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A NEW FIXED POINT THEOREM UNDER SUZUKI TYPE $\mathcal{Z}$-CONTRACTION MAPPINGS

P. KUMAM[1,2,*], D. GOPAL AND L. BUDHIA

Abstract. In this article, we introduce a new concept of Suzuki type $\mathcal{Z}$-contraction and prove a fixed point theorem which generalize $\mathcal{Z}$-contraction principle [3].

1. Introduction and Preliminaries

The study of fixed points of mappings satisfying certain contraction conditions has many applications and has been at the centre of various research activities. In this connection the classical results of Edelstien [1] and Suzuki [4] have been the source of inspiration for many researchers working in the area of metric fixed point theory (see Khamsi and Kirk [2] and references therein). Recently, Khojasteh et al. [3] introduced the notion of $\mathcal{Z}$-contraction which generalized the Banach contraction and unified some known nonlinear contractions. Following this direction of research, we introduce the notion Suzuki-type $\mathcal{Z}$-contraction and prove a fixed point theorem for such contraction.

Definition 1.1 ([3])**.** *Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a mapping, then ζ is called a simulation function if it satisfies the following conditions:*

($\zeta 1$) $\zeta(0,0) = 0$;
($\zeta 2$) $\zeta(t,s) < s - t$ *for all* $t, s > 0$;
($\zeta 3$) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ *then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

We denote the set of all simulation functions by $\mathcal{Z}$.

Example 1.2. *Let $\zeta \colon [0,\infty) \times [0,\infty) \to \mathbb{R}$, be defined by*

(1) $\zeta(t,s) = \frac{s}{s+1} - t$ *for all* $t, s \in [0,\infty)$.
(2) $\zeta(t,s) = \lambda s - t$ *for all* $t, s \in [0,\infty)$ *and* $0 < \lambda < 1$.

The above both functions ζ belongs to $\mathcal{Z}$.

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Definition 1.3 ([3])**.** *Let (X,d) be a metric space, $T\colon X\to X$ a mapping and $\zeta\in\mathcal{Z}$. Then T is called a $\mathcal{Z}$-contraction with respect to ζ if the following condition is satisfied*

$$\zeta(d(Tx,Ty),d(x,y))\geq 0 \quad \textit{for all} \quad x,y\in X.$$

2. Main results

In this section we introduce a new class of generalization of Banach contraction which is obtained by combining concepts of Suzuki type contraction and $\mathcal{Z}$-contraction mappings as follows.

Definition 2.1. *Let (X,d) be a metric space, $T\colon X\to X$ a mapping and $\zeta\in\mathcal{Z}$. Then T is called a Suzuki type $\mathcal{Z}$-contraction (in short $S\mathcal{Z}$-contraction) with respect to ζ if the following condition is satisfied*

$$\frac{1}{2}d(x,Tx)<d(x,y)\implies \zeta(d(Tx,Ty),d(x,y))\geq 0 \tag{2.1}$$

for all $x,y\in X$, with $x\neq y$.

Remark. *It is clear from the definition of simulation function that $\zeta(t,s)<s-t\leq 0$ for all $t\geq s>0$. Therefore if T is a Suzuki type $\mathcal{Z}$-contraction with respect to ζ, then*

$$\frac{1}{2}d(x,Tx)<d(x,y)\implies d(Tx,Ty)<d(x,y) \tag{2.2}$$

for all distinct $x,y\in X$. Thus every Suzuki type $\mathcal{Z}$-contraction is Suzuki type contraction.

Lemma 2.2. *Let (X,d) be a metric space and $T\colon X\to X$ be a Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta\in\mathcal{Z}$. Then the fixed point of T in X is unique, provided it exists.*

Proof. Suppose $x^*\in X$ be a fixed point of T. If possible, let $y^*\in X$ and $y^*\neq x^*$ be another fixed point of T. Since $\frac{1}{2}d(x^*,Tx^*)<d(x^*,y^*)$, by applying (2.1), we get

$$0\leq\zeta(d(Tx^*,Ty^*),d(x^*,y^*))=\zeta(d(x^*,y^*),d(x^*,y^*)).$$

The above inequality along with ($\zeta 2$) leads to a contradiction. Hence the fixed point of T is unique. □

A self map T of a metric space (X,d) is said to be asymptotically regular at point $x\in X$ if $\lim_{n\to\infty} d(T^nx,T^{n+1}x)=0$.

