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The Dattoli–Srivastava conjectures concerning generating functions involving the harmonic numbers

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A R T I C L E   I N F O

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A B S T R A C T

In a recent paper Dattoli and Srivastava [3], by resorting to umbral calculus, conjectured several generating functions involving harmonic numbers. In this sequel to their work our aim is to rigorously demonstrate the truth of the Dattoli–Srivastava conjectures by making use of simple analytical arguments. In addition, one of these conjectures is stated and proved in more general form.

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1. Introduction

The familiar harmonic numbers are defined as follows:

\[ H_0 := 0, \quad H_n := \sum_{k=1}^{n} \frac{1}{k} \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}) \] (1.1')

or, equivalently, by

\[ H_n = \psi(n + 1) - \psi(1) = \psi(n + 1) + \gamma \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) \] (1.1'')

in terms of the digamma function \( \psi(z) \) (given as the logarithmic derivative of the classical Gamma function \( \Gamma(z), z \in \mathbb{C} \setminus \mathbb{Z}_0 := \{0, -1, -2, \ldots\} \), \( \psi(z) = d \log \Gamma(z)/dz \) [1, p. 258, Entry 6.3.1]; [2, p. 13]) and the Euler-Mascheroni constant \( \gamma \).

Harmonic numbers were studied in antiquity and occur in such diverse fields as analysis of algorithms in computer science, various branches of number theory in mathematics and elementary particle physics and theoretical physics. In a recent paper, Dattoli and Srivastava [3], by utilizing an interesting approach based on the umbral calculus formalism that also includes numerical computations, proposed several generating functions involving harmonic numbers (see Eqs. (2.4), (2.7) and (2.8) below). These generating functions are rather interesting illustrations of results in “experimental mathematics” and should, therefore, be considered as hypotheses, which have passed a severe numerical verification.

In this sequel to the work of Dattoli and Srivastava [3] our aim is to rigorously demonstrate the truth of the conjectured relations by making use of simple analytical arguments.
2. Preliminaries and statement of main results

Observe that, in what follows, as customary, we set an empty sum to be zero. The Bessel functions of the first kind and order \( \nu \), denoted as \( J_\nu(z) \), are solutions of the Bessel differential equation which have the series expansion [1, p. 360, Entry 9.1.10]

\[
J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\nu+1)} \left( \frac{z}{2} \right)^{2n+\nu} \quad (\nu \in \mathbb{C}).
\]  

(2.1)

Throughout the text, \( _pF_q \), is the generalized hypergeometric function in one variable of order \((p, q)\), which is, as usual, defined by means of the hypergeometric series (see [2, p. 52] and [4, Vol. 3, Chapter 7])

\[
_{p}F_{q}\left[\begin{array}{c} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \\ z \end{array}\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}, \quad (p, q \in \mathbb{N}_0; z, \alpha_1, \ldots, \alpha_p \in \mathbb{C}; \beta_1, \ldots, \beta_q \in \mathbb{C} \setminus \mathbb{Z}_0)
\]

(2.2)

whenever this series converges and elsewhere by analytic continuation. Here \((\alpha)_n\) denotes the Pochhammer symbol defined (for \( \alpha \in \mathbb{C} \)) as \((\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)\).

The series defining \( _pF_q \) converges for all values of \( z \) and for all choices of the (numerator and denominator) parameters involved when \( p \leq q \). If \( p = q + 1 \) it converges when \(|z| < 1\) and for all choices of the parameters, when \( z = 1 \) if \( \Re(\beta_1 + \cdots + \beta_q - \alpha_1 - \cdots - \alpha_p) > 0 \) and when \( z = -1 \) if \( \Re(\beta_1 + \cdots + \beta_q - \alpha_1 - \cdots - \alpha_q) > -1 \).

The entire exponential integral \( \text{Ein}(z) \) is given by ([1, p. 228, footnote number 3]; [5, p. 516, Eq. (14)]; see also the references cited in [5])

\[
\text{Ein}(z) := \int_0^z \frac{1 - e^{-t}}{t} \, dt = -\sum_{n=1}^{\infty} \frac{(-z)^n}{n \cdot n!} = z_2F_2\left[\begin{array}{c} 1, 1; \\ 2, 2; \\ -z \end{array}\right] \quad (|z| < \infty)
\]

(2.3)

and is related as follows to both the relatively more familiar exponential integral \( E_1(z) \) and the complementary incomplete Gamma function \( \Gamma(0, z) \)

\[
\text{Ein}(z) = E_1(z) + \ln z + \gamma = \Gamma(0, z) + \ln z + \gamma,
\]

where \( \gamma \) is the Euler–Mascheroni constant and \( E_1(z) \) and \( \Gamma(0, z) \) are defined by

\[
E_1(z) := \int_z^{\infty} \frac{e^{-t}}{t} \, dt =: \Gamma(0, z).
\]

Our main results are as follows.

