1}}
\]
irreducible polynomials of degree $n$. But this ignores the fact that these $q - 1$ polynomials are not quite random, because as a random variable $u_n$ can be chosen to be zero. Thus, we need to introduce a correction factor $(q - 1)/q$, and so we probably can get

$$
\frac{\pi_q^*(n)}{(q - 1)^{n-1}} \cdot \frac{q - 1}{q} = \frac{\pi_q^*(n)}{q(q - 1)^{n-2}}
$$

irreducible polynomials of degree $n$. Hence, one could expect that

$$
(5.6) \quad I_N = (q - 1)^2 \prod_{n=2}^{N} \frac{\pi_q^*(n)}{q(q - 1)^{n-2}},
$$

where the factor $(q - 1)^2$ comes from $I_1$.

Note that (5.1) implies a heuristic upper bound for $\pi_q^*(n)$:

$$
(5.7) \quad \pi_q^*(n) \leq \frac{q(q - 1)^{n-1}}{n},
$$

because the constant term is already non-zero. In fact, regarding the prime number theorem for polynomials (see [28, Theorem 2.2]), the upper bound in (5.7) is also a good approximation of $\pi_q^*(n)$. Thus, we can get the following heuristic upper bound for $I_N$:

$$
(5.8) \quad I_N \leq \frac{(q - 1)^{N+1}}{N!}.
$$

Figure 1 illustrates the comparison between the number of consecutive irreducible polynomial sequences and the estimate (5.8) for $q = 17$, where the horizontal axis represents $N$. Figure 1 suggests that (5.8) is also a good approximation of $I_N$, and so is (5.6).

Using the standard estimate on the factorials (for example, see [27]):

$$
N! > \sqrt{2\pi N} \left(\frac{N}{e}\right)^N,
$$

where $e$ is the base of the natural logarithm, we obtain

$$(q - 1)^{N+1}/N! < 1$$

when $N \geq 3g$. Thus, under the heuristic upper bound (5.8) we have $I_N = 0$ for $N \geq 3g$, and so

$$
(5.9) \quad L(q) < 3g.
$$

This is compatible with Table 1.

Hence, heuristically there is no consecutive irreducible polynomial sequence of infinite length over $\mathbb{F}_q$.

Finally, we want to point out that in the above heuristic we assume in some sense that irreducible polynomials without zero coefficients are equidistributed.
Figure 1. Comparison between $I_N$ (dots) and the upper bound in (5.8) (stars) for $q = 17$.

Table 1. Values of $L(q)$ for small $q$

<table>
<thead>
<tr>
<th>$q$</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>17</th>
<th>19</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L(q)$</td>
<td>3</td>
<td>6</td>
<td>8</td>
<td>16</td>
<td>23</td>
<td>29</td>
<td>38</td>
<td>41</td>
<td>47</td>
</tr>
</tbody>
</table>

5.4. Lower bounds. In the proof of [5, Theorem 1.2] (noticing Theorem 1.1 and Equation (3.5) there), the authors actually have given an asymptotic formula for $I_N$ with respect to $q$:

\[
I_N = \frac{q^{N+1}}{N!} + O_N \left( q^{N+1/2} \right),
\]

where, in particular, the implied constant can be easily computed for small $N$.

The approach in [5], using reciprocal polynomials, is first fixing a consecutive irreducible sequence $u_0, \ldots, u_{n-1}$, and then searching $u_n$ such that the polynomial $(u_0X^{n-1} + \cdots + u_{n-1})X + u_n$ is irreducible. In this section, our approach is to searching $u_n$ such that the polynomial $u_0 + \cdots + u_{n-1}X^{n-1} + u_nX^n$ is irreducible. This enable us to obtain new explicit lower bounds for $I_N$ when $N$ is small, which are better than those implied in [5].

We first remark that the number of consecutive irreducible sequences of fixed length is divisible by $(q-1)^2$. Indeed, let $\{f_n\}, n \geq 1$, be a consecutive polynomial sequence defined by $\{u_n\}, n \geq 0$, in (1.1). Then, we know that for any $n \geq 1$ and $a \in \mathbb{F}_q^n$, $f_n$ is irreducible if and only if $f_n(aX)$ or $af_n(X)$ is irreducible. Thus, $\{u_n\}$ is a consecutive irreducible
sequence if and only if \( \{a^n u_n\} \) or \( \{au_n\} \) is a consecutive irreducible sequence. In particular, when \( \{u_n\} \) is a consecutive irreducible sequence, all these \((q - 1)^2\) consecutive sequences \(\{ab^n u_n\}\), where \(a, b\) run over \(\mathbb{F}_q^*\), are distinct and irreducible.

