Carleman estimates for singular parabolic equations with interior degeneracy and non smooth coefficients*

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Abstract

We establish Carleman estimates for singular/degenerate parabolic Dirichlet problems with degeneracy and singularity occurring in the interior of the spatial domain. Our results are completely new, since this situation is not covered by previous contributions for degeneracy and singularity on the boundary. In addition, we consider non smooth coefficients, thus preventing the use of standard calculations in this framework.

Keywords: Carleman estimates, singular/degenerate equations, Hardy–Poincaré inequality, Caccioppoli inequality, observability inequality, null controllability

2000AMS Subject Classification: 35Q93, 93B05, 93B07, 34H15, 35A23, 35B99

1 Introduction

Controllability issues for parabolic problems have been a mainstream issue in recent years, and several developments have been pursued: starting from the heat equation in bounded and unbounded domain, related contributions have

*The authors are member of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), and are supported by the INdAM - GNAMPA Project Systems with irregular operators
been found for more general situations. A common strategy in showing controllability results is to show that certain global Carleman estimates hold true for the operator which is the adjoint of the given one.

In this paper we focus on a class of singular parabolic operators with interior degeneracy of the form

$$u_t - (a(x)u_x)_x - \frac{\lambda}{b(x)} u,$$

(1.1)

associated to Dirichlet boundary conditions and with \((t, x) \in Q_T := (0, T) \times (0, 1), T > 0\) being a fixed number. Here \(a\) and \(b\) degenerate at the same interior point \(x_0 \in (0, 1)\), and \(\lambda \in \mathbb{R}\) satisfies suitable assumptions (see condition (2.15) below). The fact that both \(a\) and \(b\) degenerate at \(x_0\) is just for the sake of simplicity and shortness: all the stated results are still valid if they degenerate at different points. However, the prototypes are \(a(x) = |x - x_0|^{K_1}\) and \(b(x) = |x - x_0|^{K_2}\) for some \(K_1, K_2 > 0\). The main goal we have in mind is to establish global Carleman estimates for operators of the form given in (1.1).

Such estimates for uniformly parabolic operators without degeneracies or singularities have been largely developed (see, e.g., Fursikov–Imanuilov [27]). Recently, these estimates have been also studied for operators which are not uniformly parabolic. Indeed, as pointed out by several authors, many problems coming from Physics (see [30]), Biology (see [19]) and Mathematical Finance (see [29]) are described by degenerate parabolic equations. In particular, new Carleman estimates (and consequently null controllability properties) were established in [1], and also in [13], [33], for the operator

$$u_t - (au_x)_x + c(t, x)u, \quad (t, x) \in Q_T,$$

where \(a(0) = a(1) = 0\) and \(c \in L^\infty(Q_T)\) (see also [11], [12] or [23] for problems in non divergence form).

An interesting situation is the case of parabolic operators with singular lower order terms. First results in this direction were obtained in [35] for the non degenerate heat operator with singular potentials

$$u_t - \Delta u - \lambda \frac{1}{|x|^2} u, \quad (t, x) \in (0, T) \times \Omega,$$

(1.2)

with associated Dirichlet boundary conditions in a bounded domain \(\Omega\) containing \(0\) (see also [37] for the wave and Schrödinger equations and [15] for boundary degeneracy). The case \(K_2 = 2\) is the case of the so-called inverse square potential that arises for example in quantum mechanics (see, for example, [3], [17]), or in combustion problems (see, for example, [3], [9], [18], [28]). This potential is known to generate interesting phenomena. For example, in [3] and in [4] it was proved that, for all values of \(\lambda\), global positive solutions exist if \(K_2 < 2\), whereas instantaneous and complete blow-up occurs if \(K_2 > 2\). In the critical case, i.e. \(K_2 = 2\), the value of the parameter \(\lambda\) determines the behavior of the equation: if \(\lambda \leq \frac{1}{4}\) (which is the optimal constant of the Hardy inequality) global positive
solutions exist, while, if $\lambda > \frac{1}{4}$, instantaneous and complete blow-up occurs
(for other comments on this argument we refer to [36]). We recall that in [38],
Carleman estimates were established for (1.2) under the condition $\lambda \leq \frac{1}{4}$. On
the contrary, if $\lambda > \frac{1}{4}$, in [20] it was proved that null controllability fails.

Recently, in [36], Vancostenoble studied the operator that couples a degenerate diffusion coefficient with a singular potential. In particular, for $K_1 \in [0, 2)$ and $K_2 \leq 2 - K_1$, the author established Carleman estimates for the operator

$$u_t - (x^{K_1}u_x)_x - \lambda \frac{1}{x^{K_2}}u, \quad (t,x) \in Q_T,$$

unifying the results of [14] and [38] in the purely degenerate operator and in the purely singular one, respectively. This result was then extended in [21] and in [22] to the operators

$$u_t - (a(x)u_x)_x - \lambda \frac{1}{x^{K_2}}u, \quad (t,x) \in Q_T, \quad (1.3)$$

for $a \sim x^{K_1}$, $K_1 \in [0, 2)$ and $K_2 \leq 2 - K_1$. Here, as before, the function $a$ degenerates at the boundary of the space domain, and Dirichlet boundary conditions are in force.

We remark the fact that all the papers cited so far deal with a singular/degenerate operator with degeneracy or singularity at the boundary of the domain. For example, in (1.3) as $a$ one can also consider the double power function

$$a(x) = x^k(1 - x)^\kappa, \quad x \in [0, 1],$$

where $k$ and $\kappa$ are positive constants. To the best of our knowledge, [7], [24] and [25] are the first papers dealing with Carleman estimates (and, consequently, null controllability) for operators (in divergence and in non divergence form with Dirichlet or Neumann boundary conditions) with mere degeneracy at the interior of the space domain (for related systems of degenerate equations we refer to [4]).

We emphasize the fact that an interior degeneracy does not imply a simple adaptation of previous results and of the techniques for boundary ones. Indeed, imposing homogeneous Dirichlet boundary conditions, in the latter case one knows a priori that any function vanishes exactly at the degeneracy point. Now, since the degeneracy point is in the interior of the spatial domain, such information is not valid anymore, in general, and proofs cannot take advantage of this fact.

For this reason, the present paper is devoted to study the operator defined in (1.1), that couples a general degenerate diffusion coefficient with a general singular potential with degeneracy and singularity at the interior of the space domain. In particular, under suitable conditions on all the parameters of the operator, we establish Carleman estimates and, as a consequence, null controllability for the associated generalized heat problem. Clearly, this result generalizes
the one obtained in [24] or [25]: in fact, if \( \lambda = 0 \) (that is, if we consider the purely degenerate case), we recover the main contributions therein.

We also remark the fact that, though we have in mind prototypes as power functions for the degeneracy and the singularity, we do not limit our investigation to these functions, which are analytic out of their zero. Indeed, in this paper, pure powers singularities and degeneracies are considered only as a by–product of our main results, which are valid for non smooth general coefficients. This is quite a new view–point when dealing with Carleman estimates, since in this framework it is natural to assume that all the coefficients in force are quite regular. However, though this strategy has been successful for years, it is clear that also more irregular coefficients can play a rôle. For this reason, for the first time to our best knowledge, in [25] non smooth degenerate coefficients were treated. Continuing in this direction, here we consider operators which contain both degenerate and singular coefficients, as in [21], [22] and [36], but with low regularity.

The classical approach to study singular operators relies in the validity of the Hardy–Poincaré inequality

\[
\int_0^1 \frac{u^2}{x^2} dx \leq 4 \int_0^1 (u')^2 dx,
\]

which is valid for every \( u \in H^1(0,1) \) with \( u(0) = 0 \) (or of analogous ones for operators acting in higher dimensions). Similar inequalities are the starting point to prove well–posedness of the associated problems in the Sobolev spaces under consideration. In our situation, we prove an inequality related to (1.4), but with a degeneracy coefficient in the gradient term; such an estimate is valid in a suitable Hilbert space \( \mathcal{H} \) we shall introduce below, and it states the existence of \( C > 0 \) such that for all \( u \in \mathcal{H} \) we have

\[
\int_0^1 \frac{u^2}{b} dx \leq C \int_0^1 a(u')^2 dx.
\]

This inequality, which is related to another weighted Hardy–Poincaré inequality (see Proposition 2.1), is the key step for the well–posedness of (1.5). Once this is done, global Carleman estimates follow, provided that an ad hoc choice of the weight functions is made (see Theorem 3.1).

The introduction of the space \( \mathcal{H} \) (which may coincide with the usual Sobolev space in some cases) is another feature of this paper, which is completely new with respect to all the previous approaches. Indeed, including the validity of a Hardy–Poincaré inequality with double weight in the definition of \( \mathcal{H} \) has the advantage of obtaining immediately some useful functional properties, that in general could be hard to show in the usual Sobolev spaces. The choice of suitable function spaces where posing the problem was already done for the “critical” and “supercritical” cases (when \( \lambda \) equals or exceeds the best constant in the classical Hardy–Poincaré inequality) in [37] and [39] for purely singular problems. However, as already done in the purely degenerate case (4.11, 12, 14, 21, 22, 29, 34, 24, 25, 26), a weighted Sobolev space must be used.
For this reason, we believe that it is natural to unify these approaches in the singular/degenerate, as we do.

As it is well known, Carleman estimates are a crucial step in proving null controllability properties for the corresponding evolution problem

\[
\begin{cases}
  u_t - (au)_x - \frac{\lambda}{b(x)}u = h(t,x)\chi_\omega(x), \quad (t,x) \in Q_T, \\
  u(t,0) = u(t,1) = 0, \quad t \in (0,T), \\
  u(0,x) = u_0(x), \quad x \in (0,1),
\end{cases}
\]  

(1.5)

i.e. in showing that there exists \( h \in L^2(Q_T) \) such that \( u(T,x) \equiv 0 \) for \( x \in [0,1] \). Here, \( u_0 \in L^2(0,1) \), the control \( h \in L^2(Q_T) \) acts on a non empty interval \( \omega \subset (0,1) \) and \( \chi_\omega \) is the characteristic function of \( \omega \). In order to obtain such a result, the usual strategy is to use Carleman estimates to prove an observability inequality of the form

\[
\int_0^1 v^2(0,x)dx \leq C_T \int_0^T \int_\omega v^2(t,x)dxdt
\]

(1.6)

for any solution \( v \) of the adjoint problem of (1.5)

\[
\begin{cases}
  v_t + (av)_x + \frac{\lambda}{b(x)}v = 0, \quad (t,x) \in Q_T, \\
  v(t,0) = v(t,1) = 0, \quad t \in (0,T), \\
  v(T,x) = v_T(x),
\end{cases}
\]

where \( C_T > 0 \) is a universal constant. In the non degenerate case this has been obtained by a well–established procedure using Carleman and Caccioppoli inequalities. In our singular/degenerate non smooth situation, we need a new suitable Caccioppoli inequality (see Proposition 4.2), as well as global Carleman estimates in the non smooth non degenerate and non singular case (see Proposition 4.3), which will be used far away from \( x_0 \) within a localization procedure via cut–off functions. Once these tools are established, we are able to prove an observability inequality like (1.6), and then controllability results for (1.5).

