On Scott consequence systems

Dedicated to the memory of Ivan Prodanov

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Abstract

The notion of Scott consequence system (briefly, S-system) was introduced by D. Vakarelov
in [30] in an analogy to a similar notion given by D. Scott in [24]. In part one of the paper we
study the category $SSyst$ of all S-systems and all their morphisms. We show that the category
$DLat$ of all distributive lattices and all lattice homomorphisms is isomorphic to a reflective full
subcategory of the category $SSyst$. Extending the representation theory of D. Vakarelov [30]
for S-systems in P-systems, we develop an isomorphism theory for S-systems and for Tarski
consequence systems.

In part two of the paper we prove that the separation theorem for S-systems is equivalent
to some other separation principles, including the separation theorem for filters and ideals in
Boolean algebras and separation theorem for convex sets in convexity spaces.

Introduction

The notion of Scott consequence systems (briefly, S-systems) was introduced by D.
Vakarelov in [30] in an analogy to a similar notion given by D. Scott in [24]. A standard
example of an S-system consists of the set of all formulas of some formalized logical language
with consequence relation $X \vdash Y$ between sets of formulas $X$ and $Y$. A detailed study of
such consequence relations in the context of propositional languages is given by Segerberg
in [25] (see also [10]). The axioms of S-system are abstract versions of some properties of
the consequence relation $\vdash$ taken from logic. There are however many non-logical examples
of S-systems and the main aim of this paper is a study of some mathematical properties of
this notion taken in its full generality.

The paper is divided into two parts. In part one we introduce the notion of an $S$-
morphism between two S-systems, which enables us to define the category $SSyst$ of all
S-systems and all S-morphisms between them. The category $SSyst$, as well as its full
subcategory $TSyst$ of all Tarski consequence systems, are the main objects of our inves-
tigations in this part. We prove some isomorphism theorems for these categories. With
one of these theorems we extend the representation theory of S-systems in some kind of
information systems called Property systems, given by D. Vakarelov in [30]. We also show
that the categories $Bool$ (of all Boolean algebras and all Boolean homomorphisms) and

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**DLat** (of all distributive lattices and all lattice homomorphisms) are isomorphic to some reflective full subcategories of the category **SSyst**. Let’s give a more detailed description of our isomorphism theorems. We define a category **TPS**, called the category of topological property systems, and we prove that the category **SSyst** is isomorphic to a full subcategory **TPSS** of **TPS**: then we show that the restriction of this isomorphism to the full subcategory **TSyst** of **SSyst** is in fact an isomorphism between the category **TSyst** and a subcategory **T** of the category **Top** of all topological spaces and all continuous maps.

In part two of the paper we study the connections of S-systems with some other mathematical structures such as convexity spaces [5], separative algebras [19] and distributive lattices and Boolean algebras. In each of these systems a general separation principle is true. In distributive lattices and Boolean algebras this is the separation of a disjoint pair of a filter and an ideal by a disjoint pair of a prime filter and a prime ideal [26] [27] (equivalent to the well known “Prime Filter Theorem”), in convexity spaces this is the separation of a disjoint pair of convex sets by a semispace (a convex set which complement is also a convex set). The main aim of part two of the paper is a proof of the equivalence of all these principles in ZF — the Zermelo-Fraenkel Set Theory without the Axiom of Choice.

For all undefined here notions and notations, see [9], [14] and [1].

**Part I**

**The category of Scott consequence systems**

1 **Preliminaries**

We first fix some notations.

**Notation 1.1** If **C** denotes a category, we write *X* ∈ |**C**| if *X* is an object of **C**, and *f* ∈ **C**(X, Y) if *f* is a morphism of **C** with domain *X* and codomain *Y*. We denote by **Set** the category of all sets and all functions and by **Top** the category of all topological spaces and all continuous maps.

If *X* is a set then we denote by **Exp**(*X*) (resp. **Fin**(*X*)) the set of all (resp. all non-empty finite) subsets of *X*. If *f* ∈ **Set**(*X*, *Y*) and *A* ⊆ **Exp**(*Y*) then we write *f*<sup>−1</sup>(*A*) for the set \{*f*<sup>−1</sup>(*A*) : *A* ∈ *A*\}.

All lattices will be with top (=unit) and bottom (=zero) elements, denoted respectively by 1 and 0. We don’t require the elements 0 and 1 to be distinct. As usual, the lattice homomorphisms are assumed to preserve the distinguished elements 0 and 1. **DLat** will stand for the category of all distributive lattices and all lattice homomorphisms, and **Bool** for the category of all Boolean algebras and all their homomorphisms.

We shall denote by **KO**(*X*, *T*) (or, simply, by **KO**(X)) the family of all compact open subsets of a topological space (*X*, *T*) and by **D** the two-point set \{0, 1\} endowed with the discrete topology.

We give now the precise definitions of some notions mentioned in the Introduction.

**Definition 1.2** (see [25, 24, 30]) Let *W* be a non-empty set. By a Scott consequence relation in *W* we mean a binary relation ⊢ in **Exp**(W) satisfying the following conditions for any *A*, *B*, *A′*, *B′* ∈ **Exp**(W) and *x* ∈ *W*:
(Refl) If $A \cap B \neq \emptyset$ then $A \vdash B$.
(Mono) If $A \vdash B$, $A \subseteq A'$ and $B \subseteq B'$ then $A' \vdash B'$.
(Cut) If $A \vdash (B \cup \{x\})$ and $(\{x\} \cup A) \vdash B$ then $A \vdash B$.
(Fin) If $A \vdash B$ then there exist finite subsets $X \subseteq A$ and $Y \subseteq B$ such that $X \vdash Y$.

We say that $(W, \vdash)$ is a Scott consequence system, briefly, $S$-system, if $W$ is a non-empty set and $\vdash$ is a Scott consequence relation in $W$.

**Definition 1.3** (see [10, 30]) Let $S = (W, \vdash)$ be an $S$-system. We say that $\vdash$ is a Tarski consequence relation in $W$ and $S$ is a Tarski consequence system (briefly, $T$-system), if the following condition is satisfied for any $A, B \in Exp(W)$:

(TFin) If $A \vdash B$ then there exist a finite set $X \subseteq A$ and an element $b \in B$ such that $X \vdash \{b\}$.

We recall now some definitions and results from [29, 30]. They play a crucial role in our further investigations:

**Definition 1.4** ([29]) By a property system (briefly, $P$-system) we mean any triple $P = (Ob, Pr, f)$, where $Ob$ and $Pr$ are sets, $Ob \neq \emptyset$ and $f \in Set(Ob, Exp(Pr))$. The elements of $Ob$ (resp. $Pr$: $f(x)$) are called objects (resp. properties: properties of the object $x$). A $P$-system $P = (Ob, Pr, f)$ is called set-theoretical $P$-system if $Pr \subseteq Exp(Ob)$ and $f(x) = \{A \in Pr : x \in A\}$ for any $x \in Ob$.

1.5 Let $(W, \vdash)$ be an $S$-system. A subset $p \subseteq W$ is called a prime ideal in $(W, \vdash)$ if for all finite subsets $A$ and $B$ of $W$ such that $A \vdash B$, $A \cap p = \emptyset$ implies $B \setminus p \neq \emptyset$. A subset $q \subseteq W$ is called a prime filter in $(W, \vdash)$ if the set $W \setminus q$ is a prime ideal in $(W, \vdash)$. The set of all prime ideals (resp., prime filters) of $(W, \vdash)$ will be denoted by $PI(W, \vdash)$ (resp., by $PF(W, \vdash)$). Let us put $f(a) = \{p \in PI(W, \vdash) : a \notin p\}$ and $f'(a) = \{q \in PF(W, \vdash) : a \in q\}$ for all $a \in W$. Then the system $(W, PI(W, \vdash), f)$ is a $P$-system, called the canonical $P$-system over $(W, \vdash)$. It is denoted by $P(W, \vdash)$. The system $(W, PF(W, \vdash), f')$ is a set-theoretical $P$-system. It is called the canonical set-theoretical $P$-system over $(W, \vdash)$ and is denoted by $P'(W, \vdash)$.

1.6 Let $W \neq \emptyset$ be a set, $L \in [DLat]$ and $f \in Set(W, L)$. Define a binary relation $\vdash_L$ in $Exp(W)$ as follows. For any $A = \{a_i \in W : i = 1, \ldots, n\}$ and $B = \{b_j \in W : j = 1, \ldots, m\}$, put $A \vdash_L B$ iff $\bigwedge\{f(a_i) : i = 1, \ldots, n\} \leq \bigvee\{f(b_j) : j = 1, \ldots, m\}$ (here $n$ and $m$ could be equal to zero as well). For arbitrary sets $A', B' \subseteq W$ let $A' \vdash_L B'$ iff there exist finite subsets $A \subseteq A'$ and $B \subseteq B'$ such that $A \vdash B$. Then $(W, \vdash_L)$ is an $S$-system. In the special case of this construction when $W = L$ and $f = id$, the $S$-system $(L, \vdash_L)$ is denoted by $Sc(L)$.

One more special case will be used here. Let $P = (Ob, Pr, f)$ be a $P$-system. Put $W = Ob$ and $L = (Exp(Pr), \cup, \cap, \emptyset, Pr)$. By the definition of a $P$-system, we have that $f \in Set(W, L)$. Hence, applying the above construction, we obtain the $S$-system $(W, \vdash_L)$. The relation $\vdash_L$ is denoted in this case by $\vdash_P$. The $S$-system $(Ob, \vdash_P)$ is called the canonical $S$-system over $P$ and is denoted by $Sc(P)$.

**Proposition 1.7** ([30]) Let $(W, \vdash)$ be an $S$-system. Then, for any $A, B \subseteq W$, the following conditions are equivalent:

(a) $A \vdash B$;

(b) if $p$ is a prime ideal in $(W, \vdash)$ and $A \cap p = \emptyset$ then $B \setminus p \neq \emptyset$;

(c) if $p$ is a prime filter in $(W, \vdash)$ and $A \subseteq p$ then $B \cap p \neq \emptyset$. 

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Proposition 1.8 ([30]) Let \((W, \triangleright)\) be an \(S\)-system. Then:
(1) For any \(F \subseteq W\), the following conditions are equivalent:
   (a) \(F\) is a prime filter (resp., prime ideal);
   (b) \((\forall A \subseteq W)((F \triangleright A) \implies (F \cap A \neq \emptyset)) \) (resp., \((\forall A \subseteq W)((W \setminus F) \triangleright A)\) implies \((A \setminus F \neq \emptyset))\);
   (c) \(F \nmid (W \setminus F)\) (resp., \((W \setminus F) \nmid F\)).
(2) \(W\) is a prime filter (resp., prime ideal) iff \((\forall A \subseteq W)(A \nmid \emptyset)\) (resp., \((\forall A \subseteq W)(\emptyset \nmid A)\)).

We now recall the definitions of coherent spaces and coherent maps (see, for example, [14]):

1.9 Let \((X, T)\) be a topological space. A closed subset \(F\) of \(X\) is called irreducible if the equality \(F = F_1 \cup F_2\), where \(F_1\) and \(F_2\) are closed subsets of \(X\), implies that \(F = F_1\) or \(F = F_2\). The space \((X, T)\) is called sober if it is a \(T_0\)-space and for every non-void irreducible subset \(F\) of \(X\) there exists a \(x \in X\) such that \(F = cl_X \{x\}\). The space \((X, T)\) is called coherent if it is a compact sober space, the family \(KO(X, T)\) is closed under finite intersections and \(KO(X, T)\) is a base for the topology \(T\). A continuous map \(f : (X', T') \longrightarrow (X'', T'')\) is called coherent if \(U'' \in KO(X'')\) implies that \(f^{-1}(U'') \in KO(X')\).

We denote by \(Coh\text{Sp}\) the category of all coherent spaces and all coherent maps between them.

1.10 Let \(L = (L, \lor, \land, 0, 1) \in |DLat|\). Recall that (see, for example, [14]): (a) a sub-join-semilattice \(I\) of the lattice \(L\) is said to be an ideal in \(L\) if \((a \in I, b \in L\) and \(b \leq a) \Rightarrow (b \in I)\); (b) an ideal \(p\) in \(L\) is called a prime ideal if \(1 \notin p\) and \((a \land b \in p) \Rightarrow (a \in p\) or \(b \in p)\); (c) the set of all prime ideals in \(L\) is denoted by \(spec(L)\); (d) the family \(\mathcal{O} = \{O_I = \{p \in spec(L) : I \not\subseteq p\} : I\) is an ideal in \(L\}\) is a topology on the set \(spec(L)\), called Stone topology; (e) the topological space \((spec(L), \mathcal{O})\) is the classical spectrum of the lattice \(L\); it is a coherent space.

By the famous Stone representation theorem for distributive lattices (see [27]), the categories \(DLat\) and \(Coh\text{Sp}\) are dual. Let’s recall the descriptions of the duality functors \(F : Coh\text{Sp} \longrightarrow DLat\) and \(G : DLat \longrightarrow Coh\text{Sp}\). If \(X\) is a coherent space then \(F(X) = (KO(X), \cup, \cap, \emptyset, X)\); if \(f \in Coh\text{Sp}(X_1, X_2)\) then \(F(f) : F(X_2) \longrightarrow F(X_1)\) is defined by the formula \(F(f)(U) = f^{-1}(U)\) for every \(U \in KO(X_2)\); if \(L \in |DLat|\) then \(G(L) = (spec(L), \mathcal{O})\), where \((spec(L), \mathcal{O})\) is the classical spectrum of the lattice \(L\); if \(f \in DLat(L_1, L_2)\) then \(G(f) : G(L_2) \longrightarrow G(L_1)\) is defined by the formula \(G(f)(p) = f^{-1}(p)\) for every \(p \in spec(L_2)\).

2 The category of \(S\)-systems and \(S\)-morphisms

Definition 2.1 Let \((W, \triangleright)\) and \((W', \triangleright')\) be two \(S\)-systems and \(f \in Set(W, W')\). The function \(f\) is called an \(S\)-morphism if \((A \triangleright B) \Rightarrow (f(A) \triangleright' f(B))\) for any \(A, B \in Exp(W)\). We denote by \(SSyst\) the category of all \(S\)-systems and all \(S\)-morphisms between them.

