SYMMETRY MATTERS FOR SIZES OF EXTENDED FORMULATIONS

VOLKER KAIBEL, KANSTANTSIN PASHKOVICH, AND DIRK OLIVER THEIS

ABSTRACT. In 1991, Yannakakis [14] proved that no symmetric extended formulation for the matching polytope of the complete graph \( K_n \) with \( n \) nodes has a number of variables and constraints that is bounded subexponentially in \( n \). Here, symmetric means that the formulation remains invariant under all permutations of the nodes of \( K_n \). It was also conjectured in [14] that “asymmetry does not help much,” but no corresponding result for general extended formulations has been found so far. In this paper we show that for the polytopes associated with the matchings in \( K_n \) with \( \lfloor \log n \rfloor \) edges there are non-symmetric extended formulations of polynomial size, while nevertheless no symmetric extended formulation of polynomial size exists. We furthermore prove similar statements for the polytopes associated with cycles of length \( \lfloor \log n \rfloor \). Thus, with respect to the question for smallest possible extended formulations, in general symmetry requirements may matter a lot.

1. INTRODUCTION

Linear Programming techniques have proven to be extremely fruitful for combinatorial optimization problems with respect to both structural analysis and the design of algorithms. In this context, the paradigm is to represent the problem by a polytope \( P \subseteq \mathbb{R}^m \) whose vertices correspond to the feasible solutions of the problem in such a way that the objective function can be expressed by a linear functional \( x \mapsto \langle c, x \rangle \) on \( \mathbb{R}^m \) (with some \( c \in \mathbb{R}^m \)). If one succeeds in finding a description of \( P \) by means of linear constraints, then algorithms as well as structural results from Linear Programming can be exploited. In many cases, however, the polytope \( P \) has exponentially (in \( m \)) many facets, thus \( P \) can only be described by exponentially many inequalities. Also it may be that the inequalities needed to describe \( P \) are too complicated to be identified.

In some of these cases one may find an extended formulation for \( P \), i.e., a (preferably small and simple) description by linear constraints of another polyhedron \( Q \subseteq \mathbb{R}^d \) in some higher dimensional space that projects to \( P \) via some (simple) linear map \( p : \mathbb{R}^d \to \mathbb{R}^m \) with \( p(y) = Ty \) for all \( y \in \mathbb{R}^d \) (and some matrix \( T \in \mathbb{R}^{m \times d} \)). Indeed, if \( p^* : \mathbb{R}^m \to \mathbb{R}^d \) with \( p^*(x) = T^*x \) for all \( x \in \mathbb{R}^m \) denotes the linear map that is adjoint to \( p \) (with respect to the standard bases), then we have

\[
\max \{ \langle c, x \rangle : x \in P \} = \max \{ \langle p^*(c), y \rangle : y \in Q \}.
\]

As for a guiding example, let us consider the spanning tree polytope

\[
P_{\text{spt}}(n) = \text{conv}\{\chi(T) \in \{0, 1\}^{E_n} : T \subseteq E_n \text{ spanning tree of } K_n\},
\]

Date: November 19, 2009.
where \( K_n = ([n], E_n) \) denotes the complete graph with node set \([n] = \{1, \ldots, n\}\) and edge set \( E_n = \{\{v, w\} : v, w \in [n], v \neq w\}\), and \( \chi(A) \in \{0, 1\}^B \) is the characteristic vector of the subset \( A \subseteq B \) of \( B \), i.e., for all \( b \in B \), we have \( \chi(A)_b = 1 \) if and only if \( b \in A \). Thus, \( P_{\text{spt}}(n) \) is the polytope associated with the bases of the graphical matroid of \( K_n \), and we have (see [7])

\[
P_{\text{spt}}(n) = \{ x \in \mathbb{R}^E_+ : x(E_n) = n - 1, \quad x(E_n(S)) \leq |S| - 1 \text{ for all } S \subseteq [n], 2 \leq |S| \leq n - 1 \},
\]

where \( \mathbb{R}^E_+ \) is the nonnegative orthant of \( \mathbb{R}^E \), we denote by \( E_n(S) \) the subset of all edges with both nodes in \( S \), and \( x(F) = \sum_{e \in F} x_e \) for \( F \subseteq E_n \). This linear description of \( P_{\text{spt}}(n) \) has an exponential (in \( n \)) number of constraints, and as all the inequalities define pairwise disjoint facets, none of them is redundant.

The following much smaller extended formulation for \( P_{\text{spt}}(n) \) (with \( O(n^3) \) variables and constraints) appears in [5] (and a similar one in [14], who attributes it to [11]). Let us introduce additional 0/1-variables \( z_{e,v,u} \) for all \( e \in E_n, v \in e, \) and \( u \in [n] \setminus e \). While each spanning tree \( T \subseteq E_n \) is represented by its characteristic vector \( x^{(T)} = \chi(T) \) in \( P_{\text{spt}}(n) \), in the extended formulation it will be represented by the vector \( y^{(T)} = (x^{(T)}, z^{(T)}) \) with \( z_{e,v,u}^{(T)} = 1 \) (for \( e \in E_n, v \in e, u \in [n] \setminus e \)) if and only if \( e \in T \) and \( u \) is contained in the component of \( v \) in \( T \setminus e \).

The polyhedron \( Q_{\text{spt}}(n) \subseteq \mathbb{R}^d \) defined by the nonnegativity constraints \( x \geq 0, z \geq 0 \), the equations \( x(E_n) = n - 1 \),

\[
x_{\{v,w\}} - z_{\{v,w\},v,u} - z_{\{v,w\},w,u} = 0 \quad \text{for all pairwise distinct } v, w, u \in [n],
\]
as well as

\[
x_{\{v,w\}} + \sum_{u \in [n] \setminus \{v,w\}} z_{\{v,u\},w,u} = 1 \quad \text{for all distinct } v, w \in [n],
\]
satisfies \( p(Q_{\text{spt}}(n)) = P_{\text{spt}}(n) \), where \( p : \mathbb{R}^d \to \mathbb{R}^E \) is the orthogonal projection onto the \( x \)-variables. This follows from observing that, for each spanning tree \( T \subseteq E_n \), the vector \( y^{(T)} = (x^{(T)}, z^{(T)}) \) satisfies (2) and (3), and on the other hand, every nonnegative vector \( y = (x, z) \in \mathbb{R}^d_+ \) satisfying (2) and (3) also satisfies \( x(E(S)) \leq |S| - 1 \) for all \( S \subseteq [n] \) with \( |S| \geq 2 \). Indeed, to see the latter claim for \( |S| \geq 3 \) (for \( |S| = 2 \) it follows readily from (3) and \( z \geq 0 \)), one adds equations (2) for all pairwise distinct \( v, w, u \in S \) in order to obtain (after division by two and renaming summation indices)

\[
(|S| - 2)x(E(S)) = \sum_{v,w \in S, v \neq w} \sum_{u \in S \setminus \{v,w\}} z_{\{v,u\},w,u},
\]

where, due to (3) and \( z \geq 0 \), the right-hand side is bounded from above by

\[
\sum_{v,w \in S, v \neq w} (1 - x_{\{v,w\}}) = |S|(|S| - 1) - 2x(E(S)),
\]

which together with (4) (due to \(|S| - 2 > 0\)) implies \( x(E(S)) \leq |S| - 1 \).
For many other polytopes (with exponentially many facets) associated with polynomial time solvable combinatorial optimization problems polynomially sized extended formulations can be constructed as well (see, e.g., the recent survey [5]). Probably the most prominent problem in this class for which, however, no such small formulation is known, is the matching problem. In fact, Yannakakis [14] proved that no symmetric polynomially sized extended formulation of the matching polytope exists.

Here, symmetric refers to the symmetric group $\mathcal{S}(n)$ of all permutations $\pi : [n] \to [n]$ of the node set $[n]$ of $K_n$ acting on $E_n$ via $\pi \{v, w\} = \{\pi(v), \pi(w)\}$ for all $\pi \in \mathcal{S}(n)$ and $\{v, w\} \in E_n$. Clearly, this action of $\mathcal{S}(n)$ on $E_n$ induces an action on the set of all subsets of $E_n$. For instance, this yields an action on the spanning trees of $K_n$, and thus, on the vertices of $P_{spt}(n)$. The extended formulation of $P_{spt}(n)$ discussed above is symmetric in the sense that, for every $\pi \in \mathcal{S}(n)$, replacing all indices associated with edges $e \in E_n$ and nodes $v \in [n]$ by $\pi.e$ and $\pi.v$, respectively, does not change the set of constraints in the formulation. Phrased informally, all nodes of $K_n$ play the same role in the formulation. For a general definition of symmetric extended formulations see Section 2.

In order to describe the main results of Yannakakis paper [14] and the contributions of the present paper, let us denote by

$$\mathcal{M}^\ell(n) = \{M \subseteq E_n : M \text{ matching in } K_n, |M| = \ell\}$$

the set of all matchings of size $\ell$ (a matching being a subset of edges no two of which share a node), and by

$$P^\ell_{\text{match}}(n) = \text{conv}\{\chi(M) \in \{0, 1\}^{E_n} : M \in \mathcal{M}^\ell(n)\}$$

the associated polytope. According to Edmonds [6] the perfect matching polytope $P^\frac{n}{2}_{\text{match}}(n)$ (for even $n$) is described by

$$P^\frac{n}{2}_{\text{match}}(n) = \{x \in \mathbb{R}^{E_n}_+ : x(\delta(v)) = 1 \text{ for all } v \in [n],$$

$$x(E(S)) \leq (|S| - 1)/2 \text{ for all } S \subseteq [n], 3 \leq |S| \text{ odd}\} \quad (5)$$

(with $\delta(v) = \{e \in E_n : v \in e\}$). Yannakakis [14] Thm.1 and its proof] shows that there is a constant $C > 0$ such that, for every extended formulation for $P^\frac{n}{2}_{\text{match}}(n)$ (with $n$ even) that is symmetric in the sense above, the number of variables and constraints is at least $C \cdot \binom{n}{4} = 2^{\Omega(n)}$. This in particular implies that there is no polynomial size symmetric extended formulation for the matching polytope of $K_n$ (the convex hulls of characteristic vectors of all matchings in $K_n$) of which the perfect matching polytope is a face.

Yannakakis [14] moreover obtains a similar (maybe less surprising) result for traveling salesman polytopes. Denoting by

$$C^\ell(n) = \{C \subseteq E_n : C \text{ cycle in } K_n, |C| = \ell\}$$

the set of all (simple) cycles of length $\ell$ in $K_n$, and by

$$P^\ell_{\text{cycl}}(n) = \text{conv}\{\chi(C) \in \{0, 1\}^{E_n} : C \in C^\ell(n)\}$$
the associated polytopes, the traveling salesman polytope is \( P_{\text{cycl}}^n(n) \). Identifying \( P_{\text{match}}^{n/2}(n) \) (for even \( n \)) with a suitable face of \( P_{\text{cycl}}^{3n}(3n) \), Yannakakis concludes that all symmetric extended formulations for \( P_{\text{cycl}}^n(n) \) have size at least \( 2^{\Omega(n)} \) as well [14, Thm. 2 and its proof].

