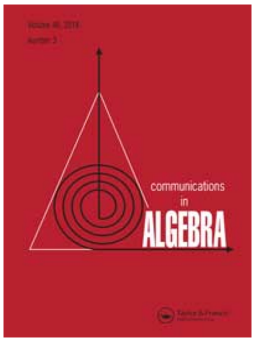

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A note on Gorenstein projective modules

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ABSTRACT
In this note, it is proven that if R is a right noetherian ring with $\mathrm{id}_{R^o} R < \infty$ and $\mathrm{Ext}_R^{i\geq 1}(M, F) = 0$ for any left R-module F with finite flat dimension, then M is Gorenstein projective; if R is a left noetherian ring with $\mathrm{id}_R R < \infty$ and M is a Gorenstein projective left R-module, then $\mathrm{Ext}_R^{i\geq 1}(M, F) = 0$ for any left R-module F with finite flat dimension.

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1. Introduction

Throughout this paper, R is an associative ring with identity, $\mathrm{Mod}R$ is the category of left R-modules. For a module $M \in \mathrm{Mod}R$, we denote the flat, injective and projective dimensions of M by $\mathrm{fd}_R M$, $\mathrm{id}_R M$ and $\mathrm{pd}_R M$, respectively.

Recall that a left R-module M is called *cotorsion* [5] if $\mathrm{Ext}_R^1(F, M) = 0$ for any flat left R-module F; and M is called *strongly cotorsion* [11] if $\mathrm{Ext}_R^1(X, M) = 0$ for any left R-module X with finite flat dimension. A right R-module N is called *strongly torsionfree* [11] if $\mathrm{Tor}_1^R(N, X) = 0$ for any left R-module X with finite flat dimension.

A complex

$$X: \quad \cdots \longrightarrow X_{i+1} \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \longrightarrow \cdots$$

is called *acyclic* if the homology complex $\mathrm{H}(X)$ is the zero complex. We use the notations $\mathrm{Z}_i(X)$ for the kernel of differential ∂_i^X and $\mathrm{C}_i(X)$ for the cokernel of the differential ∂_{i+1}^X. Recall from [6] that an acyclic complex P of projective R-modules is called a *complete projective resolution*, if the complex $\mathrm{Hom}_R(P, Q)$ is acyclic for every projective R-module Q. An R-module M is called *Gorenstein projective* if there exists a complete projective resolution P such that $\mathrm{C}_0(P) \cong M$. An acyclic complex F of flat R-modules is called a *complete flat resolution*, if the complex $I \otimes_R F$ is acyclic for every injective R-module I. An R-module N is called *Gorenstein flat* if there exists a complete flat resolution F such that $\mathrm{C}_0(F) \cong N$. A complex U of injective R-modules is called a *complete injective resolution* if it is acyclic, and the complex $\mathrm{Hom}_R(J, U)$ is acyclic for every injective R-module J. An R-module E is called *Gorenstein injective* if there exists a complete injective resolution U such that $\mathrm{Z}_0(U) \cong E$.

In [8], Huang proved that if R is a Gorenstein ring with the injective envelope of ${}_R R$ flat, then a left R-module is Gorenstein injective if and only if it is strongly cotorsion, and a right R-module is Gorenstein flat if and only if it is strongly torsionfree. In fact, in [10, Corollary 5.9], it is shown that for every module M with a left injective resolution, one has $\mathrm{Ext}_R^{i\geq 1}(F, M) = 0$ for every R-module F of finite flat dimension. In particular, every Gorenstein injective module is strongly cotorsion. Therefore, it is not hard to see that if R is a right noetherian ring with $\mathrm{id}_{R^o} R < \infty$, then a left R-module is Gorenstein injective if and only if it is strongly cotorsion, and a right R-module is Gorenstein flat if and only if it is strongly torsionfree.

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It is natural to consider the situation of Gorenstein projective modules. In this note, we prove that if R is a left noetherian ring with $\mathrm{id}_R R < \infty$ and M is a Gorenstein projective left R-module, then $\mathrm{Ext}_R^{i\geq 1}(M, F) = 0$ for any left R-module F with finite flat dimension; if R is a right noetherian ring with $\mathrm{id}_{R^o} R < \infty$ and $\mathrm{Ext}_R^{i\geq 1}(M, F) = 0$ for any left R-module F with finite flat dimension, then M is Gorenstein projective.

