A new expression for the Apery's constant (and all other values of the Riemann's zeta function $\zeta(s)$ at positive odd integers) in terms of the Bernoulli and the Stirling numbers.

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Abstract

The values of the Riemann's zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $Re(s) > 1$, at positive even integers are expressed in closed form, in terms of the Bernoulli numbers. This formula was obtained in 1740 by L. Euler, one of the greatest mathematicians of all times. The motivation for this article arose from the curiosity whether the values of the Riemann's zeta function at positive odd integers $\zeta(2k+1)$, $k = 1,2,3,\cdots$, could, as well, be expressed in terms of the Bernoulli numbers. As it is shown, the answer is affirmative. The values $\zeta(2k+1)$, $k = 1,2,3,\cdots$, can indeed be expressed in terms of the Bernoulli numbers, however this dependence is by far more complicated as compared to the corresponding expression for $\zeta(2k)$, $k = 1,2,3,\cdots$. A striking difference between $\zeta(2k)$ and $\zeta(2k+1)$ is that while $\zeta(2k)$ is expressed in terms of B_{2k} only, the derived formula for $\zeta(2k+1)$ involves all Bernoulli numbers $B_1, B_2, B_4, B_6, B_8, \cdots$.

Keywords and Phrases: Riemann Zeta Function, Apery's constant, Bernoulli numbers, Stirling numbers

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1. Introduction

Let $g(x)$ be a real valued function defined on the interval (a, ∞). Two classical operators which can apply to $g(x)$ are the shift operator

$$E\big(g(x)\big) = g(x+1) \qquad (1-1)$$

and the difference operator

$$\Delta\big(g(x)\big) = g(x+1) - g(x) = (E - I)g(x) \qquad (1-2)$$

where I is the identity operator.

Higher order operators are defined similarly:

$$E^k\big(g(x)\big) = g(x+k) \qquad (1-3)$$

$$\Delta^k\big(g(x)\big) = (E-I)^k\big(g(x)\big) = \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} E^j\big(g(x)\big)$$
$$= \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} g(x+j) \qquad (1-4)$$

where k is any positive integer.

In a series of recent articles ([13],[14]) it has been shown that for a wide class of functions $g(x)$, defined on (a,∞), the derivative operator $D = \frac{d}{dx}$ is equivalent to the operator $\ln(I+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots$, i.e.

$$D\big(g(x)\big) = \Delta\big(g(x)\big) - \frac{1}{2}\Delta^2\big(g(x)\big) + \frac{1}{3}\Delta^3\big(g(x)\big) - \frac{1}{4}\Delta^4\big(g(x)\big) + \cdots \ (1-5)$$

Definition: We say that a real valued function $g(x)$ belongs to the class K_a if

$$D\big(g(x)\big) = \left(\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots\right) g(x)$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\Delta^k\big(g(x)\big) \qquad (1-6)$$

for all $x \in (a,\infty)$.

In particular $K_{-\infty}$ will represent the class of functions for which the expansion (1-6) is valid for all real values of x, $(-\infty < x < \infty)$, while K_0 represents the class of functions satisfying (1-6) for all positive values of x, $(0 < x < \infty)$.

An important fact that will be used repeatedly in this article, is that all polynomials are in $K_{-\infty}$, ([13]). Of course, in this case the infinite series in equation (1-6) terminates and reduces to a finite sum, since all differences of order greater than the degree of the polynomial $g(x)$, are identically equal to zero.

2. The Bernoulli and the Stirling numbers

The Bernoulli numbers B_n are defined by the Taylor series expansion of the function $\frac{z}{e^z-1}$, ([1],[19]), i.e.

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n, \qquad |z|<2\pi \qquad (2-1)$$

The numbers B_n are rational numbers,

$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_3=0, B_4=-\frac{1}{30}, B_5=0, B_6=\frac{1}{42}, \cdots$$

while in general $B_{2k+1}=0,\ k=1,2,3,\cdots$.

