GREAT INTERSECTING FAMILIES OF EDGES IN HEREDITARY HYPERGRAPHS

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Chvátal stated in 1972 the following conjecture: If $\mathcal{H}$ is a hereditary hypergraph on $S$ and $\mathcal{M} \subseteq \mathcal{H}$ is a family of maximum cardinality of pairwise intersecting members of $\mathcal{H}$, then there exists an $x \in S$ such that $d_\mathcal{H}(x) = \{|H \in \mathcal{H} : x \in H| = |\mathcal{M}|$. Berge and Schönheim proved that $|\mathcal{M}| \leq \frac{1}{2}|\mathcal{H}|$ for every $\mathcal{H}$ and $\mathcal{M}$. Now we prove that if there exists an $\mathcal{M} \subseteq \mathcal{H}$, $|\mathcal{M}| = \frac{1}{2}|\mathcal{H}|$ then Chvátal's conjecture is true for this $\mathcal{H}$.

1. Introduction

Let $S$ be a set of $n$ elements and $\mathcal{H} \subseteq P(S)$ be a hypergraph on $S$, that is, $\mathcal{H}$ is a family of subsets of $S$. $\mathcal{H}$ is called a hereditary hypergraph if $A \in \mathcal{H}$ and $B \subseteq A$ imply that $B \in \mathcal{H}$. If $\mathcal{H}$ is a hypergraph on $S$, then let $d_\mathcal{H}(x) = \{|A \in \mathcal{H} : x \in A|\}$ for every $x \in S$ and let $d(\mathcal{H}) = \max\{d_\mathcal{H}(x) : x \in S\}$. A hypergraph $\mathcal{M}$ is called intersecting if $A \cap B \neq \emptyset$ holds for any pair $A, B \in \mathcal{M}$.

Chvátal conjectured [2] that if $\mathcal{H}$ is a hereditary hypergraph and $\omega(\mathcal{H}) = \max\{|\mathcal{M}| : \mathcal{M} \subseteq \mathcal{H}$ and $\mathcal{M}$ is intersecting$, then $\omega(\mathcal{H}) = d(\mathcal{H})$. It is clear that $\omega(\mathcal{H}) \geq d(\mathcal{H})$ because $\omega(\mathcal{H}) \geq d_\mathcal{H}(x)$ for every $x \in S$.


**Theorem 1** (Berge [1]). Every hereditary hypergraph $\mathcal{H} \subseteq P(S)$ is the disjoint union of pairs of disjoint subsets of $S$, together with the set $\{\emptyset\}$ if $|\mathcal{H}|$ is odd.

It immediately follows from this, that $\omega(\mathcal{H}) \leq \frac{1}{2}|\mathcal{H}|$ for every hereditary hypergraph $\mathcal{H}$. Now we prove that if $\omega(\mathcal{H}) = \frac{1}{2}|\mathcal{H}|$, then Chvátal's conjecture is true, moreover we can describe from a certain point of view all the intersecting families $\mathcal{M} \subseteq \mathcal{H}$ of maximum cardinality.

**Theorem 2.** If $\mathcal{H} \subseteq P(S)$ is a hereditary hypergraph and $\omega(\mathcal{H}) = \frac{1}{2}|\mathcal{H}|$, then $\omega(\mathcal{H}) = d(\mathcal{H})$.

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It is easy to see that the following conjecture is equivalent to Chvátal's conjecture: If \( \mathcal{H} \subset P(S) \) is a hereditary hypergraph and all intersecting families \( \mathcal{M} \subset \mathcal{H} \) of maximum cardinality contain all maximal sets of \( \mathcal{H} \), then the intersection of all maximal sets of \( \mathcal{H} \) is non-empty (\( N \in \mathcal{H} \) is a maximal set of \( \mathcal{H} \), if there is not \( H \in \mathcal{H} \) such that \( H \supseteq N \)). So in the sense of Theorem 2 and Lemma 2 (see later) if we would like to prove Chvátal's conjecture it is enough to prove the following statement: if \( \mathcal{H} \) is a hereditary hypergraph and every intersecting family \( \mathcal{M} \subset \mathcal{H} \) of maximum cardinality contains all maximal sets of \( \mathcal{H} \), then \( \omega(\mathcal{H}) = \frac{1}{2} |\mathcal{H}| \).