Finally, we remark that our studies with non smooth coefficients are particularly useful. In fact, though null controllability results could be obtained also in other ways, for example by a localization technique (at least when \( x_0 \in \omega \)), in [25] it is shown that with non smooth coefficients, even when \( \lambda = 0 \), this is not always the case. For this, our approach with observability inequalities is very general and permits to cover more involved situations.

The paper is organized in the following way: in Section 2 we study the well–posedness of problem (1.5), giving some general tools that we shall use several times. In Section 3 we provide one of the main results of this paper, i.e. Carleman estimates for the adjoint problem to (1.5). In Section 4 we apply the previous Carleman estimates to prove an observability inequality, which,
together with a Caccioppoli type inequality, lets us derive new null controllability results for the associated singular/degenerate problem, also when the degeneracy and the singularity points are inside the control region.

A final comment on the notation: by $c$ or $C$ we shall denote universal positive constants, which are allowed to vary from line to line.

**Acknowledgments.** The authors are very grateful to both the anonymous referees for the careful reading of the paper and for their comments and remarks, which lead to a much better organization of the paper.

## 2 Well-posedness

The ways in which $a$ and $b$ degenerate at $x_0$ can be quite different, and for this reason we distinguish four different types of degeneracy. In particular, we consider the following cases:

**Hypothesis 2.1.** Double weakly degenerate case (WWD): there exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ on $[0,1] \setminus \{x_0\}$, $a, b \in W^{1,1}(0,1)$ and there exists $K_1, K_2 \in (0,1)$ such that $(x-x_0)a' \leq K_1 a$ and $(x-x_0)b' \leq K_2 b$ a.e. in $[0,1]$.

**Hypothesis 2.2.** Double strongly degenerate case (SSD): there exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ on $[0,1] \setminus \{x_0\}$, $a, b \in W^{1,\infty}(0,1)$ and there exist $K_1, K_2 \in [1,2)$ such that $(x-x_0)a' \leq K_1 a$ and $(x-x_0)b' \leq K_2 b$ a.e. in $[0,1]$.

**Hypothesis 2.3.** Weakly-strongly degenerate case (WSD): there exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ on $[0,1] \setminus \{x_0\}$, $a \in W^{1,1}(0,1)$, $b \in W^{1,\infty}(0,1)$ and there exist $K_1 \in (0,1)$, $K_2 \in [1,2)$ such that $(x-x_0)a' \leq K_1 a$ and $(x-x_0)b' \leq K_2 b$ a.e. in $[0,1]$.

**Hypothesis 2.4.** Strongly-weakly degenerate case (SWD): there exists $x_0 \in (0,1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ on $[0,1] \setminus \{x_0\}$, $a \in W^{1,\infty}(0,1)$, $b \in W^{1,1}(0,1)$, and there exist $K_1 \in [1,2)$, $K_2 \in (0,1)$ such that $(x-x_0)a' \leq K_1 a$ and $(x-x_0)b' \leq K_2 b$ a.e. in $[0,1]$.

Typical examples for the previous degeneracies and singularities are $a(x) = |x-x_0|^{K_1}$ and $b(x) = |x-x_0|^{K_2}$, with $0 < K_1, K_2 < 2$.

**Remark 1.** We do not consider the cases $K_i \geq 2$, since if $a(x) = x^{K_i}$ and $K_i \geq 2$, by a standard change of variables problem [14,5] may be transformed in a non degenerate heat equation on an unbounded domain, while the control remains distributed in a bounded domain. This situation is now well-understood, even in the case $\lambda = 0$ where the lack of null controllability was proved by Micu and Zuazua in [24].

We will use the following result several times: we state it for $a$, but an analogous one holds for $b$ replacing $K_1$ with $K_2$.

6
Lemma 2.1 (Lemma 2.1, [24]). Assume that there exists \( x_0 \in (0, 1) \) such that \( a(x_0) = 0, \ a > 0 \) on \([0, 1] \setminus \{x_0\}\), and

- \( a \in W^{1,1}(0, 1) \) and there exist \( K_1 \in (0, 1) \) such that \((x - x_0)a' \leq K_1 a \) a.e. in \([0, 1]\), or
- \( a \in W^{1,\infty}(0, 1) \) and there exist \( K_1 \in [1, 2) \) such that \((x - x_0)a' \leq K_1 a \) a.e. in \([0, 1]\).

1. Then for all \( \gamma \geq K_1 \) the map

\[
x \mapsto \frac{|x - x_0|^\gamma}{a}
\]

is non increasing on the left of \( x = x_0 \) and non decreasing on the right of \( x = x_0 \), so that

\[
\lim_{x \to x_0} \frac{|x - x_0|^\gamma}{a} = 0 \quad \text{for all} \quad \gamma > K_1.
\]

2. If \( K_1 < 1 \), then \( \frac{1}{a} \in L^1(0, 1) \).

3. If \( K_1 \in [1, 2) \), then \( \frac{1}{\sqrt{a}} \in L^1(0, 1) \) and \( \frac{1}{a} \notin L^1(0, 1) \).

For the well–posedness of the problem, we start introducing the following weighted Hilbert spaces, which are suitable to study all situations, namely the \((WWD)\), \((SSD)\), \((WSD)\) and \((SWD)\) cases:

\[
H^1_{a}(0, 1) := \left\{ u \in W^{1,1}_0(0, 1) : \sqrt{a} u' \in L^2(0, 1) \right\}
\]

and

\[
H^1_{a,b}(0, 1) := \left\{ u \in H^1_{a}(0, 1) : \frac{u}{\sqrt{b}} \in L^2(0, 1) \right\},
\]

endowed with the inner products

\[
\langle u, v \rangle_{H^1_a(0, 1)} := \int_0^1 au'v' \, dx + \int_0^1 uv \, dx,
\]

and

\[
\langle u, v \rangle_{H^1_{a,b}(0, 1)} = \int_0^1 au'v' \, dx + \int_0^1 uv \, dx + \int_0^1 \frac{uw}{b} \, dx,
\]

respectively.

Note that, if \( u \in H^1_{a}(0, 1) \), then \( au' \in L^2(0, 1) \), since \( |au'| \leq (\max \sqrt{a}) \sqrt{a} |u'| \).

We recall the following weighted Hardy–Poincaré inequality, see [24 Proposition 2.6]:

7
Proposition 2.1. Assume that $p \in C([0,1])$, $p > 0$ on $[0,1] \setminus \{x_0\}$, $p(x_0) = 0$ and there exists $q > 1$ such that the function
\[
x \mapsto \frac{p(x)}{|x - x_0|^q}
\]
is non increasing on the left of $x = x_0$ and non decreasing on the right of $x = x_0$.

Then, there exists a constant $C_{HP} > 0$ such that for any function $w$, locally absolutely continuous on $[0,x_0) \cup (x_0,1]$ and satisfying
\[
w(0) = w(1) = 0 \quad \text{and} \quad \int_0^1 p(x)|w'(x)|^2 \, dx < +\infty,
\]
the following inequality holds:
\[
\int_0^1 \frac{p(x)}{(x - x_0)^2}w^2(x) \, dx \leq C_{HP} \int_0^1 p(x)|w'(x)|^2 \, dx.
\]

Remark 2. Actually, such a proposition was proved in [24] also requiring $q < 2$. However, as it is clear from the proof, the result is true without such an upper bound on $q$, that in [24] was used for other estimates.

We start with the following crucial

Lemma 2.2. If $K_1 + 2K_2 \leq 2$ and $K_2 < 1$, then there exists a constant $C > 0$ such that
\[
\int_0^1 \frac{u^2}{b} \, dx \leq C \int_0^1 a(u')^2 \, dx
\]
for every $u \in H^1_a(0,1)$.

Proof. We set $p(x) := \frac{(x - x_0)^2}{b}$, so that $p$ satisfies (2.7) with $q = 2 - K_2 > 1$ by Lemma 2.11. Thus, taken $u \in H^1_a(0,1)$, by Proposition 2.1 we get
\[
\int_0^1 \frac{u^2}{b} \, dx = \int_0^1 \frac{p(x)}{(x - x_0)^2}u^2 \, dx \leq C_{HP} \int_0^1 p(x)|u'(x)|^2 \, dx.
\]
Now, by Lemma 2.1
\[
p(x) = (x - x_0)^2 - K_1 - K_2 a(x) \frac{(x - x_0)^{K_1}}{a(x)} \frac{(x - x_0)^{K_2}}{b(x)} \leq c a(x)
\]
for some $c > 0$, and the claim follows.

Remark 3. A similar proof shows that, when $K_1 + 2K_2 \leq 2$ and $K_2 < 1/2$, then
\[
\int_0^1 \frac{u^2}{b^2} \, dx \leq C \int_0^1 a(u')^2 \, dx
\]
for every $u \in H^1_a(0,1)$. 

8
Lemma 2.3. If $K \geq 1$, then $u(x_0) = 0$ for every $u \in H^1_{a,b}(0,1)$.

Proof. Since $u \in W^{1,1}_0(0,1)$, there exists $\lim_{x \to x_0} u(x) = L \in \mathbb{R}$. If $L \neq 0$, then $|u(x)| \geq \frac{L}{4}$ in a neighborhood of $x_0$, that is
\[
\frac{|u(x)|^2}{b} \geq \frac{L^2}{4b} \notin L^1(0,1)
\]
by Lemma 2.1 and thus $L = 0$. □

We also need the following result, whose proof, with the aid of Lemma 2.3 is a simple adaptation of the one given in [26 Lemma 3.2].

Lemma 2.4. If $K \geq 1$, then
\[
H^1_1(0,1) := \left\{ u \in H^1_0(0,1) \text{ such that } \text{supp } u \subset (0,1) \setminus \{x_0\} \right\}
\]
is dense in $H^1_{a,b}(0,1)$.

