Geometric Algebra as the unified mathematical language of Physics: An introduction for advanced undergraduate students

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Abstract

In recent years, geometric algebra has emerged as the preferred mathematical framework for physics. It provides both compact and intuitive descriptions in many areas including classical and quantum mechanics, electromagnetic theory and relativity. Geometric algebra has also found prolific applications as a computational tool in computer graphics and robotics. Leading exponents of this extensive mathematical apparatus are fervently insisting its inclusion in the undergraduate physics curriculum and in this paper an introductory exposure, in familiar terms for the advanced undergraduate students, is intended.

Keywords: Vector, pseudovector, tensor, spinor, quaternion, exterior algebra, bivector, pseudoscalar and geometric algebra.

1. Introduction: Geometric algebra is an immensely powerful mathematical framework in which most of the advanced concepts of physics can be expressed. Proponents are also claiming that GA is straightforward and simple enough to be taught to school children [1]! As an initiation for the undergraduate physics students, we discuss here the basics of geometric algebra (GA) and its wide scope of applications with reference to some supplementary reading materials [2, 3].

The algebra developed by Clifford is actually an unification of the algebras of Grassmann and Hamilton into a single structure. By combining both exterior (wedge) and inner (dot) products of Grassmann algebra, Clifford's ingenuous contribution was to define a new associative product,[1] the *geometric product*:

$$\mathbf{uv} = \mathbf{u}.\mathbf{v} + \mathbf{u} \wedge \mathbf{v} = \mathrm{C} \tag{1}$$

of two vectors $\mathbf{u}$ and $\mathbf{v}$ which endows the basic vector space with an algebraic structure that embraces the vector, complex, quaternion and the spin algebras in a single formalism and sets apart Clifford algebra from others. The product C is the sum of a scalar s ($= \mathbf{u}.\mathbf{v}$) and a *bivector* $\mathbf{B}$ ($= \mathbf{u} \wedge \mathbf{v}$) represents a general multivector, termed as a *clif* or *geometric*. The scalars, bivectors are homogeneous multivectors of grade zero and two respectively. For two 3-D vectors, this product actually represents the four component quaternion defined by Hamilton.

The works of Hamilton, Grassmann, Cartan and Clifford which initiate the search for a unifying mathematical language of physics, generated considerable interest among contemporaries. The subsequent development of seemingly more straightforward *vector algebra* (VA) by Gibbs, however, almost eclipsed those important earlier studies for a long time. In fact, the reformulation of Maxwell's equations of electromagnetism by Heaviside using VA, signalled its emergence as the dominant paradigm of vector manipulations for a three dimensional world.

It should be noted that, the cross product, triple product etc. of vectors and hence the vector algebra itself can be defined in 3-D only. Generalisation to any other dimension is not possible – in two dimension, no third dimension exists to accommodate the (cross) product vector, and in higher dimensions there are too many orthogonal directions! We may recall here that, only the scalar dot product of four-vectors is required in the usual discussions of special relativity. Also, the

[1] Grassmann, in his later years, combined the inner and exterior products to form a new product very similar to eq.(1) [4]. In fact, Grassmann has also discovered the key idea of geometric product independently of Clifford and evidently somewhat before him. Many historians of mathematics have overlooked this important later work of Grassmann.

cross product is not preserved under reflection, and thus it introduces a fake chirality (handedness) to the model of reality it creates. The cross product of two vectors is a third vector and VA deals only with scalars and vectors – hence contains no geometric description beyond points and lines. All these led Tait to brand VA "a sort of hermaphrodite monster" [5]. Grassmann algebra provides an efficient and useful generalisation (to any finite dimension) by removing the inadequacies of VA. Modern physics is looking for higher dimensional (string, brane) theories beyond the usual $(3+1)$-dimensional spacetime universe. Also, in projective geometry, extra dimension is added to the reality to view it from the vantage perspective of the supernumerary dimension. Finally, "integrating and applying the large body of geometrical ideas running through mathematics and physics", Hestenes [6] and a number of other collaborators (see references in [1, 2]) have reinvented and developed Clifford's geometric algebra as a unified and versatile language of mathematical physics.

Unfortunately though, mainstream physics is yet to appropriate and embrace this development fully. Apart from the historical reasons, it appears that the introduction needs a more familiar and nonaxiomatic direct approach for the inquisitive students who are just beginning to get exposed to this algebra. In this work we discuss the formulation of GA directly from the exterior algebra and hope to equip budding scientists in working out the standard problems and to use it as the main language for expressing their work. We also intend to offer a broad overview, indicating its immense applications. The first article [2] of the supplementary materials reviews and explains, starting from the elementary vector algebra, general ideas of Grassman algebra and quaternions, right up to the geometric algebra. Multivectors accommodate multiple inner products or contractions and in GA, both the exterior and (multiple) inner products are used to define the geometric product of multivectors. In the second paper [3], following the multiplication rules of exterior and inner products, products between arbitrary multivectors of different grades are derived explicitly in terms of the components of the elements. Following an introduction to the exterior algebra in the next section, some new aspects and results are also discussed and before we engage with GA in the final section, a brief introduction to the Hamilton's quaternion is also included.