Yannakakis’ results in a fascinating way illuminate the borders of our principal abilities to express combinatorial optimization problems like the matching or the traveling salesman problem by means of linear constraints. However, they only refer to linear descriptions that respect the inherent symmetries in the problems. In fact, the second open problem mentioned in the concluding section of [14] is described as follows: “We do not think that asymmetry helps much. Thus, prove that the matching and TSP polytopes cannot be expressed by polynomial size LP’s without the asymmetry assumption.”

The contribution of our paper is to show that, in contrast to the assumption expressed in the quotation above, asymmetry can help much, or, phrased differently, that symmetry requirements on extended formulations indeed can matter significantly with respect to the minimal sizes of extended formulations. Our main results are that both \( P_{\text{match}}^{\lfloor \log n \rfloor}(n) \) and \( P_{\text{cycl}}^{\lfloor \log n \rfloor}(n) \) do not admit symmetric extended formulations of polynomial size, while they have non-symmetric extended formulations of polynomial size (see Cor. 12 and 15 for matchings, as well as Cor. 20 and 22 for cycles). The corresponding theorems from which these corollaries are derived provide some more general and more precise results for \( P_{\ell}^\text{match}(n) \) and \( P_{\ell}^\text{cycl}(n) \).

In order to establish the lower bounds for symmetric extensions, we generalize the techniques developed by Yannakakis [14]. The constructions of the compact non-symmetric extended formulations rely on small families of perfect hash functions [1, 8, 12].

The paper is organized as follows. In Section 2, we provide definitions of extensions, extended formulations, their sizes, the crucial notion of a section of an extension, and we give some auxiliary results. In Section 3, we present Yannakakis’ method to derive lower bounds on the sizes of symmetric extended formulations for perfect matching polytopes in a general setting. Here, we tried to separate as much as possible those parts of the technique that do not rely on symmetry assumptions from those that do. Thus, we hope that the presentation in Section 3 may also be useful in different contexts. In Section 4, we then exploit the method from Section 3 in order to derive lower bounds on the sizes of symmetric extended formulations for the polytopes \( P_{\ell}^\text{match}(n) \) associated with cardinality restricted matchings (for a recent survey on general cardinality restricted combinatorial optimization problems see [4]). In Section 5, we describe our non-symmetric extended formulations for these polytopes. Finally, in Section 6, we present the results on \( P_{\ell}^\text{cycl}(n) \). Some remarks conclude the paper in Section 7.

Acknowledgements. We thank Christian Bey for useful discussions on subspaces that are invariant under coordinate permutations.
2. Extended Formulations, Extensions, and Symmetry

An extension of a polytope $P \subseteq \mathbb{R}^m$ is a polyhedron $Q \subseteq \mathbb{R}^d$ together with a projection (i.e., a linear map) $p : \mathbb{R}^d \to \mathbb{R}^m$ with $p(Q) = P$; it is called a subspace extension if $Q$ is the intersection of an affine subspace of $\mathbb{R}^d$ and the nonnegative orthant $\mathbb{R}_+^d$. For instance, the polyhedron $Q_{\text{spt}}(n)$ defined in the Introduction is a subspace extension of the spanning tree polytope $P_{\text{spt}}(n)$. A (finite) system of linear equations and inequalities whose solutions are the points in an extension $Q$ of $P$ is an extended formulation for $P$. The size of an extension is the number of its facets plus the dimension of the space it lies in. The size of an extended formulation is its number of inequalities (including nonnegativity constraints, but not equations) plus its number of variables. Clearly, the size of an extended formulation is at least as large as the size of the extension it describes. Conversely, every extension is described by an extended formulation of at most its size.

Note that we do not consider the number of equations (which can be assumed to be bounded by the dimension of the ambient space) in the definition of the size of an extension or extended formulations. Actually, it would be more elegant to drop the number of variables from the definition as well. In fact, for pointed extensions the number of variables of course is bounded from above by the number of facets, and it is not hard to see that from a possibly non-pointed extension of a polytope one can derive a pointed one with the same number of facets. However, it seems unclear whether one can do this without destroying symmetry of the extension (see also Section 7). Therefore, as the lower bounds that we provide (the same applies to Yannakakis’ bounds) also refer to the ambient dimension of the symmetric extensions, we stick to the definition of size as the sum of both the number of facets and the ambient dimension. We did not include the encoding lengths of the coefficients involved into the extensions into the definition of the size, since the lower bounds that we present do not refer to them. Instead, whenever we provide upper bounds, we explicitly state that the coefficients that are involved are bounded by a constant.

Extensions or extended formulations of a family of polytopes $P \subseteq \mathbb{R}^m$ (for varying $m$) are compact if their sizes and the encoding lengths of the coefficients needed to describe them can be bounded by a polynomial in $m$ and the maximal encoding length of all components of all vertices of $P$. Clearly, the extension $Q_{\text{spt}}(n)$ of $P_{\text{spt}}(n)$ from the Introduction is compact.

In our context, sections $s : X \to Q$ play a crucial role, i.e., maps that assign to every vertex $x \in X$ of $P$ some point $s(x) \in Q \cap p^{-1}(x)$ in the intersection of the polyhedron $Q$ and the fiber $p^{-1}(x) = \{y \in \mathbb{R}^d : p(y) = x\}$ of $x$ under the projection $p$. Such a section induces a bijection between $X$ and its image $s(X) \subseteq Q$, whose inverse is given by $p$. In the spanning tree example from the Introduction, the assignment $\chi(T) \mapsto y^{(T)} = (x^{(T)}, z^{(T)})$ defined such a section. Note that, in general, sections will not be induced by linear maps. In fact, if a section is induced by a linear map $s : \mathbb{R}^m \to \mathbb{R}^d$, then the intersection of $Q$ with the affine subspace of $\mathbb{R}^d$ generated by $s(X)$ is isomorphic to $P$, thus $Q$ has at least as many facets as $P$. 
For a family $\mathcal{F}$ of subsets of $X$, an extension $Q \subseteq \mathbb{R}^d$ is said to be indexed by $\mathcal{F}$ if there is a bijection between $\mathcal{F}$ and $\{d\}$ such that (identifying $\mathbb{R}^\mathcal{F}$ with $\mathbb{R}^d$ via this bijection) the map $1_\mathcal{F} = (1_F)_{F \in \mathcal{F}} : X \to \{0, 1\}^\mathcal{F}$ whose component functions are the characteristic functions $1_F : X \to \{0, 1\}$ (with $1_F(x) = 1$ if and only if $x \in F$), is a section for the extension, i.e., $1_\mathcal{F}(X) \subseteq Q$ and $p(1_\mathcal{F}(x)) = x$ hold for all $x \in X$.

For instance, the extension $Q_{\text{spt}}(n)$ of $P_{\text{spt}}(n)$ is indexed by the family

$$\{T(e) : e \in E_n\} \cup \{T(e, v, u) : e \in E_n, v \in e, u \in [n] \setminus e\},$$

where $T(e)$ contains all spanning trees using edge $e$, and $T(e, v, u)$ consists of all spanning trees in $T(e)$ for which $u$ and $v$ are in the same component of $T \setminus \{e\}$.

In order to define the notion of symmetry of an extension precisely, let the group $\mathfrak{S}(d)$ of all permutations of $[d] = \{1, \ldots, d\}$ act on $\mathbb{R}^d$ by coordinate permutations. Thus we have $(\sigma, y)_j = y_{\sigma^{-1}(j)}$ for all $y \in \mathbb{R}^d$, $\sigma \in \mathfrak{S}(d)$, and $j \in [d]$.

Let $P \subseteq \mathbb{R}^m$ be a polytope and $G$ be a group acting on $\mathbb{R}^m$ with $\pi.P = P$ for all $\pi \in G$, i.e., the action of $G$ on $\mathbb{R}^m$ induces an action of $G$ on the set $X$ of vertices of $P$. An extension $Q \subseteq \mathbb{R}^d$ of $P$ with projection $p : \mathbb{R}^d \to \mathbb{R}^m$ is symmetric (with respect to the action of $G$), if for every $\pi \in G$ there is a permutation $\kappa_\pi \in \mathfrak{S}(d)$ with $\kappa_\pi \cdot Q = Q$ and

$$p(\kappa_\pi \cdot y) = \pi \cdot p(y) \quad \text{for all } y \in \mathbb{R}^d. \quad (6)$$

We do not restrict the notion of symmetry to actions of $G$ on $\mathbb{R}^m$ that are linear in the sense that $x \mapsto \pi.x$ is a linear map for each $\pi \in G$. Actually, for almost all parts of the paper it would even be sufficient to consider actions of a group $G$ on the set $X$ of vertices of $P$ only and to require (6) just for vectors $y \in p^{-1}(X)$ in the fibers of vertices of $P$. The slightly more restrictive notion of symmetry, however, is convenient in order to deduce lower bounds on the size of symmetric extensions for polytopes that are projections of faces of polytopes for which we have already established such bounds (Lemma 3). As this enables us both to derive more general results on matching polytopes (Theorem 11) and to transfer such results to cycle polytopes (see Theorem 19), we decided to deal with the slightly stronger notion of symmetry.

The prime examples of symmetric extensions arise from extended formulations that “look symmetric”. To be more precise, we define an extended formulation $A= y = b^=, A^\leq y \leq b^\leq$ describing the polyhedron

$$Q = \{y \in \mathbb{R}^d : A= y = b^=, A^\leq y \leq b^\leq\}$$

extending $P \subseteq \mathbb{R}^m$ as above to be symmetric (with respect to the action of $G$ on the set $X$ of vertices of $P$), if for every $\pi \in G$ there is a permutation $\kappa_\pi \in \mathfrak{S}(d)$ satisfying (6) and there are two permutations $\varphi_\pi^=$ and $\varphi_\pi^\leq$ of the rows of $(A^=, b^=)$ and $(A^\leq, b^\leq)$, respectively, such that the corresponding simultaneous permutations of the columns and the rows of the matrices $(A^=, b^=)$ and $(A^\leq, b^\leq)$ leaves them unchanged. Clearly, in this situation the permutations $\kappa_\pi$ satisfy $\kappa_\pi \cdot Q = Q$, which implies the following.
Lemma 1. Every symmetric extended formulation describes a symmetric extension.

One example of a symmetric extended formulation is the extended formulation for the spanning tree polytope described in the Introduction (with respect to the group $G$ of all permutations of the nodes of the complete graph).