Lemma 2.3. *Let (X,d) be a metric space and $T\colon X\to X$ be a Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta\in\mathcal{Z}$. Then T is asymptotically regular at every $x\in X$.*

Proof. Let $x\in X$ be arbitrary. If for some $p\in\mathbb{N}$ we have $T^px=T^{p-1}x$, that is, $Ty=y$, where $y=T^{p-1}x$, then $T^ny=T^{n-1}Ty=T^{n-1}y=\ldots=Ty=y$ for all $n\in\mathbb{N}$. Now for sufficient large $n\in\mathbb{N}$ we have

$$\begin{aligned} d(T^nx,T^{n+1}x) &= d(T^{n-p+1}T^{p-1}x,T^{n-p+2}T^{p-1}x)=d(T^{n-p+1}y,T^{n-p+2}y)\\ &= d(y,y)=0, \end{aligned}$$

Therefore, $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$.
Suppose $T^n x \neq T^{n-1}x$, for all $n \in \mathbb{N}$, then since $\frac{1}{2}d(x, Tx) < d(x, Tx)$ holds for every $x \in X$. By (2.1) we have

$$\begin{aligned} &\zeta(d(Tx, T^2x), d(x, Tx)) \geq 0 \text{ for all } x \in X. \\ i.e.\quad & 0 < d(x, Tx) - d(Tx, T^2x) \text{ by } (\zeta 2) \\ i.e.\quad & d(x, Tx) > d(Tx, T^2x) \end{aligned}$$

Also, since $\frac{1}{2}d(Tx, T^2x) < d(Tx, T^2x)$ using (2.1) we have

$$\begin{aligned} &\zeta(d(T^2x, T^3x), d(Tx, T^2x)) \geq 0 \\ i.e.\quad & d(Tx, T^2x) > d(T^2x, T^3x) \text{ by } (\zeta 2) \end{aligned}$$

Continuing this process we get

$$d(T^{n-1}x, T^n x) > d(T^n x, T^{n+1}x) \text{ for all } n \in \mathbb{N} \tag{2.3}$$

i.e. $\{d(T^n x, T^{n+1}x)\}$ is a monotonically decreasing sequence of nonnegative reals and so it must be convergent. Let $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = r \geq 0$. From (2.3) we have $\frac{1}{2}d(T^n x, T^{n+1}x) < d(T^n x, T^{n-1}x)$. Now as T is a Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ and if $r > 0$ then by $(\zeta 3)$, we have

$$0 \leq \limsup_{n\to\infty} \zeta(d(T^{n+1}x, T^n x), d(T^n x, T^{n-1}x)) < 0$$

This contradiction shows that $r = 0$, that is, $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$. Thus T is an asymptotically regular mapping at x. □

Lemma 2.4. *Let (X, d) be a metric space and $T\colon X \to X$ be a Suzuki type $\mathcal{Z}$-contraction with respect to ζ. Then the Picard sequence $\{x_n\}$ generated by T with initial value $x_0 \in X$ is a bounded sequence, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.*

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, that is, $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\}$ is not bounded. Without loss of generality we can assume that $x_{n+p} \neq x_n$ for all $n, p \in \mathbb{N}$. Since $\{x_n\}$ is not bounded, there exists a subsequence $\{x_{n_k}\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$d(x_{n_{k+1}}, x_{n_k}) > 1 \text{ and } d(x_m, x_{n_k}) \leq 1 \text{ for } n_k \leq m \leq n_{k+1} - 1. \tag{2.4}$$

Therefore by the triangular inequality we have

$$\begin{aligned} 1 < d(x_{n_{k+1}}, x_{n_k}) &\leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + d(x_{n_{k+1}-1}, x_{n_k}) \\ &\leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + 1. \end{aligned}$$

Letting $k \to \infty$ and the fact that $d(T^n x, T^{n+1}x) \to 0$ as $n \to \infty$, we obtain

$$\lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) = 1.$$

By (2.4) we have

$$\frac{1}{2}d(x_{n_k-1}, x_{n_k}) < d(x_{n_k-1}, x_{n_{k+1}-1})$$

thus by Suzuki $\mathcal{Z}-$contraction and ($\zeta 2$), we get that $d(x_{n_k}, x_{n_{k+1}}) \leq d(x_{n_k-1}, x_{n_{k+1}-1})$. Now applying triangular inequality and using (2.4), we obtain

$$\begin{aligned} 1 < d(x_{n_k}, x_{n_{k+1}}) &\leq d(x_{n_k-1}, x_{n_{k+1}-1}) \\ &\leq d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}-1}) \\ &\leq d(x_{n_k-1}, x_{n_k}) + 1. \end{aligned}$$