The following simple fact will be often used in this paper:

Proposition 2.2 Let \(f : (W, \triangleright) \longrightarrow (W', \triangleright')\) be an \(S\)-morphism and \(F' \subseteq W'\) be a prime filter (resp., prime ideal) in \((W', \triangleright')\). Then \(f^{-1}(F')\) is a prime filter (resp., prime ideal) in \((W, \triangleright)\).
Proof. Let $F' \subseteq W'$ be a prime filter in $(W', \triangleright')$, $A \subseteq W$ and $f^{-1}(F') \triangleright A$. Then $f(f^{-1}(F')) \triangleright f(A)$. Hence $F' \triangleright f(A)$. Thus, by 1.8(1), $F' \cap f(A) \neq \emptyset$. Then $A \cap f^{-1}(F') \neq \emptyset$. Therefore, by 1.8(1), $f^{-1}(F')$ is a prime filter in $(W', \triangleright')$.

The corresponding statement for the prime ideals follows directly from the just proved one. □

Definition 2.3 We will denote by $SDLat$ the full subcategory of the category $SSyst$ whose objects are of the form $Sc(L)$, where $L \in |DLat|$ (see 1.6 for the notations).

Proposition 2.4 The category $DLat$ is isomorphic to the subcategory $SDLat$ of the category $SSyst$.

Proof. We will prove that if $\varphi : DLat \rightarrow SDLat$ is defined on the objects by $\varphi(L) = Sc(L)$ and on the morphisms by $\varphi(l) = l$ then $\varphi$ is an isomorphism (see 1.6 for the notations).

We have that $a \leq b$ iff $\{a\} \triangleright L \{b\}$, for every $a, b \in L$ (see 1.6 for $\triangleright_L$). This shows that if $L, L' \in |DLat|$ then $Sc(L) = Sc(L')$ is equivalent to $L = L'$. Let now $l \in DLat(L, L')$. We will prove that $l \in SDLat(Sc(L), Sc(L'))$. Let $A, B \subseteq L$ and $A \triangleright_L B$. Then there exist finite subsets $A' = \{a_i : i = 1, \ldots, n\} \subseteq A$ and $B' = \{b_j : j = 1, \ldots, m\} \subseteq B$ such that $A' \triangleright_L B'$. This means that $\bigwedge\{a_i : i = 1, \ldots, n\} \leq \bigvee\{b_j : j = 1, \ldots, m\}$. Thus we obtain that $\bigwedge\{l(a_i) : i = 1, \ldots, n\} \leq \bigvee\{l(b_j) : j = 1, \ldots, m\}$. Hence $l(A') \triangleright_L l(B')$ and this implies that $l(A) \triangleright_L l(B)$. Therefore $l$ is an S-morphism, i.e. $l \in SDLat(Sc(L), Sc(L'))$. Conversely, if $l \in SDLat(Sc(L), Sc(L'))$, then $l \in DLat(L, L')$. Indeed, if $a, b \in L$ and $a \leq b$ then $\{a\} \triangleright_L \{b\}$ and hence $\{l(a)\} \triangleright_L \{l(b)\}$. Thus $l(a) \leq l(b)$. So, $l$ is an order-preserving map. Further, let $a \vee b = c$ in $L$. Then $\{c\} \triangleright_L \{a, b\}$. Therefore $\{l(c)\} \triangleright_L \{l(a), l(b)\}$. This implies that $l(a \vee b) = l(c) \leq l(a) \vee l(b)$. On the other hand, the inequalities $a \leq c$ and $b \leq c$ imply (since $l$ is order-preserving) that $l(a) \vee l(b) \leq l(c) = l(a \vee b)$. So, $l(a \vee b) = l(a) \vee l(b)$. Analogously we prove that $l(a \wedge b) = l(a) \wedge l(b)$. Finally, since $\emptyset \triangleright_L \{1\}$, we have that $\emptyset \triangleright_L \{l(1_L)\}$. Hence $1_L' = \bigwedge\emptyset \leq l(1_L)$. So, $l(1_L) = 1_L'$. Analogously, $\{0_L\} \triangleright_L \emptyset$ implies that $l(0_L) = 0_L'$. Therefore, $l \in DLat(L, L')$. All this shows that $\varphi$ is a functor. It is now easily seen that $\varphi$ is an isomorphism. □

We are going now to demonstrate that $SDLat$ is a reflective subcategory of the category $SSyst$. Let’s start with the following theorem which is a slight generalization of a result of Iv. Prodanov from [19]. We formulated and proved it in [8] (see there the Claim in the proof of Theorem 2.4.1).

Theorem 2.5 Let $X$ be a set and $S \subseteq Exp(X)$. Setting, for every $x \in X$, $U^-\{x\} = \{p \in S : x \notin p\}$, let $T -$ be the topology on $S$ having as a subbase the family $T^- = \{U^-\{x\} : x \in X\}$. Let $(S, T^-)$ be a coherent space and $L = F(S, T^-)$ (see 1.10 for the notations). Then $U^-\{x\} \in L$ for every $x \in X$. Define $\varphi : X \rightarrow L$ by the formula $\varphi(x) = U^-\{x\}$, for every $x \in X$. Then:

(i) the set $\varphi(X)$ generates $L; \hfill \text{(i)}$

(ii) $\varphi^{-1}(q) \in S$ for every $q \in spec(L)$ (see 1.10 for the notations); \hfill \text{(ii)}

(iii) $\Phi : spec(L) \rightarrow S, q \rightarrow \varphi^{-1}(q)$, is a CohSp-isomorphism; \hfill \text{(iii)}

(iv) if $L' \in |DLat|$ and $\theta : X \rightarrow L'$ is a function such that:

(1) $\theta^{-1}(q) \in S$ for every $q \in spec(L')$, and

(2) $\theta : spec(L') \rightarrow S, q \rightarrow \theta^{-1}(q)$, is a CohSp-morphism, \hfill \text{(iv)}

then there exists a unique lattice homomorphism $l : L \rightarrow L'$ with $l \circ \varphi = \theta; \hfill \text{(v)}$

$v : X \rightarrow L$ is an injection iff for any two different points $x$ and $y$ of $X$ there exists a $p \in S$ containing exactly one of them. \hfill \text{(v)}$

We also need the following result of Iv. Prodanov (see [8]):
Proposition 2.6 Let \(X\) be a set, \(S \subseteq \text{Exp}(X)\) and \(T^-\) be the topology on \(S\) defined in 2.5. Then the following conditions are equivalent:

(a) \((S, T^-)\) is a coherent space;

(b) \(S\) is a closed subset of the Cantor cube \(D^X\) (where \(S\) is identified with a subset of \(D^X\) in the following way: any \(A \in S\) is identified with its characteristic function \(\chi_A : X \rightarrow D\), \(\chi_A(x) = 1\) iff \(x \in A\)).

Proposition 2.7 Let \((W, \vdash)\) be an \(S\)-system. Put \(S = \Pi(W, \vdash)\) and define the topology \(T^-\) on \(S\) exactly as in 2.5. Then \((S, T^-)\) is a coherent space.

Proof. Identifying \(S\) with a subset of \(D^W\) as in 2.6, we have to prove, according to 2.6, that \(S\) is a closed subset of the Cantor cube \(D^W\).

Let \(\{p_\sigma, \sigma \in \Sigma\}\) be a net in \(S\) converging in \(D^W\) to a point \(p \in D^W\). This means that if \(f_\sigma : W \rightarrow D\) and \(f : W \rightarrow D\) are functions such that \(f^{-1}_\sigma(1) = p_\sigma\) for every \(\sigma \in \Sigma\) and \(f^{-1}(1) = p\), then \(\{f_\sigma, \sigma \in \Sigma\}\) converges to \(f\) in \(D^W\). We have to prove that \(p \in S\), i.e. that \(f^{-1}(1) \in S\).

Let \(A = \{a_i : i = 1, \ldots, n\}\) and \(B = \{b_j : j = 1, \ldots, m\}\) be two finite subsets of \(W\) and \(A \cap p = \emptyset\). We have to show that \(B \setminus p \neq \emptyset\) (see 1.5). For every \(i = 1, \ldots, n\) we have that \(f(a_i) = 0\). Let \(i \in \{1, \ldots, n\}\). Since the net \(\{f_\sigma(a_i), \sigma \in \Sigma\}\) converges to \(f(a_i)\), there exists a \(\sigma_i \in \Sigma\) such that \(f_{\sigma_i}(a_i) = 0\) for every \(\sigma \geq \sigma_i\). Let \(\sigma_0 = \sup(\{\sigma_i : i = 1, \ldots, n\})\). Then, for every \(\sigma \geq \sigma_0\) and for every \(i = 1, \ldots, n\), we have that \(f_\sigma(a_i) = 0\). Hence, for every \(\sigma \geq \sigma_0\), we get that \(A \cap p = \emptyset\). Since \(p_\sigma\) is a prime ideal, we obtain that \(B \setminus p \neq \emptyset\) for every \(\sigma \geq \sigma_0\). Consequently, for every \(\sigma \geq \sigma_0\), there exists a \(j(\sigma) \in \{1, \ldots, m\}\) such that \(b_{j(\sigma)} \notin p_\sigma\), i.e. \(f_\sigma(b_{j(\sigma)}) = 0\). Defining a function \(\alpha : \{\sigma \in \Sigma : \sigma \geq \sigma_0\} \rightarrow B\) by the formula \(\alpha(\sigma) = b_{j(\sigma)}\), for every \(\sigma \geq \sigma_0\), we get that \(f_\alpha(\alpha(\sigma)) = 0\) for every \(\sigma \geq \sigma_0\). Obviously, there exists a \(j' \in \{1, \ldots, m\}\) such that the set \(\Sigma' = \alpha^{-1}(b_{j'})\) is a cofinal subset of the directed set \((\Sigma, \leq)\). Then, for every \(\sigma' \in \Sigma'\), we have that \(f_{\sigma'}(b_{j'}) = f_{\sigma'}(\alpha(\sigma')) = 0\). Since \(\{f_{\sigma'}(b_{j'}), \sigma' \in \Sigma'\}\) is a net finer than the net \(\{f_\sigma(b_j), \sigma \in \Sigma\}\) and the last one converges to \(f(b_{j'})\), we obtain that the net \(\{f_{\sigma'}(b_{j'}), \sigma' \in \Sigma'\}\) converges also to \(f(b_{j'})\). Thus \(f(b_{j'}) = 0\), i.e. \(b_{j'} \in B \setminus p\).

Therefore, we proved that \(B \setminus p \neq \emptyset\). This implies that \(p\) is a prime ideal in \((W, \vdash)\), i.e. \(p \in S\). Hence, \(S\) is a closed subset of \(D^W\). Therefore, \((S, T^-)\) is a coherent space. \(\square\)

Theorem 2.8 Let \((W, \vdash)\) be an \(S\)-system. Then there exists a distributive lattice \((L, \lor, \land)\) with \(0\) and \(1\), and a function \(\varphi : W \rightarrow L\) such that:

(i) the set \(\varphi(W)\) generates \(L\);

(ii) for any two finite subsets \(A\) and \(B\) of \(W\) we have that \(A \vdash B\) iff \(\varphi(A) \land \varphi(B)\) (see 1.6 for the notations);

(iii) if \(L' \in \{\text{DLat}\}\) and \(\theta : (W, \vdash) \rightarrow Sc(L')\) is an \(S\)-morphism (see 1.6 for the notations) then there exists a unique lattice homomorphism \(l : L \rightarrow L'\) such that \(l \circ \varphi = \theta\).

(iv) \(\varphi : W \rightarrow L\) is an injection iff for any two different points \(x\) and \(y\) of \(W\) there exists a prime ideal \(p\) in \((W, \vdash)\) containing exactly one of them.

Proof. Put \(S = \Pi(W, \vdash)\) and let \(T^-\) be the topology on \(S\) defined exactly as in 2.5. Then, by 2.7, we have that \((S, T^-)\) is a coherent space. Hence, setting \(L = F(S, T^-)\) and \(\varphi : W \rightarrow L, x \rightarrow U^-_x = \{p \in S : x \notin p\}\) (see 1.10 for the notations and 2.5 for \(\varphi\)), we obtain, applying Theorem 2.5, that the set \(\varphi(W)\) generates \(L\). Hence, condition (i) is fulfilled. It is obvious that 2.5(v) implies our condition (iv). So, let’s prove (ii).

Let \(A = \{a_i : i = 1, \ldots, n\}\) and \(B = \{b_j : j = 1, \ldots, m\}\) be two finite subsets of \(W\). Recall that \(\varphi(A) \vdash L \varphi(B)\) iff \(\bigcap \{\varphi(a_i) : i = 1, \ldots, n\} \subseteq \bigcup \{\varphi(b_j) : j = 1, \ldots, m\}\). The following four cases are possible.
Case 1: \( n \neq 0 \) and \( m \neq 0 \).

Let \( A \vdash B \) and \( p \in \cap \{ \varphi(a_i) : i = 1, \ldots, n \} \). Then \( A \cap p = \emptyset \). Hence, by 1.5, \( B \setminus p \neq \emptyset \). Therefore \( p \in \cup \{ \varphi(b_j) : j = 1, \ldots, m \} \). Thus \( \varphi(A) \vdash_L \varphi(B) \). Conversely, let \( \varphi(A) \vdash_L \varphi(B) \). Take a \( p \in S \) such that \( A \cap p = \emptyset \). Then \( p \in \cap \{ \varphi(a_i) : i = 1, \ldots, n \} \). Thus \( p \in \cup \{ \varphi(b_j) : j = 1, \ldots, m \} \), i.e. there exists a \( j \in \{ 1, \ldots, m \} \) such that \( b_j \notin p \). Therefore \( B \setminus p \neq \emptyset \). This shows, by 1.7, that \( A \vdash B \).

Case 2: \( n = 0 \) and \( m = 0 \).