For the proof of the central result on the non-existence of certain symmetric subspace extensions (Theorem 9), a weaker notion of symmetry will be sufficient. We call an extension as above weakly symmetric (with respect to the action of $G$) if there is a section $s : X \rightarrow Q$ for which the action of $G$ on $s(X)$ induced by the bijection $s$ works by permutation of variables, i.e., for every $\pi \in G$ there is a permutation $\kappa_\pi \in S(d)$ with $s(\pi.x) = \kappa_\pi.s(x)$ for all $x \in X$. The following statement (and its proof) generalizes the construction of sections for symmetric extensions of matching polytopes described in Yannakakis paper [14, Claim 1 in the proof of Thm. 1].

Lemma 2. Every symmetric extension is weakly symmetric.

Proof. Let us first observe that a symmetric extension (with notations as above) satisfies

$$\kappa_\pi.p^{-1}(x) = p^{-1}(\pi.x) \quad \text{for all} \ \pi \in G \ \text{and} \ \ x \in X,$$

(7)

(thus, $\kappa_\pi$ permutes the fibers of points in $X$ according to $\pi$) since (6) readily implies $\kappa_\pi.p^{-1}(x) \subseteq p^{-1}(\pi.x)$, from which equality follows because both sets are affine subspaces of equal dimension (as all fibers of $p$ have the same dimension and $\kappa_\pi.p^{-1}(x)$ is an image of one of these fibers under a linear transformation).

Let $\tilde{G}$ be the subgroup of $S(d)$ generated by $\{\kappa_\pi : \pi \in G\}$. Clearly, we have

$$\sigma.Q = Q \quad \text{for all} \ \sigma \in \tilde{G}.$$  

(8)

We start the construction of a section $s : X \rightarrow Q$ establishing weak symmetry of the extension by choosing from each orbit of $X$ under the action of $G$ some $x^* \in X$ as well as an arbitrary point $y^* \in Q \cap p^{-1}(x^*)$ in the intersection of $Q$ and the fiber of $x^*$. Actually, as we can consider the orbits one by one here, we will assume in the following that there is just one of them, i.e., the action of $G$ on $X$ is transitive. Denoting by

$$\tilde{S}(x^*) = \{\sigma \in \tilde{G} : \sigma.p^{-1}(x^*) = p^{-1}(x^*)\},$$

the subgroup of $\tilde{G}$ containing all permutations that map the fiber $p^{-1}(x^*)$ to itself, we define

$$s(x^*) = \frac{1}{|\tilde{S}(x^*)|} \sum_{\sigma \in \tilde{S}(x^*)} \sigma.y^*,$$

(9)

which is a point in the convex set (polyhedron) $Q \cap p^{-1}(x^*)$, because due to (8) we have $\sigma.y^* \in Q \cap p^{-1}(x^*)$ for all $\sigma \in \tilde{S}(x^*)$. For each $x \in X$ we now choose some $\tau_x \in G$ with $\tau_x.x^* = x$ (recall that we assumed the action of $G$ to be transitive) and define

$$s(x) = \kappa_{\tau_x}.s(x^*),$$

which is contained in $Q \cap p^{-1}(x)$ due to (8) and (7).
In order to finish the proof of the lemma, it suffices to show $s(\pi.x) = \omega.s(x)$ for every $x \in X$ and $\pi \in G$. To deduce this equation, observe that due to (7) we have
\[
\begin{aligned}
&x_{\tau_{G},x}^{-1}x_{\pi}x_{\tau_{G}}.\pi^{-1}(x^*) = x_{\tau_{G},x}^{-1}((x_{\pi}x_{\tau_{G}}.\pi^{-1}(x^*))
&= x_{\tau_{G},x}^{-1}(\pi.p^{-1}(x)) = x_{\tau_{G},x}^{-1}.\pi.p^{-1}(\pi.x) = p^{-1}(x^*).
\end{aligned}
\]
Thus, $\omega = x_{\tau_{G},x}^{-1}x_{\pi}x_{\tau_{G}} \in \tilde{S}(x^*)$ holds, and in particular, $\sigma \mapsto \omega.\sigma$ defines a bijection $\tilde{S}(x^*) \rightarrow \tilde{S}(x^*)$. Therefore, we can conclude
\[
\begin{aligned}
x_{\tau_{G},x}^{-1}x_{\pi}x_{\tau_{G}}.s(x^*) &= \omega.s(x^*) = \frac{1}{|\tilde{S}(x^*)|} \sum_{\sigma \in S(x^*)} \omega.\sigma.y^* = s(x^*)
\end{aligned}
\]
from (9), which implies the equation
\[
x_{\pi}.s(x) = x_{\pi}.(x_{\tau_{G}}.s(x^*)) = x_{\pi}x_{\tau_{G}}.s(x^*) = x_{\tau_{G},x}^{-1}x_{\pi}x_{\tau_{G}}.s(x^*) = s(\pi.x)
\]
that we needed to establish. \qed

In fact, with respect to the validity of Lemma 2, we could have made more general notions of symmetric and weakly symmetric extensions by requiring, for every $\pi \in G$, instead if a coordinate permutation the existence of an arbitrary linear transformation of $\mathbb{R}^d$ satisfying the corresponding constraint that would be analogous to (6) (we need linear transformations in (10)). However, since later in the treatment we will rely heavily on the special structure of coordinate permutations, we avoided this generalization. Furthermore, at least in the applications we considered so far, the group $G$ itself usually acts by coordinate permutations, and in this case, looking only at coordinate permutations in the extended spaces seems not to be too restrictive.

Finally, the following result will turn out to be useful in order to derive lower bounds on the sizes of symmetric extensions for one polytope from bounds for another one.

**Lemma 3.** Let $Q \subseteq \mathbb{R}^d$ be an extension of the polytope $P \subseteq \mathbb{R}^m$ with projection $p : \mathbb{R}^d \rightarrow \mathbb{R}^m$, and let the face $P'$ of $P$ be an extension of a polytope $R \subseteq \mathbb{R}^k$ with projection $q : \mathbb{R}^m \rightarrow \mathbb{R}^k$. Then the face $Q' = p^{-1}(P') \cap Q \subseteq \mathbb{R}^d$ of $Q$ is an extension of $R$ via the composed projection $q \circ p : \mathbb{R}^d \rightarrow \mathbb{R}^k$.

If the extension $Q$ of $P$ is symmetric with respect to an action of a group $G$ on $\mathbb{R}^m$ (with $\pi.P = P$ for all $\pi \in G$), and a group $H$ acts on $\mathbb{R}^k$ such that, for every $\tau \in H$, we have $\tau.R = R$, and there is some $\pi_\tau \in G$ with $\pi_\tau.P' = P'$ and $q(\pi_\tau.x) = \tau.q(x)$ for all $x \in \mathbb{R}^m$, then the extension $Q'$ of $R$ is symmetric (with respect to the action of the group $H$).

**Proof.** Due to $q(p(Q')) = q(P') = R$, the polyhedron $Q'$ (together with the projection $q \circ p$) clearly is an extension of $R$. In order to prove the statement on the symmetry of this extension, let $\tau \in H$ be an arbitrary element of $H$ with $\pi_\tau \in G$ as guaranteed to exist for $\tau$ in the statement of the lemma, and let $x_{\pi_\tau} \in \mathcal{S}(d)$ be
a permutation satisfying (6) as guaranteed to exist by the symmetry of the extension $Q$ of $P$. Since we obviously have
\[ q(p(\kappa \pi \tau \cdot y)) = q(p(y)) = \tau(q(p(y))), \]
it suffices to show $\kappa \pi \tau \cdot Q' = Q'$. As $y \mapsto \kappa \pi \tau \cdot y$ defines an automorphism of $Q$ (mapping faces of $Q$ to faces of the same dimension), it suffices to show $\kappa \pi \tau \cdot Q' \subseteq Q'$. Due to $\kappa \pi \tau \cdot Q = Q$ this relation is implied by $\kappa \pi \tau \cdot p^{-1}(P') \subseteq p^{-1}(P')$, which finally follows from
\[ p(\kappa \pi \tau \cdot p^{-1}(P')) = \pi \tau \cdot p(p^{-1}(P')) = \pi \tau \cdot P' = P'. \]
\[ \square \]

3. YANNAKAKIS’ METHOD FOR BOUNDING THE SIZE OF SYMMETRIC EXTENSIONS

Here, we provide an abstract view on the method used by Yannakakis \[14\] in order to bound from below the sizes of symmetric extensions for perfect matching polytopes, without referring to these concrete poytopes. That method is capable of establishing lower bounds on the number of variables of weakly symmetric subspace extensions of certain polytopes. By the following lemma, which is basically Step 1 in the proof of \[14\] Theorem 1, such bounds imply similar lower bounds on the dimension of the ambient space and the number of facets for general symmetric extensions (that are not necessarily subspace extensions).

**Lemma 4.** If there is a symmetric extension in $\mathbb{R}^{d}$ with $f$ facets for a polytope $P$, then there is also a symmetric subspace extension in $\mathbb{R}^{d}$ with $d \leq 2\tilde{d} + f$ for $P$.

For the proof of Lemma 4, one starts with a description $A^{(1)}y = b^{(1)}, A^{(2)}y \leq b^{(2)}$ of a given symmetric extension $Q \subseteq \mathbb{R}^{d}$ of $P$ with $A^{(1)}y = b^{(1)}$ describing the affine hull of $Q$ and the system $A^{(2)}y \leq b^{(2)}$ having exactly one inequality for each facet of $Q$ with the rows of $A^{(2)}$ having length one and being parallel to the affine hull of $Q$ (these conditions make the set of rows of $A^{(2)}$ uniquely determined).

As it induces an isometry of $\mathbb{R}^{\tilde{d}}$, every $\sigma \in S(\tilde{d})$ with $\sigma.Q = Q$ permutes the facets of $Q$, maps the affine hull of $Q$ to itself, and maps length-one vectors to length-one vectors. Thus, from the uniqueness of the set of rows of $A^{(2)}$ one finds that permuting the columns of $A^{(2)}$ according to $\sigma$ yields the same matrix as one obtains from permuting the rows of $A^{(2)}$ according to the permutation of the facets of $Q$ induced by $\sigma$. Hence, one can indeed extend each such $\sigma$ to a permutation $\sigma' \in S(\tilde{d} + f)$ in such a way that we have $\sigma'.Q' = Q'$ for the polyhedron
\[ Q' = \{(y, u) \in \mathbb{R}^{\tilde{d}} \times \mathbb{R}^{f} : A^{(1)}y = b^{(1)}, A^{(2)}y + u = b^{(2)}, u \geq 0\}. \]

Hence, $Q'$ is a symmetric extension of $P$ as well. Finally, expressing every $y$-variable as the difference of two nonnegative variables one ends up with a symmetric subspace extension as required.