1.1 Grassmann algebra: The exterior or wedge product (denoted by the multiplication symbol '$\wedge$') of two vectors, introduced by Grassmann in his 'Algebra of Extension', anticommutes and creates a graded structure from the basis vectors to higher order multivector bases. From the anticommutivty, the wedge product of a vector with itself obviously, vanishes and produces a closure property. Also in an n-dimensional vector space, the exterior product of any number k, $(k \leq n)$ of linearly independent vectors is nonvanishing and the product is associative.

In addition to the scalars and vectors of ordinary vector algebra, Grassmann algebra thus introduces bivectors, trivectors etc. (multivectors of higher grade) from the wedge product of two, three vectors representing (oriented) area, (oriented) volume etc. respectively. The grade of an entity is the number of vectors wedged together to make it. The algebra, therefore, has a grade or 'rank' for its elements up to a maximum of n – the dimension of the underlying vector space. Element of a definite grade is called a blade and these elements make up distinct subspaces of the algebra and provide a more natural representation of various physical quantities.

For two distinct vectors $\mathbf{u}$ and $\mathbf{v}$ in a in two dimensional vector space $\mathcal{V}$, we get

$$\begin{aligned}
\mathbf{u} \wedge \mathbf{v} &= u_i v_j \, \hat{e}_i \wedge \hat{e}_j, \;\; i \neq j, \;\; \text{with orthonormal basis vectors } \hat{e}_i \\
&= u_1 v_2 \, \hat{e}_1 \wedge \hat{e}_2 + u_2 v_1 \, \hat{e}_2 \wedge \hat{e}_1 \\
&= (u_1 v_2 - u_2 v_1) \, \hat{e}_1 \wedge \hat{e}_2 \\
&= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{e}_1 \wedge \hat{e}_2 = -\mathbf{v} \wedge \mathbf{u}.
\end{aligned} \tag{2}$$

The determinant gives the area of the parallelogram spanned by the two vectors $\mathbf{u}$ and $\mathbf{v}$, i.e. $|\mathbf{u}||\mathbf{v}| \sin\theta$, θ being the angle between the two vectors; whereas, $\hat{e}_1 \wedge \hat{e}_2$ represents the single basis of the bivector – the highest grade multivector (of the exterior algebra $\Lambda\mathcal{V}$ in two dimensional $\mathcal{V}$). The square-magnitude of the lone basis of the bivector being -1, it represents unit *pseudoscalar* or *antiscalar* in 2-D. The unit bivector $\hat{e}_1 \wedge \hat{e}_2$ has the geometric effect of rotating the vectors $\hat{e}_1$ and $\hat{e}_2$ in its own plane by $\frac{\pi}{2}$ clockwise when multiplying them on their left. It rotates vectors by $\frac{\pi}{2}$ anticlockwise when multiplying on their right. The orientation of the basis bivector $\hat{e}_1 \wedge \hat{e}_2$ can be thought of as a 'direction of circulation' marked on the parallelogram, namely moving in the $\hat{e}_1$ direction first and then moving in the $\hat{e}_2$ direction.

Accordingly, in 3-D, $\Lambda\mathcal{V}$ has three bivector bases as:

$$\begin{aligned}
\mathbf{u}\wedge\mathbf{v} &= (u_2v_3-u_3v_2)\,\hat{e}_2\wedge\hat{e}_3+(u_3v_1-u_1v_3)\,\hat{e}_3\wedge\hat{e}_1+(u_1v_2-u_2v_1)\,\hat{e}_1\wedge\hat{e}_2\\
&= \begin{vmatrix}\hat{e}_2\wedge\hat{e}_3 & \hat{e}_3\wedge\hat{e}_1 & \hat{e}_1\wedge\hat{e}_2\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}\\
&= \begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\hat{e}_1\wedge\hat{e}_2+\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}\hat{e}_1\wedge\hat{e}_3+\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}\hat{e}_2\wedge\hat{e}_3
\end{aligned}\tag{3}$$