Lemma 2.5. If $K_1 + K_2 \leq 2$, $K_1 < 1$ and $K_2 \geq 1$, then (2.9) holds for every $u \in H^1_{a,b}(0,1)$.

Proof of Lemma 2.5. Take $u \in H^1_c(0,1)$. Then, for every $D > 0$
\[
0 \leq \int_0^1 \left( |x-x_0|^{K_1/2} u' - \frac{D}{|x-x_0|^{2-K_1/2}} (x-x_0)u \right)^2 \, dx
\]
\[
= \int_0^1 \left( |x-x_0|^{K_1} (u')^2 + \frac{D^2}{|x-x_0|^{2-K_1}} u^2 \right) \, dx - 2D \int_0^1 |x-x_0|^{K_1-2} (x-x_0)uj' \, dx.
\]
(2.10)

Now, take $x < x_0$. Then,
\[
2 \int_0^x |t-x_0|^{K_1-2} (t-x_0)u' \, dt = \int_0^x |t-x_0|^{K_1-2} (t-x_0)(u')' \, dt
\]
\[
= |x-x_0|^{K_1-2} (x-x_0)u^2(x) - (K_1-1) \int_0^x |t-x_0|^{K_1-2} u^2 \, dt.
\]
Similarly, if $y \in (x_0,1)$, we have
\[
2 \int_y^1 |t-x_0|^{K_1-2} (t-x_0)u' \, dt
\]
\[
= -|y-x_0|^{K_1-2} (y-x_0)u^2(y) - (K_1-1) \int_y^1 |t-x_0|^{K_1-2} u^2 \, dt.
\]
Now, letting $x \to x_0^-$ and $y \to x_0^+$, the sum of the two previous integrals converges to $\int_0^1$, since
\[
\lim_{x \to x_0^-} |x-x_0|^{K_1-2} (x-x_0)u^2(x) = \lim_{y \to x_0^+} |y-x_0|^{K_1-2} (y-x_0)u^2(y) = 0,
\]

9
for $u \in H^1_0(0,1)$. In conclusion, we have obtained
\[
0 \leq \int_0^1 \left( |x-x_0|^{K_1} (u')^2 + \frac{D^2 + D(K_1 - 1)}{|x-x_0|^{2-K_1}} u^2 \right) dx.
\]
Choosing $D \in (0, 1-K_1)$, we get that there exists $c > 0$ such that
\[
\int_0^1 \frac{u^2}{|x-x_0|^{2-K_1}} dx \leq c \int_0^1 |x-x_0|^{K_1} (u')^2 dx
\]
for every $u \in H^1_0(0,1)$, and by Lemma 2.4 for every $u \in H^1_{a,b}(0,1)$. Finally, by Lemma 2.1, we can estimate the right-hand-side of the previous inequality with $c \int_0^1 a(u')^2$. Moreover, since $2-K_1 \geq K_2$, we can estimate the left-hand-side from below with $c \int_0^1 u^2$.

In the (SSD) case $K_1 = K_2 = 1$ we need an additional assumption, which is obviously satisfied from the prototypes $a(x) = b(x) = |x-x_0|$, namely
\[
\text{there exists } K > 0 \text{ such that } (a')^2 \leq K \frac{a}{b} \text{ a.e. in } [0,1]. \tag{2.11}
\]

**Lemma 2.6.** Let $K_1 = K_2 = 1$ and assume (2.11). Then, (2.9) holds for all $u \in H^1_{a,b}(0,1)$.

**Proof.** By Lemma 2.3 we know that, taken $u \in H^1_{a,b}(0,1)$, then $u(x_0) = 0$. Fix $\varepsilon \in (0, \min\{x_0, 1-x_0\})$ and write
\[
\int_0^1 \frac{u^2}{b} dx = \left( \int_0^{x_0-\varepsilon} + \int_{x_0-\varepsilon}^{x_0} + \int_{x_0}^{x_0+\varepsilon} + \int_{x_0+\varepsilon}^1 \right) \frac{u^2}{b} dx.
\]
Now, by the Poincaré inequality applied to functions in $[0, x_0-\varepsilon]$ vanishing at 0, we get
\[
\int_0^{x_0-\varepsilon} \frac{u^2}{b} dx \leq \frac{1}{\min_{[0,x_0-\varepsilon]} b} \int_0^{x_0-\varepsilon} u^2 dx \leq \frac{1}{\min_{[0,x_0-\varepsilon]} b} \int_0^{x_0-\varepsilon} (u')^2 dx
\]
\[
\leq \frac{1}{\min_{[0,x_0-\varepsilon]} b} \int_0^{x_0-\varepsilon} a(u')^2 dx \leq C \int_0^1 a(u')^2 dx \tag{2.12}
\]
for some $C > 0$ independent of $u$. A similar estimate holds for $\int_{x_0+\varepsilon}^1 \frac{u^2}{b} dx$.

Moreover, by Lemma 2.1, there exists $C = C(a, b) > 0$ such that
\[
\int_{x_0-\varepsilon}^{x_0} \frac{u^2}{b} dx = \int_{x_0-\varepsilon}^{x_0} \left( \sqrt{a} u \right)^2 \frac{1}{ab} dx \leq C \int_{x_0-\varepsilon}^{x_0} \frac{(\sqrt{a} u)^2}{|x-x_0|^2} dx. \tag{2.13}
\]
Setting \( w = \sqrt{a}u \), we have \( w \in L^2(0,1) \) and \( w' = \sqrt{a}u' + \frac{a'}{2\sqrt{a}}u \); hence, by the Cauchy–Schwarz inequality, fixed \( \delta < 0 \), there exists \( C_\delta > 0 \) such that 
\[
(w')^2 \leq \frac{\delta}{4} \frac{(a')^2}{a} u^2 + C_\delta a(u')^2 \in L^2(x_0 - \varepsilon, x_0),
\]
so that \( w \in H^1(x_0 - \varepsilon, x_0) \). Being \( w(x_0) = 0 \), the classical Hardy–Poincaré inequality and (2.11) imply that 
\[
\int_{x_0 - \varepsilon}^{x_0} \frac{u^2}{|x - x_0|^2} dx \leq C \int_{x_0 - \varepsilon}^{x_0} (w')^2 dx \leq \frac{\delta C}{4} \int_{x_0 - \varepsilon}^{x_0} \frac{(a')^2}{a} u^2 dx + CC_\delta \int_{x_0 - \varepsilon}^{x_0} a(u')^2 dx.
\]
From (2.13) and (2.14), choosing \( \delta \) small enough, we immediately get 
\[
\int_{x_0 - \varepsilon}^{x_0} \frac{u^2}{b} dx \leq C \int_{0}^{1} a(u')^2 dx
\]
for some \( C > 0 \). By (2.12), and operating in a similar way in \([x_0, 1]\), the claim follows.

In view of Lemmas 2.2, 2.5 and 2.6, we introduce the space 
\[
\mathcal{H} := \left\{ \begin{array}{l}
H^1_a(0,1) & \text{if } K_1 + K_2 \leq 2 \text{ and } K_2 < 1, \\
H^1_{a,b}(0,1) & \text{if } K_1 + K_2 \leq 2, K_1 < 1 \text{ and } K_2 \geq 1, \text{ or } \\
K_1 = K_2 = 1 \text{ and (2.11) holds,} 
\end{array} \right.
\]
where the Hardy–Poincaré–type inequality (2.9) holds.

From now on, we make the following assumptions on \( a, b \) and \( \lambda \):

**Hypothesis 2.5.**
1. One among Hypothesis 2.1, 2.2 or 2.3 holds true with \( K_1 + K_2 \leq 2 \), or Hypothesis 2.4 holds with \( K_1 = K_2 = 1 \) and (2.11);
2. setting \( C^\ast \) the best constant of (2.9) in \( \mathcal{H} \), we assume that \( \lambda \neq 0 \) and 
\[
\lambda < \frac{1}{C^\ast}.
\]

Observe that the assumption \( \lambda \neq 0 \) is not restrictive since the case \( \lambda = 0 \) is considered in [24] and in [25].

Using the previous lemmas one can prove the next inequality.

**Proposition 2.2.** Assume Hypothesis 2.5. Then there exists \( \Lambda \in (0,1] \) such that for all \( u \in \mathcal{H} \)
\[
\int_{0}^{1} a(u')^2 dx - \lambda \int_{0}^{1} \frac{u^2}{b} dx \geq \Lambda \int_{0}^{1} a(u')^2 dx.
\]
Proof. If $\lambda \leq 0$, the result is obvious taking $\Lambda = 1$. Now, assume that $\lambda \in (0, C^*)$. Then

$$\int_0^1 a(u')^2 \, dx - \lambda \int_0^1 \frac{u^2}{b} \, dx \geq \int_0^1 a(u')^2 \, dx - \lambda C^* \int_0^1 a(u')^2 \, dx \geq \Lambda \int_0^1 a(u')^2 \, dx.$$

\[\square\]

Remark 4. If Hypothesis \[2.5\] holds, by Proposition \[2.2\], the standard norm $\| \cdot \|_{H^1}$ is equivalent to

$$\|u\|_{H^1}^2 := \int_0^1 a(u')^2 \, dx$$

for all $u \in H$. Indeed, for all $u \in H$, we have

$$\int_0^1 u^2 \, dx = \int_0^1 \frac{u^2}{b} \, dx \leq c \int_0^1 a(u')^2 \, dx,$$

and this is enough to conclude.

We recall the following definition:

Definition 2.1. Let $u_0 \in L^2(0,1)$ and $h \in L^2(Q_T)$. A function $u$ is said to be a (weak) solution of (1.5) if

$$u \in L^2(0,T;H) \cap C([0,T];L^2(0,1))$$

and it satisfies

$$\int_0^1 u(T,x) \varphi(T,x) \, dx - \int_0^1 u_0(x) \varphi(0,x) \, dx - \int_{Q_T} \varphi_t(t,x) u(t,x) \, dx \, dt =$$

$$- \int_{Q_T} a u_x \varphi_x \, dx \, dt + \lambda \int_{Q_T} \frac{w^2}{b} \, dx \, dt + \int_{Q_T} h(t,x) \varphi(t,x) \, dx \, dt$$

for all $\varphi \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;H)$.

Finally, we introduce the Hilbert space

$$H^2_{a,b}(0,1) := \left\{ u \in H^1_a(0,1) : au' \in H^1(0,1) \text{ and } Au \in L^2(0,1) \right\},$$

where

$$Au := (au')' + \frac{\lambda}{b} u \text{ with } D(A) = H^2_{a,b}(0,1).$$

Remark 5. Observe that if $u \in D(A)$, then $\frac{u}{b}$ and $\frac{u}{\sqrt{b}} \in L^2(0,1)$, so that $u \in H^1_{a,b}(0,1)$ and inequality \[2.9\] holds.