We have that \( A = B = \emptyset \). Let \( A \vdash B \). Then \( S = \emptyset \). Indeed, if \( p \in S \) then \( A \cap p = \emptyset \) and \( B \setminus p = \emptyset \), which is a contradiction. Hence \( S = \emptyset \). Then \( \vert L \vert = 1 \), i.e. \( 0 = 1 \). Therefore the inequality \( 1 \leq 0 \) takes place. Thus \( \emptyset \cup = 1 \leq 0 = \emptyset \). So, \( \varphi(A) \vdash_L \varphi(B) \).

Conversely, if \( \varphi(A) \vdash_L \varphi(B) \) then \( 1 \leq 0 \) and, hence, \( \vert L \vert = 1 \). This shows that \( S = \emptyset \). Now, 1.7 implies that \( A \vdash B \).

Case 3: \( n = 0 \) and \( m \neq 0 \).

Let \( A \vdash B \). We will prove that \( \cap \{ \varphi(b_j) : j = 1, \ldots, m \} = S \). Suppose that there exists a \( p \in S \) such that \( p \notin \cup \{ \varphi(b_j) : j = 1, \ldots, m \} \). Then \( B \subseteq p \). This is a contradiction because \( A \cap p = \emptyset \). Hence \( \cap \{ \varphi(b_j) : j = 1, \ldots, m \} = S \). Therefore \( \varphi(A) \vdash_L \varphi(B) \).

Conversely, let \( \varphi(A) \vdash_L \varphi(B) \). Then \( \cap \{ \varphi(b_j) : j = 1, \ldots, m \} = S \). Let \( p \in S \). Then \( A \cap p = \emptyset \) and \( B \setminus p \neq \emptyset \). This shows, by 1.7, that \( A \vdash B \).

Case 4: \( n \neq 0 \) and \( m = 0 \).

Let \( A \vdash B \). We will prove that \( \cap \{ \varphi(a_i) : i = 1, \ldots, n \} = \emptyset \). Suppose that there exists a \( p \in \cap \{ \varphi(a_i) : i = 1, \ldots, n \} \). Then \( A \cap p = \emptyset \). Hence, by 1.7, \( B \setminus p \neq \emptyset \). This is a contradiction because \( B \setminus p = \emptyset \). Therefore \( \cap \{ \varphi(a_i) : i = 1, \ldots, n \} = \emptyset \). Thus \( \varphi(A) \vdash_L \varphi(B) \).

Conversely, let \( \varphi(A) \vdash_L \varphi(B) \). Then \( \cap \{ \varphi(a_i) : i = 1, \ldots, n \} = \emptyset \). Let \( p \in S \). Suppose that \( A \cap p = \emptyset \). Then \( p \in \cap \{ \varphi(a_i) : i = 1, \ldots, n \} \), which is a contradiction. Hence, for every \( p \in S \), we have that \( A \cap p \neq \emptyset \). Now, 1.7 implies that \( A \vdash B \). So, (ii) is proved.

We prove (iii) now. Let \( \theta : W \rightarrow L' \) be as in (iii). Obviously, it is enough to show that \( \theta \) satisfies conditions (1) and (2) of 2.5(iv). In order to check condition (1) of 2.5(iv), let’s take a \( q \in \text{spec}(L') \). We have to prove that \( p = \theta^{-1}(q) \in S \).

Suppose that \( p \notin S \). Then there exist two finite subsets \( A \) and \( B \) of \( W \) such that \( A \vdash B \), \( A \cap p = \emptyset \) and \( B \subseteq p \). Then \( \theta(A) \cap q = \emptyset \) and \( \theta(B) \subseteq q \). Let \( A = \{ a_i : i = 1, \ldots, n \} \) and \( B = \{ b_j : j = 1, \ldots, m \} \). Since \( A \vdash B \), we have that \( \lambda^\prime \{ \theta(a_i) : i = 1, \ldots, n \} \leq \lambda^\prime \{ \theta(b_j) : j = 1, \ldots, m \} \). The equality \( \theta(A) \cap q = \emptyset \) and the fact that \( q \) is a prime ideal imply that \( \lambda^\prime \{ \theta(a_i) : i = 1, \ldots, n \} \subseteq q \). Hence \( \lambda^\prime \{ \theta(b_j) : j = 1, \ldots, m \} \subseteq q \). But this is impossible, since \( \theta(B) \) is a subset of \( q \) and, therefore, \( \lambda^\prime \{ \theta(b_j) : j = 1, \ldots, m \} \subseteq q \) (because \( q \) is an ideal). So, we got a contradiction. Hence \( p = \theta^{-1}(q) \in S \). Therefore, condition (1) of 2.5(iv) is fulfilled.

Now, we will show that condition (2) of 2.5(iv) is fulfilled, i.e. we will prove that the function \( \Theta : \text{spec}(L') \rightarrow S \), \( q \rightarrow \theta^{-1}(q) \), is a \( \text{CohSp} \)-morphism. Let’s show first that \( \Theta : (\text{spec}(L'), O') \rightarrow (S, T^-) \) is a continuous map (here \( O' \) is the Stone topology on \( \text{spec}(L') \) (see 1.10)). Recall that the family \( \mathcal{P}^- = \{ U^-_x : x \in W \} \), where \( U^-_x = \{ p \in S : x \notin p \} \) for every \( x \in W \), is a subbase of the topology \( T^- \) on \( S \). Hence, we have to prove that \( \Theta^{-1}(U^-_x) \subseteq O' \) for every \( x \in W \).

Let \( x \in W \). Then \( \Theta^{-1}(U^-_x) = \{ q \in \text{spec}(L') : \Theta(q) \notin U^-_x \} = \{ q \in \text{spec}(L') : \theta^{-1}(q) \notin U^-_x \} = \{ q \in \text{spec}(L') : \theta(x) \notin q \} = \{ q \in \text{spec}(L') : I(\theta(x)) \subseteq q \} = O_1(\theta(x)) \) (see 1.10 for the notations), where \( I(\theta(x)) = \{ l \in L' : l \leq \theta(x) \} \). Since \( I(\theta(x)) \) is an ideal in \( L' \), we obtain that \( \Theta^{-1}(U^-_x) \subseteq O' \). Therefore, \( \Theta \) is a continuous map.

Let \( K \) be a compact open subset of \( (S, T^-) \). Then, obviously, \( K \) is a finite union of elements of the family \( \mathcal{B}^- \) of all finite intersections of the elements of \( \mathcal{P}^- \). Hence, for
showing that \( \Theta^{-1}(K) \) is a compact subset of \( \text{spec}(L') \), it is enough to show that \( \Theta^{-1}(U^-_x) \) is a compact subset of \( \text{spec}(L') \) for every \( x \in W \). (Here we use the fact that the family \( KO(\text{spec}(L')) \) of all compact open subsets of \( \text{spec}(L') \) is closed under finite intersections. It is so because the space \( \text{spec}(L') \) is coherent (see 1.10)). Let \( x \in X \). As we have shown, \( \Theta^{-1}(U^-_x) = O_I(\theta(x)) \). Since \( O_I(\theta(x)) \) is a compact set (see [27]), the proof is completed. □

This theorem implies the following result:

**Theorem 2.9** The category \( \text{DLat} \) is isomorphic to a reflective full subcategory of the category \( \text{SSyst} \) of all S-systems and their morphisms.

**Proof.** In 2.4, we proved that the category \( \text{DLat} \) is isomorphic to the full subcategory \( \text{SDLat} \) of the category \( \text{SSyst} \). Let’s show that \( \text{SDLat} \) is a reflective subcategory of \( \text{SSyst} \). Take an S-system \((W, \vdash)\). Then, by 2.8, there exists an \( L \in [\text{DLat}] \) and a function \( \varphi : W \rightarrow L \) which, by 2.8(ii), is an S-morphism between \((W, \vdash)\) and \( \text{Sc}(L) \). So \( \varphi \in \text{SSyst}(W, \text{Sc}(L)) \). Now 2.8(iii) and the fact that \( l \in \text{SDLat}(L, L') \) implies \( l \in \text{SSyst}(\text{Sc}(L), \text{Sc}(L')) \) (see the proof of 2.4) show that \( \varphi \) is an \( \text{SDLat} \)-reflection arrow. Therefore, \( \text{SDLat} \) is a reflective subcategory of \( \text{SSyst} \). □

Since the category \( \text{Bool} \) is a reflective full subcategory of the category \( \text{DLat} \) (see [16] or [14](Exercise 4.5)), we obtain immediately (using also 4G from [1]) the following corollary:

**Corollary 2.10** The category \( \text{Bool} \) is isomorphic to a reflective full subcategory of the category \( \text{SSyst} \) of all S-systems and their morphisms.

Theorem 2.8 implies also the following two results of D. Vakarelov [30]:

**Corollary 2.11** ([30]) Let \((W, \vdash)\) be an S-system, satisfying the following additional condition:

(Antisymm) if \( \{a\} \vdash \{b\} \) and \( \{b\} \vdash \{a\} \) then \( a = b \) \((a, b \in W)\).

Then there exists a distributive lattice \((L, \lor, \land)\) with 0 and 1, and an injection \( \varphi : W \rightarrow L \) such that:

(i) the set \( \varphi(W) \) generates \( L \);

(ii) for any two finite subsets \( A = \{a_i : i = 1, \ldots, n\} \) and \( B = \{b_j : j = 1, \ldots, m\} \) of \( W \) we have that \( A \vdash B \) iff \( \lor\{\varphi(a_i) : i = 1, \ldots, n\} \leq \lor\{\varphi(b_j) : j = 1, \ldots, m\} \).

**Proof.** By 2.8, there exist a distributive lattice \( L \) and a function \( \varphi : W \rightarrow L \) which satisfy conditions (i) and (ii). We have only to show that the function \( \varphi \) is an injection.

Let \( a, b \in W \) and \( a \neq b \). Suppose that every prime ideal \( p \) in \((W, \vdash)\) contains (resp. doesn’t contain) both \( a \) and \( b \). Then, by the definition of \( \varphi \) (see the proof of 2.8), we get that \( \varphi(a) = 0 = \varphi(b) \) (resp. \( \varphi(a) = 1 = \varphi(b) \)). Hence, by (ii), we obtain, in both cases, that \( \{a\} \vdash \{b\} \) and \( \{b\} \vdash \{a\} \). Now, condition (Antisymm) implies that \( a = b \), which is a contradiction. Therefore, there exists a prime ideal \( p \) in \((W, \vdash)\) containing exactly one of the points \( a \) and \( b \). We thus get, using 2.8(iv), that \( \varphi \) is an injection. □

**Corollary 2.12** (Representation theorem for S-systems in P-systems)([30]) Let \((W, \vdash)\) be an S-system. Then \((W, \vdash) = \text{Sc}(P(W, \vdash)) = \text{Sc}(P'(W, \vdash)) \) (see 1.5 and 1.6 for the notations).
Proof. Let $S = PI(W, \vdash)$. The definition of the function $\varphi$ from the proof of 2.8 and Definition 1.5 show that $P(W, \vdash) = (W, S, \varphi)$. Put $P = (W, S, \varphi)$. Then, using the notations of 1.6, we obtain, by 2.8(ii), that $P = P$. Therefore, $(W, \vdash) = Sc(P) = Sc(P(W, \vdash))$. Put $P' = P'(W, \vdash)$. It is easy to see, using only the definitions of the relevant notions and notations, that $P = P$. Thus $Sc(P(W, \vdash)) = Sc(P'(W, \vdash))$. □

In the next section we will extend this representation theorem to an isomorphism theorem (see 3.4).

3 Some isomorphism theorems

Definition 3.1 (a) Let $(W, V, f)$ and $(W', V', f')$ be two set-theoretical P-systems and $\varphi \in \text{Set}(W, W')$. The function $\varphi$ is called a $P$-morphism if $\varphi^{-1}(V') \subseteq V$. We denote by $\text{SPS}$ the category of all set-theoretical P-systems and P-morphisms.

(b) We denote by $\text{TPS}$ the category whose objects are all pairs $(X, P)$, where $X$ is a non-empty set and $P \subseteq \text{Exp}(X)$, and, for any $(X, P), (X', P') \in [\text{TPS}]$, the set $\text{TPS}((X, P), (X', P'))$ consists of all $f \in \text{Set}(X, X')$ such that $f^{-1}(P') \subseteq P$. The objects of the category $\text{TPS}$ are called topological property systems.

Remark 3.2 (a) The full subcategory $\text{Top}'$ of $\text{Top}$, consisting of all non-empty topological spaces, is a full subcategory of $\text{TPS}$.

(b) $\text{SPS}$ and $\text{TPS}$ are isomorphic categories.

For proving (b), define two functors $G : \text{SPS} \rightarrow \text{TPS}$ and $H : \text{TPS} \rightarrow \text{SPS}$ by $G(W, V, f) = (W, V)$ (on objects of $\text{SPS}$), $G\varphi = \varphi$ (on morphisms of $\text{SPS}$), $H(X, P) = (X, P, f)$, where $f \in \text{Set}(X, \text{Exp}(P))$, $f(x) = \{A \in P : x \in A\}$ (on objects of $\text{TPS}$) and $H(\varphi) = \varphi$ (on morphisms of $\text{TPS}$). Then it is easy to see that $G \circ H = \text{Id}_{\text{TPS}}$ and $H \circ G = \text{Id}_{\text{SPS}}$. Hence, $\text{SPS}$ and $\text{TPS}$ are isomorphic categories. □

Definition 3.3 We denote by $\text{TPSS}$ the full subcategory of $\text{TPS}$ whose objects are all $(X, P) \in [\text{TPS}]$ which satisfy the following condition (*):

(*) If $V \subseteq X$ is such that for any two finite sets $F \subseteq V$ and $\Phi \subseteq X \setminus V$ there exists a $U \in P$ with $F \subseteq U$ and $U \cap \Phi = \emptyset$, then $V \in P$.

Theorem 3.4 The categories $\text{SSyst}$ and $\text{TPSS}$ are isomorphic.

Proof. The proof will consist of several steps.

Step 1. In this step we will define two functors $T : \text{SSyst} \rightarrow \text{TPS}$ and $S : \text{TPS} \rightarrow \text{SSyst}$.