Next, we give some estimates for such sequences of length 2, 3 and 4 which are compatible with the above heuristic.

**Theorem 5.5.** The following hold:

1. \( I_2 = \frac{1}{2}(q - 1)^3 \);
2. \( I_3 \geq \frac{1}{12}(q - 1)^2(q - 7)(2q - 3\sqrt{q} - 5) \) when \( q > 7 \).

**Proof.** (1) As we have noted at the beginning of this section, if \( \{u_n\} \) is a consecutive irreducible sequence, \( \{ab^n u_n\} \) are all distinct and consecutive irreducible sequences when \(a, b\) run through \(\mathbb{F}_q^*\). Therefore, we can fix \(u_0 = u_1 = 1\). A quadratic polynomial \(u_2 X^2 + X + 1\) is irreducible if and only if the discriminant is not a quadratic residue in \(\mathbb{F}_q^*\). Since the discriminant is \(1 - 4u_2\), we have

\[
I_2 = \frac{\#\{u_2 \in \mathbb{F}_q^* \mid 1 - 4u_2 \text{ is not a quadratic residue of } \mathbb{F}_q\}}{(q - 1)^2},
\]

which, by noticing that 1 is not contained in the image of \(1 - 4u_2\) for \(u_2 \in \mathbb{F}_q^*\), in fact is equal to \((q - 1)/2\). This gives us the desired result.

(2) For \( I_3 \), first fix \(u_2\), and \(u_1 = u_0 = 1\) such that the polynomial \(u_2 X^2 + X + 1\) is irreducible, and we proceed to count how many of the polynomials

\[
f_3 = u_3 X^3 + u_2 X^2 + X + 1,
\]

are irreducible when \(u_3\) runs over \(\mathbb{F}_q^*\). The first thing to notice is that if \(u_3 \neq u_3'\), then

\[
gcd(u_3 X^3 + u_2 X^2 + X + 1, u_3' X^3 + u_2 X^2 + X + 1) = gcd(u_3 X^3 + u_2 X^2 + X + 1, (u_3' - u_3) X^3) = 1,
\]

which means that these two polynomials have different irreducible factors. By simple calculation, the discriminant of \(f_3 = u_3 X^3 + u_2 X^2 + X + 1\) is equal to

\[
Disc(f_3) = -27u_3^2 + (18u_2 - 4)u_3 - 4u_2^3 + u_2^2.
\]

Notice that this discriminant can be viewed as a polynomial in \(u_3\), and it has no multiple roots if and only if its discriminant

\[
-432u_2^3 + 432u_2^2 - 144u_2 + 16 \neq 0.
\]

Let \(\chi\) be the multiplicative quadratic character of \(\mathbb{F}_q\). Now, under the condition (5.13), which means that \(Disc(f_3)\) is not a square polynomial
in $u_3$ up to a multiplicative constant, we estimate the number of $u_3$ such that $\text{Disc}(f_3)$ is a square element in the following:

$$\# \{ u_3 \in \mathbb{F}_q^* \mid \chi(\text{Disc}(f_3)) = 1 \} \geq \# \{ u_3 \in \mathbb{F}_q \mid \chi(\text{Disc}(f_3)) = 1 \} - 1$$

$$= \frac{1}{2} \sum_{u_3 \in \mathbb{F}_q} (1 + \chi(\text{Disc}(f_3))) - 1$$

$$\geq \frac{q}{2} - \frac{1}{2} \sum_{u_3 \in \mathbb{F}_q} \chi(\text{Disc}(f_3)) - 1$$

$$\geq \frac{q}{2} - \sqrt{q}/2 - 1,$$

where the last inequality comes from Lemma 2.3. Thus, using Lemma 2.1, we get that under the condition (5.13), for at least $q/2 - \sqrt{q}/2 - 1$ values of $u_3$ the polynomial $f_3$ has an odd number of distinct irreducible factors.