We also recall the following integration by parts with functions in the reference spaces:
Lemma 2.7 (Green formula, [26], Lemma 2.3). Assume one among the Hypotheses 2.1, 2.2, 2.3, 2.4. Then, for all \((u, v) \in H^2_{a,b}(0,1) \times H^1_{a}(0,1)\) the following identity holds:

\[
\int_0^1 (au')'vdx = -\int_0^1 au'vdx. \tag{2.16}
\]

Observe that in the non degenerate case, it is well known that the heat operator with an inverse–square singular potential

\[ u_t - \Delta u - \frac{\lambda}{|x|^2} u \]

gives rise to well–posed Cauchy-Dirichlet problems if and only if \(\lambda\) is not larger than the best Hardy inequality (see [4], [10], [39]). For this reason, it is not strange that we require an analogous condition for problem (1.5), by invoking Hypothesis 2.5; as a consequence, using the standard semigroup theory, we have that (1.5) is well–posed; indeed, by [32, Theorem 3.4.1 and Remark 3.4.3], we immediately have

**Theorem 2.1.** Assume Hypothesis 2.5. For every \(u_0 \in L^2(0,1)\) and \(f \in L^2(Q_T)\), there exists a unique solution \(u \in L^2(0,1) \cap C([0,T];L^2(0,1))\) of problem (1.5). In particular, if \(u_0 \in D(A)\) and \(f \in W^{1,1}(0,T;L^2(0,1))\), then

\[ u \in C^1(0,T;L^2(0,1)) \cap C([0,T];D(A)). \]

**Proof.** Observe that \(D(A)\) is dense in \(L^2(0,1)\).

\(A\) is symmetric. Indeed, for any \(u, v \in D(A)\), by Lemma 2.7 we have

\[
\langle v, Au \rangle_{L^2(0,1)} = \int_0^1 (au')'vdx + \lambda \int_0^1 \frac{uv}{b}dx = -\int_0^1 au'vdx + \lambda \int_0^1 \frac{uv}{b}dx = \int_0^1 (av')udx + \lambda \int_0^1 \frac{uv}{b}dx = \langle Av, u \rangle_{L^2(0,1)}.
\]

Observe that, since \(u, v \in H\), the integral \(\int_0^1 \frac{uv}{b}dx\) is convergent.

\(A\) is non positive. By Proposition 2.2 and Lemma 2.7, for all \(u \in D(A)\) we have

\[
-(Au, u)_{L^2(0,1)} = -\int_0^1 \left( a(u')' + \frac{\lambda}{b} u \right)udx = \int_0^1 a(u')^2dx - \lambda \int_0^1 \frac{u^2}{b}dx \geq C\|u\|^2_{H}.
\]

\(A\) is self–adjoint. Let \(T : L^2(0,1) \to L^2(0,1)\) be the mapping defined in the following usual way: to each \(f \in L^2(0,1)\) associate the weak solution

\[ u = T(f) \in H \]

of

\[
\int_0^1 (au'v - \lambda \frac{uv}{b}) dx = \int_0^1 fvdx
\]
for every $v \in \mathcal{H}$. Note that $T$ is well defined by the Lax–Milgram Lemma via Proposition 2.2, which also implies that $T$ is continuous. Now, it is easy to see that $T$ is injective and symmetric. Thus it is self–adjoint. As a consequence, $A = T^{-1} : D(A) \to L^2(0,1)$ is self–adjoint (for example, see Proposition A.8.2).

$A$ is $m$–dissipative. Being $A$ self–adjoint, this is a straightforward consequence of Corollary 2.4.8. Then $(A, D(A))$ generates a cosine family and an analytic contractive semigroup of angle $\frac{\pi}{2}$ on $L^2(0,1)$ (see, for example, Example 3.14.16 and 3.7.5).

The additional regularity is a consequence of Lemma 4.1.5 and Proposition 4.1.6.

3 Carleman estimates for singular/degenerate problems

In this section we prove one of the main result of this paper, i.e. a new Carleman estimate with boundary terms for solutions of the singular/degenerate problem

\[
\begin{align*}
v_t + (av_x)_x + \frac{\lambda}{b(x)}v &= h(t,x), \quad (t,x) \in Q_T, \\
v(t,0) &= v(t,1) = 0, \quad t \in (0,T), \\
v(T,x) &= \nu_T(x),
\end{align*}
\]

which is the adjoint of problem (1.5).

On the degenerate function $a$ we make the following assumption:

**Hypothesis 3.1.** Hypothesis 2.5 holds. Moreover, if $K_1 > \frac{4}{3}$, then there exists a constant $\theta \in (0, K_1]$ such that

\[
x \mapsto \frac{a(x)}{|x - x_0|^\theta}
\]

is non increasing on the left of $x = x_0$, is non decreasing on the right of $x = x_0$. (3.18)

In addition, when $K_1 > \frac{3}{2}$ the function in (3.18) is bounded below away from 0 and there exists a constant $\Sigma > 0$ such that

\[
|a'(x)| \leq \Sigma |x - x_0|^{2\theta - 3} \text{ for a.e. } x \in [0,1].
\]

Remark 6. If $a(x) = |x - x_0|^{K_1}$, then (3.18) is clearly satisfied with $\theta = K_1$. Moreover, the additional requirements for the sub-case $K_1 > \frac{3}{2}$ are technical ones and are introduced in 25 to guarantee the convergence of some integrals (see 25 Appendix). Of course, the prototype $a(x) = |x - x_0|^{K_1}$ satisfies again such conditions with $\theta = K_1$. 

14
To prove Carleman estimate, let us introduce the function \( \varphi := \Theta \psi \), where

\[
\Theta(t) := \frac{1}{t(T-t)^4} \quad \text{and} \quad \psi(x) := c_1 \left[ \int_{x_0}^x \frac{y-x_0}{a(y)} dy - c_2 \right], \tag{3.20}
\]

where \( c_2 > \sup_{[0,1]} \int_{x_0}^x \frac{y-x_0}{a(y)} dy \) and \( c_1 > 0 \) (for the observability inequality \( c_1 \) will be taken sufficiently large, see Lemma 4.1). Observe that \( \Theta(t) \to +\infty \) as \( t \to 0^+, T^- \), and clearly \(- c_1 c_2 \leq \psi < 0\).

The main result of this section is the following

**Theorem 3.1.** Assume Hypothesis \ref{hypothesis}. Then, there exist two positive constants \( C \) and \( s_0 \), such that every solution \( v \) of (3.17) in

\[
\mathcal{V} := L^2(0, T; H^2_{a,b}(0,1)) \cap H^1(0, T; H) \tag{3.21}
\]
satisfies, for all \( s \geq s_0 \),

\[
\int_{Q_T} \left( s \Theta a(v_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} dx dt \\
\leq C \left( \int_{Q_T} h^2 e^{2s\varphi} dx dt + sc_1 \int_0^T \left[ a \Theta e^{2s\varphi(t,x)} (x-x_0)(v_x)^2 dt \right]^{x=1}_{x=0} \right).
\]

**Remark 7.** In [38] the authors prove a related Carleman inequality for the non degenerate singular 1-D system

\[
\begin{cases}
v_t + v_{xx} + \frac{\mu}{x^2} + \frac{\lambda}{x^2} v = h & (t, x) \in Q_T, \\
v(t, 0) = v(t, 1) = 0 & t \in (0, T), \\
v(T, x) = v_T(x) & x \in (0, 1),
\end{cases} \tag{3.22}
\]

where \( \beta \in [0, 2) \). In our case, i.e. when \( \mu = 0 \) and \( x_0 = 0 \), such an inequality reads as follows:

\[
\int_{Q_T} \left( s^3 \Theta^3 x^2 v^2 + s^2 \Theta \frac{v^2}{x^2} + s \Theta \frac{v^2}{x^{2/3}} \right) e^{2s\varphi} dx dt \leq \frac{1}{2} \int_{Q_T} h^2 e^{2s\varphi} dx dt,
\]

where \( \Psi(x) = \frac{x^2}{2} - 1 < 0 \) in \([0,1]\]. Actually, such an inequality is proved for solutions \( v \) such that

\[
v(t, x) = 0 \text{ for all } (t, x) \in (0, T) \times (1-\eta, 1) \text{ for some } \eta \in (0, 1). \tag{3.23}
\]

However, in [38] Remark 3.5] the authors say that Carleman estimates can be proved also for all solutions of (3.22) not satisfying (3.23). We think that this latter situation is much more interesting, since by the Carleman estimates, if \( h = 0 \), then \( v \equiv 0 \) even if (3.23) does not hold.
The proof of Theorem 3.1 is quite long, and several intermediate lemmas will be used. First, for \( s > 0 \), define the function 
\[
w(t, x) := e^{s\varphi(t, x)} v(t, x),
\]
where \( v \) is any solution of (3.17) in \( V \); observe that, since \( v \in V \) and \( \varphi < 0 \), then \( w \in V \) and satisfies

\[
\begin{aligned}
\left\{
\begin{array}{ll}
(e^{-s\varphi}w)_t + (a(e^{-s\varphi}w)_x)_x + \lambda \frac{e^{-s\varphi}w}{b} = h, & \quad (t, x) \in (0, T) \times (0, 1), \\
w(t, 0) = w(t, 1) = 0, & \quad t \in (0, T), \\
w(T, x) = w(0, x) = 0, & \quad x \in (0, 1).
\end{array}
\right.
\]
\]

(3.24)

As usual, we re-write the previous problem as follows: setting 
\[Lv := v_t + (av)_x + \frac{\lambda}{b} v\]
and \( L_s w = e^{s\varphi} L(e^{-s\varphi}w) \), then (3.24) becomes 

\[
\begin{aligned}
\left\{
\begin{array}{ll}
L_s w = e^{s\varphi} h, & \\
w(t, 0) = w(t, 1) = 0, & \quad t \in (0, T), \\
w(T, x) = w(0, x) = 0, & \quad x \in (0, 1).
\end{array}
\right.
\]

Computing \( L_s w \), one has 
\[
L_s w = L^+_s w + L^-_s w,
\]
where 
\[
L^+_s w := (aw)_x + \frac{\lambda w}{b} - s\varphi_t w + s^2 a\varphi^2_x w,
\]
and 
\[
L^-_s w := w_t - 2sa\varphi_x w_x - s(a\varphi_x)_x w.
\]