For any $(W, \vdash) \in [\text{SSyst}]$, put $T(W, \vdash) = (W, PF(W, \vdash))$ (see 1.5 for the notations) and let $T(\varphi) = \varphi$ on the morphisms of $\text{SSyst}$. It is easily seen, using 2.2, that $T$ is a functor.

For any $(X, P) \in [\text{TPS}]$, put $S(X, P) = (X, \vdash_P)$, where the binary relation $\vdash_P$ in $\text{Exp}(X)$ is defined as follows: if $A$ and $B$ are two finite subsets of $X$ then $A \vdash_P B$ iff $(\forall U \in P)((A \subseteq U) \Rightarrow (U \cap B \neq \emptyset))$; if $A$ and $B$ are two arbitrary subsets of $X$ then $A \vdash_P B$ iff there exist finite subsets $A' \subseteq A$ and $B' \subseteq B$ such that $A' \vdash_P B'$. Put $S(\varphi) = \varphi$ on the morphisms of $\text{TPS}$. Let’s show that $T$ is a functor from the category $\text{TPS}$ to the category $\text{SSyst}$. Take a $(X, P) \in [\text{TPS}]$. Define $f : X \rightarrow \text{Exp}(P)$ putting $f(x) = \{U \in P : x \in U\}$, for every $x \in X$. Then $P = (X, P, f)$ is a (set-theoretical) P-system such that $S(X, P) = Sc(P)$ (see 1.6 for $Sc(P)$). For proving this take two finite subsets $A = \{a_i \in X : i = 1, \ldots, n\}$ and $B = \{b_j \in X : j = 1, \ldots, m\}$ of $X$. Let
$A \vdash \exists B$. This means that $\bigcap \{f(a_i) : i = 1, \ldots, n\} \subseteq \bigcup \{f(b_j) : j = 1, \ldots, m\}$. Let $U \in \mathcal{P}$ and $A \subseteq U$. Then $U \in \bigcap \{f(a_i) : i = 1, \ldots, n\}$. Hence $U \in \bigcup \{f(b_j) : j = 1, \ldots, m\}$. Thus $B \cap U \neq \emptyset$. So, we have proved that $A \vdash \exists B$. Conversely, let $A \vdash \exists B$. Take a $U \in \bigcap \{f(a_i) : i = 1, \ldots, n\}$. Then $A \subseteq U$. Now, the definition of the relation $\vdash \exists$ implies that $U \cap B \neq \emptyset$. Thus $U \in \bigcup \{f(b_j) : j = 1, \ldots, m\}$. So, $\bigcap \{f(a_i) : i = 1, \ldots, n\} \subseteq \bigcup \{f(b_j) : j = 1, \ldots, m\}$, i.e. $A \vdash \exists B$. Therefore the relations $\vdash \exists$ and $\vdash \exists$ coincide on the finite subsets of $X$. Then, by their definitions, they coincide on arbitrary subsets of $X$. So, $S(X, \mathcal{P}) = Sc(P)$. Since, by 1.6, $Sc(P)$ is an $S$-system, we get that the images of the objects of the category $\mathcal{T}PS$ by $S$ are objects of the category $SSyst$. Let’s show that the images of the morphisms of the category $\mathcal{T}PS$ by $S$ are morphisms of the category $SSyst$. Indeed, let $\varphi \in \mathcal{T}PS((X, \mathcal{P}), (X', \mathcal{P}'))$. Take two finite subsets $A$ and $B$ of $X$ such that $A \vdash \exists B$. We have to prove that $\varphi(A) \vdash \exists \varphi(B)$. Let $U' \in \mathcal{P}'$ be such that $\varphi(A) \subseteq U'$. Then $\varphi^{-1}(U') \in \mathcal{P}$ and $A \subseteq \varphi^{-1}(U')$. Since $A \vdash \exists B$, we obtain that $B \cap \varphi^{-1}(U') \neq \emptyset$. Thus $U' \cap \varphi(B) \neq \emptyset$. So, $\varphi(A) \vdash \exists \varphi(B)$. Therefore $\varphi$ is an $S$-morphism.

Step 2. We will prove that the functor $S \circ T'$ coincides with the identity functor $Id_{SSyst}$ of the category $SSyst$.

Let $(W, \vdash) \in \mathcal{SSyst}$. Then $(S \circ T)(W, \vdash) = S(W, \mathcal{P}W) = (W, \vdash p_w)$ (using the notations of Step 1 and denoting by $\mathcal{P}W$ the family $PF(W, \vdash)$). For any two finite subsets $A$ and $B$ of $W$, we have, by 1.7, that $A \vdash p_w B$ if $A \vdash \exists B$. This implies that the same is valid for arbitrary subsets of $W$. So, $(S \circ T)(W, \vdash) = (W, \vdash)$, i.e. $S \circ T$ and $Id_{SSyst}$ coincide on the objects of $SSyst$. Since, they obviously, coincide on the morphisms of $SSyst$, the equality $S \circ T = Id_{SSyst}$ is proved.

Step 3. We will prove that:

(a) if $C = (T \circ S)(\mathcal{T}PS)$ then $C = (T \circ S)(C)$ and $C = (T \circ S)(C)$ for every $C \in |C|$;

(b) the subcategory $C$ of $\mathcal{T}PS$ is isomorphic to $SSyst$.

Using Step 2, we obtain that $T(\mathcal{SSyst}) = (T \circ S \circ T)(\mathcal{SSyst}) \subseteq (T \circ S)(\mathcal{T}PS) = C$. Hence the functor $T : \mathcal{SSyst} \rightarrow \mathcal{T}PS$ can be regarded also as a functor from $\mathcal{SSyst}$ to $\mathcal{C}$. Denote this functor by $T'$ (i.e. $T' : \mathcal{SSyst} \rightarrow \mathcal{C}$). Denote by $S'$ the restriction of the functor $S$ to the subcategory $C$ of the category $\mathcal{T}PS$ (i.e. $S' : C \rightarrow \mathcal{SSyst}$). Then, by Step 2, $S' \circ T' = Id_{SSyst}$. Further, we will show that $T' \circ S' = Id_{C}$. This is obviously true on the morphisms of $C$. Let $C \in |C|$. Then $C = (T \circ S)(D)$ for some $D \in \mathcal{T}PS$. Using again Step 2, we obtain that $(T' \circ S')(C) = (T \circ S \circ T \circ S')(D) = (T \circ S)(D) = C$. So, $T' \circ S' = Id_{C}$. Hence, $C = (T \circ S)(C)$ and $C$ is isomorphic to $SSyst$.

Step 4. We will prove that the subcategories $\mathcal{T}PS$ and $C$ (see Step 3 for $C$) of the category $\mathcal{T}PS$ coincide.

Let $(X, \mathcal{P}) \in |C|$. We will show that $(X, \mathcal{P}) \in \mathcal{T}PS$. Let $V \subseteq X$ be such that for any two finite sets $F \subseteq V$ and $\Phi \subseteq X \setminus V$ there exists a $U \in \mathcal{P}$ with $F \subseteq U$ and $U \cap \Phi = \emptyset$. By Step 3, we have that $(X, \mathcal{P}) = (T \circ S)(X, \mathcal{P})$. Hence $\mathcal{P} = PF(X, \vdash)$ (see Step 1 for the notations). So, we have to prove that $V$ is a prime filter in $(X, \vdash)$. By 1.8(1), it is enough to show that $V \vdash (X \setminus V)$. Let $A$ be a finite subset of $V$ and $B$ be a finite subset of $X \setminus V$. Then, by our hypothesis, there exists a $U \in \mathcal{P}$ with $A \subseteq U$ and $U \cap B = \emptyset$. This means that $A \vdash (X \setminus V)$. So, we have proved that $(X, \mathcal{P}) \in \mathcal{T}PS$.

Let $(X, \mathcal{P}) \in \mathcal{T}PS$. We will show that $(X, \mathcal{P}) \in |C|$ by proving that $(X, \mathcal{P}) = (T \circ S)(X, \mathcal{P})$. Since $(T \circ S)(X, \mathcal{P}) = (X, PF(X, \vdash))$, we have to prove that $\mathcal{P} = PF(X, \vdash)$. Let $V \in PF(X, \vdash)$ and let $F$ be a finite subset of $V$. Since, by 1.8(1), $V \vdash F \subseteq (X \setminus V)$, we obtain that $F \vdash G$, for every finite subset $G$ of $X \setminus V$. Hence, by the definition of the relation $\vdash$ (see Step 1), for every finite subset $G$ of $X \setminus V$ there exists an element $U$
of $\mathcal{P}$ such that $F \subseteq U$ and $U \cap G = \emptyset$. Since $(X, \mathcal{P})$ satisfies condition (*) from 3.3, we obtain that $V \in \mathcal{P}$. So, we have proved that $PF(X, \vdash \mathcal{P}) \subseteq \mathcal{P}$. Conversely, let $V \in \mathcal{P}$. Then the definition of the relation $\vdash \mathcal{P}$ implies that if $F$ is a finite subset of $V$ and $G$ is a finite subset of $X \setminus V$ then $F \vdash \mathcal{P} G$. Hence $V \vdash \mathcal{P} (X \setminus V)$. Thus, by 1.8(1), $V$ is a prime filter in $(X, \vdash \mathcal{P})$. So, $\mathcal{P} \subseteq PF(X, \vdash \mathcal{P})$. Hence, $\mathcal{P} = PF(X, \vdash \mathcal{P})$. Therefore $(X, \mathcal{P}) \in \mathcal{C}_f$. So, the subcategories $TPSS$ and $\mathcal{C}$ of the category $TPS$ coincide.

Now, we complete the proof of our theorem combining the results obtained in Step 3 and Step 4. □

Let’s remark that in Step 4 of the proof of 3.4 we obtained, in fact, the following result:

**Proposition 3.5** Let $(X, \mathcal{P}) \in |TPS|$ and $\mathcal{P}' = PF(X, \vdash \mathcal{P})$ (see Step 1 in the proof of 3.4 for the notations). Then a subset $V$ of $X$ belongs to $\mathcal{P}'$ iff for any two finite sets $F \subseteq V$ and $G \subseteq X \setminus V$ there exists an element $U$ of $\mathcal{P}$ such that $F \subseteq U$ and $U \cap G = \emptyset$. In particular, $\mathcal{P} \subseteq \mathcal{P}'$.

As a special case of this proposition, we obtain immediately the following corollary:

**Corollary 3.6** Let $(X, \mathcal{P}) \in |TPS|$, $\mathcal{P}' = PF(X, \vdash \mathcal{P})$ (see Step 1 in the proof of 3.4 for the notations) and $\mathcal{P} = PF(X, \vdash \mathcal{P})$ (see 1.5 for the notations). Then

a) $X \in \mathcal{P}'$ iff $\mathcal{P}$ is an $\omega$-cover of $X$ (i.e., for every finite subset $F$ of $X$ there exists an element $U$ of $\mathcal{P}$ containing $F$);

b) $\emptyset \in \mathcal{P}'$ iff for every finite subset $F$ of $X$ there exists an element $U$ of $\mathcal{P}$ such that $U \cap F = \emptyset$.

We have also the following result:

**Proposition 3.7** Let $(X, \mathcal{P}) \in |TPS|$, $\mathcal{P}' = PF(X, \vdash \mathcal{P})$ (see Step 1 in the proof of 3.4 for the notations) and let $\mathcal{P}$ be closed under finite intersections. Then $\mathcal{P}'$ is closed under arbitrary intersections.

**Proof.** First of all, using the fact that $\mathcal{P}$ is closed under finite intersections, we will prove that for a subset $V$ of $X$ the following conditions are equivalent:

1. $V \in \mathcal{P}'$;
2. if $F$ is a finite subset of $V$ then $\bigcap \{U \in \mathcal{P} : F \subseteq U\} \subseteq V$.

The implication (1) $\Rightarrow$ (2) follows immediately from 3.5. Let’s show that (2) $\Rightarrow$ (1). Let $F$ be a finite subset of $V$ and $G = \{g_i : i = 1, \ldots, n\} \subseteq X \setminus V$. Then, by (2), for every $i \in \{1, \ldots, n\}$ there exists an element $U_i$ of $\mathcal{P}$ such that $F \subseteq U_i$ and $g_i \notin U_i$. Putting $U = \bigcap \{U_i : i = 1, \ldots, n\}$, we obtain that $U \in \mathcal{P}$, $F \subseteq U$ and $U \cap G = \emptyset$. Hence, by 3.5, $V \in \mathcal{P}'$. So, the conditions (1) and (2) are equivalent.

Let $A$ be a set and, for every $\alpha \in A$, $V_\alpha$ be an element of $\mathcal{P}'$. Put $V = \bigcap \{V_\alpha : \alpha \in A\}$. We will check that $V$ satisfies (2). Take a finite subset $F$ of $V$. Then $F \subseteq V_\alpha$, for every $\alpha \in A$. Since $V_\alpha \in \mathcal{P}'$, we have, by (2), that $\bigcap \{U \in \mathcal{P} : F \subseteq U\} \subseteq V_\alpha$. Hence $\bigcap \{U \in \mathcal{P} : F \subseteq U\} \subseteq \bigcap \{V_\alpha : \alpha \in A\} = V$. So, condition (2) is fulfilled. Thus, $V \in \mathcal{P}'$. □

Let’s now concentrate on $T$-systems.

**Proposition 3.8** Let $(W, \vdash)$ be a $T$-system and $\mathcal{P} = PF(W, \vdash)$ (see 1.5 for the notations). Then:

a) for a $V \subseteq W$, we have that $V \in \mathcal{P}$ iff $\bigcap \{U \in \mathcal{P} : F \subseteq U\} \subseteq V$ for every finite subset $F$ of $V$;

b) $\mathcal{P}$ is closed under arbitrary intersections.
Definition 3.13

Let $V$ be a subset of $W$. Using consecutively 1.8(1), (TFin) (see 1.3) and 1.7, we obtain that $(V \in \mathcal{P}) \iff (V \not\subseteq W \setminus V) \iff (\forall w \in (W \setminus V) \text{ and } \forall \text{ finite subset } F \subseteq \mathcal{P}) \text{ we have that } F \not\subseteq U \iff (\forall w \in (W \setminus V) \text{ and } \forall \text{ finite subset } F \subseteq \mathcal{P}) \text{ such that } F \subseteq U \text{ and } w \not\in U \iff (\forall \text{ finite subset } F \subseteq \mathcal{P}) \forall \text{ finite subset } F \subseteq \mathcal{P}$.