If polynomial $f_3$ is reducible and has an odd number of distinct irreducible factors, it must have three distinct roots in $\mathbb{F}_q$. By (5.12), there are at most $(q - 1)/3$ such polynomials $f_3$. But here we can get a better estimate. Assume that $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}_q^*$ are three distinct roots of $f_3$, namely

$$f_3 = u_3X^3 + u_2X^2 + X + 1 = u_3(X - \alpha_1)(X - \alpha_2)(X - \alpha_3).$$

So, we get

$$(5.14) \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = -u_2/u_3, \\
\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = 1/u_3, \\
\alpha_1\alpha_2\alpha_3 = -1/u_3. \end{cases}$$

Put $\beta_i = \alpha_i^{-1}, i = 1, 2, 3$. By (5.14), we have

$$(5.15) \begin{cases} \beta_1 + \beta_2 + \beta_3 = -1, \\
\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3 = u_2. \end{cases}$$

Note that $u_2$ is fixed. If we fix $\beta_1$ (that is, $\alpha_1$), then $\beta_2$ and $\beta_3$ (that is, $\alpha_2$ and $\alpha_3$) are uniquely determined by (5.15). Thus, by symmetry there are at most $(q - 1)/(3!)$ such polynomials $f_3$. So, it happens for at most $(q - 1)/6$ values of $u_3$, and thus at least

$$q/2 - \sqrt{q}/2 - 1 - (q - 1)/6 = q/3 - \sqrt{q}/2 - 5/6$$

values of $u_3$ give an irreducible polynomial $f_3$ if $u_2$ satisfies (5.13).

In view of (5.11) and (5.13), there are at least $I_3/(q - 1)^2 - 3$ choices of $u_2$ such that the polynomial $u_2X^2 + X + 1$ is irreducible and the
condition (5.13) is not satisfied. Thus, we deduce that
\[ I_3 \geq (I_2/(q-1)^2 - 3)(q/3 - \sqrt{q}/2 - 5/6)(q-1)^2, \]
which, together with the first result (1), implies the desired result. Note that, to ensure each part on the right-hand side of (5.16) is positive, we need \( q > 7 \).

The strategy to estimate \( I_4 \) is the same as in the proof of Theorem 5.5, but the deductions are much more complicated.

**Theorem 5.6.** Assume that the characteristic of \( \mathbb{F}_q \) is not 2 or 3. Then, for \( q \geq 23 \) we have
\[ I_4 \geq \frac{1}{48}(q-1)^2(q-22\sqrt{q}-4)(2q^2-3q\sqrt{q}-25q+21\sqrt{q}+41). \]

**Proof.** The lower bound for \( I_4 \) can be found in a very similar way. Again, we fix the values \( u_3, u_2 \) such that the polynomials \( u_2X^2 + X + 1 \) and \( u_3X^3 + u_2X^2 + X + 1 \) are irreducible, and consider the polynomial
\[ f_4 = u_4X^4 + u_3X^3 + u_2X^2 + X + 1, \quad u_4 \in \mathbb{F}_q^*. \]
In this case, the discriminant is equal to
\[ \text{Disc} (f_4) = 256u_4^3 - 192u_3^2u_3 - 128u_3^2u_2^2 + 144u_2^3u_2 - 27u_4^2 \]
\[ + 144u_4u_3^2u_2 - 6u_4u_3^2 - 80u_4u_3u_2^2 + 18u_4u_3u_2 + 16u_4u_2^3 \]
\[ - 4u_4u_2^3 - 27u_3^2 + 18u_2^3 - 4u_3^2 - 4u_2^2 - u_2^2. \]
In view of the term \( 256u_4^3 \) in (5.18), \( \text{Disc} (f_4) \) is not a square polynomial in \( u_4 \) up to a multiplicative constant. Using Lemma 2.1, when \( \text{Disc} (f_4) \neq 0 \) (that is, \( f_4 \) is square-free), we have that \( \text{Disc} (f_4) \) is not a square element if and only if either \( f_4 \) is irreducible, or it has two different non-zero roots in \( \mathbb{F}_q \) and it is divisible by an irreducible polynomial of degree 2.

Let \( \chi \) be the multiplicative quadratic character of \( \mathbb{F}_q \). We first count the number of values of \( u_4 \) such that \( \text{Disc} (f_4) \) is non-zero and is not a square element in \( \mathbb{F}_q \). This number is at least
\[ \frac{1}{2} \sum_{u_4 \in \mathbb{F}_q} (1 - \chi(\text{Disc} (f_4))) - 1 \]
\[ \geq \frac{q}{2} - \frac{1}{2} \left| \sum_{u_4 \in \mathbb{F}_q} \chi(\text{Disc} (f_4)) \right| - 1 \]
\[ \geq q/2 - \sqrt{q} - 1, \]
where the last inequality follows from Lemma 2.3. Note that the term 
"−1" in (5.19) follows from the fact that we originally want $u_4 \in \mathbb{F}_q^*$.