Of course,
\[
2\langle L^+_s w, L^-_s w \rangle \leq 2\langle L^+_s w, L^-_s w \rangle + \|L^+_s w\|_{L^2(Q_T)}^2 + \|L^-_s w\|_{L^2(Q_T)}^2
= \|L_s w\|_{L^2(Q_T)}^2 = \|he^{s\varphi}\|_{L^2(Q_T)}^2,
\]
(3.25)

where \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( L^2(Q_T) \). As usual, we will separate the scalar product \( \langle L^+_s w, L^-_s w \rangle \) in distributed terms and boundary terms.
Lemma 3.1. The following identity holds:

\[
\langle L^+_s w, L^-_s w \rangle = \frac{s}{2} \int_{Q_T} \varphi_t w^2 dx dt - 2s^2 \int_{Q_T} a\varphi_x \varphi_{tx} w^2 dx dt + s \int_{Q_T} (2a^2 \varphi_{xx} + aa' \varphi_x)(w_x)^2 dx dt + s^3 \int_{Q_T} (2a \varphi_{xx} + a' \varphi_x)a(\varphi_x)^2 w^2 dx dt - s\lambda \int_{Q_T} \frac{a\varphi_x b'}{b^2} w^2 dx dt \]

\[
\begin{aligned}
&+ \int_0^T [aw_x w_t]_{x=1}^1 dt - s \int_0^1 [w^2 \varphi_t]_{t=0}^T dx + \frac{s^2}{2} \int_0^1 [a(\varphi_x)^2 w^2]_{t=0}^T dt \\
&+ \int_0^T [-s\varphi_x (aw_x)^2 + s^2 a\varphi_{xx} w^2 - s^3 a(\varphi_x)^3 w^2 - s\lambda \frac{a\varphi_x}{b} w^2]_{x=0}^{x=1} dt \}
\end{aligned}
\]

\[
\begin{aligned}
&+ \int_0^T [-sa(a\varphi_x)_x w w_x]_{x=0}^{x=1} dt - \frac{1}{2} \int_0^1 [a(w^2) - \lambda \frac{1}{2b^2} w^2]_{t=0}^T dx \}
\end{aligned}
\]

(3.26)

Proof. Computing \(\langle L^+_s w, L^-_s w \rangle\), one has that

\[
\langle L^+_s w, L^-_s w \rangle = I_1 + I_2 + I_3 + I_4,
\]

where

\[
I_1 := \int_{Q_T} ((aw)_x - s\varphi_t w + s^2 a(\varphi_x)^2 w) w_t dx dt,
\]

\[
I_2 := \int_{Q_T} ((aw)_x - s\varphi_t w + s^2 a(\varphi_x)^2 w)(-2sa\varphi_x w_x) dx dt,
\]

\[
I_3 := \int_{Q_T} ((aw)_x - s\varphi_t w + s^2 a(\varphi_x)^2 w)(-s(a\varphi_x)_x w) dx dt,
\]

and

\[
I_4 := \lambda \int_{Q_T} \frac{w_t}{b} (w - 2sa\varphi_x w_x - s(a\varphi_x)_x w) dx dt.
\]

By several integrations by parts in space and in time (see [1] Lemma 3.4, [24] Lemma 3.1 or [25] Lemma 3.1), and observing that \(\int_{Q_T} a(\varphi_x)_{xx} w w_x dx dt = 0\)
(by the very definition of \( \varphi \)), we get

\[
I_1 + I_2 + I_3 = \int_{Q} \frac{s}{2} \varphi_t w^2 dx dt - 2s^2 \int_{Q} a \varphi_x \varphi_{tx} w^2 dx dt + s \int_{Q} (2a^2 \varphi_{xx} + a' \varphi_x) (w_x)^2 dx dt + s^3 \int_{Q} (2a \varphi_{xx} + a' \varphi_x) a(\varphi_x)^2 w^2 dx dt \\
+ \int_{0}^{T} \left| aw_x w_t \right|_{x=0}^{T} dt - s \int_{0}^{T} \left| w^2 \varphi_t \right|_{t=0}^{T} dx + \frac{s^2}{2} \int_{0}^{T} \left| (\varphi_x)^2 w^2 \right|_{t=0}^{T} dt \\
+ \int_{0}^{T} \left[ -s \varphi_x (aw_x)^2 + s^2 a \varphi_t \varphi_x w^2 - s^3 a^2 (\varphi_x)^3 w^2 \right]_{x=0}^{T} dt \\
+ \int_{0}^{T} \left[ -s a (\varphi_x) x w w_x \right]_{x=0}^{T} dt - \frac{1}{2} \int_{0}^{1} \left[ a (w_x)^2 \right]_{t=0}^{T} dx.
\]

Next, we compute \( I_4 \):

\[
I_4 = \lambda \left( \int_{Q} \frac{1}{2b} (w^2)_t dx dt - 2s \int_{Q} \frac{a}{b} \varphi_x w_x w dx dt \\
- s \int_{Q} \frac{(a \varphi_x)_x}{b} w^2 dx dt \right) \\
= \lambda \left( \int_{0}^{1} \frac{1}{2b} (w^2)_{x=0}^{T} dx - s \int_{Q} \frac{a}{b} \varphi_x (w^2)_x dx dt - s \int_{Q} \frac{(a \varphi_x)_x}{b} w^2 dx dt \right) \\
= \lambda \left( \int_{0}^{1} \frac{1}{2b} (w^2)_{x=0}^{T} dx - s \int_{0}^{T} \left[ \frac{a}{b} \varphi_x w^2 \right]_{x=0}^{T} dt \\
+ s \int_{Q} \frac{(a \varphi_x)_x}{b} w^2 dx dt - s \int_{Q} \frac{(a \varphi_x)_x}{b} w^2 dx dt \right) \\
= \lambda \left( \int_{0}^{1} \frac{1}{2b} (w^2)_{x=0}^{T} dx - s \int_{0}^{T} \left[ \frac{a \varphi_x}{b} w^2 \right]_{x=0}^{T} dt - s \int_{0}^{T} \left[ \frac{a \varphi_x}{b} w^2 \right]_{x=0}^{T} dt \right). (3.27)
\]

Adding (3.27) - (3.28), (3.26) follows immediately. \( \square \)

For the boundary terms in (3.26), we have:

**Lemma 3.2.** The boundary terms in (3.26) reduce to

\[-s \int_{0}^{T} \left[ \Theta (aw_x) (w^2) \right]_{x=0}^{T} dt.\]

**Proof.** As in [24] or [25], using the definition of \( \varphi \) and the boundary conditions
on $w$, one has that
\[
\int_0^T [aw_xw_t]_{x=0}^{x=1} dt - \frac{s}{2} \int_0^1 [w^2 \varphi_t]_{t=0}^{t=T} dx + \frac{s^2}{2} \int_0^1 [a(\varphi_x)^2 w^2]_{t=0}^{t=T} dt \\
+ \int_0^T [-s \varphi_x (aw_x)^2 + s^2 a \varphi_x \varphi_x w^2 - s^3 a^2 (\varphi_x)^3 w^2]_{x=0}^{x=1} dt \\
+ \int_0^T [-s a (\varphi_x)_x w w_x]_{x=0}^{x=1} dt - \frac{1}{2} \int_0^1 [a(w_x)^2]_{t=0}^{t=T} dx = -s \int_0^T [\Theta (aw_x)^2 \psi']_{x=0}^{x=1} dt.
\]
Moreover, since $w \in V$, $w \in C([0, T]; H)$; thus $w(0, x)$, $w(T, x)$ are well defined, and using the boundary conditions of $w$, we get that
\[
\int_0^1 \left[ \frac{1}{2b} w^2 \right]_{t=0}^{t=T} dx = 0.
\]

Now, consider the last boundary term $s \lambda \int_0^T \left[ \frac{a \varphi_x w^2}{b} \right]_{x=0}^{x=1} dt$. Using the definition of $\varphi$, this term becomes $s \lambda \int_0^T \left[ \frac{a \psi' w^2}{b} \right]_{x=0}^{x=1} dt$. By definition of $\psi$ and using Hypothesis 2.5.2, the function $\frac{a \psi'}{b}$ is bounded on $[0, 1]$. Thus, by the boundary conditions on $w$, one has
\[
s \lambda \int_0^T \left[ \frac{\Theta \psi'}{b} w^2 \right]_{x=0}^{x=1} dt = 0.
\]

Now, the crucial step is to prove the following estimate:

**Lemma 3.3.** Assume Hypothesis 3.1. Then there exist two positive constants $s_0$ and $C$ such that for all $s \geq s_0$ the distributed terms of (3.26) satisfy the estimate
\[
\frac{s}{2} \int_{Q_T} \varphi_t w^2 dx dt - 2s^2 \int_{Q_T} a \varphi_x \varphi_x w^2 dx dt \\
+ s \int_{Q_T} (2a^2 \varphi_{xx} + aa' \varphi_x)(w_x)^2 dx dt \\
+ s^3 \int_0^T \int_0^1 (2a \varphi_{xx} + a' \varphi_x) a(\varphi_x)^2 w^2 dx dt - s \lambda \int_{Q_T} \frac{a \varphi_x b'}{b^2} w^2 dx dt \\
\geq \frac{C}{2} s \int_{Q_T} \Theta a(w_x)^2 dx dt + \frac{C^3}{2} s^3 \int_{Q_T} \Theta^3 (x - x_0)^2 \frac{a}{w^2} dx dt.
\]

**Proof.** Proceeding as in [24, Lemma 3.2] or in [25, Lemma 4.1], one can prove
that, for $s$ large enough,
\[
\frac{s}{2} \int_{Q_T} \varphi_t w^2 dxdt - 2s^2 \int_{Q_T} a \varphi_x \varphi_{tx} w^2 dxdt \\
+ s \int_{Q_T} (2a^2 \varphi_{xx} + aa' \varphi_x) (w_x)^2 dxdt \\
+ s^3 \int_{Q_T} (2a \varphi_{xx} + a' \varphi_x) a(\varphi_x)^2 w^2 dxdt \\
\geq \frac{3C}{4} s \int_{Q_T} \Theta (w_x)^2 dxdt + \frac{C^3}{2} s^3 \int_{Q_T} \Theta^3 \frac{(x - x_0)^2}{a} w^2 dxdt,
\]
where $C$ is a positive constant. Let us remark that one can assume $C$ as large as desired, provided that $s_0$ increases as well. Indeed, taken $k > 0$, from $CsA_1 + C^3 s^3 A_2 = kC^s A_1 + k^3 C^3 s^3 A_2,$ we can choose $s_0' = ks_0$ and $C' = kC$ large as needed.