In this article we introduce two more operators, namely the shift operator $\tilde{E}$ and the difference operator $\tilde{\Delta}$, which operate on the indices of the Bernoulli numbers, i.e.

$$\tilde{E}B_n=B_{n+1}\ \ and\ \ \tilde{\Delta}B_n=B_{n+1}-B_n=\left(\tilde{E}-I\right)B_n,\ \ n=0,1,2,\cdots \qquad (2-2)$$

Higher order operators are defined similarly,

$$\tilde{E}^k B_n=B_{n+k} \qquad (2-3)$$

$$\tilde{\Delta}^k B_n=\left(\tilde{E}-I\right)^k B_n=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\tilde{E}^j B_n=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}B_{n+j} \quad (2-3)$$

where k is any positive integer.

Finally we consider the factorial function $x^{[m]}$ defined by

$$x^{[m]}=x(x-1)(x-2)\cdots\left(x-(m-1)\right), \qquad m=1,2,3,\cdots \quad (2-4)$$

consisting of m factors. Equation (2-4) can be written as

$$x^{[m]}=\sum_{k=1}^{m}s_k^m x^k=s_1^m x+s_2^m x^2+\cdots+s_m^m x^m \qquad (2-5)$$

where the coefficients s_k^m are the Stirling numbers of the first kind ([17]). Obviously $s_m^m = 1$, while $s_k^m = 0$ for $k \leq 0,\ k \geq m+1,\ m$ being any positive integer.

3. Some preliminary remarks about the Riemann's zeta function $\zeta(s)$

In a recent article (Theorem 10, in [12]) it was shown that the Riemann's zeta function $\zeta(s),\ s \in \mathbb{C} - \{1\}$ can be expressed as

$$\zeta(s) = \frac{1}{s-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \Delta^{n-1}(x^{-s+1})\Big|_{x=1} \qquad (3-1)$$

where the Δ operator operates on the variable x, while the differences are evaluated at $x = 1$. Formula (3-1) was first derived by Helmut Hasse in 1930, ([9]). In the same Theorem 10, in [12], it was further shown that for real $s > 1$, the expression for $\zeta(s)$ in equation (3-1) reduces to the usual definition of the function $\zeta(s)$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \quad , \qquad s \ \ real > 1 \quad (3-2)$$

The values of $\zeta(s)$ at positive even integers are given by the formula

$$\zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2\,(2k)!} B_{2k}\,, \quad k = 1,2,3,\cdots \qquad (3-3)$$

This formula, derived by Euler in 1740, expresses $\zeta(2k),\ k = 1,2,3,\cdots$ as a rational multiple of π^{2k}.

It is also known ([12]) that for $k = 1,2,3,\cdots$

$$\zeta(-2k) = 0, \quad \zeta(-2k+1) = -\frac{B_{2k}}{2k} \qquad (3-4)$$

Unfortunately there is no analogous closed form expression for $\zeta(2k+1)$, i.e. for the values of the Riemann's zeta function at odd positive integers. Instead various integral and/or series representations for $\zeta(2k+1)$ have been derived, ([2],[3],[4],[5],[6],[7],[8],[10],[15],[16],[18]).

In a recent article ([11]) a universal formula for $\zeta(2k+1)$ is derived. This formula will be the starting point for our subsequent analysis. As it will be shown, a proper transformation of this universal formula leads to an expression for $\zeta(2k+1),\ k=1,2,3,\cdots$, in terms of the Bernoulli and the Stirling numbers of the first kind.

4. An expression for $\boldsymbol{\zeta(2k+1),\ k=1,2,3,\cdots}$, in terms of the Bernoulli and the Stirling numbers of the first kind

As it was shown in [11], the values of the Riemann's zeta function at positive odd integers $\zeta(2k+1)$, are given by the formula

$$\zeta(2k+1)=(-1)^{k-1}\frac{2^{2k+1}}{(2k+1)!}\pi^{2k}\left\{\sum_{j=1}^{\infty}\Phi(x,j,k)\Delta^{j}(\ln x)\right\}\Bigg|_{x=1} \qquad (4-1)$$

where the function $\Phi(x,j,k)$ is a polynomial in x of degree $(2k+1)$, expressed by the formula

$$\Phi(x,j,k)=(-1)^{j-1}\sum_{\rho=0}^{2k+1}\frac{(-1)^{\rho}}{\rho+j+1}\binom{\rho+j}{\rho}\Delta^{\rho}\left\{(x+j)^{2k+1}\right\} \qquad (4-2)$$

(In equation (4-2), the Δ operator operates on the variable x).