**Proposition 1.** Let \( \text{Ein}^{(n)}(z) \) \((n \in \mathbb{N}_0)\), \( \text{Ein}^{(0)}(z) := \text{Ein}(z) \) be the \( n \)th derivative of the function \( \text{Ein}(z) \) defined in (2.3) and let \( H_n \) be the harmonic numbers. Then

\[
\sum_{n=0}^{\infty} \frac{z^n}{n!} H_n = e^z \text{Ein}(z).
\]

(2.5)

In particular, we have

\[
\sum_{n=0}^{\infty} \frac{z^n}{n!} H_n = e^z \text{Ein}(z).
\]

**Proposition 2.** In terms of the \( _2F_2 \) hypergeometric function, we have

\[
\sum_{n=0}^{\infty} \frac{z^n}{n!} H_n = ze^z \sum_{k=1}^{\infty} \frac{1}{k} \, _2F_2\left[\begin{array}{c} 1, \\ 2, \frac{k}{2} + 1; \\ -z \end{array}\right] \quad (|z| < \infty; \ell \in \mathbb{N}).
\]

(2.6)

As an immediate consequence of **Proposition 2**, we have the following:

**Corollary 1.** We have

\[
\sum_{n=0}^{\infty} \frac{z^{2n}}{n!} H_{2n} = e^z z^2 \sum_{k=1}^{\infty} \frac{1}{k} \, _2F_2\left[\begin{array}{c} 1, 1; \\ 2, \frac{k}{2} + 1; \\ -z \end{array}\right] \quad (|z| < \infty).
\]

(2.7)

**Proposition 3.** Let \( L_n(x; [H]) \) be the polynomials

\[
L_n(x; [H]) := n! \sum_{k=0}^{n} \frac{(-x)^n H_{n-k}}{(n-k)! (k)!^2}
\]
where \( H_n \) are the harmonic numbers. In terms of the entire exponential integral and Bessel function of the first kind and of order zero, \( \text{Ein}(z) \) and \( J_0(z) \), we have

\[
\sum_{n=0}^{\infty} \frac{z^n}{n!} \text{Ein}(z) = e^z J_0(2\sqrt{z}) \text{Ein}(z).
\]

(2.8)

**Remark 1.** The exponential generating function for the harmonic numbers \( H_n \) given by (2.5) is known [3, p. 687]. Dattoli and Srivastava conjectured the summation formulae (2.4) and (2.8) and they gave them in slightly different form (the function \( \Phi(z) \) is used instead \( \text{Ein}(z) \)), however note that \( \text{Ein}(z) = -\Phi(z) \) [3, p. 687, Eq. (8) and p. 698, Eq. (25)]. The summation (2.6), in such generality, was not investigated by Dattoli and Srivastava and it appears to be new, but they conjectured it for the special case when \( \ell = 2 \) (see (2.7)) [3, p. 688, Eq. (13)]. Note that although (2.7) and Eq. (13) in [3] are different in form, they are equivalent and it can be shown by equalizing the right hand sides of these equations.

3. Proof of the results

In this section, short simple direct proofs of the above-given results are provided.

**Proof of Proposition 1.** First, we deduce the following formula

\[
\sum_{n=0}^{\infty} \frac{z^n}{n!} (n + \lambda - \lambda) = e^z \Phi(z, \lambda) \quad \text{with} \quad \Phi(z, \lambda) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} (\lambda)_n,
\]

(3.1)

which is obtained in different way in [3, p. 691, Eq. (43)] and it is here given and proved for the sake of completeness. To prove (3.1) we shall need the next result (see, e.g., [6, p. 126, Entry (6.6.34)] and the references cited there)

\[
\psi(x + y) - \psi(x) = -\sum_{k=1}^{\infty} \frac{(-y)_k}{k (x)_k} \quad \text{with} \quad \psi(x) = \ln(x).
\]

(3.2)

as well as the simple summation

\[
\sum_{n=0}^{\infty} (-n)_k \frac{z^n}{n!} = (-z)^k e^z \quad (z \in \mathbb{C}; \ k \in \mathbb{N}_0).
\]

(3.3)

Observe that the expansion (3.3) is readily available

\[
\sum_{n=0}^{\infty} (-n)_k \frac{z^n}{n!} = (-1)^k \sum_{n=0}^{\infty} \frac{n!}{(n-k)!} \frac{z^n}{n!} = (-1)^k \sum_{n=0}^{\infty} \frac{z^n}{(n-k)!} = (-z)^k e^z,
\]

with the aid of

\[
(-1)_k = (-1)^k \frac{l!}{(l-k)!} \quad (l \in \mathbb{N}_0; k = 0, 1, \ldots, l).
\]

Now, in view of (3.2) and (3.3), the proposed formula (3.1) follows since we have

\[
\sum_{n=0}^{\infty} \psi(n + \lambda - \lambda) = \sum_{n=0}^{\infty} \frac{(-n)_k}{k (\lambda)_k} \frac{z^n}{n!} = -e^z \sum_{k=1}^{\infty} \frac{(-z)^k}{k (\lambda)_k}.
\]

Next, keeping in the mind the definition of the harmonic numbers \( H_n \) in (1.1') and since we have

\[
\text{Ein}(z) := -\Phi(z, 1).
\]