The following simple lemma provides the strategy for Yannakakis’ method, which we need to extend slightly by allowing restrictions to affine subspaces.
Lemma 5. Let $Q \subseteq \mathbb{R}^d$ be a subspace extension of the polytope $P \subseteq \mathbb{R}^m$ with vertex set $X \subseteq \mathbb{R}^m$, and let $s : X \to Q$ be a section for the extension. If $S \subseteq \mathbb{R}^m$ is an affine subspace, and, for some $X^* \subseteq X \cap S$, the coefficients $c_x \in \mathbb{R} (x \in X^*)$ yield an affine combination of a nonnegative vector

$$\sum_{x \in X^*} c_x s(x) \geq 0_d \quad \text{with} \quad \sum_{x \in X^*} c_x = 1,$$

(11)

from the section images of the vertices in $X^*$, then $\sum_{x \in X^*} c_x x \in P \cap S$ holds.

Proof. Since $Q$ is a subspace extension, we obtain $\sum_{x \in X^*} c_x s(x) \in Q$ from $s(x) \in Q$ (for all $x \in X^*$). Thus, if $p : \mathbb{R}^d \to \mathbb{R}^m$ is the projection of the extension, we derive

$$P \ni p(\sum_{x \in X^*} c_x s(x)) = \sum_{x \in X^*} c_x p(s(x)) = \sum_{x \in X^*} c_x x.$$  

(12)

As $S$ is an affine subspace containing $X^*$, we also have $\sum_{x \in X^*} c_x x \in S$. \hfill \Box

Due to Lemma 5, one can prove that subspace extensions of some polytope $P$ with certain properties do not exist by finding, for such a hypothetical extension, a subset $X^*$ of vertices of $P$ and an affine subspace $S$ containing $X^*$, for which one can construct coefficients $c_x \in \mathbb{R}$ satisfying (11) such that $\sum_{x \in X^*} c_x x$ violates some inequality that is valid for $P \cap S$.

Actually, following Yannakakis, we will not apply Lemma 5 directly to a hypothetical small weakly symmetric subspace extension, but we will rather first construct another subspace extension from the one assumed to exist that is indexed by some convenient family $\mathcal{F}$. We say that an extension $Q$ of a polytope $P$ is consistent with a family $\mathcal{F}$ of subsets of the vertex set $X$ of $P$ if there is a section $s : X \to Q$ for the extension such that, for every component function $s_j$ of $s$, there is a subfamily $\mathcal{F}_j$ of $\mathcal{F}$ such that $s_j$ is constant on every set in $\mathcal{F}_j$, and the sets in $\mathcal{F}_j$ partition $X$. In this situation, we also call the section $s$ consistent with $\mathcal{F}$.

Lemma 6. If $P \subseteq \mathbb{R}^m$ is a polytope and $\mathcal{F}$ is a family of vertex sets of $P$ for which there is some extension $Q$ of $P$ that is consistent with $\mathcal{F}$, then there is some extension $Q'$ for $P$ that is indexed by $\mathcal{F}$. If $Q$ is a subspace extension, then $Q'$ can be chosen to be a subspace extension as well.

Lemmas 5 and 6 suggest the following strategy for proving that subspace extensions of some polytope $P$ with certain properties (e.g., being weakly symmetric and using at most $B$ variables) do not exist by (a) exhibiting a family $\mathcal{F}$ of subsets of the vertex set $X$ of $P$ with which such an extension would be consistent and (b) determining a subset $X^* \subseteq X$ of vertices and an affine subspace $S$ containing $X^*$, for which one can construct coefficients $c_x \in \mathbb{R}$ satisfying

$$\sum_{x \in X^*} c_x 1_\mathcal{F}(x) \geq 0_\mathcal{F} \quad \text{with} \quad \sum_{x \in X^*} c_x = 1,$$

(13)

such that $\sum_{x \in X^*} c_x x$ violates some inequality that is valid for $P \cap S$. 

Proof. In order to prove Lemma 6, let \( Q = \{ y \in \mathbb{R}^d : Ay \leq b \} \) with projection \( p : \mathbb{R}^d \to \mathbb{R}^m \) be some extension of the polytope \( P \subseteq \mathbb{R}^m \) with vertex set \( X \) that is consistent with the family \( F \) of subsets of \( X \), and let \( s : X \to Q \) be some section as required in the definition of consistency of \( Q \) with \( F \). In order to construct an extension \( Q' \) of \( P \) that is indexed by \( F \), let us write each component function \( s_j \) of \( s \) as

\[
s_j = \sum_{F \in F} r_{j,F} \cdot 1_F,
\]

where the coefficient \( r_{j,F} \) in the linear combination is the constant value of \( s_j \) on the set \( F \), if \( F \in F_j \) (referring to the notation in the definition of consistency), and \( r_{j,F} = 0 \) otherwise. With the matrix \( R = (r_{j,F}) \in \mathbb{R}^{d \times F} \) we thus have

\[
s(x) = R \cdot 1_F(x) \quad \text{for all} \ x \in X.
\]

We use \( Q' = \{ y' \in \mathbb{R}^F : ARy' \leq b \} \) as the polyhedron and \( p' : \mathbb{R}^F \to \mathbb{R}^d \) with \( p'(y') = p(Ry') \) for all \( y' \in \mathbb{R}^F \) as the projection of the subspace extension we aim at. Indeed, we then clearly have \( p'(Q') \subseteq P \). Furthermore, (14) implies

\[
p'(1_F(x)) = p(s(x)) = x \quad \text{for all} \ x \in X
\]

and

\[
AR \cdot 1_F(x) = As(x) \leq b \quad \text{for all} \ x \in X.
\]

Thus, from (16) we conclude

\[
1_F(X) \subseteq Q',
\]

and from this, using (15),

\[
p'(Q') \supseteq p'(1_F(X)) = p(s(X)) = X.
\]

Hence, we find \( p'(Q') = P \), i.e., \( Q' \) together with the projection \( p' \) is an extension of \( P \), which, due to (15) and (17), is indexed by \( F \).

If we start with a subspace extension \( Q = \{ y \in \mathbb{R}^d_+ : Ay = b \} \), we can use \( Q' = \{ y' \in \mathbb{R}^F_+ : ARy' = b \} \) in the proof, since then \( R \) is nonnegative (implying \( Ry' \geq 0 \) for all \( y' \in \mathbb{R}^F_+ \)) and, of course, we have \( 1_F(x) \in \mathbb{R}^F_+ \) for all \( x \in X \). This also proves the statement on subspace extensions.

Let us finally investigate more closely the sections that come with weakly symmetric extensions. In particular, we will discuss an approach to find suitable families \( F \) within the strategy mentioned above in the following setting. Let \( Q \subseteq \mathbb{R}^d \) be a weakly symmetric extension of the polytope \( P \subseteq \mathbb{R}^m \) (with respect to an action of the group \( G \) on the vertex set \( X \) of \( P \)) along with a section \( s : X \to Q \) such that for every \( \pi \in G \) there is a permutation \( \pi_s \in \mathfrak{S}(d) \) that satisfies \( s(\pi \cdot x) = \pi_s \cdot s(x) \) for all \( x \in X \) (with \( \pi_s \cdot s(x)_{(j) = s_{\pi_s^{-1}(j)}(x)} \)).

In this setting, we can define an action of \( G \) on the set \( \mathcal{S} = \{ s_1, \ldots, s_d \} \) of the component functions of the section \( s : X \to Q \) with \( \pi \cdot s_j = s_{\pi_s^{-1}(j)} \in \mathcal{S} \) for each \( j \in [d] \). In order to see that this definition indeed is well-defined (note
Lemma 8. In the setting described above, we have

\[ s \text{ isotropy group of the element } s_j \in S \]

Proof. This follows readily from the fact that the index \( d \) from above by

\[ s \text{ of } \pi \text{ from } (18) \]

from which one deduces \( 1.s_j = s_j \) for the one-element 1 in \( G \) as well as \( (\pi \pi').s_j = \pi.(\pi'.s_j) \) for all \( \pi, \pi' \in G \). The isotropy group of \( s_j \in S \) under this action is

\[ \text{iso}_G(s_j) = \{ \pi \in G : \pi.s_j = s_j \} . \]

From (18) one sees that, for all \( x \in X \) and \( \pi \in \text{iso}_G(s_j) \), we have \( s_j(x) = s_j(\pi^{-1}.x) \). Thus, \( s_j \) is constant on every orbit of the action of the subgroup \( \text{iso}_G(s_j) \) of \( G \) on \( X \). We conclude the following.

Observation 7. In the setting described above, if \( \mathcal{F} \) is a family of subsets of \( X \) such that, for each \( j \in [d] \), there is a sub-family \( \mathcal{F}_j \) partitioning \( X \) and consisting of vertex sets each of which is contained in an orbit under the action of \( \text{iso}_G(s_j) \) on \( X \), then \( s \) is consistent with \( \mathcal{F} \).

In general, it will be impossible to identify the isotropy groups \( \text{iso}_G(s_j) \) without more knowledge on the section \( s \). However, for each isotropy group \( \text{iso}_G(s_j) \), one can at least bound its index \( (G : \text{iso}_G(s_j)) \) in \( G \).

Lemma 8. In the setting described above, we have \( (G : \text{iso}_G(s_j)) \leq d \).

Proof: This follows readily from the fact that the index \( (G : \text{iso}_G(s_j)) \) of the isotropy group of the element \( s_j \in S \) under the action of \( G \) on \( S \) equals the cardinality of the orbit of \( s_j \) under that action, which due to \( |S| \leq d \), clearly is bounded from above by \( d \).

The bound provided in Lemma 8 can become useful, in case one is able to establish a statement like “if \( \text{iso}_G(s_j) \) has index less than \( \tau \) in \( G \) then it contains a certain subgroup \( H_j \)”.

Choosing \( \mathcal{F}_j \) as the family of orbits of \( X \) under the action of the subgroup \( H_j \) of \( G \), then \( \mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_d \) is a family as in Observation 7. If this family (or any refinement of it) can be used to perform Step (b) in the strategy outlined in the paragraph right after the statement of Lemma 6 then one can conclude the lower bound \( d \geq \tau \) on the number of variables \( d \) in an extension as above.

4. Bounds on Symmetric Extensions of \( P_{\text{match}}^\ell(n) \)

In this section, we use Yannakakis’ method described in Section 5 to prove the following result.

Theorem 9. For every \( n \geq 3 \) and odd \( \ell \) with \( \ell \leq n/2 \), there exists no weakly symmetric subspace extension for \( P_{\text{match}}^\ell(n) \) with at most \( \left( \binom{n}{\ell-1} / 2 \right) \) variables (with respect to the group \( G(n) \) acting via permuting the nodes of \( K_n \) as described in the Introduction).