Theorem 1: *The polynomial $\Phi(x,j,k)$ in equation (4-2) may be expressed equivalently as*

$$\Phi(x,j,k)=\frac{(-1)}{j!}\frac{d^{j}}{d\Delta^{j}}\left(\frac{ln(I+\Delta)}{\Delta}\right)\left\{(x+j)^{2k+1}\right\} \qquad (4-3)$$

where j and k are positive integers.

Note 1: We note that even though the operator $\frac{d^j}{d\Delta^j}\left(\frac{\ln(I+\Delta)}{\Delta}\right)$, in general contains an infinite number of terms, in reality, when this operator operates on the polynomial $\{(x+j)^{2k+1}\}$, (which is a polynomial in x of degree $(2k+1)$), only a finite number of terms will survive, since all differences of order greater than $(2k+1)$ are identically equal to zero. Taking this note into consideration, and in order to simplify the calculations, we may in general write

$$\frac{\ln(I+\Delta)}{\Delta}=\sum_{\rho=1}^{\infty}\frac{(-1)^{\rho-1}}{\rho}\Delta^{\rho-1}=I-\frac{1}{2}\Delta+\frac{1}{3}\Delta^2-\frac{1}{4}\Delta^3+\cdots \quad (4-4)$$

where, however, we understand that when this operator operates on the polynomial $\{(x+j)^{2k+1}\}$, all terms with $\rho-1>(2k+1)$, vanish (will be identically equal to zero).

Proof: Let j be any fixed positive integer. Taking the j^{th} derivative of both sides of equation (4-4) with respect to Δ, yields,

$$\frac{d^j}{d\Delta^j}\left(\frac{\ln(I+\Delta)}{\Delta}\right) = (-1)^j j!\left\{\frac{1}{j+1}I-\frac{1}{j+2}\frac{(j+1)!}{1!\,j!}\Delta+\frac{1}{j+3}\frac{(j+2)!}{2!\,j!}\Delta^2 - \frac{1}{j+4}\frac{(j+3)!}{3!\,j!}\Delta^3+\cdots\right\} \Longrightarrow$$

$$\frac{d^j}{d\Delta^j}\left(\frac{\ln(I+\Delta)}{\Delta}\right) = (-1)^j\, j!\,\left\{\frac{1}{j+1}I-\frac{1}{j+2}\binom{j+1}{1}\Delta+\frac{1}{j+3}\binom{j+2}{2}\Delta^2 - \frac{1}{j+4}\binom{j+3}{3}\Delta^3+\cdots\right\}$$

or equivalently,

$$\frac{(-1)^j}{j!}\,\frac{d^j}{d\Delta^j}\left(\frac{\ln(I+\Delta)}{\Delta}\right)=\sum_{\rho=0}^{\infty}\frac{(-1)^\rho}{\rho+j+1}\binom{\rho+j}{\rho}\Delta^\rho \qquad (4-5)$$

By virtue of equation (4-5), the expression for the polynomial $\Phi(x,j,k)$ in (4-2) becomes,

$$\Phi(x,j,k)=\frac{(-1)}{j!}\,\frac{d^j}{d\Delta^j}\left(\frac{\ln(I+\Delta)}{\Delta}\right)\{(x+j)^{2k+1}\}$$

and this completes the proof.

The polynomial $\Phi(x,j,k)$ (in (4-2)), is expressed as a finite sum of successive differences of the polynomial $(x+j)^{2k+1}$. Next Theorem provides an alternative expression of $\Phi(x,j,k)$ in terms of successive derivatives of the polynomial $(x+j)^{2k+1}$.

Theorem 2: *The polynomial $\Phi(x,j,k)$ in equation (4-2) can be expressed equivalently as*

$$\Phi(x,j,k)=\frac{(-1)}{j!}\,\tilde{E}^{[j]}\,e^{(\tilde{E}-jI)D}\,\{B_0(x+j)^{2k+1}\} \qquad (4-6)$$

where the $\tilde{E}$ operator is defined in equation (2-2), (operates on the indices of the Bernoulli numbers), the $D=\frac{d}{dx}$ is as usual the derivative operator, operating on the x variable and

$$\tilde{E}^{[j]}=\tilde{E}\big(\tilde{E}-I\big)\big(\tilde{E}-2I\big)\big(\tilde{E}-3I\big)\cdots\big(\tilde{E}-(j-1)I\big) \qquad (4-7)$$

(according to equation (2-4)), where j is any positive integer.