Proof. Let $V$ be a subset of $W$. Using consecutively 1.8(1), (TFin) (see 1.3) and 1.7, we obtain that $(V \in \mathcal{P}) \iff (V \not\subseteq W \setminus V) \iff (\forall w \in (W \setminus V) \text{ and } \forall \text{ finite subset } F \subseteq \mathcal{P}) \text{ we have that } F \not\subseteq U \iff (\forall w \in (W \setminus V) \text{ and } \forall \text{ finite subset } F \subseteq \mathcal{P}) \text{ such that } F \subseteq U \text{ and } w \not\in U \iff (\forall \text{ finite subset } F \subseteq \mathcal{P}) \forall \text{ finite subset } F \subseteq \mathcal{P}$.

b) Let $A$ be a set, and, for every $\alpha \in A$, $V_\alpha$ be an element of $\mathcal{P}$. Put $V = \bigcap\{V_\alpha : \alpha \in A\}$. Let $F$ be a finite subset of $V$. Then $F \subseteq V_\alpha$, for every $\alpha \in A$. By a), we obtain that $\bigcap\{U \subseteq \mathcal{P} : F \subseteq U\} \subseteq V_\alpha$, for every $\alpha \in A$. Hence $\bigcap\{U \subseteq \mathcal{P} : F \subseteq U\} \subseteq \bigcap\{V_\alpha : \alpha \in A\} = V$. Now, a) implies that $V \in \mathcal{P}$. □

Corollary 3.9

Let $(W, \triangleright)$ be a $T$-system and $\mathcal{P} = PF(W, \triangleright)$ (see 1.5 for this notation). Then, for $A \subseteq W$, we have that $V \in \mathcal{P}$ iff for every finite subset $F$ of $V$ there exists an element $U$ of $\mathcal{P}$ such that $F \subseteq U \subseteq V$.

Proof. It follows directly from a) and b) of 3.8. □

Definition 3.10

(2) A topological space $X$ is called a $P_\infty$-space if the intersection of any family of open subsets of $X$ is an open subset of $X$.

Remark 3.11

The $P_\infty$-spaces were introduced by P.S. Alexandroff in [2] under the name of discrete spaces. They are also known as Alexandroff spaces or, shortly, $A$-spaces (see [12]). We call them $P_\infty$-spaces because the term “$A$-space” or “Alexandroff space” is used in the literature with another meaning (see, for example, [11]).

Definition 3.12

([6]) (see also [7]) Let $X$ be a set, $\mathcal{M} \subseteq Exp(X)$ and $\mathcal{O}$ be a topology on the set $\mathcal{M}$. We say that $\mathcal{O}$ is a topology of Tychonoff type on $\mathcal{M}$ if the family $\mathcal{O} \cap \{A^+_M : A \subseteq X\}$, where $A^+_M = \{M \in \mathcal{M} : M \subseteq A\}$, is an open base of $\mathcal{O}$. In what follows, we will denote by $\mathcal{O}_M$ the family $\{A \subseteq X : A^+_M \in \mathcal{O}\}$. When $\mathcal{M} = Fin(X)$ (see 1.1 for this notation), we will write simply $\mathcal{O}$ instead of $\mathcal{O}_{Fin(X)}$.

Definition 3.13

We denote by $\mathcal{T}'$ the category whose objects are all $P_\infty$-spaces of the form $(Fin(X), \mathcal{O})$, where $X$ is a non-empty set and $\mathcal{O}$ is a topology of Tychonoff type on $Fin(X)$, and, for any two objects $(Fin(X), \mathcal{O})$ and $(Fin(X'), \mathcal{O}')$ of $\mathcal{T}'$, the set $\mathcal{T}'((Fin(X), \mathcal{O}), (Fin(X'), \mathcal{O}'))$ consists of all $f \in Set(X, X')$ for which the map $f_{Fin} : Fin(X) \to Fin(X')$, defined by $f_{Fin}(F) = f(F)$ for any $F \in Fin(X)$, is a continuous map between $(Fin(X), \mathcal{O})$ and $(Fin(X'), \mathcal{O}')$.

Proposition 3.14

Let $f \in Set(X, X')$ and $\mathcal{O}$ (resp., $\mathcal{O}'$) be a topology of Tychonoff type on $Fin(X)$ (resp., on $Fin(X')$). Then the following are equivalent:

a) $f_{Fin} \in Top((Fin(X), \mathcal{O}), (Fin(X'), \mathcal{O}'))$ (see 3.13 for the definition of $f_{Fin}$);

b) $f \in TPS((X, \mathcal{O}), (X', \mathcal{O}'))$ (see 3.12 for the notations).

Proof. Let’s first remark that if $A \subseteq X'$ then

\[(**): f_{Fin}^{-1}(A') = (f^{-1}(A))^{++}\]

(here and below we write, for short, $A^{++}$ instead of the full notation $A^{++}_{Fin(X)}$ fixed in 3.12).

Indeed, if $F$ is a finite subset of $X$ then we have: $(F \in f_{Fin}^{-1}(A')) \iff (f_{Fin}(F) \in A') \iff (f(F) \subseteq A) \iff (F \subseteq f^{-1}(A)) \iff (F \in (f^{-1}(A))^{++})$. So, $f_{Fin}^{-1}(A') = (f^{-1}(A))^{++}$.

We now prove that a) $\Rightarrow$ b). Take an $A \in \mathcal{P}_{O'}$. Then $A^+ \in \mathcal{O}'$ and hence $f_{Fin}^{-1}(A^+) \in \mathcal{O}$. Thus, by (**), $(f^{-1}(A))^{++} \in \mathcal{O}$. This implies that $f^{-1}(A) \in \mathcal{P}_O$. So, $f^{-1}(\mathcal{P}_O) \subseteq \mathcal{P}_O$.

Let’s prove that b) $\Rightarrow$ a). Since, by the definition of Tychonoff type topology (see 3.12), the family $\mathcal{P}^+_{O'} = \{A^+ : A \in \mathcal{P}_O\} = \mathcal{O}' \cap \{A^+ : A \subseteq X'\}$ is an open base of the...
topology $\mathcal{O}'$, it is enough to show that $f_{\text{Fin}}^{-1}(A^+) \subseteq \mathcal{O}$ for every $A \in \mathcal{P}_\mathcal{O}$. So, let $A \in \mathcal{P}_\mathcal{O}$. Then, by (**) $f_{\text{Fin}}^{-1}(A^+) = (f^{-1}(A))^+$. Since $f^{-1}(A) \in \mathcal{P}_\mathcal{O}$, we obtain that $(f^{-1}(A))^+ \in \mathcal{O}$. Hence, $f_{\text{Fin}}$ is a continuous function. □

**Notation 3.15** We denote by $\mathcal{T}_{\text{Syst}}$ the full subcategory of $\mathcal{SSyst}$ whose objects are all Tarski consequence systems.

**Theorem 3.16** The categories $\mathcal{T}_{\text{Syst}}$ and $\mathcal{T}'$ are isomorphic.

Proof. If $(W,\vdash) \in |\mathcal{SSyst}|$ then we will write $\mathcal{P}_W$ instead of $\mathcal{P}F(W,\vdash)$; if $X$ is a set then $U^+$ will stand for $U^+_{\text{Fin}(X)}$ (see 1.5 and 3.12 for the notations).

The proof of the theorem will be carried out in several steps.

**Step 1.** In this step we define a functor $\mathcal{T}'' : \mathcal{T}_{\text{Syst}} \longrightarrow \mathcal{T}'$.

For every $(W,\vdash) \in |\mathcal{T}_{\text{Syst}}|$, put $\mathcal{T}''(W,\vdash) = (\mathcal{F}(W),\mathcal{O})$, where $\mathcal{O}$ is the topology on the set $\mathcal{F}(W)$ having as a base the family $\mathcal{P}_W^+ = \{U^+ : U \in \mathcal{P}_W\}$. If $f \in \mathcal{T}_{\text{Syst}}((W,\vdash),(W',\vdash'))$ then put $\mathcal{T}''(f) = f$. Let’s show that $\mathcal{T}''$ is a functor from the category $\mathcal{T}_{\text{Syst}}$ to the category $\mathcal{T}'$. Take a $(W,\vdash) \in |\mathcal{T}_{\text{Syst}}|$. Then, by 1.8(2) and $(\text{TFin})$ (see 1.3), we obtain that $W \in \mathcal{P}_W$. Further, the family $\mathcal{P}_W$ is closed under arbitrary intersections (by (3.8(b))). Since, obviously,

$$(\#) \quad (\bigcap \{U_\alpha : \alpha \in A\})^+ = \bigcap \{U^+_\alpha : \alpha \in A\}, \quad \text{for every set } A,$$

we get that $\mathcal{P}_W^+$ is closed under arbitrary intersections. So, the family $\mathcal{P}_W^+$ can be taken as a base of a topology $\mathcal{O}$ on $\mathcal{F}(W)$. Evidently, $(\mathcal{F}(W),\mathcal{O})$ is a $\mathcal{P}_\infty$-space. Since $\mathcal{O} \cap \{A^+ : A \subseteq W\} \supseteq \mathcal{P}_W^+$, we obtain that $\mathcal{O}$ is a topology of Tychonoff type on $\mathcal{F}(W)$. Thus, $\mathcal{T}''(W,\vdash) \in |\mathcal{T}'|$.

Let’s prove that

$$(2) \quad \mathcal{O} \cap \{A^+ : A \subseteq W\} = \mathcal{P}_W^+.$$

We will use this equality in Step 3 below. Obviously, it will suffice to demonstrate that $\mathcal{O} \cap \{A^+ : A \subseteq W\} \subseteq \mathcal{P}_W^+$. Let $A \subseteq W$ and $A^+ \in \mathcal{O}$. Take an $F \in A^+$. Since $A^+ \in \mathcal{O}$ and $\mathcal{P}_W^+$ is a base of $\mathcal{O}$, there exists an element $U \in \mathcal{P}_W$ such that $F \in U^+ \subseteq A^+$. This implies that $F \subseteq U \subseteq A$. Now, by 3.9, we obtain that $A \in \mathcal{P}_W$. Hence, $A^+ \in \mathcal{P}_W^+$. So, the equality (2) is proved.

Take now a morphism $f \in \mathcal{T}_{\text{Syst}}((W,\vdash),(W',\vdash'))$. We have to show that $\mathcal{T}''(f) \in \mathcal{T}'(\mathcal{T}''(W,\vdash),\mathcal{T}''(W',\vdash'))$. By 3.14, it is enough to prove that $f^{-1}(\mathcal{P}_W^+) \subseteq \mathcal{P}_W$. Since this follows directly from 2.2, we get that $\mathcal{T}''(f) \in \mathcal{T}'(\mathcal{T}''(W,\vdash),\mathcal{T}''(W',\vdash'))$. It is now easily seen that $\mathcal{T}''$ is a functor from the category $\mathcal{T}_{\text{Syst}}$ to the category $\mathcal{T}'$.

**Step 2.** We will define a functor $\mathcal{S}'' : \mathcal{T}' \longrightarrow \mathcal{T}_{\text{Syst}}$.

If $(\mathcal{F}(X),\mathcal{O}) \in |\mathcal{T}'|$ then we put $\mathcal{S}''(\mathcal{F}(X),\mathcal{O}) = \mathcal{S}(X,\mathcal{P}_\mathcal{O})$, where $\mathcal{S}$ is the functor defined in Step 1 of the proof of 3.4 (see also 3.12, 3.13 and 1.1 for the notations). (Hence, $\mathcal{S}''(\mathcal{F}(X),\mathcal{O}) = (X,\vdash_{\mathcal{P}_\mathcal{O}})$. If $f \in \mathcal{T}'((\mathcal{F}(X),\mathcal{O}),((\mathcal{F}(X'),\mathcal{O}'))$ then we put $\mathcal{S}''(f) = f$. We will show that $\mathcal{S}''$ is a functor from the category $\mathcal{T}$ to the category $\mathcal{T}_{\text{Syst}}$. Let $(\mathcal{F}(X),\mathcal{O}) \in |\mathcal{T}'|$. Then $(X,\mathcal{P}_\mathcal{O}) \in \mathcal{T}_{\text{PS}}$ and, by the proof of 3.4, $\mathcal{S}(X,\mathcal{P}_\mathcal{O})$ is an $\mathcal{S}$-system. Hence $\mathcal{S}''(\mathcal{F}(X),\mathcal{O})$ is an $\mathcal{S}$-system and we have to prove only that it satisfies condition (TFin) from 1.3. So, let $A$ and $B$ be two finite subsets of $X$ and $A \vdash_{\mathcal{P}_\mathcal{O}} B$. Then, by the definition of $\vdash_{\mathcal{P}_\mathcal{O}}$ (see Step 1 of the proof of 3.4 for it), $(\forall U \in \mathcal{P}_\mathcal{O})(((A \subseteq U) \Rightarrow (U \cap B \neq \emptyset))$. Since the family $\mathcal{O}$ is closed under arbitrary intersections, the equality (\#) implies that $\mathcal{P}_\mathcal{O}$ is also closed under arbitrary intersections. Hence $U_0 = \bigcap \{U \in \mathcal{P}_\mathcal{O} : A \subseteq U\}$ is an element of $\mathcal{P}_\mathcal{O}$. Thus $U_0 \cap B \neq \emptyset$. Let $b \in U_0 \cap B$. Then, for every $U \in \mathcal{P}_\mathcal{O}$ such that $A \subseteq U$, we have that $b \in U_0 \cap B \subseteq U \cap B$. So $b \in U$, for every $U \in \mathcal{P}_\mathcal{O}$ such that $A \subseteq U$. This means that $A \vdash_{\mathcal{P}_\mathcal{O}} \{b\}$. Hence, condition (TFin) from 1.3 is fulfilled. Thus,
then there exist convex sets

principle for convex sets in convexity spaces [28], [5]:

in separative algebra will be given in the next section.) Let

(Sep DL) [27], Boolean algebras (Sep BA) [27] and separative algebras (Sep SA), the later

proposition 1.7.