Now, we estimate the term $-s \lambda \int_{Q_T} \frac{a \varphi_x b'}{b^2} w^2 dxdt$. Using the definition of $\varphi$ and the assumption on $b$, one has
\[
-s \lambda \int_{Q_T} \frac{a \varphi_x b'}{b^2} w^2 dxdt = -s \lambda \int_{Q_T} \Theta \frac{a b' b'}{b^2} w^2 dxdt \\
= -s \lambda c_1 \int_{Q_T} \Theta \frac{(x - x_0)b'}{b^2} w^2 dxdt \\
\geq -s \lambda c_1 K_2 \int_{Q_T} \frac{\Theta}{b} w^2 dxdt.
\]
Since $w(t, \cdot) \in H$ for every $t \in [0, 1]$, for $w \in V$, we get
\[
\int_{Q_T} \frac{\Theta}{b} w^2 dxdt \leq C^* \int_{Q_T} \Theta a(w_x)^2 dxdt.
\]
Hence,
\[
-s \lambda \int_{Q_T} \frac{a \varphi_x b'}{b^2} w^2 dxdt \geq -s \lambda c_1 K_2 C^* \int_{Q_T} \Theta a(w_x)^2 dxdt,
\]
and we can assume, in view of what remarked above, that this last quantity is smaller than $-s \frac{C}{4} \int_{Q_T} \Theta a(w_x)^2 dxdt.$

Summing up, the distributed terms of $\int_{Q_T} L^+ w L^- w dxdt$ can be estimated as
\[
\{\text{D.T.}\} \geq \frac{C}{2} s \int_{Q_T} \Theta a(w_x)^2 dxdt + \frac{C^3}{2} s^3 \int_{Q_T} \Theta^3 \frac{(x - x_0)^2}{a} w^2 dxdt,
\]
for $s$ large enough and $C > 0.$
From Lemma 3.1, Lemma 3.2 and Lemma 3.3, we deduce immediately that there exist two positive constants $C$ and $s_0$, such that for all $s \geq s_0$,

$$\int_{Q_T} L_s^+ w L_s^- w dx dt \geq C \int_{Q_T} \Theta a(w_x)^2 dx dt$$

$$+ C s^3 \int_{Q_T} \Theta^3 \frac{(x-x_0)^2}{a} w^2 dx dt - s \int_0^T \left[ \Theta a^2 w_x^2 \psi' \right]_{x=0}^{x=1} dt.$$  

(3.30)

Thus, a straightforward consequence of (3.25) and of (3.30) is the next result.

**Lemma 3.4.** Assume Hypothesis 3.1. Then, there exist two positive constants $C$ and $s_0$, such that for all $s \geq s_0$,

$$s \int_{Q_T} \Theta a(w_x)^2 dx dt + s^3 \int_{Q_T} \Theta^3 \frac{(x-x_0)^2}{a} w^2 dx dt$$

$$\leq C \left( \int_{Q_T} h^2 e^{2s\psi(t,x)} dx dt + s \int_0^T \left[ \Theta a^2 w_x^2 \psi' \right]_{x=0}^{x=1} dt. \right).$$  

(3.31)

Recalling the definition of $w$, we have $v = e^{-s\psi} w$ and $v_x = -s \Theta \psi' e^{-s\psi} w + e^{-s\psi} w_x$. Thus, substituting in (3.31), Theorem 3.1 follows.

### 4 Observability results and application to null controllability

In this section we shall apply the just established Carleman inequalities to observability and controllability issues. For this, we assume that the control set $\omega$ satisfies the following assumption:

**Hypothesis 4.1.** The subset $\omega$ is such that

(i) it is an interval which contains the degeneracy point:

$$\omega = (\alpha, \beta) \subset (0, 1)$$

is such that $x_0 \in \omega$,  

(4.32)

or

(ii) it is an interval lying on one side of the degeneracy point:

$$\omega = (\alpha, \beta) \subset (0, 1)$$

is such that $x_0 \not\in \bar{\omega}$.  

(4.33)

On the function $a$ we make the following assumptions:

**Hypothesis 4.2.** Hypothesis 3.1 is satisfied. Moreover, if Hypothesis 2.1 or 2.3 holds, then there exist two functions $g \in L^\infty_{loc}([0, 1] \setminus \{x_0\})$, $h \in W^{1,\infty}_{loc}([0, 1] \setminus \{x_0\})$, and $\omega$ satisfies the following assumption:
\{x_0\}) and two strictly positive constants \(g_0, h_0\) such that \(g(x) \geq g_0\) for a.e. \(x\) in \([0,1]\) and
\[
- \frac{a'(x)}{2\sqrt{a(x)}} \left( \int_x^B g(t) dt + h_0 \right) + \sqrt{a(x)g(x)} = h(x,B) \quad \text{for a.e. } x, B \in [0,1]
\]
with \(x < B < x_0\) or \(x_0 < x < B\).

Remark 8. Since we require identity (4.34) far from \(x_0\), once \(a\) is given, it is easy to find \(g, h, g_0\) and \(h_0\) with the desired properties. For example, if \(a(x) := |x - x_0|^{\alpha}, \alpha \in (0,1)\), we can take \(g(x) \equiv g_0 = h_0 = 1\) and
\[
h(x,B) = \left| x - x_0 \right|^{\frac{\alpha}{2} - 1} \left[ \frac{\alpha}{2} \text{sign}(x - x_0)(B + 1 - x) + |x - x_0| \right], \text{ for all } x \text{ and } B \in [0,1],
\]
with \(x < B < x_0\) or \(x_0 < x < B\). Clearly, \(g \in L^\infty_{\text{loc}}([0,1] \setminus \{x_0\})\) and \(h \in W^{1,\infty}_{\text{loc}}([0,1] \setminus \{x_0\}; L^\infty(0,1))\).

Now, we associate to problem (1.5) the homogeneous adjoint problem
\[
\begin{cases}
\begin{align*}
& v_t + (av_x)_x + \frac{\lambda}{b(x)} v = 0, \quad (t,x) \in Q_T, \\
& v(t,0) = v(t,1) = 0, \quad t \in (0,T), \\
& v(T,x) = v_T(x),
\end{align*}
\end{cases}
\]
where \(T > 0\) is given and \(v_T(x) \in L^2(0,1)\). By the Carleman estimate in Theorem 3.1, we will deduce the following observability inequality for all the degenerate cases:

**Proposition 4.1.** Assume Hypotheses 4.1 and 4.2. Then there exists a positive constant \(C_T\) such that every solution \(v \in C([0,T]; L^2(0,1)) \cap L^2(0,T; \mathcal{H})\) of (4.35) satisfies
\[
\int_0^1 v^2(0,x) dx \leq C_T \int_0^T \int_\omega v^2(t,x) dx dt.
\]

Using the observability inequality (4.36) and a standard technique (e.g., see [31, Section 7.4]), one can prove the null controllability result for the linear degenerate problem (1.3):

**Theorem 4.1.** Assume Hypotheses 4.1 and 4.2. Then, given \(u_0 \in L^2(0,1)\), there exists \(h \in L^2(Q_T)\) such that the solution \(u\) of (1.5) satisfies
\[
u(T,x) = 0 \quad \text{for every } x \in [0,1].
\]

Moreover
\[
\int_{Q_T} h^2 dx dt \leq C \int_0^1 u_0^2(x) dx,
\]
for some positive constant \(C\).
4.1 Proof of Proposition 4.1

In this subsection we will prove, as a consequence of the Carleman estimate proved in Section 3, the observability inequality (4.36). For this purpose, we will give some preliminary results. As a first step, consider the adjoint problem

\[
\begin{cases}
v_t + Av = 0, & (t, x) \in Q_T, \\
v(t, 0) = v(t, 1) = 0, & t \in (0, T), \\
v(T, x) = v_T(x) \in D(A^2),
\end{cases}
\tag{4.37}
\]

where

\[D(A^2) = \left\{ u \in D(A) : Au \in D(A) \right\}\]

and \(Au := (au_x)_x + \lambda \frac{u}{b}\). Observe that \(D(A^2)\) is densely defined in \(D(A)\) for the graph norm (see, for example, [8, Lemma 7.2]) and hence in \(L^2(0, 1)\). As in [11], [12], [23] or [24], define the following class of functions:

\[\mathcal{W} := \left\{ v \text{ is a solution of (4.37)} \right\}\]

Obviously (see, for example, [8, Theorem 7.5])

\[\mathcal{W} \subset C^1([0, T] ; H^2_{a,b}(0, 1)) \subset V \subset U\]

where, \(V\) is defined in (3.21) and

\[U := C([0, T] ; L^2(0, 1)) \cap L^2(0, T ; \mathcal{H})\].

We start with

**Proposition 4.2** (Caccioppoli’s inequality). Let \(\omega'\) and \(\omega\) be two open subintervals of \((0, 1)\) such that \(\omega' \subset \omega \subset (0, 1)\) and \(x_0 \notin \omega'\). Let \(\varphi(t, x) = \Theta(t) \Upsilon(x)\), where \(\Theta\) is defined in (3.20) and

\[\Upsilon \in C([0,1], (-\infty,0)) \cap C^1([0,1] \setminus \{x_0\}, (-\infty,0))\]

is such that

\[|\Upsilon_x| \leq \frac{c}{\sqrt{a}} \text{ in } [0,1] \setminus \{x_0\}\]  

for some \(c > 0\). Then, there exist two positive constants \(C\) and \(s_0\) such that every solution \(v \in \mathcal{W}\) of the adjoint problem (4.37) satisfies

\[\int_0^T \int_{\omega'} (v_x)^2 e^{2s^2} dx dt \leq C \int_0^T \int_{\omega} v^2 dx dt,\]

for all \(s \geq s_0\).
Of course, our prototype for $\psi$ is the function $\psi$ defined in (3.20), since
\[ |\psi'(x)| = c_1 \sqrt{\frac{|x - x_0|^2}{a(x)}} \frac{1}{\sqrt{a(x)}} \leq c \frac{1}{\sqrt{a(x)}} \]
by Lemma 2.1.

Proof. The proof follows the one of [24, Proposition 4.2], but it is different for the presence of the singular term.