Note 2: Equation (4-6) actually converts a finite sum of differences (equation (4-2)) into a finite sum of derivatives. Even though the Taylor series expansion of the operator $e^{(\tilde{E}-jI)D}$ (in equation (4-6)) contains infinitely many terms, in reality the infinite series terminates and becomes a finite sum, since all the derivatives of order higher than $(2k+1)$, (the degree of the polynomial $(x+j)^{2k+1}$) are identically equal to zero.

Proof: a) Application of equation (4-3) with $j=1$ yields,

$$\Phi(x,1,k)=(-1)\frac{d}{d\Delta}\left(\frac{\ln(I+\Delta)}{\Delta}\right)\{(x+1)^{2k+1}\} \qquad (4-8)$$

Since all polynomials are in $K_{-\infty}$, (by virtue of Theorem 2 in [13]), the operator $\ln(I+\Delta)$ is equivalent to the operator $D=\frac{d}{dx}$, (or alternatively Δ is equivalent to $e^{D}-I$), regarding their application on any polynomial of x, and therefore from equation (4-8) we have,

$$\frac{d}{d\Delta}\left(\frac{\ln(I+\Delta)}{\Delta}\right)\{(x+1)^{2k+1}\} = \frac{d}{d\Delta}\left(\frac{D}{e^D - I}\right)\{(x+1)^{2k+1}\}$$
$$= \frac{d}{dD}\left(\frac{D}{e^D - I}\right)\frac{dD}{d\Delta}\{(x+1)^{2k+1}\}$$
$$= \frac{d}{dD}\left(\frac{D}{e^D - I}\right)e^{-D}\,\{(x+1)^{2k+1}\} \qquad (4-9)$$

or making use of equation (2-1),

$$\frac{d}{d\Delta}\left(\frac{\ln(I+\Delta)}{\Delta}\right)\{(x+1)^{2k+1}\} = \frac{d}{dD}\left(\sum_{n=0}^{\infty}\frac{B_n}{n!}D^n\right)\,e^{-D}\{(x+1)^{2k+1}\}$$
$$= \left(\frac{B_1}{0!}I + \frac{B_2}{1!}D + \frac{B_3}{2!}D^2 + \frac{B_4}{3!}D^3 + \cdots\right)e^{-D}\{(x+1)^{2k+1}\}$$
$$= \left(\frac{B_1}{0!}I + \frac{\tilde{E}\,B_1}{1!}D + \frac{\tilde{E}^2 B_1}{2!}D^2 + \frac{\tilde{E}^3 B_1}{3!}D^3 + \cdots\right)e^{-D}\{(x+1)^{2k+1}\}$$
$$= \left(\frac{I}{0!} + \frac{\tilde{E}\,D}{1!} + \frac{\left(\tilde{E}\,D\right)^2}{2!} + \frac{\left(\tilde{E}\,D\right)^3}{3!} + \cdots\right)\,B_1\,e^{-D}\{(x+1)^{2k+1}\}$$
$$= e^{\tilde{E}D}e^{-D}[B_1\{(x+1)^{2k+1}\}] = e^{(\tilde{E}-I)D}\left[\left(\tilde{E}B_0\right)\{(x+1)^{2k+1}\}\right]$$
$$= \tilde{E}e^{(\tilde{E}-I)D}\{B_0(x+1)^{2k+1}\} \qquad (4-10)$$

and this, in view of equation (4-8), proves Theorem 2 for $j = 1$.