2. Maximum intersecting families in \( \mathcal{H} \), when \( \omega(\mathcal{H}) = \frac{1}{2} |\mathcal{H}| \)

Theorem 2 follows from the stronger theorem below by Lemma 1.

**Theorem 3.** Let \( \mathcal{H} \subset P(S) \) be a hereditary hypergraph, satisfying \( \omega(\mathcal{H}) = \frac{1}{2} |\mathcal{H}| \) and let \( N_1, N_2, \ldots, N_p \) be all the maximal sets of \( \mathcal{H} \). If \( |\mathcal{H}| \) is even, then \( \cap_{i=1}^{p} N_i = M \neq \emptyset \) and every intersecting family \( \mathcal{M} \subset \mathcal{H} \) of maximum cardinality arises in the following way: Take a maximum intersecting family \( \mathcal{M} \) in \( P(M) \) and let \( M = \{A \in \mathcal{H} : \exists B \in \mathcal{M}, B \subset A \} \). If \( |\mathcal{H}| \) is odd and \( \mathcal{M} \subset \mathcal{H} \) is an intersecting family of maximum cardinality, then there exists either a maximal set \( N_i \) of \( \mathcal{M} \) such that \( \mathcal{M} \subset \mathcal{H} \setminus \{N_i\} \) or an \( N_{p+1} \subset S \), \( N_{p+1} \notin \mathcal{H} \) such that \( \mathcal{M} \cup \{N_{p+1}\} \) is an intersecting family and \( \mathcal{H} \cup \{N_{p+1}\} \) is a hereditary hypergraph (i.e. either \( |\mathcal{M}| = \frac{1}{2} |\mathcal{H} \setminus \{N_i\}| \) or \( |\mathcal{M} \cup \{N_{p+1}\}| = \frac{1}{2} |\mathcal{H} \cup \{N_{p+1}\}| \).

**Lemma 1** (Schönheim [9]). If \( N_1, \ldots, N_p \) are all the maximal sets in a hereditary hypergraph \( \mathcal{H} \subset P(S) \) and \( \cap_{i=1}^{p} N_i = M \neq \emptyset \), then \( \omega(\mathcal{H}) = \frac{1}{2} |\mathcal{H}| \) holds.

**Proof.** It follows from Theorem 1 that \( d(\mathcal{H}) \leq \omega(\mathcal{H}) \leq \frac{1}{2} |\mathcal{H}| \). Take now an element \( x \) of \( M \) and form disjoint pairs from all sets of \( \mathcal{H} \): \( \{A, A \setminus \{x\}\} \). We have to prove that \( \{A, A \setminus \{x\}\} = \{A \setminus \{x\}, (A \setminus \{x\}) \setminus \{x\}\} \) and \( A \in \mathcal{H} \) implies \( A \setminus \{x\} \in \mathcal{H} \). However \( (A \setminus \{x\}) \setminus \{x\} = A \) and \( A \in \mathcal{H} \) implies that there exists an \( N_i \) such that \( A \subset N_i \in \mathcal{H} \), consequently \( A \cup \{x\} \in \mathcal{H} \) holds. Hence follows that there is exactly one member of each pair containing \( x \) i.e. \( d(\mathcal{H}) = \frac{1}{2} |\mathcal{H}| \) and \( d(\mathcal{H}) = \omega(\mathcal{H}) = \frac{1}{2} |\mathcal{H}| \) hold.

**Lemma 2.** If \( \mathcal{H} \subset P(S) \) is a hereditary hypergraph and there exists an \( x \in S \) such that \( d(x) = \frac{1}{2} |\mathcal{H}| \), then \( x \in \cap_{i=1}^{p} N_i \) when \( N_1, N_2, \ldots, N_p \) are all the maximal sets in \( \mathcal{H} \). Moreover if there exists an \( x \in S \) such that \( d(x) = \frac{1}{2} |\mathcal{H}| - \frac{1}{2} \), then there exists exactly one \( N_i \) not containing \( x \).