(3.4)

Eq. (3.1) with \( \lambda = 1 \) gives the required series summation in (2.5) (in (2.4) also, in the case when \( \ell = 0 \)). Further, we prove (2.4) for \( \ell \in \mathbb{N} \) as follows. Upon differentiating both sides of Eq. (3.1) \( \ell \) times with respect to \( \lambda \), we arrive at

\[
\sum_{n=0}^{\infty} \frac{\psi(n + \lambda - \lambda)}{\lambda} \left( \frac{n!}{\ell!} \right) = -\frac{\partial^\ell}{\partial \lambda^\ell} e^z \Phi(z, \lambda) \quad (\ell \in \mathbb{N}_0),
\]

so that, by series rearrangement of the left-hand side and making use of the familiar Leibniz rule for the \( \ell \)th derivative of a product of two functions on the right-hand side of the latter expression, we have

\[
\sum_{n=0}^{\infty} \frac{\psi(n + \ell + \lambda) - \psi(\ell + \lambda)}{\lambda} \left[ \frac{n!}{\ell!} \right] = -e^z \sum_{k=0}^{\ell} \frac{\ell!}{k!} \frac{\partial^k}{\partial \lambda^k} \Phi(z, \lambda) \quad (\ell \in \mathbb{N}_0).
\]

(3.5)

Finally, in view of the definition (1.1) and relation (3.4), the required generating function given by (2.4) follows at once from (3.5).
Proof of Proposition 2. First, observe that

\[ H_{n} = \sum_{k=1}^{n} \frac{1}{O \cdot n + k} = \sum_{k=1}^{n} \frac{1}{(\ell - 1) \cdot n + k} = \sum_{m=0}^{\ell - 1} \sum_{k=1}^{n} \frac{1}{m \cdot n + k} \quad (\ell, n \in \mathbb{N}). \]

However, we have

\[ \sum_{m=0}^{\ell - 1} \sum_{k=1}^{n} \frac{1}{m \cdot n + k} = \sum_{m=0}^{\ell - 1} \sum_{k=1}^{n} \frac{1}{m \cdot \ell + k} \quad (\ell, n \in \mathbb{N}) \]

and because of the well-known result [4, Vol. 1, p. 600, Entry 4.1.3.(2)]

\[ \sum_{m=0}^{n-1} \frac{1}{m \cdot \ell + k} = \frac{1}{\ell} \left[ \psi \left( n + \frac{k}{\ell} \right) - \psi \left( \frac{k}{\ell} \right) \right], \]

we finally obtain

\[ H_{n} = \frac{1}{\ell} \sum_{k=1}^{\ell} \left[ \psi \left( n + \frac{k}{\ell} \right) - \psi \left( \frac{k}{\ell} \right) \right] \quad (\ell, n \in \mathbb{N}). \quad (3.6) \]

Next, it should be noticed that Eq. (3.1) may be rewritten as follows:

\[ \sum_{n=0}^{\infty} \left[ \psi(z + n) - \psi(z) \right] \frac{z^n}{n!} = \frac{1}{z} \sum_{k=1}^{\infty} \frac{(-z)^k}{k(k+1)\lambda^k} \]

Indeed, what is needed is to transform the right-hand side of (3.1) in the following way

\[ -e^{-z} \sum_{k=1}^{\infty} \frac{(-z)^k}{k(k+1)\lambda^k} = \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{(-z)^k}{k(k+1)\lambda^k} \]

Now, by making use of Eqs. (3.6) and (3.7), we, without difficulty, deduce the desired result

\[ \sum_{n=0}^{\infty} \frac{z^n}{n!} H_{\ell, n} = \frac{1}{\ell} \sum_{k=1}^{\ell} \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} \left[ \psi \left( n + \frac{k}{\ell} \right) - \psi \left( \frac{k}{\ell} \right) \right] \right) = \frac{1}{\ell} \sum_{k=1}^{\ell} \frac{z^k}{k} F_{2} \left[ 1, 1; \frac{1}{2}, \frac{1}{\lambda} + 1; -z \right]. \]

\[ \square \]

Proof of Proposition 3. The proof requires use of the following elementary double series identity (see, for instance, [7, p. 100, Eq. (1)])

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{n} A(m, n - m) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} A(m, n) \]

so that, in the light of (2.5) and upon recalling the series representation of the Bessel function in (2.1), we need only to verify the following straightforward evaluation:

\[ \sum_{n=0}^{\infty} \frac{z^n}{n!} L_{n}(x, |H|) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-x)^k}{(n-k)!} \frac{H_{n-k}}{(k!)^2} = \sum_{k=0}^{\infty} \frac{(-x)^k}{(k!)^2} \sum_{n=0}^{\infty} \frac{z^n}{n!} H_{n} = e^{x} \text{Ein}(z) \sum_{m=0}^{\infty} \frac{(-x)^k}{(k!)^2} = e^{x} \text{Ein}(z) J_{0}(2\sqrt{2x}). \]

\[ \square \]

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