b) Application of equation (4-3) with $j = 2$, and reasoning as in part (a) and making use of equation (4-10), implies:

$$\Phi(x,2,k) = \frac{(-1)}{2!}\,\frac{d^2}{d\Delta^2}\left(\frac{\ln(I+\Delta)}{\Delta}\right)\{(x+2)^{2k+1}\}$$
$$= \frac{(-1)}{2!}\frac{d}{d\Delta}\left\{\left[\frac{d}{d\Delta}\left(\frac{\ln(I+\Delta)}{\Delta}\right)\right]\{(x+2)^{2k+1}\}\right\}$$
$$= \frac{(-1)}{2!}\frac{d}{d\Delta}\left\{\left[\tilde{E}e^{(\tilde{E}-I)D}\right]\{B_0(x+2)^{2k+1}\}\right\}\xrightarrow{(e^D\Leftrightarrow\Delta+I)}$$
$$= \frac{(-1)}{2!}\frac{d}{d\Delta}\,\left\{\tilde{E}(\Delta+I)^{(\tilde{E}-I)}\right\}\{B_0(x+2)^{2k+1}\}$$
$$= \frac{(-1)}{2!}\left\{\tilde{E}\left(\tilde{E}-I\right)(\Delta+I)^{(\tilde{E}-2I)}\right\}\{B_0(x+2)^{2k+1}\}\xrightarrow{(e^D\Leftrightarrow\Delta+I)}$$
$$= \frac{(-1)}{2!}\tilde{E}\left(\tilde{E}-I\right)e^{(\tilde{E}-2I)D}\,\{B_0(x+2)^{2k+1}\} \qquad (4-11)$$

and this proves Theorem 2 for $j = 2$.

c) Proceeding similarly, step by step, one may prove Theorem 2 for $j = 3$, then for $j = 4$, etc, and this shows the validity of Theorem 2 for all positive integers j.

Next Theorem is the main Theorem in this article. By virtue of this Theorem the values of $\zeta(2k+1), k = 1,2,3,\cdots$ may be expressed in terms of the Bernoulli and the Stirling numbers of the first kind.

Theorem 3: *The values of* $\zeta(2k+1), k = 1,2,3,\cdots$ *are given by the formula*

$$\zeta(2k+1) = (-1)^k \frac{2^{2k+1}}{(2k+1)!}\pi^{2k} \cdot \sum_{j=1}^{\infty} \frac{1}{j!}\{s_1^j \Omega_1^k + s_2^j \Omega_2^k + s_3^j \Omega_3^k + \cdots + s_j^j \Omega_j^k\}\left\{ \Delta^j (ln\, x)\big|_{x=1}\right\} \qquad (4-12)$$

where $s_1^j, s_2^j, s_3^j, \cdots, s_j^j$ *are the Stirling numbers of the first kind, and*

$$\Omega_\lambda^k = \binom{2k+1}{0} B_\lambda + \binom{2k+1}{1} B_{\lambda+1} + \cdots + \binom{2k+1}{2k+1} B_{\lambda+(2k+1)} \quad (4-13)$$

where $\lambda = 1,2,3,\cdots, j$ *and* B_λ *are the Bernoulli numbers.*

Proof: (a) The starting point in our proof is equation (4-1), where the polynomial $\Phi(x, j, k)$ is given by equation (4-2), ([11]). Taking into account the equivalent expression for $\Phi(x, j, k)$, as given in equation (4-6), we have:

$$\zeta(2k+1) = (-1)^k \frac{2^{2k+1}}{(2k+1)!}\pi^{2k} \cdot \sum_{j=1}^{\infty} \frac{1}{j!} \tilde{E}^{[j]}\left\{ e^{(\tilde{E}-jI)D} \left[B_0(x+j)^{2k+1}\right]\big|_{x=1}\right\}\left\{\Delta^j (\ln x)\big|_{x=1}\right\} \qquad (4-14)$$

Let us consider the function $Q(x)$, defined as follows:

$$Q(x) = e^{(\tilde{E}-jI)D}\{B_0(x+j)^{2k+1}\}$$

Expanding $e^{(\tilde{E}-jI)D}$ in a Taylor series and noting that all derivatives of order higher than $(2k+1)$ vanish identically, (since the degree of the polynomial $(x+j)^{2k+1}$ is $(2k+1)$), we obtain,

$$Q(x)=\left(I+\frac{(\tilde{E}-jI)}{1!}D+\frac{(\tilde{E}-jI)^2}{2!}D^2+\frac{(\tilde{E}-jI)^3}{3!}+\cdots\right.$$
$$\left.+\frac{(\tilde{E}-jI)^{2k+1}}{(2k+1)!}D^{2k+1}\right)\{B_0(x+j)^{2k+1}\}\Rightarrow$$