**Proof.** If \( d(x) = \frac{1}{2} |\mathcal{H}| \) and \( x \notin N_i \) hold for some \( 1 \leq i \leq p \), then \( d(\mathcal{H} \setminus \{N_i\}) > \frac{1}{2} |\mathcal{H} \setminus \{N_i\}| \), a contradiction.
If \( d(x) = \frac{1}{2} |\mathcal{H}| - \frac{1}{2} \) and \( x \in \bigcap_{i=1}^p N_i \), then \( d(x) = \frac{1}{2} |\mathcal{H}| \) by Lemma 1, a contradiction. On the other hand, if \( x \notin N_i, N_j \) holds for some \( 1 \leq i, j \leq p \) then \( d(x) = \frac{1}{2} |\mathcal{H}| - \frac{1}{2} > \frac{1}{2} |\mathcal{H}| - \{|N_i, N_j|\} \), a contradiction again.

### 3. Proof of Theorem 3

First we prove that the families \( \mathcal{M} \subset \mathcal{H} \) described in the theorem are intersecting families of maximum cardinality. It is easy to see that \( \mathcal{M} \) is an intersecting family. We prove that \( |\mathcal{M}| = \frac{1}{2} |\mathcal{H}| \). By the argument used in Lemma 1, we can form disjoint pairs from the sets of \( \mathcal{H} \): \{A, A \Delta M\}. Obviously, there exists a \( B \in \mathcal{M} \) such that either \( B \subset A \) or \( B \subset A \Delta M \), so either \( A \in \mathcal{M} \) or \( A \Delta M \in \mathcal{M} \). Then \( |\mathcal{M}| \geq \frac{1}{2} |\mathcal{H}| \) holds and this implies \( |\mathcal{M}| = \frac{1}{2} |\mathcal{H}| \) by Theorem 1.

We will prove the other part of the theorem by induction on \( |\mathcal{H}| \). The following induction step was used by Daykin, Hilton and myself [4] for a simple proof of Theorem 1. Let \( T \) be a subset of \( S \) of minimum cardinality for which there exists an \( H \in \mathcal{H} \) such that \( S \setminus (H \cup T) \in \mathcal{H} \). It is easy to see that there exists such a (possibly empty) \( T \). Let \( \mathcal{A} = \{A \in \mathcal{H} : S \setminus (A \cup T) \in \mathcal{H} \} \). Then one can prove that \( \mathcal{H} \setminus \mathcal{A} \) is also a hereditary hypergraph and \( \mathcal{A} \) is the disjoint union of disjoint pairs \( \{A, B\} \) with \( B = S \setminus (A \cup T) \). It is easy to see that \( A = S \setminus (B \cup T) \) and \( A \in \mathcal{A} \) implies \( B \in \mathcal{A} \). So the union of our pairs is \( S \setminus T \).

First we carry on proving the case \( \omega(\mathcal{H}) = \frac{1}{2} |\mathcal{H}| \). If \( \mathcal{M} \subset \mathcal{H} \) is an intersecting family of maximum cardinality, that is \( |\mathcal{M}| = \frac{1}{2} |\mathcal{H}| \), then \( |\mathcal{M} \cap \mathcal{A}| = \frac{1}{2} |\mathcal{A}| \). Here \( |\mathcal{M} \cap \mathcal{A}| \leq \frac{1}{2} |\mathcal{A}| \) is trivial and if \( |\mathcal{M} \cap \mathcal{A}| < \frac{1}{2} |\mathcal{A}| \), then \( \mathcal{M} \setminus \mathcal{A} \) is an intersecting family in the hereditary hypergraph \( \mathcal{H} \setminus \mathcal{A} \) such that \( |\mathcal{M} \setminus \mathcal{A}| = |\mathcal{M}| - |\mathcal{M} \cap \mathcal{A}| > \frac{1}{2} |\mathcal{H}| - \frac{1}{2} |\mathcal{A}| = \frac{1}{2} |\mathcal{H} \setminus \mathcal{A}| \), which contradicts Theorem 1.