Letting $k \to \infty$ and using the fact that $d(T^n x, T^{n+1} x) \to 0$ as $n \to \infty$, we obtain

$$\lim_{k\to\infty} d(x_{n_{k+1}-1}, x_{n_k-1}) = 1.$$

Further, since $\frac{1}{2} d(x_{n_{k+1}-1}, x_{n_k-1}) < d(x_{n_{k+1}-1}, x_{n_k-1})$ for sufficiently large k, therefore by (2.1) and ($\zeta 3$), we have

$$\begin{aligned} 0 &\leq \zeta(d(Tx_{n_{k+1}-1}, Tx_{n_k-1}), d(x_{n_{k+1}-1}, x_{n_k-1})) \\ &\leq \limsup_{k\to\infty} \zeta(d(Tx_{n_{k+1}-1}, Tx_{n_k-1}), d(x_{n_{k+1}-1}, x_{n_k-1})) \\ &= \limsup_{k\to\infty} \zeta(d(x_{n_{k+1}}, x_{n_k}), d(x_{n_{k+1}-1}, x_{n_k-1})) < 0 \end{aligned}$$

which is absurd. Hence $\{x_n\}$ is bounded. □

Now, we need the following property for our main result.

Definition 2.5. *Let $T : (X, d) \to (X, d)$ be a mapping and $x_0 \in X$ be arbitrary. Then T is said to possess property (K) if for a bounded Picard sequence $\{x_n = Tx_{n-1}; n \in \mathbb{N}\}$ there exists subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = C > 0$ where $m_k > n_k > k,\ k \in \mathbb{N}$ then*

$$\frac{1}{2} d(x_{m_k-1}, x_{m_k}) < d(x_{m_k-1}, x_{n_k-1})$$

holds.

Example 2.6. *Let $X = \{1, 2, 3\}$ and $T : X \to X$ be defined as $T(1) = 2, T(2) = 3, T(3) = 1$. Let $x_0 = 1$ and define $x_n = Tx_{n-1}$; $n = 1, 2, 3, \cdots$. Then*

$$x_{3n} = 1, x_{3n+1} = 2, x_{3n+2} = 3 \textit{ for all } n \in \mathbb{N} \cup \{0\}.$$

Consider the subsequences $\{x_{m_k}\} = \{x_{3k+2}\}$ and $\{x_{n_k}\} = \{x_{3k+1}\}$ then $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = 1 > 0$ where $m_k = 3k+2 > n_k = 3k+1 > k,\ k \in \mathbb{N}\cup\{0\}$. Also $\frac{1}{2} d(x_{m_k-1}, x_{m_k}) = \frac{1}{2}$ and $d(x_{m_k-1}, x_{n_k-1}) = 1$. Thus

$$\frac{1}{2} d(x_{m_k-1}, x_{m_k}) < d(x_{m_k-1}, x_{n_k-1})$$

gets satisfied for our choice of subsequences. Hence property (K) is satisfied.

In the above example the Picard sequence is not Cauchy (and hence not convergent) and so the mapping T has no fixed point.

The following theorem assure the existence of fixed point for Suzuki type $\mathcal{Z}$-contraction under the additional assumption that T has property (K).

Theorem 2.7. *Let (X, d) be a complete metric space and $T\colon X \to X$ be a Suzuki type $\mathcal{Z}$-contraction with respect to ζ. Then T has a unique fixed point x^* in X and for every $x_0 \in X$ the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point of T, provided T has property (K).*

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, that is, $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. We shall show that this sequence is a Cauchy sequence. For this, let

$$C_n = \sup\{d(x_i, x_j) : i, j \geq n\}.$$

Note that the sequence $\{C_n\}$ is a monotonically decreasing sequence of positive reals and by Lemma-2.4 the sequence $\{x_n\}$ is bounded, therefore $C_n < \infty$ for all $n \in \mathbb{N}$. Thus $\{C_n\}$ is monotonic bounded sequence, therefore convergent, that is, there exists $C \geq 0$ such that $\lim_{n\to\infty} C_n = C$. We shall show that $C = 0$. If $C > 0$ then by the definition of C_n, for every $k \in \mathbb{N}$ there exists n_k, m_k such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < d(x_{m_k}, x_{n_k}) \leq C_k.$$

Hence

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = C. \tag{2.5}$$

By triangular inequality we have

$$d(x_{m_k-1}, x_{n_k-1}) \leq d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}) \text{ and}$$
$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})$$

Using Lemma 2.3, (2.5) and letting $k \to \infty$ in the above inequalities, we obtain

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = C. \tag{2.6}$$

Then by property (K), for $m_k > n_k > k,\ k \in \mathbb{N}$ we have

$$\frac{1}{2} d(x_{m_k-1}, x_{m_k}) < d(x_{m_k-1}, x_{n_k-1}).$$

Using (2.1), equations (2.5), (2.6), ($\zeta 3$) and $C > 0$, we have

$$0 \quad \leq \quad \limsup_{k\to\infty} \zeta(d(x_{m_k}, x_{n_k}), d(x_{m_k-1}, x_{n_k-1})) < 0.$$