Let us consider a smooth function $\xi : [0, 1] \to \mathbb{R}$ such that
\[
\begin{cases}
0 \leq \xi(x) \leq 1, & \text{for all } x \in [0, 1], \\
\xi(x) = 1, & x \in \omega', \\
\xi(x) = 0, & x \in [0, 1] \setminus \omega.
\end{cases}
\]

Since $v$ solves (4.37), we have
\[
0 = \int_0^T \frac{d}{dt} \left( \int_0^1 \xi^2 e^{2s\varphi} v^2 dx \right) dt = \int_0^T 2s e^2 \varphi \xi^2 e^{2s\varphi} v^2 + 2\xi^2 e^{2s\varphi} v v_t dx dt
\]
\[
= 2 \int_{Q_T} \xi^2 s \varphi v e^{2s\varphi} v^2 dx dt + 2 \int_{Q_T} \xi^2 e^{2s\varphi} v \left(-\frac{\lambda}{b} - (av)_x \right) dx dt
\]
\[
= 2 \int_{Q_T} \xi^2 s \varphi v e^{2s\varphi} v^2 dx dt - 2 \lambda \int_{Q_T} \xi^2 e^{2s\varphi} v^2 dx dt + 2 \int_{Q_T} (\xi^2 e^{2s\varphi} v)_x av v_x dx dt.
\]
(4.41)

If $\lambda \leq 0$, then, differentiating the last term in (4.41), we get
\[
2 \int_{Q_T} \xi^2 e^{2s\varphi} v^2 dx \leq 2 \lambda \int_{Q_T} \xi^2 e^{2s\varphi} v^2 dx - 2 \int_{Q_T} (\xi^2 e^{2s\varphi})_x avv_x dx dt
\]
\[
\leq -2 \int_{Q_T} \xi^2 s \varphi v e^{2s\varphi} v^2 dx dt - 2 \int_{Q_T} (\xi^2 e^{2s\varphi})_x avv_x dx dt,
\]
and then one can proceed as for the proof of [24, Proposition 4.2], obtaining the claim.

Otherwise, if $\lambda > 0$, fixed $\varepsilon > 0$, by the Cauchy–Schwarz inequality, we have
\[
\int_0^1 \xi^2 e^{2s\varphi} v^2 dx \leq C' \int_0^1 a(w_x)^2 dx
\]
\[
\leq C \int_0^1 a[(\xi e^{s\varphi})_x]^2 v^2 dx + \varepsilon \int_0^1 \xi^2 e^{2s\varphi} a(v_x)^2 dx
\]
for some $C_\varepsilon > 0$. Moreover,
\[
[(\xi e^{s\varphi})_x]^2 \leq C \chi_\omega (e^{2s\varphi} + s^2 (\varphi_x)^2 e^{s\varphi}) \leq C \chi_\omega \left(1 + \frac{1}{a}\right)
\]

24
for some positive constant $C$. Indeed, $e^{2s\varphi} < 1$, while $s^2(\varphi_x)^2e^{2s\varphi}$ can be estimated with
\[
\frac{c}{(-\max_{\mathcal{T}})^2(T_x)^2} \leq \frac{c}{a}
\]
by (4.39), for some constants $c > 0$. Thus
\[
2\lambda \int_{Q_T} \xi^2 e^{2s\varphi} \frac{\xi^2}{b} dx dt \leq 2\lambda C\varepsilon \int_{Q_T} a[(\xi e^{s\varphi})_x]^2 v^2 dx dt + 2\lambda \varepsilon \int_{Q_T} \xi^2 e^{2s\varphi} a(v_x)^2 dx dt \leq C \int_T^T \int_{\omega} v^2 dx dt + 2\lambda \varepsilon \int_{Q_T} \xi^2 e^{2s\varphi} a(v_x)^2 dx dt,
\]
for a positive constant $C$ depending on $\varepsilon$. Hence, differentiating the last term in (4.41) and using (4.42), we get
\[
2 \int_{Q_T} \xi^2 e^{2s\varphi} a(v_x)^2 dx dt = 2\lambda \int_{Q_T} \xi^2 e^{2s\varphi} \frac{\xi^2}{b} dx dt - 2 \int_{Q_T} (\xi^2 e^{2s\varphi})_x a v v_x dx dt
\]
\[
\leq C \int_0^T \int_{\omega} v^2 dx dt + 2\lambda \varepsilon \int_{Q_T} \xi^2 e^{2s\varphi} a(v_x)^2 dx dt - 2 \int_{Q_T} (\xi^2 e^{2s\varphi})_x a v v_x dx dt.
\]
Thus, applying again the Cauchy-Schwarz inequality, we get
\[
(2 - 2\lambda\varepsilon) \int_{Q_T} \xi^2 e^{2s\varphi} a(v_x)^2 dx dt \leq C \int_0^T \int_{\omega} v^2 dx dt + 2 \int_{Q_T} \xi^2 s \varphi e^{2s\varphi} v^2 dx dt
\]
\[
- 2 \int_{Q_T} (\xi^2 e^{2s\varphi})_x a v v_x dx dt
\]
\[
\leq C \int_0^T \int_{\omega} v^2 dx dt + 2\lambda \varepsilon \int_{Q_T} \xi^2 s \varphi e^{2s\varphi} v^2 dx dt + 2\varepsilon \int_0^T \int_{\omega} (\sqrt{a} xe^{s\varphi} v_x)^2 dx dt
\]
\[
+ D\varepsilon \int_0^T \int_{\omega} \left(\frac{(\xi^2 e^{2s\varphi})_x}{\xi e^{s\varphi}} v\right)^2 dx dt
\]
\[
= C \int_0^T \int_{\omega} v^2 dx dt + 2\lambda \varepsilon \int_{Q_T} \xi^2 s \varphi e^{2s\varphi} v^2 dx dt + 2\varepsilon \int_0^T \int_{\omega} \xi^2 e^{2s\varphi} a(v_x)^2 dx dt
\]
\[
+ D\varepsilon \int_0^T \int_{\omega} \frac{|(\xi^2 e^{2s\varphi})_x|^2}{\xi^2 e^{2s\varphi}} a v^2 dx dt.
\]
for some $D_\varepsilon > 0$. Hence,

$$2(1 - \varepsilon - \lambda \varepsilon) \int_0^T \int_\omega \xi^2 e^{2s\varphi} a(v_x)^2 \, dx \, dt \leq C \int_0^T \int_\omega v^2 \, dx \, dt$$

$$- 2 \int_0^T \int_\omega \xi^2 s\varphi_t e^{2s\varphi} v^2 \, dx \, dt + D_\varepsilon \int_0^T \int_\omega \frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}} a v^2 \, dx \, dt.$$

Since $x_0 \notin \bar{\omega}'$, then

$$2(1 - \varepsilon - \lambda \varepsilon) \inf_{\omega'} a(x) \int_0^T \int_{\omega'} e^{2s\varphi} (v_x)^2 \, dx \, dt$$

$$\leq 2(1 - \varepsilon - \lambda \varepsilon) \int_0^T \int_\omega \xi^2 e^{2s\varphi} a(v_x)^2 \, dx \, dt$$

$$\leq 2(1 - \varepsilon - \lambda \varepsilon) \int_0^T \int_\omega \xi^2 e^{2s\varphi} a(v_x)^2 \, dx \, dt$$

$$\leq C \int_0^T \int_\omega v^2 \, dx \, dt - 2 \int_0^T \int_\omega \xi^2 s\varphi_t e^{2s\varphi} v^2 \, dx \, dt + D_\varepsilon \int_0^T \int_\omega \frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}} a v^2 \, dx \, dt.$$

Finally, we show that there exists a positive constant $C$ (still depending on $\varepsilon$) such that

$$- 2 \int_0^T \int_\omega \xi^2 s\varphi_t e^{2s\varphi} v^2 \, dx \, dt + D_\varepsilon \int_0^T \int_\omega \frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}} a v^2 \, dx \, dt$$

$$\leq C \int_0^T \int_\omega v^2 \, dx \, dt,$$

so that the claim will follow. Indeed,

$$|s\varphi_t e^{2s\varphi}| \leq c s^{1/4} \frac{1}{s_0^{1/4} (-\max \Upsilon)^{1/4}},$$

$$|\dot{\Theta}| \leq c \Theta^{5/4}$$

and

$$|s\varphi_t e^{2s\varphi}| \leq c s(-\Upsilon) \Theta^{5/4} e^{2s\varphi} \leq \frac{c}{(s(-\Upsilon))^{5/4}}$$

for some constants $c > 0$ which may vary at every step.

On the other hand, $\frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}}$ can be estimated by

$$C(e^{2s\varphi} + s^2 (\varphi_x)^2 e^{2s\varphi}) \chi_\omega,$$

and proceeding as before, we get the claim, choosing $\varepsilon$ small enough, namely $\varepsilon < (1 + \lambda)^{-1}$.

We shall also use the following
Lemma 4.1. Assume Hypotheses 4.1 and 4.2. Then there exist two positive constants $C$ and $s_0$ such that every solution $v$ of (4.37) satisfies, for all $s \geq s_0$,
\[
\int_{Q_T} \left( s\Theta (v_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} \,dxdt \leq C \int_0^T \int_{(0, T) \times (A, B)} e^{2s\varphi} \,dxdt.
\]
Here $\Theta$ and $\varphi$ are as in (4.20) with $c_1$ sufficiently large.

Using the following non degenerate classical Carleman estimate, one has that the proof of the previous lemma is a simple adaptation of the proof of Lemma 5.1 and 5.2, to which we refer, also to explain why $c_1$ must be large.

**Proposition 4.3 (Nondegenerate nonsingular Carleman estimate).** Let $z$ be the solution of
\[
\begin{cases}
  z_t + (az_x)_x + \frac{z}{b} = h \in L^2((0, T) \times (A, B)), \\
  z(t, A) = z(t, B) = 0, \quad t \in (0, T),
\end{cases}
\]
(4.43)
where $b \in C([A, B])$ is such that $b \geq b_0 > 0$ in $[A, B]$ and $a$ satisfies

$(a_1)$ $a \in W^{1,1}(A, B)$, $a \geq a_0 > 0$ in $(A, B)$ and there exist two functions $g \in L^1(A, B)$, $h \in W^{1,1}(A, B)$ and two strictly positive constants $g_0$, $h_0$ such that $g(x) \geq g_0$ for a.e. $x$ in $[A, B]$ and
\[
- \frac{a'(x)}{2\sqrt{a(x)}} \left( \int_x^B g(t) \,dt + h_0 \right) + \sqrt{a(x)} g(x) = b(x) \quad \text{for a.e. } x \in [A, B];
\]
or

$(a_2)$ $a \in W^{1,\infty}(A, B)$ and $a \geq a_0 > 0$ in $(A, B)$.