Hence we know that

\[
|\mathcal{M} \setminus \mathcal{A}| = \frac{1}{2} |\mathcal{H} \setminus \mathcal{A}|, \quad \omega(\mathcal{H} \setminus \mathcal{A}) = \frac{1}{2} |\mathcal{H} \setminus \mathcal{A}| \quad \text{and} \quad |\mathcal{H} \setminus \mathcal{A}| < |\mathcal{H}|.
\]

Therefore we may suppose that the statement of the theorem is true for \( \mathcal{H} \setminus \mathcal{A} \) (if \( \mathcal{H} \neq \mathcal{A} \)). Let \( M' \) be the non-empty intersection of the maximal sets of \( \mathcal{H} \setminus \mathcal{A} \). Then all intersecting families \( \mathcal{M}^* \subset \mathcal{H} \setminus \mathcal{A} \) of maximum cardinality contain \( M' \), consequently \( M' \in \mathcal{M} \setminus \mathcal{A} \). If \( M' \subset T \) holds, then \( A \cap M' = \emptyset \) holds for all \( A \in \mathcal{A} \). But \( |\mathcal{M} \cap \mathcal{A}| = \frac{1}{2} |\mathcal{A}| \geq 1 \) and this implies \( M' \subset (S \setminus T) \neq \emptyset \). Use the notation \( M = M' \setminus (S \setminus T) \). If \( x \in M \), then it is easy to see that \( d(x) = \frac{1}{2} |\mathcal{H}| \) and if \( x \notin M \), then \( d(x) < \frac{1}{2} |\mathcal{H}| \). Using Lemma 1 and Lemma 2 we obtain that \( \mathcal{M} \) is the intersection of all maximal sets of \( \mathcal{H} \).

Since \( \mathcal{M} \setminus \mathcal{A} \) is an intersecting family in \( \mathcal{H} \setminus \mathcal{A} \) of maximum cardinality, it follows that \( \mathcal{M} \setminus \mathcal{A} \) is like we described in the theorem and \( \mathcal{M} \setminus \mathcal{A} \) can be completed to an intersecting family of cardinality \( \frac{1}{2} |\mathcal{H}| \) from the pairs of \( \mathcal{A} \). If \( M' \subset S \setminus T \), i.e. \( M = M' \), then \( \mathcal{M} \) is like we described in the theorem. Suppose now \( M' \cap T \neq \emptyset \) and let \( M_1, M_2 \subset M \), \( M_1 \cup M_2 = M \), \( M_1 \cap M_2 = \emptyset \). If \( \mathcal{M} \setminus \mathcal{A} \) does not contain \( M_1 \) and \( M_2 \), then \( M_1 \cup (M' \cap T) \) and \( M_2 \cup (M' \cap T) \) are elements of \( \mathcal{M} \setminus \mathcal{A} \). This is true since
M \setminus \mathcal{A}$ contains exactly one set from the two complementary sets of $M'$. But it is easy to see, that there exists a pair of sets from the above described $\mathcal{A}$ such that one member of this pair intersects $M$ in $M_1$, so it does not intersect $M_2 \cup (M' \cap T)$ and at the same time the other member of the pair intersects $M$ in $M_2$, consequently it does not intersect $M_1 \cup (M' \cap T)$. Then $M$ does not contain the members of this pair and this contradicts $|M| = \frac{1}{2} |\mathcal{H}|$. So $M \setminus \mathcal{A}$ contains exactly one of $M_1$ and $M_2$, i.e. $M$ contains a maximum intersecting family of edges of $P(M)$.

Let us denote it by $M'$. We know that $M \supseteq \{A \in \mathcal{H} : \exists B \in M', B \subset A\}$, but here the cardinality of the right hand side is $\frac{1}{2} |\mathcal{H}|$, consequently $M = \{A \in \mathcal{H} : \exists B \in M', B \subset A\}$.

Finally we have to settle the case $\mathcal{H} = \mathcal{A}$. But then it is easy to see that $\mathcal{H} = P(S)$ and the statement is trivial.