From Theorem 9, we can derive the following more general lower bounds. Since we need it in the proof of the next result, and also for later reference, we state a simple fact on binomial coefficients first.
Lemma 10. For each constant \( b \in \mathbb{N} \) there is some constant \( \beta > 0 \) with
\[
\left( \frac{M - b}{N} \right) \geq \beta \left( \frac{M}{N} \right)
\]
for all large enough \( M \in \mathbb{N} \) and \( N \leq \frac{M}{2} \).

Theorem 11. There is a constant \( C > 0 \) such that, for all \( n \) and \( 1 \leq \ell \leq \frac{n}{2} \), the size of every extension for \( P^\ell_{\text{match}}(n) \) that is symmetric (with respect to the group \( \mathcal{G}(n) \) acting via permuting the nodes of \( K_n \) as described in the Introduction) is bounded from below by
\[
C \cdot \left( \left\lfloor \frac{n}{|\ell - 1|/2} \right\rfloor \right).
\]

Proof. For odd \( \ell \), this follows from Theorem 9 using Lemmas 12 and 4. For even \( \ell \), the polytope \( P^{\ell-1}_{\text{match}}(n-2) \) is (isomorphic to) a face of \( P^\ell_{\text{match}}(n) \) defined by \( x_e = 1 \) for an arbitrary edge \( e \) of \( K_n \). From this, as \( \ell - 1 \) is odd and not larger than \( (n-2)/2 \) with \( |(\ell-2)/2| = |(\ell-1)/2| \), and due to Lemma 10, the theorem follows by Lemma 3.

For even \( n \) and \( \ell = n/2 \), Theorem 11 provides a similar bound to Yannakakis result (see Step 2 in the proof of [14, Theorem 1]) that no weakly symmetric subspace extension of the perfect matching polytope of \( K_n \) has a number of variables that is bounded by \( (n/2) \) for any \( k < n/4 \).

Theorem 11 in particular implies that the size of every symmetric extension for \( P^\ell_{\text{match}}(n) \) with \( \Omega(\log n) \leq \ell \leq n/2 \) is bounded from below by \( n^{\Omega(\log n)} \), which has the following consequence.

Corollary 12. For \( \Omega(\log n) \leq \ell \leq n/2 \), there is no compact extended formulation for \( P^\ell_{\text{match}}(n) \) that is symmetric (with respect to the group \( G = \mathcal{G}(n) \) acting via permuting the nodes of \( K_n \) as described in the Introduction).

The rest of this section is devoted to prove Theorem 9. Throughout, with \( \ell = 2k + 1 \), we assume that \( Q \subseteq \mathbb{R}^d \) with \( d \leq \binom{n}{k} \) is a weakly symmetric subspace extension of \( P^{2k+1}_{\text{match}}(n) \) for \( 4k + 2 \leq n \). We will only consider the case \( k \geq 1 \), as for \( \ell = 1 \) the theorem trivially is true (note that we restrict to \( n \geq 3 \)). Weak symmetry is meant with respect to the action of \( G = \mathcal{G}(n) \) on the set \( X \) of vertices of \( P^{2k+1}_{\text{match}}(n) \) as described in the Introduction, and we assume \( s : X \rightarrow Q \) to be a section as required in the definition of weak symmetry. Thus, we have
\[
X = \{ \chi(M) \in \{0,1\}^{E_n} : M \in \mathcal{M}^{2k+1}(n) \},
\]
where \( \mathcal{M}^{2k+1}(n) \) is the set of all matchings \( M \subseteq E_n \) with \( |M| = 2k + 1 \) in the complete graph \( K_n = (V,E) \) (with \( V = [n] \)), and
\[
(\pi \cdot \chi(M))_{\{v,w\}} = \chi(M)_{\{\pi^{-1}(v),\pi^{-1}(w)\}}
\]
holds for all \( \pi \in \mathcal{G}(n) \), \( M \in \mathcal{M}^{2k+1}(n) \), and \( \{v,w\} \in E \).

In order to identify suitable subgroups of the isotropy groups \( \text{iso}_\mathcal{G}(n)(s_j) \) (see the remarks at the end of Section 3), we use the following result on subgroups of the symmetric group \( \mathcal{G}(n) \), where \( \mathfrak{A}(n) \subseteq \mathcal{G}(n) \) is the alternating group formed
by all even permutations of \([n]\). This result is Claim 2 in the proof of Thm. 1 of Yannakakis paper [14]. Its proof relies on a theorem of Bochert’s [3] stating that any subgroup of \(\mathfrak{S}(m)\) that acts primitively on \([m]\) and does not contain \(\mathfrak{A}(m)\) has index at least \([(m+1)/2]\) in \(\mathfrak{S}(m)\) (see [13, Thm. 1.4.2]).

**Lemma 13.** For each subgroup \(U\) of \(\mathfrak{S}(n)\) with \((\mathfrak{S}(n) : U) \leq \binom{n}{k}\) for \(k < \frac{n}{4}\), there is some \(W \subseteq \left\{V_{\pi} : \pi \in \mathfrak{A}(n) \right\}\) such that

\[
H_j = \{ \pi \in \mathfrak{A}(n) : \pi(v) = v \text{ for all } v \in W \} \subseteq U
\]

holds. As we assumed \(d \leq \left(\frac{n}{4}\right)\) (with \(k < \frac{n}{4}\) due to \(4k + 2 \leq n\)), Lemmas 8 and 13 imply \(H_j \subseteq \left\{V_{\pi} \right\}\) for all \(j \in [d]\). For each \(j \in [d]\), two vertices \(\chi(M)\) and \(\chi(M')\) of \(\mathcal{P}_{\text{match}}(n)\) (with \(M, M' \in \mathcal{M}_{2k+1}(n)\)) are in the same orbit under the action of the group \(H_j\) if and only if we have

\[
M \cap E(V_j) = M' \cap E(V_j) \quad \text{and} \quad V_j \setminus M = V_j \setminus M'. \tag{19}
\]

Indeed, it is clear that (19) holds if we have \(\chi(M') = \pi.\chi(M)\) for some permutation \(\pi \in H_j\). In turn, if (19) holds, then there clearly is some permutation \(\pi \in \mathfrak{S}(n)\) with \(\pi(v) = v\) for all \(v \in V_j\) and \(M' = \pi.M\). Due to \(|M| = 2k + 1 > 2|V_j|\) there is some edge \(\{u, w\} \in M\) with \(u, w \notin V_j\). Denoting by \(\tau \in \mathfrak{S}(n)\) the transposition of \(u\) and \(w\), we thus also have \(\pi \tau(v) = v\) for all \(v \in V_j\) and \(M' = \pi \tau.M\). As one of the permutations \(\pi\) and \(\pi \tau\) is even, say \(\pi'\), we find \(\pi' \in H_j\) and \(M' = \pi'.M\), proving that \(M\) and \(M'\) are contained in the same orbit under the action of \(H_j\).

As it will be convenient for Step (b) (referring to the strategy described after the statement of Lemma 6), we will use the following refinements of the partitionings of \(X\) into orbits of \(H_j\) (as mentioned at the end of Section 3). Clearly, for \(j \in [d]\) and \(M, M' \in \mathcal{M}_{2k+1}(n)\),

\[
M \setminus E(V \setminus V_j) = M' \setminus E(V \setminus V_j) \tag{20}
\]

implies (19). Thus, for each \(j \in [d]\), the equivalence classes of the equivalence relation defined by (20) refine the partitioning of \(X\) into orbits under \(H_j\), and we may use the collection of all these equivalence classes (for all \(j \in [d]\)) as the family \(\mathcal{F}\) in Observation 7. With

\[
\Lambda = \{(A, B) : A \subseteq E \text{ matching and there is some } j \in [d] \text{ with } A \subseteq E \setminus E(V \setminus V_j), B = V_j \setminus V(A)\},
\]

(with \(V(A) = \bigcup_{a \in A} a\)) we hence have \(\mathcal{F} = \{F(A, B) : (A, B) \in \Lambda\}\), where

\[
F(A, B) = \{\chi(M) : M \in \mathcal{M}_{2k+1}(n), A \subseteq M \subseteq E(V \setminus B)\}.
\]

In order to construct a subset \(X^* \subseteq X\) which will be used to derive a contradiction as mentioned after Equation (13), we choose two arbitrary disjoint subsets \(V_\ast, V^* \subseteq V\) of nodes with \(|V_\ast| = |V^*| = 2k + 1\), and define

\[
\mathcal{M}^* = \{M \in \mathcal{M}_{2k+1}(n) : M \subseteq E(V_\ast \cup V^*)\}.
\]
as well as

\[ X^* = \{ \chi(M) : M \in \mathcal{M}^* \} . \]

Thus, \( \mathcal{M}^* \) is the set of perfect matchings on \( K(V_\ast \cup V^*) \). Clearly, \( X^* \) is contained in the affine subspace \( S \) of \( \mathbb{R}^E \) defined by \( x_e = 0 \) for all \( e \in E \setminus E(V_\ast \cup V^*) \).

In fact, \( X^* \) is the vertex set of the face \( \mathbb{P}^{2k+1}_{\text{match}}(n) \cap S \) of \( \mathbb{P}^{2k+1}_{\text{match}}(n) \), and for this face the inequality \( x(V_\ast : V^*) \geq 1 \) is valid (where \( (V_\ast : V^*) \) is the set of all edges having one node in \( V_\ast \) and the other one in \( V^* \)), since every matching \( M \in \mathcal{M}^* \) intersects \( (V_\ast : V^*) \) in an odd number of edges. Therefore, in order to derive the desired contradiction, it suffices to find \( c_x \in \mathbb{R} \) (for all \( x \in X^* \)) with

\[ \sum_{x \in X^*} c_x = 1, \]

\[ \sum_{x \in X^*} c_x \cdot 1_F(x) \geq 0_F, \] (22)

and

\[ \sum_{x \in X^*} c_x \sum_{e \in (V_\ast \cup V^*)} x_e = 0 . \] (23)

For each \( i \in [2k+1]_{\text{odd}} = \{1, 3, 5, \ldots, 2k+1\} \), let

\[ \mathcal{M}^*_i = \{ M \in \mathcal{M}^* : |M \cap (V_\ast : V^*)| = i \} \]

(thus, \( \mathcal{M}^* = \bigcup_{i \in [2k+1]_{\text{odd}}} \mathcal{M}^*_i \)) and

\[ \bar{y}^{(i)} = \frac{1}{|\mathcal{M}^*_i|} \sum_{M \in \mathcal{M}^*_i} \chi(M) . \]