$$Q(x)=\left(\{I(x+j)\}^{2k+1}+\frac{(\tilde{E}-jI)}{1!}(2k+1)\{I(x+j)\}^{2k}\right.$$
$$+\frac{(\tilde{E}-jI)^2}{2!}(2k+1)(2k)\{I(x+j)\}^{2k-1}$$
$$+\frac{(\tilde{E}-jI)^3}{3!}(2k+1)(2k)(2k-1)\{I(x+j)\}^{2k-2}+\cdots$$
$$\left.+\frac{(\tilde{E}-jI)^{2k+1}}{(2k+1)!}(2k+1)(2k)(2k-1)\cdots\cdot 2\cdot 1\right)(B_0)\Rightarrow$$

$$Q(x)=\left(\{I(x+j)\}+\{\tilde{E}-jI\}\right)^{2k+1}=\left(Ix+\tilde{E}\right)^{2k+1} \qquad (4-15)$$

and hence

$$Q(x=1)=\left(I+\tilde{E}\right)^{2k+1}(B_0) \qquad (4-16)$$

We note that since $Q(x=1)$ depends on k, we may set

$$Q^k=Q(x=1)=\left(I+\tilde{E}\right)^{2k+1}(B_0)\Rightarrow$$

$$Q^k=\left\{\binom{2k+1}{0}I+\binom{2k+1}{1}\tilde{E}+\binom{2k+1}{2}\tilde{E}^2+\cdots+\binom{2k+1}{2k+1}\tilde{E}^{2k+1}\right\}(B_0)$$

In view of this expression for Q^k, the expression

$$e^{(\tilde{E}-jI)D}\,[B_0(x+j)^{2k+1}]\Big|_{x=1}=Q(x=1)=Q^k$$

appearing in (4-14) takes the equivalent form,

$$Q^k=\binom{2k+1}{0}B_0+\binom{2k+1}{1}B_1+\binom{2k+1}{2}B_2+\cdots+\binom{2k+1}{2k+1}B_{2k+1}$$

$$(4-17)$$

(b) Making use of equation (2-5) the operator $\tilde{E}^{[j]}$ appearing in (4-14) can be expanded in terms of the Stirling numbers of the first kind, i.e.

$$\tilde{E}^{[j]} = s_1^j \tilde{E} + s_2^j \tilde{E}^2 + s_3^j \tilde{E}^3 + \cdots + s_j^j \tilde{E}^j \qquad (4-18)$$

and hence equation (4-14) implies,

$$\zeta(2k+1) = (-1)^k \frac{2^{2k+1}}{(2k+1)!} \pi^{2k} \cdot \sum_{j=1}^{\infty} \frac{1}{j!} \{ s_1^j (\tilde{E} Q^k) + s_2^j (\tilde{E}^2 Q^k) + + \cdots + s_\lambda^j (\tilde{E}^\lambda Q^k) + \cdots + s_j^j (\tilde{E}^j Q^k) \} \left\{ \Delta^j (\ln x) \big|_{x=1} \right\}$$

and this formula reduces easily to equation (4-12) since

$$\Omega_\lambda^k = (\tilde{E}^\lambda Q^k), \qquad \lambda = 1, 2, 3, \cdots, j$$

and the proof is thus completed.

5. The Apery constant $\zeta(3)$

The Apery constant $\zeta(3)$ is obtained from equation (4-12) if we set $k = 1$.

$$\zeta(3) = -\frac{4\pi^2}{3} \left\{ \frac{1}{1!} [s_1^1 \Omega_1^1][\Delta(\ln x)|_{x=1}] + \frac{1}{2!} [s_1^2 \Omega_1^1 + s_2^2 \Omega_2^1][\Delta^2(\ln x)|_{x=1}] + \frac{1}{3!} [s_1^3 \Omega_1^1 + s_2^3 \Omega_2^1 + s_3^3 \Omega_3^1][\Delta^3(\ln x)|_{x=1}] + \frac{1}{4!} [s_1^4 \Omega_1^1 + s_2^4 \Omega_2^1 + s_3^4 \Omega_3^1 + s_4^4 \Omega_4^1][\Delta^4(\ln x)|_{x=1}] + \cdots \right\} \qquad (5-1)$$