Let us turn to the case $\omega(\mathcal{H}) = \frac{1}{2} |\mathcal{H}| - \frac{1}{2}$. Suppose that $\mathcal{H}$ is an intersecting family in $\mathcal{H}$ of maximum cardinality and let $T \subset S$ and $\mathcal{A} \subset \mathcal{H}$ be the same as in the case $\omega(\mathcal{H}) = \frac{1}{2} |\mathcal{H}|$. Now we can see that $|\mathcal{H} \cap \mathcal{A}| = \frac{1}{2} |\mathcal{A}|$ and $\omega(\mathcal{H} \setminus \mathcal{A}) = |M \setminus \mathcal{A}| = \frac{1}{2} |\mathcal{H} \setminus \mathcal{A}| - \frac{1}{2}$ by the argument used in the previous case. We may suppose that the statement of the theorem is true for $\mathcal{H} \setminus \mathcal{A}$ and $M \setminus \mathcal{A}$ because $\mathcal{H} \setminus \mathcal{A}$ contains at least one element (and if $\mathcal{H} \setminus \mathcal{A} = \emptyset$, then the statement is trivial). First we show that Chvátal's conjecture is true for this $\mathcal{H}$, i.e. there exists an $x \in S$ such that $d_{\mathcal{H}}(x) = \frac{1}{2} |\mathcal{H}| - \frac{1}{2}$ holds. We know that there exists either a maximal set of $\mathcal{H} \setminus \mathcal{A}$ (let us denote it by $N^*$) such that $\mathcal{M} \setminus \mathcal{A}$ is an intersecting family in $(\mathcal{H} \setminus \mathcal{A}) \setminus \{N^*\}$ or an $N^{**} \subset S$, $N^{**} \notin \mathcal{H} \setminus \mathcal{A}$ such that $(\mathcal{M} \setminus \mathcal{A}) \cup \{N^{**}\}$ is an intersecting family in the hereditary hypergraph $(\mathcal{H} \setminus \mathcal{A}) \cup \{N^{**}\}$. In both cases we can find an $x \in S \setminus T$ such that $d_{\mathcal{H} \setminus \mathcal{A}}(x) = \frac{1}{2} |\mathcal{H} \setminus \mathcal{A}| - \frac{1}{2}$. On the other hand $x \in S \setminus T$ implies $d_{\mathcal{A}}(x) = \frac{1}{2} |\mathcal{A}|$ and hence $d_{\mathcal{H}}(x) = \frac{1}{2} |\mathcal{H}| - \frac{1}{2}$ follows. Now if $M$ does not contain all maximal sets of $\mathcal{H}$, then let $N_i \notin \mathcal{M}$ and hence $M \subset \mathcal{H} \setminus \{N_i\}$ trivially holds. If $M$ contains all maximal sets of $\mathcal{H}$, then let $x \in S \setminus T$ satisfying $d_{\mathcal{H}}(x) = \frac{1}{2} |\mathcal{H}| - \frac{1}{2}$. By Lemma 2 there exists a maximal set $N_i$ of $\mathcal{H}$ such that $x \notin N_i$. It is trivial that $\mathcal{M} \cup \{N_i \cup \{x\}\}$ is an intersecting family and we will prove that $\mathcal{H} \cup \{N_i \cup \{x\}\}$ is a hereditary hypergraph. Suppose that it is not the case. Then $\mathcal{H}$ does not contain the subsets $H_1$, $H_2$, ..., $H_t$ of $N_i \cup \{x\}$ ($t \geq 1$). But $\mathcal{H}$ contains all subsets of $N_i$ thus $x \in H_i$ ($1 \leq i \leq t$). It is easy to see that $\mathcal{H} = (\mathcal{H} \cup \{N_i \cup \{x\}\}) \cup \{H_1, \ldots, H_t\}$ is a hereditary hypergraph and $d_{\mathcal{H}}(x) = d_{\mathcal{H}}(x) + t + 1 = \frac{1}{2} (|\mathcal{H}| + 2t + 1) > \frac{1}{2} (|\mathcal{H}| + t + 1) = \frac{1}{2} |\mathcal{H}|$ which contradicts Theorem 1. $\mathcal{H} \cup \{N_i \cup \{x\}\}$ is a hereditary hypergraph and we have proved the statement of the theorem.

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References