Every nonnegative affine combination of the \( \bar{y}^{(i)} \) with coefficients \( \bar{c}_i \) yields a nonnegative affine combination \( \bar{y} \) with coefficients

\[ c_x = \frac{\bar{c}_i}{|\mathcal{M}^*_i|} \quad \text{for all } x = \chi(M) \in X^* \text{ with } M \in \mathcal{M}^*_i . \] (24)

In order to investigate the components \( \bar{y}_{F(A,B)} \) (for \( (A, B) \in \Lambda \)) of such an affine combination

\[ \bar{y} = \sum_{i \in [2k+1]_{\text{odd}}} \bar{c}_i \bar{y}^{(i)} \quad \text{with} \quad \sum_{i \in [2k+1]_{\text{odd}}} \bar{c}_i = 1 , \]

we first observe (for all \( i \in [2k+1]_{\text{odd}} \))

\[ \bar{y}_{F(A,B)}^{(i)} = \frac{1}{|\mathcal{M}^*_i|} \left| \{ M \in \mathcal{M}^*_i : A \subseteq M \subseteq E((V_\ast \cup V^*) \setminus B) \} \right| \]

for all \( (A, B) \in \Lambda \). In particular, we have \( \bar{y}_{F(A,B)}^{(i)} = 0 \) (thus \( \bar{y}_{F(A,B)} = 0 \)) for all \( (A, B) \notin \Lambda^* \) with

\[ \Lambda^* = \{ (A, B) \in \Lambda : A \subseteq E(V_\ast \cup V^*) \text{ and } B \subseteq V \setminus (V_\ast \cup V^*) \} . \]

We partition \( \Lambda^* \) into the subsets \( \Lambda(a_\ast, a^*, a_\ast^*, b) \) containing all \( (A, B) \in \Lambda^* \) with

\[ |A \cap E(V_\ast)| = a_\ast , \quad |A \cap E(V^*)| = a^* , \quad |A \cap (V_\ast : V^*)| = a_\ast^* \text{, and } |B| = b \] .
Let us consider the point \( \bar{y}^{(i)} \) for some \( i \in [2k + 1]_{\text{odd}} \). Since, for all \( M \in \cM^* \), the number of \( (A, B) \in \Lambda(a_*, a^*, a_*^*, b) \) with \( A \subseteq M \) equals
\[
\mu(i, a_*, a^*, a_*^*, b) = \binom{2k + 1 - i}{a_*} \binom{2k + 1 - i}{a^*} \binom{i}{a_*^*} \binom{n - (4k + 2)}{b},
\]
we find that the sum of the coordinates \( \bar{y}^{(i)} \) for all \( (A, B) \in \Lambda(a_*, a^*, a_*^*, b) \) equals \( \mu(i, a_*, a^*, a_*^*, b) \). Since due to the symmetry of the complete graph we have \( \bar{y}^{(i)}_{F(A, B)} = \bar{y}^{(i)}_{F(A', B')} \) for all \( (A, B), (A', B') \in \Lambda(a_*, a^*, a_*^*, b) \), we hence conclude
\[
\bar{y}^{(i)}_{F(A, B)} = \frac{1}{|\Lambda(a_*, a^*, a_*^*, b)|} \mu(i, a_*, a^*, a_*^*, b)
\]
for all \( i \in [2k + 1]_{\text{odd}} \) and \( (A, B) \in \Lambda(i, a_*, a^*, a_*^*, b) \).

In particular, any coefficients \( \bar{c}_i \in \mathbb{R} \) (for \( i \in [2k + 1]_{\text{odd}} \)) that satisfy
\[
\sum_{i \in [2k + 1]_{\text{odd}}} \bar{c}_i = 1 \tag{25}
\]
and
\[
\sum_{i \in [2k + 1]_{\text{odd}}} \bar{c}_i \mu(i, a_*, a^*, a_*^*, b) \geq 0 \quad \text{for all } (a_*, a^*, a_*^*, b) \tag{26}
\]
via (24) yield an affine combination satisfying (21) and (22).

Due to \( |V_j| \leq k \) for all \( j \in [n] \), we have \( |A| \leq k \) (even \( |A| + |B| \leq k \)) for all \( (A, B) \in \Lambda \). Thus, we can restrict condition (26) to those quadruples of nonnegative integers with \( a_* + a^* + a_*^* \leq k \). For such a quadruple,
\[
f_{a_*, a^*, a_*^*, b}(t) = \binom{2k + 1 - t}{a_*} \binom{2k + 1 - t}{a^*} \binom{t}{a_*^*} \binom{n - (4k + 2)}{b}
\]
is a polynomial of degree at most \( k \) in variable \( t \) (expanding \( \binom{\eta}{s} = (\eta(\eta - 1) \cdots (\eta - s + 1) / s! \) for all \( \eta \in \mathbb{R} \) and \( s \in \mathbb{N} \)).

Let \( \bar{c}_i \in \mathbb{R} \) (for \( i \in [2k + 1]_{\text{odd}} \)) be the (unique) solution of the following system having a Vandermonde (thus regular) coefficient matrix:
\[
\sum_{i \in [2k + 1]_{\text{odd}}} \bar{c}_i = 1
\]
\[
\sum_{i \in [2k + 1]_{\text{odd}}} t^\ell \bar{c}_i = 0 \quad \text{for all } \ell \in [k]
\]
Then, for any polynomial
\[
p(t) = \sum_{\ell=0}^{k} \alpha_\ell t^\ell
\]
(with coefficients \( \alpha_0, \ldots, \alpha_k \in \mathbb{R} \)) of degree at most \( k \), we have
\[
\sum_{i \in [2k + 1]_{\text{odd}}} \bar{c}_i p(i) = \sum_{\ell=0}^{k} \alpha_\ell \sum_{i \in [2k + 1]_{\text{odd}}} t^\ell \bar{c}_i = \alpha_0 = p(0).
\]
In particular, with the constant polynomial $p(t) = 1$, this implies (25), and with $p(t) = f_{a^*,a^*,a^*,b}(t)$ it yields

$$
\sum_{i \in [2k+1]_{\text{odd}}} \bar{c}_i \mu(i, a^*, a^*, a^*) = \sum_{i \in [2k+1]_{\text{odd}}} \bar{c}_i f_{a^*,a^*,a^*,b}(i) = f_{a^*,a^*,a^*,b}(0),
$$

(27)

for all $(a^*, a^*, a^*, b)$ with $a^* + a^* + a^* \leq k$. Since, for these quadruples, we have $f_{a^*,a^*,a^*,b}(0) \geq 0$, (26) thus holds for all relevant quadruples.

Actually, (27) also implies (23), which finally concludes the proof of Theorem 9. To see this, let us calculate the left-hand-side

$$
\sum_{x \in X_*} \sum_{e \in (V_* : V_*)} x_e = \sum_{i \in [2k+1]_{\text{odd}}} \bar{c}_i \sum_{e \in (V_* : V_*)} \left\{M \in \mathcal{M}_i^*: e \in M\right\}
$$

of (23). Since we have

$$
\sum_{e \in (V_* : V_*)} \left\{M \in \mathcal{M}_i^*: e \in M\right\} = |\mathcal{M}_i^*| \cdot \mu(i, 0, 0, 1, 0),
$$

we deduce from (27) (recall that we may assume $k \geq 1$, as indicated at the beginning of the proof) that the left-hand-side of (23) equals $f_{0,0,1,0}(0) = 0$.

5. A NON-SYMMETRIC EXTENSION FOR $P_\ell^\text{match}(n)$

We shall establish the following result on the existence of extensions for cardinality restricted matching polytopes in this section.

**Theorem 14.** For all $n$ and $\ell$, there are extensions for $P_\ell^\text{match}(n)$ whose sizes can be bounded by $2^{O(\ell)} n^2 \log n$ (and for which the encoding lengths of the coefficients needed to describe the extensions by linear systems can be bounded by a constant).

In particular, Theorem 14 implies the following, although, according to Corollary 12, no compact symmetric extended formulations exist for $P_\ell^\text{match}(n)$ with $\ell = \Theta(\log n)$.

**Corollary 15.** For all $n$ and $\ell \leq O(\log n)$, there are compact extended formulations for $P_\ell^\text{match}(n)$.

The proof of Theorem 14 relies on the following result on the existence of small families of perfect-hash functions, which is from [1, Sect. 4]. Its proof is based on results from [8, 12].

**Theorem 16** (Alon, Yuster, Zwick [1]). There are maps $\phi_1, \ldots, \phi_{q(n,r)} : [n] \to [r]$ with $q(n,r) \leq 2^{O(r)} \log n$ such that, for every $W \subseteq [n]$ with $|W| = r$, there is some $i \in [q(n,r)]$ for which the map $\phi_i$ is bijective on $W$.

Furthermore, we will use the following two auxiliary results. The first one (Lemma 17) provides a construction of an extension of a polytope that is specified as the convex hull of some polytopes of which extensions are already available. In fact, in this section it will be needed only for the case that these extensions are the polytopes themselves (this is a special case of a result of Balas’, see [2, Thm.2.1]).
However, we will face the slightly more general situation in our treatment of cycle polytopes in Section 6.

**Lemma 17.** If the polytopes $P_i \subseteq \mathbb{R}^m$ (for $i \in [q]$) have extensions $Q_i$ of size $s_i$, respectively, then

$$P = \text{conv}(P_1 \cup \cdots \cup P_q)$$

has an extension of size $\sum_{i=1}^q(s_i + 2) + 1$.

**Proof.** For each $i \in [q]$, let

$$Q_i = \{ y \in \mathbb{R}^{d_i} : A^{(i)} y \leq b^{(i)} \}$$

be an extension of $P_i$ with projection $p_i : \mathbb{R}^{d_i} \to \mathbb{R}^m$. Then the polyhedron

$$Q = \{ (y^{(1)}, \ldots, y^{(q)}, z) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_q} \times \mathbb{R}^q :$$

$$A^{(i)} y^{(i)} \leq z_i b^{(i)} \text{ for all } i \in [q], \sum_{i \in [q]} z_i = 1, z \geq 0 \}$$

with projection $p : \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_q} \times \mathbb{R}^q \to \mathbb{R}^m$ defined via

$$p(y^{(1)}, \ldots, y^{(q)}, z) = p_1(y^{(1)}) + \cdots + p_q(y^{(q)})$$

is an extension of $P$ of size $\sum_{i=1}^q s_i + 2q + 1$.