2. Main results

We list the following lemma for later use.

Lemma 2.1.

(1) *[9, Proposition 1] For a right noetherian ring R,*

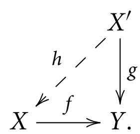
$$\mathrm{id}_{R^o} R = \sup\{\mathrm{fd}_R I \mid I \in \mathrm{Mod}\, R \text{ is injective}\}.$$

(2) *[3, Theorem 3.8] For a left noetherian ring R,*

$$\mathrm{id}_R R = \sup\{\mathrm{id}_R M \mid M \in \mathrm{Mod}\, R \text{ with } \mathrm{fd}_R M < \infty\}.$$

We also need to recall the following definitions.

Definition 2.2. [4] Let $\mathcal{A}$ be a full subcategory of Mod$R$. A morphism $f : X \to Y$ in Mod$R$ with $X \in \mathcal{A}$ is called an *$\mathcal{A}$-precover* of Y if for any morphism $g : X' \to Y$ in Mod$R$ with $X' \in \mathcal{A}$, there exists a morphism $h : X' \to X$ such that the following diagram commutes:



A homomorphism $f : X \to Y$ is said to be *right minimal* if an endomorphism $h : X \to X$ is an automorphism whenever $f = fh$. An $\mathcal{A}$-precover $f : X \to Y$ is called an $\mathcal{A}$-cover if f is right minimal; $\mathcal{A}$ is called *covering* if every module in Mod$R$ has an $\mathcal{A}$-cover. Dually, the notions of an *$\mathcal{A}$-preenvelope*, a *left minimal homomorphism* and an *$\mathcal{A}$-envelope* are defined.

2.3 ([7, Proposition 2.3]). Every R-module has a projective resolution, so to prove that a module M is Gorenstein projective it suffices to verify the following:

(1) $\mathrm{Ext}_R^{i\geq 1}(M, P) = 0$ for every projective R-module P.

(2) M has a co-proper right projective resolution. That is, there exists an acyclic complex of R-modules $X = 0 \to M \to P^1 \to P^2 \to \cdots$ with each P^i projective, such that $\mathrm{Hom}_R(X, Q)$ is acyclic for every projective R-module Q.

Theorem 2.4.

(1) *Let R be a left noetherian ring with $\mathrm{id}_R R < \infty$. If M is a Gorenstein projective left R-module, then $\mathrm{Ext}_R^{i\geq 1}(M, F) = 0$ for any left R-module F with finite flat dimension.*

(2) *Let R be a right noetherian ring with $\mathrm{id}_{R^o} R < \infty$ and M a left R-module. If $\mathrm{Ext}_R^{i\geq 1}(M, F) = 0$ for any left R-module F with finite flat dimension, then M is Gorenstein projective.*

Proof.

(1) Let F be a left R-module with finite flat dimension. By Lemma 2.1(2), one has $\mathrm{id}_R F < \infty$. Hence the result follows from [2, Lemma 2.1].

(2) Let R be a right noetherian ring with $\mathrm{id}_{R^o}R < \infty$ and M a left R-module. There exists an exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow X \longrightarrow 0$$

such that E is an injective module and $X = \mathrm{Coker}(M \to E)$. Since every module has a flat cover [1, Theorem 3], one has the following exact sequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\alpha} E \longrightarrow 0,$$

where $\alpha : F \to E$ is a flat cover and $K = \mathrm{Ker}\,\alpha$. Now consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & D & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & E & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X & = & X & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By Lemma 2.1(1), one has $\mathrm{fd}_R E < \infty$ and so $\mathrm{fd}_R K < \infty$ by the exactness of the middle row in the above diagram. Hence $\mathrm{Ext}^1_R(M, K) = 0$. It yields that the exact sequence

$$0 \longrightarrow K \longrightarrow D \longrightarrow M \longrightarrow 0$$

in the above diagram splits. Thus one get a monomorphism $M \to F$. Consequently, it follows from [6, Proposition 6.5.1] that there exists an exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi} F' \longrightarrow Y \longrightarrow 0$$

such that $\varphi : M \to F'$ is a flat preenvelope and $Y = \mathrm{Coker}\,\varphi$. Next we show that φ is an $\mathcal{F}(R)$-preenvelope, where $\mathcal{F}(R)$ denotes the full subcategory consisting of modules of finite flat dimensions.