Notice that the components of this bivector are the same as the components of the *pseudovector* given by the 'cross product' defined in 3-dimensional VA. These are actually the projected areas of the parallelogram on each of the three coordinate planes. The square root of the sum of the squares of its components, representing the area of the parallelogram spanned by the vectors **u** and **v** as it lies in the three-dimensional space. Clearly, the bivector produced by the wedge product of two vectors **u** and **v** represents their common plane. Usually, a general bivector is represented as: $\mathbf{B}=B_{ij}\,\hat{e}_i\wedge\hat{e}_j;\quad i\neq j$, with B_{ij} representing the components of the bivector. But, since $\hat{e}_i\wedge\hat{e}_j=-\hat{e}_j\wedge\hat{e}_i$, actually only three components $(B_{ij}-B_{ji})$ of the bivector are independent in 3-D. Subsequently we will learn further, how bivectors are used to generate rotations and the number of independent bivector bases in any dimension gives the number of rotational degrees of freedom. Finally, $\Lambda\mathcal{V}$ has one additional trivector basis $\hat{e}_1\wedge\hat{e}_2\wedge\hat{e}_3$ in 3-D and this highest grade element is given by:

$$\begin{aligned}
\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w} &= u_1(v_2w_3-v_3w_2)\,\hat{e}_1\wedge\hat{e}_2\wedge\hat{e}_3+u_2(v_3w_1-v_1w_3)\,\hat{e}_2\wedge\hat{e}_3\wedge\hat{e}_1+u_3(v_1w_2\\
&- v_2w_1)\,\hat{e}_3\wedge\hat{e}_1\wedge\hat{e}_2\\
&= \begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}\hat{e}_1\wedge\hat{e}_2\wedge\hat{e}_3
\end{aligned}\tag{4}$$

produces a pseudoscalar in (3-D) exterior algebra. The determinant gives the volume of the parallelepiped spanned by the three vectors **u**, **v** and **w** (the scalar triple product of the vectors in VA) and $\hat{e}_1\wedge\hat{e}_2\wedge\hat{e}_3$ represents the unit pseudoscalar. It may be noted that $\hat{e}_i\wedge\hat{e}_j\wedge\hat{e}_k$ provides a representation of the Levi-Civita index ϵ_{ijk}. In higher dimensional ($n>3$) spaces, the components of a trivector are the projections of the volume of a parallelepiped onto the coordinate three-spaces, as it is oriented in a higher n-dimensional space. The wedge or exterior product (also called *progressive* in projective geometry) 'joins' or 'expands' the subspaces to form an upgraded subspace.

After introducing the exterior product, Grassmann [4] has also introduced the dot or inner product of two vectors, defined similarly as incorporated in ordinary vector algebra (VA) but, always to be carried out first in a sequence, i.e $\mathbf{u}\cdot\mathbf{v}\wedge\mathbf{w}\equiv(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$. For any arbitrary vector **v** and any bivector **B**: $\mathbf{v}.\mathbf{B}=v_i\,\hat{e}_i.\,B_{jk}\,\hat{e}_j\wedge\hat{e}_k=v_i\,B_{ik}\,\hat{e}_k-v_i\,B_{ji}\,\hat{e}_j=v_i(B_{ij}-B_{ji})\,\hat{e}_j\ (i\neq j),=-\mathbf{B}.\mathbf{v}=\mathbf{v}'$, a new vector. Also, $\mathbf{v}\wedge\mathbf{B}=v_i\,\hat{e}_i\wedge B_{jk}\,\hat{e}_j\wedge\hat{e}_k=v_i\,B_{jk}\,\hat{e}_i\wedge\hat{e}_j\wedge\hat{e}_k=\mathbf{B}\wedge\mathbf{v}$ – a trivector [2, 3]. The notion of inner product in exterior algebra will be discussed further subsequently.

By definition the wedge product implies closure property. For example, a bivector $\mathbf{v}_1\wedge\mathbf{v}_2$ in two dimensions and a trivector $\mathbf{v}_1\wedge\mathbf{v}_2\wedge\mathbf{v}_3$ in three dimensions, both having only one component each that flips sign under reflection, represent the highest grade element and pseudoscalar of the respective dimensions. But a trivector $\mathbf{v}_1\wedge\mathbf{v}_2\wedge\mathbf{v}_3$ in 2-D and a quadrivector $\mathbf{v}_1\wedge\mathbf{v}_2\wedge\mathbf{v}_3\wedge\mathbf{v}_4$ in 3-D collapse back down to scalar zero so as to prevent construction of any element of grade higher than the dimensionality of the space. The wedge product is, by definition, *associative* in the sense: $\mathbf{v}_1\wedge(\mathbf{v}_2\wedge\mathbf{v}_3)=(\mathbf{v}_1\wedge\mathbf{v}_2)\wedge\mathbf{v}_3=\mathbf{v}_1\wedge\mathbf{v}_2\wedge\mathbf{v}_3$.