A

The following statement is called separation principle (or separation theorem) for S-systems:

Equivalence of the separation principle for

Part II

Equivalence of the separation principle for

S-systems with some other separation principles

The following statement is called separation principle (or separation theorem) for S-systems:

(Sep S) If $A \not\subseteq B$ then there exist in $S$ a prime filter $\Phi$ and a prime ideal $I$ such that

$A \subseteq \Phi, B \subseteq I$ and $\Phi \cap I = \emptyset$

(Sep S) is proved in [30] by using the Zorn Lemma and in this paper it is a part of

proposition 1.7.

(Sep S) is similar to the separation theorem for filters and ideals in distributive lattices

(Sep DL) [27], Boolean algebras (Sep BA) [27] and separative algebras (Sep SA), the later introduced by Prodanov in [19]. (The definition of separative algebra and filters and ideals in separative algebra will be given in the next section.) Let $X$ be a distributive lattice

(Boolean algebra, separative algebra). Then the statements (Sep DL), (Sep BA) and (Sep SA) have the following common formulation:

If $\Phi_0$ is a filter in $X$, $I_0$ is an ideal in $X$, then there exist a prime filter $\Phi$ and prime ideal $I$ in $X$ such that $\Phi_0 \subseteq \Phi$, $I_0 \subseteq I$ and $\Phi \cap I = \emptyset$.

Strange enough is that (Sep S) has close connection with the following separation principle for convex sets in convexity spaces [28], [5]:

(Sep CS) If $A_0$ and $B_0$ are convex sets in a convexity space $X$ such that $A_0 \cap B_0 = \emptyset$, then there exist convex sets $A$ and $B$ such that $A_0 \subseteq A$, $B_0 \subseteq B$, $A \cap B = \emptyset$ and $A \cup B = X$.  

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Note that (SeP CS) is an abstract version of the separation of convex sets in vector spaces given in Kakutani’s theorem [15].

In this part we will prove in ZF (Zermelo-Fraenkel Set Theory without the Axiom of Choice) that (Sep S) is equivalent to (Sep DL), (Sep BA), (Sep SA) and (Sep CS). The proof will be done in the next sections following this chain of implications:

\[(SepS) \Rightarrow (SepSA) \Rightarrow (SepCS) \Rightarrow (SepDL) \Rightarrow (SepBA) \Rightarrow (SepS)\]

The structure of Part two is the following. In section 4 we introduce the notions of separative algebra and convexity space, the latter being a special case of separative algebra. This yields the implication (SepSA) \Rightarrow (SepCS). In section 5 we prove the implication (SepS) \Rightarrow (SepSA). In section 6 we estabilishes the implications (SepCS) \Rightarrow (SepDA) \Rightarrow (SepBA). Section 7 is devoted to the final implication (SepBA) \Rightarrow (SepS). Section 8 contains the Main Theorem of Part Two and some conclusions and open problems.

4 Convexity spaces and separative algebras

The notion of convexity space is a result of an axiomatic approach to the convexity. The main idea is the following. Suppose \(X\) is a nonempty set with a multivalued binary operation “.” in \(X\), called convexity. This means that for each \(a, b \in X\), \(a.b\), denoted also by \(a b\), is a subset of \(X\). The geometrical meaning of \(a b\) is the following: \(x \in a b\) means that \(x\) is between \(a\) and \(b\). Now the definition of convex sets is the following:

- A subset \(A \subseteq X\) is convex if for any \(a, b \in A\) we have \(a b \subseteq A\).
- A set \(A \subseteq X\) is called a prime filter in \(X\) if \(AA \subseteq A\).

If we extend “.” for subsets \(A, B\) of \(X\) setting \(A.B = AB = \bigcup_{a \in A, b \in B} ab\) then \(A\) is a convex subset of \(X\) iff \(AA \subseteq A\).

The first who use this axiomatic approach to convexity is Prenowitz [17], [18]. See also [5], [4], [28], [20, 21, 19]. Bryant and Webster in [5] prove the separation theorem for convexity spaces using the axiomatic system of Prenowitz. Another proof of (Sep CS) based on weaker set of axioms has been given by Tagamlitzki in [28]. A topological generalizations of Tagamlitzki’s theorem are given by Prodanov [20], [21]. In [19] Prodanov extend the Tagamlitzki’s definition of convexity space in order to obtain a separation theorem covering the separation principles for convexity spaces, distributive lattices and Boolean algebras. The resulting notion is called separative algebra. The basic idea is this. In distributive lattices and Boolean algebras filters and ideals have properties similar to those of convex sets. So it is natural to consider a set \(X\) with two convexities, say “.” and “+” and convex sets with respect to the convexity “.” to call filters and convex sets with respect to “+” to call ideals. The formal definition of separative algebra can be obtained in two steps: first is defined the notion of preseparative algebra and then adding one more axiom — the notion of separative algebra.

**Definition 4.1** (Prodanov [19]) A system \(X = (X, ., +)\) is called a preseparative algebra if \(W\) is a non-empty set with two binary multivalued operations “.” and “+” satisfying the following axioms for arbitrary \(a, b, c, x \in X\)

(i) \(ab = ba\),
(ii) \(a(bc) = (ab)c\),
(iii) \(a + b + c = (a + b) + c\),
(iv) \(a + b = b + a\).

A set \(A \subseteq X\) is called a filter in \(X\) if \(AA \subseteq A\), \(A\) is called an ideal in \(X\) if \(A + A \subseteq A\), \(A\) is a prime filter in \(X\) if \(A\) is a filter and \(X \setminus A\) is an ideal in \(X\), dually, \(A\) is a prime ideal in \(X\) if \(A\) is an ideal in \(X\) and \(X \setminus A\) is a filter in \(X\). If \(A \subseteq X\) we denote by \(\mu(A)\) the smallest filter containing \(A\) and by \(\alpha(A)\) the smallest ideal containing \(A\). If \(A = \{x\}\) then instead of \(\mu(\{x\})\) we will write \(\mu(x)\), and similarly for \(\alpha(x)\).
Let for $x, y \in X$ define $x \leq y$ iff $\mu(x) \cap \alpha(y) \neq \emptyset$. Then $X$ is called a separative algebra if the following axiom is satisfied:

(iv) $x \leq y$ and $y \leq z$ then $x \leq z$ — transitivity of $\leq$.

The following theorem is given by Prodanov in [19]:

**Theorem 4.2** (Prodanov [19]) (Separation theorem for separative algebras) Let $X = (X, +, \cdot)$ be a separative algebra. Then $X$ satisfies the separation principle (Sep SA), namely:

If $\Phi_0$ is a filter in $X$, $I_0$ is an ideal in $X$ and $\Phi_0 \cap I_0 = \emptyset$, then there exist a prime filter $\Phi$ and prime ideal $I$ in $X$ such that $\Phi_0 \subseteq \Phi$, $I_0 \subseteq I$ and $\Phi \cap I = \emptyset$.

**Proof.** The proof by application of the Zorn Lemma is given in [8]. □

The following is equivalent to the Tagamlitzki’s definition of convexity space.

**Definition 4.3** (Prodanov [19]) A separative algebra $X = (X, \cdot, +)$ is called a convexity space if the operations “.” and “+” coincide. Thus, $X = (X, \cdot)$ is a convexity space if “.$” is a multyvalued binary operation in $X$ satisfying the following axioms:

(i) $ab = ba$,
(ii) $(ab)c = (ab)c$,
(iii) from $a \in (bx)$ and $c \in (dx)$ it follows that $(ab) \cap (bc) \neq \emptyset$.
(iv) The relation $\leq$, defined by the following equivalence $x \leq y$ iff $\mu(x) \cap \mu(y) \neq \emptyset$ is transitive.

Note that filters and ideals in convexity spaces coincide and are called convex sets, and prime filters are semispaces, namely those convex sets whose complement is also a convex set.

Let us remind the separation principle for convexity spaces:

(Sep CS) If $A_0$ and $B_0$ are convex sets in a convex space $X$ such that $A_0 \cap B_0 = \emptyset$, then there exist convex sets $A$ and $B$ such that $A_0 \subseteq A$, $B_0 \subseteq B$, $A \cap B = \emptyset$ and $A \cup B = X$.

**Corollary 4.4** (SepSA) ⇒ (SepCS), namely:

The separation principle for separative algebras (Sep SA) implies in ZF the separation principle for convexity spaces (Sep CS).

**Proof.** The implication is trivial, because convexity spaces are separative algebras. □

**Corollary 4.5** (Tagamlitzki [28] Separation theorem for convexity spaces)

If $X = (X, \cdot)$ is a convexity space then $X$ satisfies the separation principle (Sep CS).

**Proof.** This is a corollary of theorem 4.2 and corollary 4.4. Direct proof by an application of the Zorn Lemma is given by Tagamlitzki in [28]. □

In the next section we will prove in ZF that the separation principle for S-systems implies the separation principle for separative algebras: (Sep S)⇒(Sep SA). To this end we will need some technical definitions and lemmas from [8] for preseparative and separative algebras, which we will formulate here without proofs.

Let $(X, +, \cdot)$ be a preseparative algebra. By means of the operations $\cdot$ and $+$ we introduce two new operations in $X$ as follows:

- $a/b = \{x \in X : a \in b.x\}$,
- $a - b = \{x \in X : a \in b + x\}$

We extend these operations for arbitrary subsets putting:

- $A/B = \bigcup_{a \in A, b \in B} a/b$,
- $A - B = \bigcup_{a \in A, b \in B} a - b$.

Sometimes instead of $A/B$ we will write $A : B$ or $\frac{A}{B}$.
Lemma 4.6 Let \( \bullet \) be any of the operations \( ., + \) and \( - \). Then the following conditions are true:

(i) \( A \bullet \emptyset = \emptyset = A \emptyset \),
(ii) If \( A \subseteq A' \) and \( B \subseteq B' \) then \( A \bullet B \subseteq A' \bullet B' \),
(iii) \( A \bullet (B \cup C) = (A \bullet B) \cup (A \bullet C) \), and similarly for infinite sums,
(iv) \( (A/B) \cap C \neq \emptyset \) iff \( A \cap (BC) \neq \emptyset \),
(iv') \( (A - B) \cap C \neq \emptyset \) iff \( A \cap (B + C) \neq \emptyset \),
(v) \( AB = BA \),
(v') \( A + B = B + A \),
(vi) \( A(BC) = (AB)C \), \( (vi') A + (B + C) = (A + B) + C \).

Let \( A^n = AA \ldots A \) \( n \)-times and \( nA = A + A + \ldots A \) \( n \)-times, putting \( A^1 = 1A = A \).

Then:

(vii) \( (A \cup B)^2 = A^2 \cup AB \cup B^2 \), \( (viii) 2(A \cup B) = 2A \cup (A + B) \cup 2B \),
(viii') \( (A \cup B \cup C)^2 = A^2 \cup B^2 \cup C^2 \cup AB \cup AC \cup BC \),
(viii'') \( 2(A \cup B \cup C) = 2A \cup 2B \cup 2C \cup (A + B) \cup (A + C) \cup (B + C) \),
(ix) Each of the conditions \((*)\), \((**)*\) is equivalent to the axiom (iii) of preseparative algebra.

\[ (*): \frac{a + b}{c} \subseteq \frac{a + b}{c}, \]
\[ (**): \frac{a(b - c)}{c} \subseteq \frac{(ab) - c}{c}. \]
\[ (x): A + \frac{B}{c} \subseteq \frac{A + B}{c}, \]
\[ (xi): A(B - C) \subseteq (AB) - C, \]
\[ (xii): (A/B)/C = A/(BC), \]
\[ (xiii): (A - B) - C = A - (B + C), \]
\[ (xiv): \frac{A}{B} + \frac{C}{D} \subseteq \frac{A + C}{B + D}, \]
\[ (xv): (A - B), (C - D) \subseteq (AC) - (B + D). \]

The next lemma summarizes some properties of filters and ideals in preseparative algebras. Let us remind that \( \mu(A) \) is the smallest filter containing the set \( A \) and \( \alpha(A) \) is the smallest ideal containing the set \( A \).

Lemma 4.7

(i) \( \mu(A) = \bigcup_{i=1}^{\infty} A^i \), \( (i') \alpha(A) = \bigcup_{i=1}^{\infty} iA \),
(ii) \( a \). If \( \Phi \) is a filter then \( \mu(\Phi) = \Phi \),
\( b \). If \( A \subseteq B \) then \( \mu(A) \subseteq \mu(B) \),
\( c \). \( A \subseteq \mu(A) \),
\( d \). \( \mu(\mu(A)) = \mu(A) \),
\( e \). \( \mu(A \cup B) = \mu(A) \cup \mu(B) \cup \mu(B) \).

(ii') Similarly for ideals and the corresponding conditions a-e for the operation \( \alpha \).

(iii) Let \( \Phi \) be a filter and \( I \) be an ideal. Then:
\( a \). \( \Phi - I \) is a filter and \( \frac{1}{\Phi} \) is an ideal,
\( b \). If \( I \cap (\Phi - I) \neq \emptyset \) then \( \Phi \cap I \neq \emptyset \),
\( c \). If \( (\Phi - I) \cap I \neq \emptyset \) then \( \Phi \cap I \neq \emptyset \).

Proof. The lemma can be proved using lemma 4.6, see also [8]. \( \square \)

5 S-systems and separative algebras

The main aim of this section is to prove in ZF the implication \((\text{SepS}) \Rightarrow (\text{SepSA})\). To this end we will give a construction which turns each separative algebra into an S-system.