Let $\psi : M \to L$ be an R-homomorphism such that $\mathrm{fd}_R L$ is finite. Consider the short exact sequence $0 \to K \to F'' \xrightarrow{\pi} L \to 0$ in which $\pi : F'' \to L$ is a flat cover. Clearly, K is of finite flat dimension and so $\mathrm{Ext}^{i\geq 1}_R(M, K) = 0$ by hypothesis. Thus, one has the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, K) \longrightarrow \mathrm{Hom}_R(M, F'') \longrightarrow \mathrm{Hom}_R(M, L) \longrightarrow 0.$$

Therefore, there exists an R-homomorphism $h : M \to F''$ such that $\pi h = \psi$. Since $\varphi : M \to F'$ is a flat preenvelope, there is an R-homomorphism $g : F' \to F''$ such that $h = g\varphi$. Thus, one has $\pi g \varphi = \psi$ and so φ is an $\mathcal{F}(R)$-preenvelope.

Next we show that there exists a monic $\mathcal{P}(R)$-preenvelope $M \to P$ with P projective, where $\mathcal{P}(R)$ denotes the full subcategory consisting of modules of finite projective dimensions. Let $0 \to A \to P \to F' \to 0$ be an exact sequence such that P is projective and $A = \mathrm{Ker}(P \to F')$. Clearly, one has $\mathrm{fd}_R A < \infty$ and so $\mathrm{Ext}^{i\geq 1}_R(M, A) = 0$. Hence one has the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \longrightarrow \mathrm{Hom}_R(M, P) \longrightarrow \mathrm{Hom}_R(M, F') \longrightarrow 0.$$

Therefore, there exists a monic $\mathcal{P}(R)$-preenvelope $M \to P$ with P projective.

Now consider the following exact sequence

$$0 \longrightarrow M \xrightarrow{\beta} P \longrightarrow C \longrightarrow 0,$$

where β is a $\mathcal{P}(R)$-preenvelope, P is a projective R-module and $C = \text{Coker}\,\beta$. Let Q be a projective R-module. Applying the functor $\text{Hom}_R(-, Q)$ to the above exact sequence, one has $\text{Ext}_R^{i\geq 1}(C, Q) = 0$ as $\beta : M \to P$ is a $\mathcal{P}(R)$-preenvelope and $\text{Ext}_R^{i\geq 1}(M, Q) = 0$.

It remains to show that $\text{Ext}_R^{i\geq 1}(C, B) = 0$ for any left R-module B with finite flat dimension. Let B be a left R-module with $\text{fd}_R B < \infty$. Applying the functor $\text{Hom}_R(-, Q)$ to the exact sequence $0 \to M \to P \to C \to 0$, one has the following exact sequence

$$0 \longrightarrow \text{Ext}_R^1(M, B) \longrightarrow \text{Ext}_R^2(C, B) \longrightarrow 0.$$

By assumption, one has $\text{Ext}_R^1(M, B) = 0$ and so $\text{Ext}_R^{i\geq 2}(C, B) = 0$. Consider the following exact sequence

$$0 \longrightarrow Z \longrightarrow P' \xrightarrow{\gamma} B \longrightarrow 0,$$

where P' is a projective module and $Z = \text{Ker}\,\gamma$. Note that $\text{fd}_R Z < \infty$. Applying the functor $\text{Hom}_R(C, -)$ to the above exact sequence, one has the following exact sequence

$$0 = \text{Ext}_R^1(C, P') \longrightarrow \text{Ext}_R^1(C, B) \longrightarrow \text{Ext}_R^2(C, Z) = 0.$$

Thus $\text{Ext}_R^1(C, B) = 0$. Now proceeding in this manner, one could get the desired co-proper right projective resolution of M. This completes the proof. □

Recall that a left and right noetherian ring R is called *Gorenstein* if $\text{id}_R R < \infty$ and $\text{id}_{R^o} R < \infty$. Now the following result is an immediate consequence of Theorem 2.4.

Corollary 2.5. *Let R be a Gorenstein ring. Then a left R-module M is Gorenstein projective if and only if $\text{Ext}_R^{i\geq 1}(M, F) = 0$ for any left R-module F with finite flat dimension.*

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