Now, for our purpose, it remains to estimate the number of values
of $u_4$ such that the polynomials $f_4$ has the form
\begin{equation}
(5.20) \quad f_4 = (X + a)(X + b)(cX^2 + dX + e),
\end{equation}
for some $a, b, c, d, e \in \mathbb{F}_q$ with $abc \neq 0$ and $a \neq b$, where the polynomial $cX^2 + dX + e$ is irreducible. If there is no value of $u_4$ satisfying (5.20), then this will yield a better bound for $I_4$, which is
$$I_4 \geq I_3(q/2 - \sqrt{q} - 1).$$

In the following, we suppose that there indeed exist values of $u_4$ such
that $f_4$ has the form (5.20). In fact, it is equivalent to count the number
of values of $u_4$ such that the reciprocal polynomial of $f_4$,
$$g_4 = X^4 + X^3 + u_2X^2 + u_3X + u_4,$$
has two different non-zero roots in $\mathbb{F}_q$ and a quadratic irreducible factor. Replace $X$ by $(Y - 1/4)$ in $g_4$, we get
$$h_4 = Y^4 + \alpha Y^2 + \beta Y + \gamma,$$
where
\begin{equation}
(5.21) \quad \begin{cases}
\alpha = u_2 - 3/8, \\
\beta = u_3 - u_2/2 + 1/8, \\
\gamma = u_4 - u_3/4 + u_2/16 - 3/256.
\end{cases}
\end{equation}

Then, the cubic resolvent of $h_4$ is
$$R_4 = Y^3 + 2\alpha Y^2 + (\alpha^2 - 4\gamma)Y - \beta^2.$$ 
Since $h_4$ has two roots in $\mathbb{F}_q$, the sum of these two roots is also in $\mathbb{F}_q$. By Lemma 2.6 this means that $R_4$ has a root $y$ which is a square element in $\mathbb{F}_q$, where we need to use the assumption that the characteristic of $\mathbb{F}_q$ is not 2 or 3. If $\beta = u_3 - u_2/2 + 1/8 \neq 0$, then $y$ is non-zero. Note that the number of values of $(u_2, u_3)$ such that $\beta = 0$ does not exceed the number of all possible choices of $u_2$ (such that the polynomial $u_2X^2 + X + 1$ is irreducible), so we have
\begin{equation}
(5.22) \quad \#\{(u_2, u_3) \mid \beta = 0\} \leq I_2/(q - 1)^2,
\end{equation}
which implies that
\begin{equation}
(5.23) \quad \#\{(u_2, u_3) \mid \beta \neq 0\} \geq I_3/(q - 1)^2 - I_2/(q - 1)^2,
\end{equation}
Now, assume that $\beta = u_3 - u_2/2 + 1/8 \neq 0$. Since $y \neq 0$ and
$$y^3 + 2\alpha y^2 + (\alpha^2 - 4\gamma)y - \beta^2 = 0,$$
we obtain
\begin{equation}
(5.24) \quad u_4 = (y^3 + 2\alpha y^2 + (\alpha^2 + u_3 - u_2/4 + 3/64)y - \beta^2)/(4y).
\end{equation}
So, for each value of $u_4$ satisfying (5.20), there exists a square element $y$ in $F_q^*$ such that $u_4$ can be recovered by (5.24). Substituting (5.24) into (5.18), we get
\[
\text{Disc } (f_4) = t/(4y)^3,
\]
where $t$ is a polynomial in $y$ and has coefficients only depending on $u_2, u_3$. Note that as a polynomial in $y$, the leading term of $t$ is $256y^9$, and so $\deg t = 9$.