Let \( X = (X, +) \) be a separative algebra. For \( A, B \subseteq X \) define the relation \( A \vdash X B \) iff \( \mu(A) \cap \alpha(B) \neq \emptyset \).
Note that by this definition we have
\[ x \vdash_X y \text{ iff } x \leq y. \]

**Theorem 5.1** Let \( X = (X, \cdot, +) \) be a separative algebra. Then \( S(X) = (X, \vdash_X) \) is an S-system, called the S-system over \( X \).

**Proof.** The proof will follow from the next four lemmas. \( \square \)

**Lemma 5.2** The relation \( \vdash_X \) satisfies the axiom
(Refl) If \( A \cap B \neq \emptyset \) then \( A \vdash_X B \)

**Proof.** By lemma 4.7 (ii)c and (ii')c \( A \subseteq \mu(A) \) and \( B \subseteq \alpha(B) \). Then \( A \cap B \neq \emptyset \) implies \( \mu(A) \cap \alpha(B) \neq \emptyset \), which shows \( A \vdash_X B \). \( \square \)

**Lemma 5.3** The relation \( \vdash_X \) satisfies the axiom
(Mono) If \( A \subseteq A' \), \( B \subseteq B' \) and \( A \vdash_X B \) then \( A' \vdash_X B' \).

**Proof.** Suppose \( A \subseteq A' \), \( B \subseteq B' \) and \( A \vdash_X B \). Then we have \( \mu(A) \cap \mu(B) \neq \emptyset \), which by lemma 4.7 (ii)b and (ii')b implies \( A' \vdash_X B' \). \( \square \)

**Lemma 5.4** The relation \( \vdash_X \) satisfies the axiom
(Cut) If \( A \vdash_X B \cup \{x\} \) and \( \{x\} \cup A \vdash_X B \) then \( A \vdash_X B \).

**Proof.** Suppose that (Cut) does not hold and proceed to obtain a contradiction. Then for some \( A, B \subseteq X \) and \( x \in X \) we have \( A \vdash B \cup x \) and \( x \cup A \vdash B \) but \( A \not\vdash B \) (for simplicity we write \( x \) instead of \( \{x\} \) and omit the subscript \( X \) in \( \vdash \)). From here we obtain
1. \( \mu(A) \cap \alpha(B \cup x) \neq \emptyset \),
2. \( \mu(x \cup A) \cap \alpha(B) \neq \emptyset \) and
3. \( \mu(A) \cap \alpha(B) = \emptyset \).

By lemma 4.7 (ii)e and (ii')e we obtain
4. \( \mu(x \cup A) = \mu(x) \cup \mu(x), \mu(A) \cup \mu(A) \) and
5. \( \alpha(B \cup x) = \alpha(B) \cup (\alpha(B) + \alpha(x)) \cup \alpha(x) \).

From (1), (3) and (5) we obtain either
(a) \( \mu(A) \cap (\alpha(B) + \alpha(x)) \neq \emptyset \) or
(b) \( \mu(A) \cap \alpha(x) \neq \emptyset \)

From (2), (3) and (4) we obtain either
(a') \( (\mu(A), \mu(x)) \cap \alpha(B) \neq \emptyset \) or
(b') \( \mu(x) \cap \alpha(B) \neq \emptyset \).

So we have to consider the following combinations of cases and to obtain a contradiction from each of them: \( (aa') \), \( (ab') \), \( (ba') \) and \( (bb') \). As an example we shall treat of only the case \( (aa') \). For the sake of brevity we put \( \Phi = \mu(A) \), \( I = \alpha(B) \), \( \Phi \) is a filter, \( I \) is an Ideal. Then (a) and (a') become:
(a) \( \Phi \cap (I + \alpha(x)) \neq \emptyset \) and
(a') \( \Phi \cap (I, \mu(x)) \neq \emptyset \).

Applying lemma 4.7 (iii) to (a) and (a') we obtain
5. \( \mu(x) \cap I \neq \emptyset \) and
6. \( \alpha(x) \cap (\Phi - I) \neq \emptyset \).

By (5) we conclude that there exists \( y \in X \) such that
7. \( y \in \mu(x) \) and
8. \( y \in I \).

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Proposition 5.6

Then for some \( \alpha \), \( \mu \) and \( x \), \( z \) are elements of \( x \) for which

(10) \( z \in \Phi - I \).

Since \( y \subseteq \alpha(z) \), then from (11) we get

(12) \( \mu(x) \cap \alpha(y) \neq \emptyset \).

Then from (12) we get

(13) \( \mu(z) \cap \alpha(x) \neq \emptyset \) and consequently \( z \leq y \).\n
By lemma 4.7 (iii) \( \Phi - I \) is a filter. Then by (10) we get

(16) \( \mu(z) \subseteq \Phi - I \).

Then from (12) we get

(17) \( \alpha(y) \subseteq \frac{I}{\Phi} \).

From (16) and (17) we get

(18) \( \mu(z) \cap \alpha(y) \subseteq (\Phi - I) \cap \frac{I}{\Phi} \).

By (15) and (18) we obtain

(19) \( (\Phi - I) \cap \frac{I}{\Phi} \neq \emptyset \).

Applying lemma 4.7 (iii) to (19) we obtain \( \Phi \cap I \neq \emptyset \) and hence \( \mu(A) \cap \alpha(B) \neq \emptyset \) which contradicts (3). This completes the proof. \( \square \)

Lemma 5.5 The relation \( \vdash_x \) satisfies the axiom (Fin) If \( A \vdash_x B \) then there exist finite subsets \( A' \subseteq A \) and \( B' \subseteq B \) such that \( A' \vdash_x B' \).

Proof. Suppose \( A \vdash_x B \), then we have

(20) \( \mu(A) \cap \alpha(B) \neq \emptyset \).

By lemma 4.7 (i) and (i') we have

(21) \( \mu(A) = \bigcup_{i=1}^{\infty} A^i \) and

(22) \( \alpha(B) = \bigcup_{j=1}^{\infty} jB \).

From (20), (21) and (22) we obtain that for some \( x \in X \), \( x \in \bigcup_{i=1}^{\infty} A^i \) and \( x \in \bigcup_{j=1}^{\infty} jB \). Then for some \( i, j \) we have

(23) \( x \in A^i \) and

(24) \( x \in jB \).

It follows from (23) that there exist a finite subset \( A' = \{a_1, a_2, \ldots, a_i\} \subseteq A \) such that \( x \in a_1, a_2 \ldots, a_i \). From here we obtain that \( a_1, a_2 \ldots, a_i \subseteq \mu(A') \) and consequently

(25) \( x \in \mu(A') \subseteq \mu(A) \).

In an analogous way we obtain from (24) that there exists a finite subset \( B' = \{b_1, b_2, \ldots, b_j\} \subseteq B \) such that \( x \in a_1, a_2 \ldots, a_i \). From here we obtain that \( a_1, a_2 \ldots, a_i \subseteq \mu(A') \) and consequently

(26) \( x \in \alpha(B') \subseteq \alpha(B) \).

Then from (20), (25) and (26) we obtain

(27) \( \mu(A') \cap \alpha(B') \neq \emptyset \) and hence

(28) \( A' \vdash_x B' \).

Thus, for some finite subsets \( A' \subseteq A \) and \( B' \subseteq B \) we have \( A' \vdash_x B' \). \( \square \)

Proposition 5.6 Let \( \mathcal{X} = (X, +) \) be a separative algebra and \( S(\mathcal{X}) = (X, \vdash) \) be the corresponding S-system over \( \mathcal{X} \). Then:

(i) \( \Phi \) is a prime filter in \( \mathcal{X} \) iff \( \Phi \) is a prime filter in \( S(\mathcal{X}) \).

(ii) \( I \) is a prime ideal in \( \mathcal{X} \) iff \( I \) is a prime ideal in \( S(\mathcal{X}) \).
Proof. (i)(⇒) Let $\Phi$ be a prime filter in $X$. Then by lemma 4.7 (ii)a and (ii')a we have $\Phi = \mu(\Phi)$ and $\alpha(X \setminus \Phi) = X \setminus \Phi$ and consequently $\mu(\Phi) \cap \alpha(X \setminus \Phi) = \emptyset$. This shows that $\Phi \not\vdash_X X \setminus \Phi$. Then by proposition 1.8 (1c) $\Phi$ is a prime filter in $S(X)$.

(⇐) Let $\Phi$ be a prime filter in $S(X)$, then $\Phi \not\vdash_X X \setminus \Phi$. and consequently

$$(29) \mu(\Phi) \cap \alpha(X \setminus \Phi) = \emptyset.$$ 

From (29) we obtain

$$(30) \mu(\Phi) \subseteq X \setminus \alpha(X \setminus \Phi).$$ 

By lemma 4.7 (ii') we have $X \setminus \Phi \subseteq \alpha(X \setminus \Phi)$ and from here

$$(31) X \setminus \alpha(X \setminus \Phi) \subseteq \Phi.$$ 

From (30) and (31) we conclude that

$$(32) \mu(\Phi) \subseteq \Phi.$$ 

By lemma 4.7 (ii) we have $\Phi \subseteq \mu(\Phi)$ and by (32) we obtain

$$(33) \Phi = \mu(\Phi), \text{ hence } \Phi \text{ is a filter in } X.$$ 

In a similar way we can conclude from (29) that

$$(34) X \setminus \Phi = \alpha(X \setminus \Phi), \text{ which shows that } X \setminus \Phi \text{ is an ideal in } X.$$ 

From (33) and (34) we obtain that $\Phi$ is a prime filter in $X$.

(ii) The proof is similar to that of (i). □

Corollary 5.7 (SepS) ⇒ (SepSA), namely:

The separation principle for $S$-systems (Sep S) implies in ZF the separation principle for separative algebras (Sep SA).

Proof. Suppose that (Sep S) holds for any $S$-system and let $X = (X, +)$ be a separative algebra. To prove (Sep SA) for $X$ suppose $\Phi_0$ is a filter and $I_0$ is an ideal in $X$ and $\Phi_0 \cap I_0 = \emptyset$. Since $\Phi_0 = \mu(\Phi_0)$ and $I_0 = \alpha(I_0)$ we have $\mu(\Phi_0) \cap \alpha(I_0) = \emptyset$ and hence $\Phi_0 \not\vdash_X I_0$ in the $S$-system $S(X)$ over $X$. Then by (Sep S) there exist a prime filter $\Phi$ and a prime ideal $I$ in $S(X)$ such that $\Phi_0 \subseteq \Phi$, $I_0 \subseteq I$, $\Phi \cap I = \emptyset$. By lemma 5.6 $\Phi$ is a prime filter in $X$ and $I$ is a prime ideal in $X$. This shows that (Sep SA) is fulfilled in $X$. □

Let $X = (X,.,)$ be a convexity space. Since convexity spaces are separative algebras we may apply the construction of building an $S$-system over $X$. This system satisfies the following condition of symmetry (Sym ⊢) If $A \vdash B$ then $B \vdash A$.

S-systems satisfying (Sym ⊢) are called symmetrical S-systems. Let us note that each $S$-system $(W, \vdash)$ defines a symmetrical $S$-system in $W$ by $A \vdash B$ iff $A \vdash B$ and $B \vdash A$. Convexity spaces give a good example of symmetrical $S$-systems.

6 Covexity spaces and distributive lattices

In this section we will prove that the separation theorem for convexity spaces (Sep CS) implies in ZF the separation theorem for distributive lattices (Sep DL). For that purpose we will use a construction given by Prodanov in [21], which endows each distributive lattice with a structure of convexity space. Namely we have the following proposition:

Proposition 6.1 (Prodanov [21])

Let $D = (D, 0, 1, \land, \lor)$ be a distributive lattice with zero 0 and unit 1 and lattice ordering $\leq$. For $a, b \in D$ define a multivalued operation $a \cdot b$ as follows:

$$a \cdot b = \{x \in D : a \land b \leq x \leq a \lor b\}$$

Then the structure $CS(D) = (D,.)$ is a convexity space (over $D$).
Proposition 6.2 Let \( D = (D, 0, 1, \land, \lor) \) be a distributive lattice and \( CS(D) \) be the convexity space over \( D \). Then:

(i) \( \Phi \) is a filter in \( D \) iff \( \Phi \) is a convex set in \( CS(D) \) and \( 1 \in \Phi \),
(ii) \( I \) is an ideal in \( D \) iff \( I \) is a convex set in \( CS(D) \) and \( 0 \in I \).

Proof. (i) (\( \Rightarrow \)) Suppose \( \Phi \) is a filter in \( D \). Then \( 1 \in \Phi \). It remains to show that \( \Phi \) is a convex set in \( CS(D) \). Suppose

(1) \( a, b \in \Phi \) and
(2) \( x \in a.b \).

Since \( \Phi \) is a filter we obtain from (1) that

(3) \( a \land b \in \Phi \).

From (2) we get

(4) \( a \land b \leq x \leq a \lor b \).

From the first inequality of (4) and (3) we obtain that \( x \in \Phi \), which shows that \( \Phi \) is a convex set in \( CS(D) \).

(\( \Leftarrow \)) Suppose \( 1 \in \Phi \) and \( \Phi \) is a convex set in \( CS(D) \). We have to show that \( \Phi \) is a filter, namely

(f1) \( 1 \in \Phi \),
(f2) If \( a \in \Phi \) and \( a \leq b \) then \( b \in \Phi \),
(f3) If \( a, b \in \Phi \) then \( a \land b \in \Phi \).

Condition (f1) is fulfilled. For (f2) suppose

(5) \( a \in \Phi \) and
(6) \( a \leq b \).

We obtain from (6) the following obvious inequalities:

(7) \( a \land 1 \leq a \leq b \leq 1 = a \lor 1 \).

From (7) we get that \( b \in a.1 \). Since \( a, 1 \in \Phi \) and \( \Phi \) is a convex set we conclude that \( b \in \Phi \).

For the condition (f3) suppose

(8) \( a, b \in \Phi \).

We have the following obvious inequality

(9) \( (a \land b) \leq (a \land b) \leq (a \lor b) \).

From (9) we obtain that \( a \land b \in a.b \). Then from (8) and from the fact that \( \Phi \) is a convex set we conclude that \( a \land b \in \Phi \).