Indeed, for every convex combination $x = \sum_{i=1}^q \lambda_i x^{(i)}$ of points $x^{(i)} \in P_i$ with $\sum_{i=1}^q \lambda_i = 1$ and $\lambda \geq 0$, we find that $(\lambda_1 y^{(1)}, \ldots, \lambda_q y^{(q)}, \lambda)$ with $y^{(i)} \in p_i^{-1}(x^{(i)})$ chosen arbitrarily is contained in $Q$ and projects to $x$ via $p$. Conversely, if $(y^{(1)}, \ldots, y^{(q)}, z) \in Q$, then we have $p_i(y^{(i)}) = 0$ for all $i \in [q]$ with $z_i = 0$ (since $A^{(i)} y^{(i)} \leq 0$ implies that $p_i(y^{(i)})$ is contained in the recession cone $\{0\}$ of the polytope $P_i$). Therefore, we obtain

$$p(y^{(a)}, \ldots, y^{(q)}, z) = \sum_{i \in [q] : z_i \neq 0} p_i(y^{(i)}) = \sum_{i \in [q] : z_i \neq 0} z_i p_i(\frac{1}{z_i} y^{(i)}) \in P,$$

since the latter sum is a convex combination of points $p_i(\frac{1}{z_i} y^{(i)}) \in P_i$, as, for each $i \in [q]$ with $z_i \neq 0$, the point $\frac{1}{z_i} y^{(i)}$ is contained in $Q_i$ due to $A^{(i)} y^{(i)} \leq z_i b^{(i)}$. □

The second auxiliary result that we need deals with describing a 0/1-polytope that is obtained by splitting variables of a 0/1-polytope of which a linear description is already available.

**Lemma 18.** Let $S$ be a set of subsets of $[t]$,

$$P = \text{conv}\{ \chi(S) \in \{0,1\}^t : S \in S \} \subseteq \mathbb{R}^t,$$

the corresponding 0/1-polytope, $J = J(1) \cup \cdots \cup J(t)$ a disjoint union of finite sets $J(i)$,

$$S^* = \{ S^* \subseteq J : \text{There is some } S \in S \text{ with }$$

$$|S^* \cap J(i)| = 1 \text{ for all } i \in S, |S^* \cap J(i)| = 0 \text{ for all } i \notin S \}, \quad (28)$$

and

$$P^* = \text{conv}\{ \chi(S^*) \in \{0,1\}^J : S^* \in S^* \}.$$
If $P = \{ y \in [0, 1]^t : Ay \leq b \}$ for some $A \in \mathbb{R}^{s \times t}$ and $b \in \mathbb{R}^s$, then

$$P^* = \{ x \in [0, 1]^t : \sum_{i=1}^t A_{s,i} \cdot \sum_{j \in J(i)} x_j \leq b_i \text{ for all } i \in [t] \}.$$  \hfill (29)

Proof. Let us first observe that, for each $S^* \in S^*$, there is a unique $S \in S$ as in (28), which we denote by $S(S^*)$.

Clearly, $P^*$ is contained in the polytope on the right hand side of (29). In order to show the reverse inclusion, let $x \in [0, 1]^t$ be some point that satisfies $\sum_{i=1}^t A_{s,i}x(J(i)) \leq b_i$ for all $i \in [s]$ (with $x(J(i)) = \sum_{j \in J(i)} x_j$). Thus, we have $\tilde{x} = (x(J(i)))_{i \in [t]} \in P$, and hence there are coefficients $\lambda_S \geq 0$ for all $S \in S$ with

$$\tilde{x} = \sum_{S \in S} \lambda_S \chi(S) \quad \text{and} \quad \sum_{S \in S} \lambda_S = 1.$$  \hfill (30)

Denoting, for each $i \in [t]$, by $S(x)$ the set of all $S \in S$ with $x(J(i)) \neq 0$, we find $\lambda_S = 0$ for all $S \in S \setminus S(x)$, which implies

$$x(J(i)) = \sum_{S \in S(x)} \lambda_S.$$  \hfill (31)

We will complete the proof by showing that the coefficients

$$\lambda_{S^*} = \frac{\prod_{j \in S^*} x_j}{\prod_{i \in S(S^*)} x(J(i))} \lambda_{S^*} \geq 0$$

for all $S^* \in S^*$ with $S(S^*) \in S(x)$ and $\lambda_{S^*} = 0$ for all other $S^* \in S^*$ satisfy

$$\sum_{S^* \in S^*} \lambda_{S^*} \chi(S^*) = x \quad \text{and} \quad \sum_{S^* \in S^*} \lambda_{S^*} = 1.$$  \hfill (31)

The second equation of (31) follows readily from $\sum_{S \in S} \lambda_S = 1$ and the fact that, for all $S \in S(x)$, we have

$$\sum_{S^*: S(S^*) = S} \lambda_{S^*}x = \frac{\lambda_S}{\prod_{i \in S} x(J(i))} \sum_{S^*: S(S^*) = S} \prod_{j \in J(i)} x_j$$

$$= \frac{\lambda_S}{\prod_{i \in S} x(J(i))} \prod_{i \in S} \sum_{j \in J(i)} x_j = \frac{\lambda_S}{\prod_{i \in S} x(J(i))} \prod_{i \in S} x(J(i)) = \lambda_S.$$
We establish the first equation of (31) by considering each component \( j \in J \) with \( j \in J(i) \) for some \( i \in [t] \) separately:

\[
\sum_{S^* \in S^*} \lambda_{S} \cdot \chi(S^*)_{j} = \sum_{S \in S(x)} \sum_{S^*} \lambda_{S} \cdot \prod_{i \in S} x(J(i)) \prod_{\ell \in S^*} x_{\ell} \\
= \sum_{S \in S(x)} \lambda_{S} \cdot \prod_{i \in S} x(J(i)) \prod_{\ell \in S^*} x_{\ell} \\
= x_{j} \cdot \sum_{S \in S(x)} \lambda_{S} \cdot \prod_{i \in S} x(J(i)) \prod_{\ell \in S^*} x_{\ell} \\
= x_{j} \cdot \sum_{S \in S(x)} \lambda_{S} \cdot \prod_{i \in S} x(J(i)) \\
\]

where the second but last equation follows from \( (30) \).

In order to prove Theorem 14, let \( \phi_1, \ldots, \phi_q \) be maps as guaranteed to exist by Theorem 16 with \( r = 2\ell \) and \( q = q(n, 2\ell) \leq 2^{O(\ell)} \log n \), and denote \( \mathcal{M}_i = \{ M \in \mathcal{M}^{\ell}(n) : \phi_i \text{ is bijective on } V(M) \} \) for each \( i \in [q] \). By Theorem 16 we have \( \mathcal{M}^{\ell}(n) = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_q \). Consequently,

\[
P_{\text{match}}^{\ell}(n) = \text{conv}(P_1 \cup \cdots \cup P_q) \tag{32}
\]

with \( P_i = \text{conv}\{ \chi(M) : M \in \mathcal{M}_i \} \) for all \( i \in [q] \), where we have

\[
P_i = \{ x \in \mathbb{R}^{E_+^k} : x_{E \setminus E_i} = 0, x(\delta(\phi_i^{-1}(s))) = 1 \text{ for all } s \in [2\ell], \\
x(E_i(\phi_i^{-1}(S))) \leq (|S| - 1)/2 \text{ for all } S \subseteq [2\ell], |S| \text{ odd} \},
\]

where \( E_i = E \setminus \bigcup_{j \in [2\ell]} E(\phi_i^{-1}(j)) \). This follows by Lemma 18 from Edmonds’ linear description (5) of the perfect matching polytope \( P_{\text{match}}^{\ell}(2\ell) \) of \( K_{2\ell} \). As the sum of the number of variables and the number of inequalities in the description of \( P_i \) is bounded by \( 2^{O(\ell)} n^2 \) (the summand \( n^2 \) coming from the nonnegativity constraints on \( x \in \mathbb{R}^{E_+^k} \) and the constant in \( O(\ell) \) being independent of \( i \)), we obtain an extension of \( P_{\text{match}}^{\ell}(n) \) of size \( 2^{O(\ell)} n^2 \log n \) by Lemma 17. This proves Theorem 14.

6. Extensions for Cycle Polytopes

By a modification of Yannakakis’ construction for the derivation of lower bounds on the sizes of symmetric extensions for traveling salesman polytopes from the corresponding lower bounds for matching polytopes [14 Thm. 2], we obtain lower bounds on the sizes of symmetric extensions for \( P_{\text{cycl}}^{\ell}(n) \). The lower bound \( \ell \geq 42 \)
in the statement of the theorem is convenient with respect to both formulating the bound and proving its validity.

**Theorem 19.** There is a constant $C' > 0$ such that, for all $n$ and $42 \leq \ell \leq n$, the size of every extension for $P^\ell_{\text{cycl}}(n)$ that is symmetric (with respect to the group $\mathfrak{S}(n)$ acting via permuting the nodes of $K_n$ as described in the Introduction) is bounded from below by

$$C' n \left( \left\lfloor \frac{n}{3} \right\rfloor - 1 \right) / 2. $$

**Proof.** For $\ell \leq n$, let us define

$$\bar{\ell} = \ell \mod 6, \quad n' = \left\lfloor \frac{n - \bar{\ell}}{3} \right\rfloor, \quad \text{and} \quad \ell' = \frac{\ell - \bar{\ell}}{6}. $$

For later reference, let us argue that we have

$$\ell' \leq \frac{n'}{2}. \quad (33)$$

In order to establish (33), we have to show

$$\frac{\ell - \bar{\ell}}{3} \leq \left\lfloor \frac{n - \bar{\ell}}{3} \right\rfloor, \quad (34)$$

which follows readily for $\ell \leq n - 2$ (due to $\lfloor a/3 \rfloor \geq (a - 2)/3$ for all $a \in \mathbb{Z}$). For $\ell \geq n - 2$ (thus $0 \leq n - \ell \leq 2$) we have

$$(n - \bar{\ell}) \mod 3 = ((n - \bar{\ell}) \mod 6) \mod 3 = (n - \ell) \mod 3 = n - \ell,$$

and thus (34) in this case is satisfied due to

$$\left\lfloor \frac{n - \bar{\ell}}{3} \right\rfloor = \frac{n - \bar{\ell}}{3} - \frac{1}{3} (n - \bar{\ell}) \mod 3 = \frac{n - \bar{\ell}}{3} - \frac{n - \ell}{3} = \frac{\ell - \bar{\ell}}{3}. $$

As we have $3n' + \bar{\ell} \leq n$, we can find four pairwise disjoint subsets $S, T, R, U$ of nodes of the complete graph $K_n = (V, E)$ on $n$ nodes with $|S| = |T| = |U| = n'$ and $|R| = \ell$ (see Fig. 6). We denote the elements of these sets as follows:

$$S = \{s_1, \ldots, s_{n'}\} \quad T = \{t_1, \ldots, t_{n'}\} \quad U = \{u_1, \ldots, u_{n'}\} \quad R = \{r_1, \ldots, r_{\ell}\}$$

Define the subset

$$E_0 = (S : U) \cup (S : R) \cup \{(t_i, v) \in E : i \in [n'], v \in V \setminus \{s_i, u_i\}\}$$

of edges of $K_n$, and denote by $F$ the face of $P^\ell_{\text{cycl}}(n)$ that is defined by $x_e = 0$ for all $e \in E_0$.