Thus, under the condition $\beta \neq 0$, the number of values of $u_4$ satisfying (5.20) is at most
\[
\frac{1}{4} \sum_{y \in F_q^*} (1 + \chi(y)) (1 - \chi(t/(4y)^3))
\]
\[
= \frac{1}{4} \sum_{y \in F_q^*} (1 + \chi(y)) (1 - \chi((4y)^9t))
\]
\[
= \frac{1}{4} \sum_{y \in F_q^*} (1 + \chi(y)) (1 - \chi(ty)),
\]
where the last identity comes from the fact that $q$ is odd and $\chi$ is a multiplicative character. Notice that there already exists value of $u_4$ such that $\text{Disc } (f_4)$ is not a square element, this means that there exists value of $y$ such that $ty$ is not a square element. We also note that the leading term of $ty$ is a square (which is $256y^9$). So, we must have that both $ty$ and $ty^2$ are not a square polynomial in $y$ up to a constant. Besides, as a polynomial in $y$, each of them has at most 10 distinct roots. Now as before, employing Lemma 2.3, we get
\begin{equation}
(5.25) \quad \frac{1}{4} \sum_{y \in F_q} (1 + \chi(y)) (1 - \chi(ty)) \leq \frac{1}{4} \sum_{y \in F_q} (1 + \chi(y)) (1 - \chi(ty)) \leq \frac{q}{4} + \frac{1}{4} \left| \sum_{y \in F_q} \chi(ty) \right| + \frac{1}{4} \left| \sum_{y \in F_q} \chi(ty^2) \right| \leq q/4 + 9\sqrt{q}/2,
\end{equation}
where one should note that $t$ is a polynomial in $y$ and $\sum_{y \in F_q} \chi(y) = 0$. 
Therefore, combining (5.19) with (5.25), fix $u_2, u_3$ such that $\beta = u_3 - u_2/2 + 1/8 \neq 0$, the number of values of $u_4$ such that $f_4$ is irreducible is at least

\[ \frac{q}{2} - \sqrt{q} - 1 - (q/4 + 9\sqrt{q}/2) = q/4 - 11\sqrt{q}/2 - 1. \]

So, in view of (5.23), we deduce that

\[ I_4 \geq \left( \frac{I_3}{(q - 1)^2} - \frac{I_2}{(q - 1)^2} \right) (q/4 - 11\sqrt{q}/2 - 1)(q - 1)^2, \]

which, together with Theorem 5.5, concludes the proof. Note that, to ensure $q/4 - 11\sqrt{q}/2 - 1 > 0$, we need $q \geq 23$. \qed

We want to remark that the method we use here will become much more complicated in bounding $I_N$ explicitly for $N \geq 5$, and thus it might be not applicable.

6. Open Questions

The results in this paper about consecutive polynomial sequences give some insights to understand their factorization feature, but definitely there is a long way ahead. Here, we pose some related questions which might be of interest to be studied. Certainly, there are many other things remaining to be explored.

Let $\{f_n\}, n \geq 1$, be a consecutive polynomial sequence defined by a sequence $\{u_n\}, n \geq 0$, in (1.1).

**Question 6.1.** Whether does there exist consecutive irreducible sequence $\{u_n\}$ of infinite length?

In view of the heuristic upper bound of $L(q)$ in (5.9) and Table 1, the answer to this question seems to be no for finite fields. Unfortunately, this question seems to be beyond reach. Thus, we propose the following problem.

**Question 6.2.** Construct consecutive polynomial sequence $\{f_n\}$ such that there are infinitely many irreducible polynomials in the sequence.

Here, aside from the existence, we also ask for closed formulas to construct such sequences. Results in [23] almost show the existence of such sequences and that the irreducible elements are quite scattered, because in our case we need that all the coefficients are non-zero. At the end of Section 3, when $q$ is an odd power of $p$ and under Artin’s conjecture, the sequence with all the coefficients equal to 1 contains infinitely many irreducible polynomials. Here, what we want is an unconditional result.

**Question 6.3.** Is the lower bound for sequence $\{f_n\}$ of infinite length in Corollary 3.3 optimal?
The specific example showed in Theorem 3.4 suggests that maybe the lower bound in Corollary 3.3 can be improved.

**Question 6.4.** Find upper bound of $\omega(f_{m+1} \cdots f_{m+H})$ for $H$ consecutive terms.

Note that for any polynomial $g(X) \in \mathbb{F}_q[X]$ of degree $n \geq 2$, to get large $\omega(g)$ it is required that $g$ only has irreducible factors of low degree. Then, it is easy to check that $\omega(g) \leq c(q)n / \log n$, where $c(q)$ is some function with respect to $q$. Thus, for integer $n \geq 2$ we have $\omega(f_n) \leq c(q)n / \log n$. Now, the problem is whether we can get better upper bounds for $\omega(f_{m+1} \cdots f_{m+H})$.

We say that a term $f_n$ has a **primitive irreducible divisor** if there exists an irreducible polynomial $g \in \mathbb{F}_q[X]$ such that $g \mid f_n$, but $g \nmid f_i$ for $i < n$.

**Question 6.5.** Can we show that almost all terms in $\{f_n\}$ have primitive irreducible divisors?

This question is a natural analogue of the study on the existence of primitive prime divisors in sequences of integers (such as linear recurrences of integers [3, 8], and sequences generated in arithmetic dynamics [9, 26]).

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