This contradiction proves that $C = 0$ and hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.
Now, we claim that

$$\frac{1}{2} d(x_n, Tx_n) < d(x_n, x^*) \text{ or } \frac{1}{2} d(Tx_n, T^2 x_n) < d(Tx_n, x^*) \tag{2.7}$$

for all $n \in \mathbb{N}$. If not assume that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2} d(x_m, Tx_m) \geq d(x_m, x^*) \text{ and } \frac{1}{2} d(Tx_m, T^2 x_m) \geq d(Tx_m, x^*). \tag{2.8}$$

Therefore $2d(x_m, x^*) \leq d(x_m, Tx_m) \leq d(x_m, x^*) + d(x^*, Tx_m)$ which implies that

$$d(x_m, x^*) \leq d(x^*, Tx_m). \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$d(x_m, x^*) \leq d(x^*, Tx_m) \leq \frac{1}{2} d(Tx_m, T^2 x_m). \tag{2.10}$$

Since, $\frac{1}{2} d(x_m, Tx_m) < d(x_m, Tx_m)$, then by (2.1) we have

$$0 \leq \zeta(d(Tx_m, T^2 x_m), d(x_m, Tx_m)).$$

From ($\zeta 2$), we get

$$0 \leq \zeta(d(Tx_m, T^2 x_m), d(x_m, Tx_m)) < d(x_m, Tx_m) - d(Tx_m, T^2 x_m)$$

$$\begin{aligned} i.e.\ d(Tx_m, T^2x_m) &< d(x_m, Tx_m) \\ &< d(x_m, x^*) + d(x^*, Tx_m) \\ &\leq \frac{1}{2}d(Tx_m, T^2x_m) + \frac{1}{2}d(Tx_m, T^2x_m) \\ &= d(Tx_m, T^2x_m) \end{aligned}$$

a contradiction.
Hence (2.7) holds. So from (2.7), since T is Suzuki type $\mathcal{Z}$-contraction for every $n \in \mathbb{N}$ either

$$0 \leq \zeta(d(Tx_n, Tx^*), d(x_n, x^*)) \text{ or } 0 \leq \zeta(d(T^2x_n, Tx^*), d(Tx_n, x^*)).$$

Now, we consider

$$\begin{aligned} 0 &\leq \zeta(d(T^2x_n, Tx^*), d(Tx_n, x^*)) \\ &\leq \lim_{n\to\infty} \sup \zeta(d(T^2x_n, Tx^*), d(Tx_n, x^*)) \\ &\leq \lim_{n\to\infty} \sup(d(Tx_n, x^*) - d(T^2x_n, Tx^*)) \\ &= -d(x^*, Tx^*). \end{aligned}$$

Hence $d(x^*, Tx^*) = 0$, *i.e.*, $x^* = Tx^*$. Thus x^* is a fixed point of T. Similarly it can be shown that x^* will remain fixed point of T when we consider $0 \leq \zeta(d(Tx_n, Tx^*), d(x_n, x^*))$. Uniqueness of fixed point follows from Lemma 2.2. □

3. Illustrative example

Example 3.1. *Consider $X = \{1, 3, 5, 7\}$ which is complete under the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define the mapping $T : X \to X$ by*

$$T(1) = 1 = T(3) = T(5) = 1 \quad \text{and } T(7) = 3.$$

Then, T is Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ where $\zeta(t, s) = \lambda s - t$. for all $s, t \in [0, \infty)$ and $\lambda = \frac{1}{2}$. Hence, all the conditions of Theorem 2.1 satisfied and T has a unique fixed point $x = 1$. However, T is not contractive mapping and so not $\mathcal{Z}$-contraction.

Here note that the property (K) is vacuously satisfied.

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P. Kumam
[1]KMUTTFixed Point Research Laboratory, Department of Mathematics, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand and [2]KMUTT-Fixed Point Theory and Applications Research Group (KMUTT-FPTA), Theoretical and Computational Science Center (TaCS), Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand
E-mail address: *Corresponding author:poom.kum@kmutt.ac.th

D. Gopal
Department of Applied Mathematics & Humanities, S. V. National Institute of Technology, Surat-395007, India
E-mail address: gopal.dhananjay@rediffmail.com

L. Budhia
Department of Applied Mathematics & Humanities, S. V. National Institute of Technology, Surat-395007, India
E-mail address: lokesh86budhia@gmail.com