For the systematization of the computations we may consider the following table, (for the first four terms $\Omega_1^1, \Omega_2^1, \Omega_3^1, \Omega_4^1$):

$$\textbf{\textit{Table}}\ \mathbf{1} \quad (\boldsymbol{k}=\mathbf{1})$$

Ω_1^1	$\binom{3}{0}B_1+\binom{3}{1}B_2+\binom{3}{2}B_3+\binom{3}{3}B_4=B_1+3B_2+B_4$
Ω_2^1	$\binom{3}{0}B_2+\binom{3}{1}B_3+\binom{3}{2}B_4+\binom{3}{3}B_5=B_2+3B_4$
Ω_3^1	$\binom{3}{0}B_3+\binom{3}{1}B_4+\binom{3}{2}B_5+\binom{3}{3}B_6=3B_4+B_6$
Ω_4^1	$\binom{3}{0}B_4+\binom{3}{1}B_5+\binom{3}{2}B_6+\binom{3}{3}B_7=B_4+3B_6$

Taking into consideration Table 1, the Apery's constant (formula (5-1) takes the following form,

$$\zeta(3)=-\frac{4\pi^2}{3}\left\{\frac{1}{1!}s_1^1(B_1+3B_2+B_4)[\Delta(\ln x)|_{x=1}]\right.$$
$$+\frac{1}{2!}[s_1^2(B_1+3B_2+B_4)+s_2^2(B_2+3B_4)][\Delta^2(\ln x)|_{x=1}]$$
$$+\frac{1}{3!}[s_1^3(B_1+3B_2+B_4)+s_2^3(B_2+3B_4)$$
$$+s_3^3(3B_4+B_6)][\Delta^3(\ln x)|_{x=1}]$$
$$+\frac{1}{4!}[s_1^4(B_1+3B_2+B_4)+s_2^4(B_2+3B_4)+s_3^4(3B_4+B_6)$$
$$\left.+s_4^4(B_4+3B_6)][\Delta^4(\ln x)|_{x=1}]+\cdots\right\} \qquad (5-2)$$

Formula (5-2) gives an expression of the Apery's constant in terms of the Bernoulli and the Stirling numbers of the first kind. Substituting numerical values for the Bernoulli and the Stirling numbers (and expanding the differences of the function $\ln x$, evaluated at $x=1$, according to formula (1-4)) we obtain:

$$\zeta(3)=\frac{4\pi^2}{3}\left\{\frac{1\cdot 2}{3\cdot 4\cdot 5}\ln 2+\frac{2\cdot 3}{4\cdot 5\cdot 6}\ln\left(\frac{2^2}{3}\right)+\frac{3\cdot 4}{5\cdot 6\cdot 7}\ln\left(\frac{2^3\cdot 4}{3^3}\right)\right.$$
$$\left.+\frac{4\cdot 5}{6\cdot 7\cdot 8}\ln\left(\frac{2^4\cdot 4^4}{5\cdot 3^6}\right)+\cdots\right\} \qquad (5-3)$$

This expression of the Apery's constant is identical to the one obtained in [11].

6. An expression for $\zeta(5)$

An expression for $\zeta(5)$ is obtained from equation (4-12) if we set $k = 2$.

$$\zeta(5) = \frac{4\pi^4}{15}\left\{\frac{1}{1!}s_1^1\Omega_1^2[\Delta(\ln x)|_{x=1}] + \frac{1}{2!}[s_1^2\Omega_1^2 + s_2^2\Omega_2^2][\Delta^2(\ln x)|_{x=1}] + \frac{1}{3!}[s_1^3\Omega_1^2 + s_2^3\Omega_2^2 + s_3^3\Omega_3^2][\Delta^3(\ln x)|_{x=1}] + \frac{1}{4!}[s_1^4\Omega_1^2 + s_2^4\Omega_2^2 + s_3^4\Omega_3^2 + s_4^4\Omega_4^2][\Delta^4(\ln x)|_{x=1}] + \cdots\right\} \quad (6-1)$$