Every cycle $C \in \mathcal{C}^\ell(n)$ with $C \cap E_0 = \emptyset$ satisfies $|V(C) \cap T| \leq \lfloor \ell/6 \rfloor$, because $C$ visits at least two nodes (from $V \setminus T$) between any two visits to $T$, and $|V(C) \cap T|$ is even. Therefore, denoting

$$\tilde{C} = \{C \in \mathcal{C}^\ell(n) : C \cap E_0 = \emptyset, |V(C) \cap T| = \lfloor \ell/6 \rfloor\},$$

we find that

$$\tilde{F} = \text{conv}\{\chi(C) : C \in \tilde{C}\} = \{x \in F : x(\delta(T)) = 2\lfloor \ell/6 \rfloor\}$$
is a face of $F$. Moreover, for every $C \in \tilde{C}$, we have $|C \cap E(S)| \geq \lfloor \ell/6 \rfloor$ (denoting by $E(S)$ the set of edges of $K_n$ with both nodes in $S$). Thus, with

$$C' = \{ C \in \tilde{C} : |C \cap E(S)| = \lfloor \ell/6 \rfloor \}$$

we find that

$$P' = \text{conv}\{ \chi(C) : C \in C' \} = \{ x \in \tilde{F} : x(E(S)) = \lfloor \ell/6 \rfloor \}$$

is a face of $\tilde{F}$. It is the face

$$P' = \{ x \in P^\ell_{\text{cycl}}(n) : x(E(S)) = \lfloor \ell/6 \rfloor, x(E_0) = 0 \}$$

of $P^\ell_{\text{cycl}}(n)$.

**Figure 1.** A cycle of length $\ell = 15$ in $K_{21}$ inducing a matching of size 2 in $K_5$.

Since a cycle $C \in C^\ell(n)$ is contained in $C'$ if and only if $C \cap E(S)$ is a matching of size $\ell' = \lfloor \ell/6 \rfloor$, we find that via the orthogonal projection $q : \mathbb{R}^E \to \mathbb{R}^{E(S)}$ we have

$$q(P') = P'^\ell_{\text{match}}(n')$$

after identification of $S$ with the node set of $K_{n'}$ via $s_i \mapsto i$ for all $i \in [n']$. Moreover, for every $\tau \in \mathcal{S}(n')$ the permutation $\pi \in \mathcal{S}(n)$ with

$$\pi(s_i) = s_{\tau(i)}, \quad \pi(t_i) = t_{\tau(i)}, \quad \pi(u_i) = u_{\tau(i)}$$

for all $i \in [n']$, and $\pi(r) = r$ for all $r \in R$ satisfies $\pi.P' = P'$ and

$$q(\pi.x) = \tau.q(x) \quad \text{for all } x \in \mathbb{R}^{E_{n'}}.$$

Hence, due to Lemma 3, a symmetric extension of $P^\ell_{\text{cycl}}(n)$ of size $s$ yields a symmetric extension of $P'^\ell_{\text{match}}(n')$ of size at most $s + n^2$ (as one can define the
face $P'$ of $P_{\text{cyc}}^\ell(n)$ by $2 + |E_0| \leq n^2$ equations), which, due to (33) and Theorem 11 implies (with the constant $C > 0$ from Theorem 11)

$$s \geq \frac{C}{2} \cdot \left( \frac{\lfloor \frac{n-\ell}{3} \rfloor}{\lfloor \frac{\ell}{n} \rfloor - 1} / 2 \right)$$

(35)

for large enough $n$ (since, due to $\ell \geq 42$, the binomial coefficient in (35) grows at least cubically in $n$). Because of $\ell \leq 5$, Lemma 10 implies the existence of a constant $C' > 0$ as claimed in the theorem. □

**Corollary 20.** For $\Omega(\log n) \leq \ell \leq n$, there is no compact extended formulation for $P_{\text{cyc}}^\ell(n)$ that is symmetric (with respect to the group $\mathfrak{S}(n)$ acting via permuting the nodes of $K_n$ as described in the Introduction).

On the other hand, if we drop the symmetry requirement, we find extensions of the following size.

**Theorem 21.** For all $n$ and $\ell$, there are extensions for $P_{\text{cyc}}^\ell(n)$ whose sizes can be bounded by $2^{O(\ell)} n^3 \log n$ (and for which the encoding lengths of the coefficients needed to describe the extensions by linear systems can be bounded by a constant).

Before we prove Theorem 21 we state a consequence that is similar to Corollary 12 for matching polytopes. It shows that, despite the non-existence of symmetric extensions for the polytopes associated with cycles of length $\Theta(\log n)$ (Corollary 20), there are non-symmetric compact extensions of these polytopes.

**Corollary 22.** For all $n$ and $\ell \leq O(\log n)$, there are compact extended formulations for $P_{\text{cyc}}^\ell(n)$.

The rest of the section is devoted to prove Theorem 21 i.e., to construct an extension of $P_{\text{cyc}}^\ell(n)$ whose size is bounded by $2^{O(\ell)} n^3 \log n$. We proceed similarly to the proof of Theorem 14 (the construction of extensions for matching polytopes), this time starting with maps $\phi_1, \ldots, \phi_q$ as guaranteed to exist by Theorem 16 with $r = \ell$ and $q = q(n, \ell) \leq 2^{O(\ell)} \log n$, and defining

$$C_i = \{ C \in C^\ell(n) : \phi_i \text{ is bijective on } V(C) \}$$

for each $i \in [q]$. Thus, we have $C^\ell(n) = C_1 \cup \cdots \cup C_q$, and hence,

$$P_{\text{cyc}}^\ell(n) = \text{conv}(P_1 \cup \cdots \cup P_q)$$

(36)

with $P_i = \text{conv}\{ \chi(C) : C \in C_i \}$ for all $i \in [q]$. Due to Lemma 17 it suffices to exhibit, for each $i \in [q]$, an extension of $P_i$ of size bounded by $O(2^\ell \cdot n^3)$ (with the constant independent of $i$). Towards this end, let for $i \in [q]$

$$V_i = \phi_i^{-1}(e) \quad \text{for all } e \in [\ell],$$

and define, for each $v^* \in V_i$,

$$P_i(v^*) = \text{conv}\{ \chi(C) : C \in C_i, v^* \in V(C) \}.$$

Thus, we have

$$P_i = \text{conv} \bigcup_{v^* \in V_i} P_i(v^*),$$
and hence, again due to Lemma 17, it suffices to construct extensions of the $P_i(v^*)$, whose sizes are bounded by $O(2^\ell \cdot n^2)$.

In order to derive such extensions define, for each $i \in [q]$ and $v^* \in V_\ell$, a directed acyclic graph $D$ with nodes

$$(A, v) \quad \text{for all } A \subseteq [\ell - 1] \text{ and } v \in \phi_i^{-1}(A),$$

as well as two additional nodes $s$ and $t$, and arcs

$$(s, \{\phi_i(v)\}, v)) \quad \text{and} \quad (([\ell - 1], v), t)$$

for all $v \in \phi_i^{-1}([\ell - 1])$, as well as

$$((A, v), (A \cup \{\phi_i(w)\}, w))$$

for all $A \subseteq [\ell - 1]$, $v \in \phi_i^{-1}(A)$, and $w \in \phi_i^{-1}([\ell - 1] \setminus A)$. This is basically the dynamic programming digraph (using an idea going back to [10]) from the color-coding method for finding paths of prescribed lengths described in [1]. Each $s$-$t$-path in $D$ corresponds to a cycle in $C_i$ that visits $v^*$, and each such cycle, in turn, corresponds to two $s$-$t$-paths in $D$ (one for each of the two directions of transversal).

Defining $Q_i(v^*)$ as the convex hull of the characteristic vectors of all $s$-$t$-paths in $D$ in the arc space of $D$, we find that $P_i(v^*)$ is the image of $Q_i(v^*)$ under the projection whose component function corresponding to the edge $\{v, w\}$ of $K_n$ is given by the sum of all arc variables corresponding to arcs $((A, v), (A', w))$ (for $A, A' \subseteq [\ell - 1]$ if $v^* \notin \{v, w\}$, and by the sum of the two arc variables corresponding to $(s, \{\phi_i(w)\}, w))$ and $(([\ell - 1], v), t)$ in case of $v = v^*$. Clearly, $Q_i(v^*)$ can be described by the nonnegativity constraints, the flow conservation constraints for all nodes in $D$ different from $s$ and $t$, and by the equation stating that there must be exactly one flow-unit leaving $s$. As the number of arcs of $D$ is bounded by $O(2^\ell \cdot n^2)$, we thus have found an extension of $P_i(v^*)$ of the desired size.

7. CONCLUSIONS

The results presented in this paper demonstrate that there are polytopes which have compact extended formulations though they do not admit symmetric ones. These polytopes are associated with matchings (or cycles) of some prescribed cardinalities (see [4] for a recent survey on general cardinality restricted combinatorial optimization problems). Nevertheless, whether there are compact extended formulations for general matching polytopes (or for perfect matching polytopes) or not, remains one of the most interesting open question here. In fact, it is even unknown whether there are any (non-symmetric) extended formulations of these polytopes of size $2^{o(n)}$.

Actually, it seems that there are almost no lower bounds known on the sizes of extensions, except for the one obtained by the observation that every extension $Q$ of a polytope $P$ with $f$ faces (vertices, edges, ..., facets) has at least $f$ faces itself, thus $Q$ has at least $\log f$ facets (since a face is uniquely determined by the subset of facets it is contained in) [9]. It would be most interesting to obtain other
lower bounds, including special ones for 0/1-polytopes. We refer once more to Yannakakis’ paper [14] for a very interesting interpretation of the smallest possible size of an extension of a polytope $P$ to the “positive rank” of a slack matrix associated to $P$.

We finally would like to mention a question with respect to the definition of the size of an extension. As indicated in Section 2 it would be elegant to ignore the number of variables here, since a non-trivial lineality space of the extension may seem to be unnecessary, because it can easily be removed by taking intersection with the orthogonal complement of the lineality space and representing the intersection by means of coordinates with respect to some basis of that orthogonal complement (note that the lineality space of an extension of a polytope is contained in the kernel of the projection). However, it is unclear whether this can always be done in such a way that the symmetry of the extension is preserved. Thus, another open question is whether lineality spaces can really help in constructing small symmetric extended formulations.

REFERENCES

Volker Kaibel
E-mail address: kaibel@ovgu.de

Kanstantsin Pashkovich
E-mail address: pashkovi@imo.math.uni-magdeburg.de

Dirk Oliver Theis
E-mail address: dirk.theis@ovgu.de