(ii) — The proof is similar to that of (i). \( \square \)

Corollary 6.3 (i) \( (SepCS) \Rightarrow (SepDL) \), namely:

The separation principle for convexity spaces \( (Sep\ CS) \) implies in ZF the separation principle of distributive lattices \( (Sep\ DL) \).

(ii) \( (SepDL) \Rightarrow (SepBA) \)

Proof. (i) Suppose \( (Sep\ CS) \) holds for convexity spaces and let \( D = (D, 0, 1, \land, \lor) \) be a distributive lattice. Let \( \Phi_0 \) be a filter and \( I_0 \) be an ideal in \( D \) such that \( \Phi_0 \cap I_0 = \emptyset \). Then by proposition 6.2 \( \Phi_0 \) and \( I_0 \) are convex sets in \( CS(D) \). Then by \( (Sep\ CS) \) there exist convex sets \( \Phi \) and \( I \) such that \( \Phi_0 \subseteq \Phi \), \( I_0 \subseteq I \), \( \Phi \cap I = \emptyset \), and \( \Phi \cup I = D \). Since \( 1 \in \Phi_0 \subseteq \Phi \) we get \( 1 \in \Phi \) and hence by proposition 6.2 \( \Phi \) is a filter. Analogously \( I \) is an ideal. Conditions \( \Phi \cap I = \emptyset \) and \( \Phi \cup I = D \) show that \( \Phi \) is a prime filter and \( I \) is a prime ideal. Thus \( (Sep\ DL) \) is fulfilled.

(ii) This implication is trivial, because every Boolean algebra is a distributive lattice.

\( \square \)
7 S-systems and Boolean algebras

In this section we shall prove that the separation theorem for Boolean algebras (Sep BA) implies in ZF the separation theorem for S-systems (Sep S). To this end we will use classical propositional calculus PC and its connections with the theory of Boolean algebras. Standard reference book here is Rasiowa and Sikorski [23]. Let us note that from (Sep BA) one can simply deduce the so called

"Prime Filter Theorem": Every proper filter in a Boolean algebra can be extended to a prime filter.

So we have the following

Fact 7.1 (Sep BA) implies in ZF the "Prime Filter Theorem".

The following fact will play a crucial role in the next.

Fact 7.2 (see [23]) The "Prime Filter Theorem" implies in ZF the completeness theorem of Classical Propositional Calculus.

We assume that PC has an infinite set VAR of propositional variables and the standard set of the Boolean connectives — propositional constants ⊥ (false) and ⊤ (truth), ¬ (negation), ∧ (conjunction), ∨ (disjunction), ⇒ (implication) and ⇔ (equivalence). We assume also that PC has some Hilbert-style axiomatization with some set of axiom schemes and a rule of inference Modus Ponens (MP) \( \frac{A \rightarrow B}{A, B} \). The notion of a proof of a formula A from a set of formulas X (notation \( X \rightarrow A \)) is the standard one: \( X \rightarrow A \) iff there exists a finite sequence of formulas \( A_1, \ldots, A_n = A \) such that each \( A_i \) is either an axiom of PC, or an element of X, or can be obtained by MP from some \( A_k \) and \( A_l \) by MP with \( k, l \leq i \). A formula A is a theorem of PC if \( \emptyset \rightarrow A \).

We will use the following properties of the relation \( X \rightarrow A \)

Fact 7.3 (i) Compactness: If \( X \rightarrow A \) then for some finite set \( X_0 \subseteq X \) \( X_0 \rightarrow A \).

(ii) If \( A \in X \) or \( A \) is a theorem of PC then \( X \rightarrow A \).

(iii) Deduction Theorem: \( X, A \rightarrow B \) iff \( X \rightarrow A \Rightarrow B \).

We will use the standard two-valued interpretation of PC in the truth-value set \{0, 1\}. A valuation \( v \) of the set of formulas of PC to the set \{0, 1\} is called a model of a formula A if \( v(A) = 1 \), \( v \) is a model of a set of formulas X if for each formula \( A \in X \) we have \( v(A) = 1 \).

Let \( S = (W, \vdash) \) be an S-system. We may assume that \( W \cap VAR = \emptyset \). Denote by PC(S) the propositional calculus with propositional variables \( VAR \cup W \). By a \( W \)-formula of PC(S) we mean a formula having only variables from \( W \). The relation of semantical consequence of a formula A from a set of formulas X is defined as follows:

\( X \models A \) iff every model of X is a model of A.

The next equivalence is called the completeness theorem for PC:

(Completeness theorem for PC) \( X \rightarrow A \) iff \( X \models A \)

The following extended form of (Cut) in S-systems will be needed later on.

Lemma 7.4 Let \( S = (W, \vdash) \) be an S-system. The following is true:

(Extended Cut ) If \( A_1 \cup \{x\} \vdash B_1 \) and \( A_2 \vdash \{x\} \cup B_2 \) then \( A_1 \cup A_2 \vdash B_1 \cup B_2 \).
Let \( S = (W_S, \vdash_S) \) be any S-system. Till the end of the section we will assume \( S \) to be fixed. We associate with \( S \) a propositional calculus \( PC(S) \) having a set of propositional variables \( \text{VAR}(S) = \text{VAR} \cup W_S \). A formula \( A \) in \( PC(S) \) is called an S-formula if it has variables only from \( W_S \).

If \( A = \{a_1, \ldots, a_n\} \subseteq W_S \) then we will use the following notations:
\[
\land(A) = a_1 \land \ldots \land a_n, \quad \land(\emptyset) = \top,
\]
\[
\lor(A) = a_1 \lor \ldots \lor a_n, \quad \lor(\emptyset) = \bot.
\]

We introduce the following set of S-formulas \( T(S) = \{\land(A) \Rightarrow \lor(B) | A \vdash_S B\} \).

**Proposition 7.5** (Characterization theorem of \( \vdash_S \))

For any finite subsets \( A, B \) of \( W_S \) the following equivalence holds:
\[ A \vdash_S B \iff T(S) \Dasharrow \land(A) \Rightarrow \lor(B). \]

**Proof.**

(\( \Rightarrow \)) Suppose \( A \vdash_S B \) for some finite subsets \( A, B \) of \( W_S \). Then \( \land(A) \Rightarrow \lor(B) \in T(S) \) and then by 7.3 (ii) \( T(S) \Dasharrow \land(A) \Rightarrow \lor(B) \).

(\( \Leftarrow \)) Suppose \( T(S) \Dasharrow \land(A) \Rightarrow \lor(B) \). Then by the compactness of \( \Dasharrow \) there exists a finite subfamily \( X \subseteq T(S) \) such that \( X \Dasharrow \land(A) \Rightarrow \lor(B) \). We shall prove by induction on the cardinality of \( |X| \) of \( X \).

**Basis:** \( |X| = 0 \). Then \( \land(A) \Rightarrow \lor(B) \) is a theorem of \( PC \). Since \( \land(A) \) is a conjunction of variables and \( \lor(B) \) is a disjunction of variables then the only possibility is the case \( A \cap B \neq \emptyset \). Then by the axiom (Refl) we have \( A \vdash_S B \).

**Induction hypothesis.** Let for \( |X| = k \) the statement is true and suppose that \( |X| = k + 1 \). Then \( X = X' \cup \{\land(P) \Rightarrow \lor(Q)\} \) with \( |X'| = k \) and some \( P = \{p_1, \ldots, p_m\} \) and \( Q = \{q_1, \ldots, q_n\} \). Then by the Deduction Theorem we obtain \( X' \Dasharrow C \) where \( C = (\land(P) \Rightarrow \lor(Q)) \Rightarrow (\land(A) \Rightarrow \lor(B)) \). Let for simplicity assume \( m = n = 2 \) (the general case is similar), then \( C = ((p_1 \land p_2) \Rightarrow (q_1 \lor q_2)) \Rightarrow (\land(A) \Rightarrow \lor(B)) \). By PC the formula \( C \) is equivalent to the following conjunction \( D_1 \land D_2 \land D_3 \land D_4 \), where
\[
D_1 = \land(A) \Rightarrow \lor(B) \lor p_1, \quad D_2 = \land(A) \Rightarrow \lor(B) \lor p_2,
\]
\[
D_3 = \land(A) \land q_1 \Rightarrow \lor(B), \quad D_4 = \land(A) \land q_2 \Rightarrow \lor(B).
\]

Then for each \( i = 1 \rightarrow 4 \) we have \( X' \Dasharrow D_i \). By the induction hypothesis we obtain that the following four relations are fulfilled:

(*) \( A \vdash_S \{p_1\} \cup B, \quad A \vdash_S \{p_2\} \cup B, \quad A \cup \{q_1\} \vdash_S B, \quad A \cup \{q_2\} \vdash_S B. \)

Since \( p_1 \land p_2 \Rightarrow q_1 \lor q_2 \in T(S) \) we have

(**) \( \{p_1, p_2\} \vdash_S \{q_1, q_2\} \).

Applying several times (Extended Cut) to (*) and (**) we obtain \( A \vdash_S B \), which have to be proved. \( \Box \)

**Theorem 7.6** (SepBA) \( \Rightarrow \) (SepS), namely

the separation principle for Boolean algebras (Sep BA) implies in ZF the separation principle for S-systems (Sep S).

**Proof.** Suppose (Sep BA) holds in the class of Boolean algebras. Then by 7.1 and 7.2 we obtain that the Completeness Theorem for \( PC(S) \) is true and can be used.

To prove (Sep S) suppose that for some finite subsets \( A, B \subseteq W_S \) we have \( A \nvdash_S B \). Then by the characterization theorem for \( \vdash_S \) (Theorem 7.5) we have that \( T(S) \nDasharrow \land(A) \Rightarrow \lor(B) \). By the completeness theorem for \( PC(S) \) we obtain that \( T(S) \nvdash \land(A) \Rightarrow \lor(B) \). Hence there exists a valuation \( v \) which is a model for \( T(S) \) but not for \( \land(A) \Rightarrow \lor(B) \), i.e. \( v(\land(A)) = 1 \) and \( v(\lor(B)) = 0 \).

Define \( \Gamma = \{a \in W_S : v(a) = 1\} \) and \( \Delta = \{a \in W_S : v(a) = 0\} \).
Obviously \( A \subseteq \Gamma, B \subseteq \Delta, \Gamma \cap \Delta = \emptyset \) and \( \Gamma \cup \Delta = W_S \). It remains to be proved that \( \Gamma \) is a prime filter in \( S \) and \( \Delta \) is a prime ideal in \( S \) (see definition 1.5).

Let \( P, Q \) be finite subsets of \( W_S \), \( P \subseteq \Gamma \) and \( P \vdash_S Q \). We will prove that \( Q \cap \Gamma \neq \emptyset \), which will show that \( \Gamma \) is a prime filter in \( S \). Suppose for the sake of contradiction \( Q \cap \Gamma = \emptyset \). Then \( v(\vee(Q)) = 0 \). From \( P \subseteq \Gamma \) we obtain that \( v(\wedge(P)) = 1 \). Consequently

1. \( v(\wedge(P) \Rightarrow \vee(Q)) = 0 \).

By \( P \vdash_S Q \) we obtain that \( \wedge(P) \Rightarrow \vee(Q) \in T(S) \). Since \( v \) is a model for \( T(S) \) we obtain

2. \( v(\wedge(P) \Rightarrow \vee(Q)) = 1 \),

which contradicts (1). This contradiction shows that \( Q \cap \Gamma \neq \emptyset \) and completes the proof that \( \Gamma \) is a prime filter in \( S \). Since \( \Delta = W_S \setminus \Gamma \) it follows that \( \Delta \) is a prime ideal in \( S \).

Thus we have shown that there exist a prime filter \( \Gamma \), a prime ideal \( \Delta \) such that \( \Gamma \cap \Delta = \emptyset, \Gamma \cup \Delta = W_S, A \subseteq \Gamma \) and \( B \subseteq \Delta \), which ends the proof. \( \square \)

### 8 The Main Theorem

We conclude Part two of the paper with the following

**Theorem 8.1 (The main Theorem)**

The separation principles for \( S \)-systems (\( \text{Sep} \ S \)), separative algebras (\( \text{Sep} \ SA \)), convexity spaces (\( \text{Sep} \ CS \)), distributive lattices (\( \text{Sep} \ DL \)) and Boolean algebras (\( \text{Sep} \ BA \)) are equivalent in \( ZF \).

**Proof.** The proof is contained in the following implications:

1. \( \text{Sep} S \Rightarrow \text{Sep} SA \) — by corollary 5.7,
2. \( \text{Sep} SA \Rightarrow \text{Sep} CS \) — by corollary 4.4,
3. \( \text{Sep} CS \Rightarrow \text{Sep} DL \) — by corollary 6.3(i),
4. \( \text{Sep} DL \Rightarrow \text{Sep} BA \) — by corollary 6.3(ii),
5. \( \text{Sep} BA \Rightarrow \text{Sep} S \) — by theorem 7.6. \( \square \)

**Corollary 8.2** All separation principles (\( \text{Sep} S \)), (\( \text{Sep} SA \)), (\( \text{Sep} CS \)), (\( \text{Sep} DL \)) and (\( \text{Sep} BA \)) are equivalent to the "Prime Filter Theorem" for Boolean algebras and hence all of them are weaker than the Axiom of Choice.

**Proof.** By the fact 7.1 and the Main Theorem we have that all the above separation principles are equivalent to the "Prime Filter Theorem" for Boolean algebras and the later is weaker than the Axiom of Choice (see [13]). \( \square \)

A remarkable part of this corollary is the equivalence of the separation principle for convexity spaces (\( \text{Sep} CS \)), which has a geometrical nature, with the "Prime Filter Theorem". This is in a contrast with another geometrical theorem — "Hahn–Banach Extension Theorem", which is weaker than the "Prime Filter Theorem" (see [13]). Let us note that (\( \text{Sep} CS \)) implies in \( ZF \) the Kakutani’s separation theorem for convex sets in real vector spaces (see [28]). In connection with this we formulate the following problem.

**Problems 8.3** Is Kakutani’s theorem imply in \( ZF \) the separation theorem for convex spaces (\( \text{Sep} CS \))?
References