For the systematization of the computations we may consider the following Table, (for the first four terms $\Omega_1^2, \Omega_2^2, \Omega_3^2, \Omega_4^2$):

$$\boldsymbol{Table\ 2} \quad (\boldsymbol{k = 2})$$

Ω_1^2	$\binom{5}{0}B_1 + \binom{5}{1}B_2 + \binom{5}{2}B_3 + \binom{5}{3}B_4 + \binom{5}{4}B_5 + \binom{5}{5}B_6 = B_1 + 5B_2 + 10B_4 + B_6$
Ω_2^2	$\binom{5}{0}B_2 + \binom{5}{1}B_3 + \binom{5}{2}B_4 + \binom{5}{3}B_5 + \binom{5}{4}B_6 + \binom{5}{5}B_7 = B_2 + 10B_4 + 5B_6$
Ω_3^2	$\binom{5}{0}B_3 + \binom{5}{1}B_4 + \binom{5}{2}B_5 + \binom{5}{3}B_6 + \binom{5}{4}B_7 + \binom{5}{5}B_8 = 5B_4 + 10B_6 + B_8$
Ω_4^2	$\binom{5}{0}B_4 + \binom{5}{1}B_5 + \binom{5}{2}B_6 + \binom{5}{3}B_7 + \binom{5}{4}B_8 + \binom{5}{5}B_9 = B_4 + 10B_6 + 5B_8$

Taking into consideration Table 2, we find the following value for $\zeta(5)$:

$$\zeta(5) = \frac{4\pi^4}{15}\left\{\frac{1}{1!}s_1^1(B_1 + 5B_2 + 10B_4 + B_6)[\Delta(\ln x)|_{x=1}]\right.$$
$$+ \frac{1}{2!}[s_1^2(B_1 + 5B_2 + 10B_4 + B_6)$$
$$+ s_2^2(B_2 + 10B_4 + 5B_6)][\Delta^2(\ln x)|_{x=1}]$$
$$+ \frac{1}{3!}[s_1^3(B_1 + 5B_2 + 10B_4 + B_6) + s_2^3(B_2 + 10B_4 + 5B_6)$$
$$+ s_3^3(5B_4 + 10B_6 + B_8)][\Delta^3(\ln x)|_{x=1}]$$
$$+ \frac{1}{4!}[s_1^4(B_1 + 5B_2 + 10B_4 + B_6) + s_2^4(B_2 + 10B_4 + 5B_6)$$
$$+ s_3^4(5B_4 + 10B_6 + B_8) + s_4^4(B_4 + 10B_6 + 5B_8)][\Delta^4(\ln x)|_{x=1}]$$
$$\left. + \cdots \right\}$$

and substituting numerical values for the Bernoulli and the Stirling numbers, we obtain,

$$\zeta(5) = \frac{4\pi^4}{15}\left\{\frac{1\cdot 2\cdot 30}{3\cdot 4\cdot 5\cdot 6\cdot 7}\ln 2 + \frac{2\cdot 3\cdot 40}{4\cdot 5\cdot 6\cdot 7\cdot 8}\ln\left(\frac{2^2}{3}\right)\right.$$
$$\left. + \frac{3\cdot 4\cdot 48}{5\cdot 6\cdot 7\cdot 8\cdot 9}\ln\left(\frac{2^3\cdot 4}{3^3}\right) + \frac{4\cdot 5\cdot 54}{6\cdot 7\cdot 8\cdot 9\cdot 10}\ln\left(\frac{2^4\cdot 4^4}{5\cdot 3^6}\right) + \cdots\right\}$$

and this is identical to the value of $\zeta(5)$ derived in [11].

7. Concluding Remarks

Making extensive use of Operational Calculus and the fact that the operators $D = \frac{d}{dx}$ and $\ln(I + \Delta)$ are equivalent when applied to any polynomial of x, we derive a new expression for the values of the Riemann's Zeta function $\zeta(2k + 1), k = 1,2,3,\cdots$, in terms of the Bernoulli and the Stirling numbers. In particular, for $k = 1$, we obtain an expression for the Apery's constant $\zeta(3)$, while for $k = 2$ we obtain an expression for $\zeta(5)$.

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