Then, for all $\lambda \in \mathbb{R}$, there exist three positive constants $C$, $r$ and $s_0$ such that for any $s > s_0$
\[
\int_0^T \int_A^B \left( s\Theta (z_x)^2 + s^3 \Theta^3 z^2 \right) e^{2s\varphi} \,dxdt \leq C \left( \int_0^T \int_A^B h^2 e^{2s\varphi} \,dxdt - (B.T.) \right),
\]
(4.44)
where
\[
(B.T.) = \left\{ \begin{array}{ll}
  s \int_0^T \left[ a^{3/2} e^{2s\Phi} \Theta \left( \int_x^B g(\tau) \,d\tau + h_0 \right) (z_x)^2 \right]_{x=A}^{x=B} \,dt, & \text{if } (a_1) \text{ holds}, \\
  s \int_0^T \left[ a e^{2s\Phi} \Theta e^{r(x_2)} \right]_{x=A}^{x=B} \,dt, & \text{if } (a_2) \text{ holds}.
\end{array} \right.
\]

Here the function $\Phi$ is defined as $\Phi(t, x) := \Theta(t) \rho(x)$, where $\Theta$ is as in (3.20),
\[
\rho(x) := \left\{ \begin{array}{ll}
  -r \left[ \int_x^A \frac{1}{\sqrt{a(t)}} \int_t^B g(s) \,ds \,dt + \int_A^x \frac{h_0}{\sqrt{a(t)}} \,dt \right] - c, & \text{if } (a_1) \text{ holds}, \\
  e^{r(x_2)} - c, & \text{if } (a_2) \text{ holds}.
\end{array} \right.
\]
(4.45)
and
\[ \zeta(x) = \mathfrak{d} \int_x^B \frac{1}{a(t)} dt. \]

Here \( \mathfrak{d} = \|a'\|_{L^\infty(A,B)} \) and \( c > 0 \) is chosen in the second case in such a way that \( \max_{[A,B]} \rho < 0 \).

Proof. Rewrite the equation of (4.43) as \( z_t + (az_x)_x = \bar{h} \), where \( \bar{h} := h - \frac{\lambda z}{b} \). Then, applying [25, Theorem 3.1], there exist two positive constants \( C \) and \( s_0 > 0 \), such that
\[ \int_0^T \int_A^B (s\Theta(z_x)^2 + s^3\Theta^3 z^2) e^{2s\Phi} dxdt \leq C \left( \int_0^T \int_A^B \bar{h}^2 e^{2s\Phi} dxdt - (B,T) \right), \]
(4.46)
for all \( s \geq s_0 \). Using the definition of \( \bar{h} \), the term \( \int_0^T \int_A^B e^{2s\Phi} \bar{h}^2 dxdt \) can be estimated in the following way:
\[ \int_0^T \int_A^B \bar{h}^2 e^{2s\Phi} dxdt \leq 2 \int_0^T \int_A^B h^2 e^{2s\Phi} dxdt + 2\lambda^2 \int_0^T \int_A^B \frac{z^2}{b^2} e^{2s\Phi} dxdt. \]
(4.47)
Applying the classical Poincaré inequality to \( w(t,x) := e^{s\Phi} z(t,x) \) and observing that \( 0 < \inf \Theta \leq \Theta \leq c\Theta^2 \), one has
\[ 2\lambda^2 \int_0^T \int_A^B \frac{z^2}{b^2} e^{2s\Phi} dxdt = 2\lambda^2 \int_0^T \int_A^B \frac{w^2}{b^2} dxdt \leq 2\lambda^2 \frac{C}{b_0} \left( \int_0^T \int_A^B (w_x)^2 dxdt \right), \]
\[ \leq C \int_0^T \int_A^B (s^2\Theta^2 \lambda^2 + (z_x)^2) e^{2s\Phi} dxdt \]
\[ \leq \int_0^T \int_A^B \frac{s^2}{2} \Theta(z_x)^2 e^{2s\Phi} dxdt + \int_0^T \int_A^B \frac{s^3}{2} \Theta^3 z^2 e^{2s\Phi} dxdt, \]
for \( s \) large enough. Using this last inequality in (4.47), we have
\[ \int_0^T \int_A^B \bar{h}^2 e^{2s\Phi} dxdt \leq 2 \int_0^T \int_A^B h^2 e^{2s\Phi} dxdt + \int_0^T \int_A^B \frac{s^2}{2} \Theta(z_x)^2 e^{2s\Phi} dxdt \]
\[ + \int_0^T \int_A^B \frac{s^3}{2} \Theta^3 z^2 e^{2s\Phi} dxdt. \]
(4.48)
Using this inequality in (4.46), (4.44) follows immediately. \( \square \)

In order to prove Proposition 4.1, the last result that we need is the following:

**Lemma 4.2.** Assume Hypotheses 4.1 and 4.2. Then there exists a positive constant \( C_T \) such that every solution \( v \in W \) of (4.37) satisfies
\[ \int_0^1 v^2(0,x)dx \leq C_T \int_0^T \int_\omega v^2(t,x)dxdt. \]

28
Proof. Multiplying the equation of (4.37) by $v_t$ and integrating by parts over $(0,1)$, one has

$$0 = \int_0^1 \left( v_t + (av_x)_x + \lambda \frac{v}{b} \right) v_t \, dx = \int_0^1 \left( v_t^2 + (av_x)_x v_t + \lambda \frac{v v_t}{b} \right) \, dx = \int_0^1 v_t^2 \, dx + [av_x v_t]_{x=0}^{x=1}$$

$$- \int_0^1 av_x v_{xx} \, dx + \lambda \frac{d}{dt} \int_0^1 \frac{v^2}{b} \, dx = \int_0^1 v_t^2 \, dx - \frac{1}{2} \frac{d}{dt} \int_0^1 a(v_x)^2 \, dx + \frac{\lambda}{2} \frac{d}{dt} \int_0^1 \frac{v^2}{b} \, dx$$

$$\geq -\frac{1}{2} \frac{d}{dt} \int_0^1 a(v_x)^2 \, dx + \frac{\lambda}{2} \frac{d}{dt} \int_0^1 \frac{v^2}{b} \, dx.$$

Thus, the function

$$t \mapsto \int_0^1 a(v_x)^2 \, dx - \lambda \int_0^1 \frac{v^2}{b} \, dx$$

is non decreasing for all $t \in [0,T]$. In particular,

$$\int_0^1 a(v_x)^2(0,x) \, dx - \lambda \int_0^1 \frac{v^2(0,x)}{b(x)} \, dx \leq \int_0^1 a(v_x)^2(t,x) \, dx - \lambda \int_0^1 \frac{v^2(t,x)}{b(x)} \, dx$$

$$\leq (1 + |\lambda| C^*) \int_0^1 a(v_x)^2(t,x) \, dx.$$

Integrating the previous inequality over $[\frac{T}{4}, \frac{3T}{4}]$, $\Theta$ being bounded therein, we find

$$\int_0^1 a(x)(v_x)^2(0,x) \, dx - \lambda \int_0^1 \frac{v^2(0,x)}{b(x)} \, dx$$

$$\leq \frac{2}{T} (1 + |\lambda| C^*) \int_T^{\frac{2T}{3}} \int_0^1 a(v_x)^2 \, dx \, dt$$

$$\leq C_T \int_T^{\frac{2T}{3}} \int_0^1 s \Theta a(v_x)^2 e^{2s} \, dx \, dt.$$

Hence, from the previous inequality and Lemma 4.11 if $\lambda \leq 0$

$$\int_0^1 a(v_x)^2(0,x) \, dx \leq \int_0^1 a(v_x)^2(0,x) \, dx - \lambda \int_0^1 \frac{v^2(0,x)}{b(x)} \, dx \leq C \int_0^T \int_\omega v^2 \, dxdt$$

for some positive constant $C > 0$.

If $\lambda > 0$, using again Lemma 4.11 and (4.49), one has

$$\int_0^1 a(v_x)^2(0,x) \, dx - \lambda \int_0^1 \frac{v^2(0,x)}{b(x)} \, dx \leq C \int_0^T \int_\omega v^2 \, dxdt.$$  (4.50)

Hence, by (2.9) and (4.50), we have

$$\int_0^1 a(v_x)^2(0,x) \, dx \leq \lambda \int_0^1 \frac{v^2(0,x)}{b(x)} \, dx + C \int_0^T \int_\omega v^2 \, dxdt$$

$$\leq \lambda C^* \int_0^1 a(v_x)^2(0,x) \, dx + C \int_0^T \int_\omega v^2 \, dxdt.$$
Thus

\[(1 - \lambda C^*) \int_0^1 a(v_x)^2(0, x) dx \leq C \int_0^T \int_\omega v^2 dxdt,\]

for a positive constant \(C\). In every case, there exists \(C > 0\) such that

\[\int_0^1 a(v_x)^2(0, x) dx \leq C \int_0^T \int_\omega v^2 dxdt. \tag{4.51}\]

The Hardy-Poincaré inequality (see Proposition 2.1) and (4.51) imply that

\[
\int_0^1 \left( \frac{a}{(x-x_0)^2} \right)^{1/3} v^2(0, x) dx \leq \int_0^1 \frac{p}{(x-x_0)^2} v^2(0, x) dx \\
\leq C_{HP} \int_0^1 p(v_x)^2(0, x) dx \\
\leq cC_{HP} \int_0^1 a(v_x)^2(0, x) dx \\
\leq C \int_0^T \int_\omega v^2 dxdt,
\]

for a positive constant \(C\). Here \(p(x) = (a(x)|x - x_0|^4)^{1/3}\) if \(K_1 > \frac{4}{3}\), while \(p(x) = |x - x_0|^{4/3} \max a\) otherwise, and \(c, C\) are obtained by Lemma 2.1.

Again by Lemma 2.1, we have

\[
\left( \frac{a(x)}{(x-x_0)^2} \right)^{1/3} \geq C_3 := \min \left\{ \left( \frac{a(1)}{(1-x_0)^2} \right)^{1/3}, \left( \frac{a(0)}{x_0} \right)^{1/3} \right\} > 0.
\]

Hence

\[C_3 \int_0^1 v(0, x)^2 dx \leq C \int_0^T \int_\omega v^2 dxdt\]

and the claim follows.

\[\square\]

**Proof of Proposition 4.1.** It follows by a density argument as for the proof of [24, Proposition 4.1].

**References**