In higher dimensions, one can accordingly define quadrivector, pentavector etc. by taking wedge product among four, five or more vectors and in general, with k ($<n$) vectors a 'k-blade' may be formed. Thus, each dimension is accordingly represented in exterior algebra. In the reduced form any k-fold ($k<n$) wedge product can be expressed as:

$$\begin{aligned}
\mathbf{v}_1\wedge\mathbf{v}_2\wedge\,...\,\wedge\mathbf{v}_k &= \sum_{i_1=1}^{n}v_{1i_1}\,\hat{e}_{i_1}\wedge\sum_{i_2=1}^{n}v_{2i_2}\,\hat{e}_{i_2}\,...\,\wedge\sum_{i_k=1}^{n}v_{ki_k}\,\hat{e}_{i_k}\\
&= \sum_{i_1<i_2\,...\,<i_k}(-1)^{\gamma}v_{1i_1}\,v_{2i_2}\,...\,v_{ki_k}\,\hat{e}_{i_1}\wedge\hat{e}_{i_2}\,...\,\wedge\hat{e}_{i_k},
\end{aligned}\tag{5}$$

where γ denotes the number of transpositions required to obtain $i_1 < i_2 ... < i_k$.

The wedge product thus, provides a natural extension to higher dimensions ($n > 3$) and the product of any number k ($< n$) of independent vectors is usually called simple multivector *blade* of grade k or a simple k-blade in short. It lives in a geometrical space known as the k-th *exterior power*. The magnitude of the resulting k-blade is the volume of the k-dimensional parallelotope (a generalisation of the parallelepiped in higher dimensions). According to this terminology for the elements of the algebra, a scalar is of grade zero, a vector has grade 1, bivector has grade 2 and a trivector is assigned grade 3 etc. In n dimensions, both vectors and $(n-1)$-blades have n components and the $(n-1)$-blades are called *antivectors* or *pseudovectors*.

Finally following eq.(5), the n-fold wedge product ($k = n$) in $\mathcal{V}^n$ is given by:

$$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge ... \wedge \mathbf{v}_n = \begin{vmatrix} v_{11} & v_{12} & ... & v_{1n} \\ v_{21} & v_{22} & ... & v_{2n} \\ ... & ... & ... & ... \\ v_{n1} & v_{n2} & ... & v_{nn} \end{vmatrix} \hat{e}_1 \wedge \hat{e}_2 ... \wedge \hat{e}_n \ . \tag{6}$$

The determinant (Δ), obtained from the n components of each of the n vectors gives the volume of the n-parallelotope in the n-dimensional vector space and the lone component of the highest n-grade element of this algebra. Evidently from the expression (eq.6), it flips sign under reflection – hence follows the name pseudoscalar (or antiscalar).

For a k-blade and an $(n-k)$-blade, the number of basis elements, given by the binomial coefficients $\binom{n}{k}$ and $\binom{n}{n-k}$ respectively, are the same. This produces an exact symmetry and the total number of 'multivector' basis elements is given by:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Whereas VA deals with scalars and vectors only, the wedge product expands the vector space with the subspaces of multivector blades of all grades from 0 to n which contain a total of 2^n basis elements. Thus, the grade structure of exterior algebra follows the pattern of so called Pascal's triangle. For any two blades $\mathbf{A}_p$ and $\mathbf{A}_r$, which are of grade p and r respectively, it simply follows that, the grade of $\mathbf{A}_p \wedge \mathbf{A}_r$ will be $p + r$ (unless the product happens to be zero, in which case its grade is zero). Finally, the further expanded *Clifford space* of GA using the geometric product contains all the 2^n basis elements plus any arbitrary linear sum over the basis elements, the so called clif or geometric.

Dot product of two vectors produces a scalar, and hence the name scalar product in VA. In tensor algebra, this product is termed as inner product or contraction which reduces the total rank by 2 and the product is not always a scalar. For the higher rank tensors, it is extended with the provision of multiple inner products or contractions, producing different lower rank tensors in addition to scalars. Similarly, the exterior algebra also creates the possibility of accommodating multiple inner products or contractions of two multivector blades. For example, with $\mathbf{A}_p$ and $\mathbf{A}_r$ ($r > p$) one can get p multivector blades of all grades from $r-p$ to $r+p-2$ in steps of $+2$. In fact, the geometric product between $\mathbf{A}_p$ and $\mathbf{A}_r$ is defined to contain all these (p) terms in addition to the highest grade ($p+r$) term from the wedge product. For $p = r$, the full contraction, producing the lowest 0 grade scalar element, is obviously symmetric. The symmetry of the successive higher blades alternate. Both exterior and inner products have definite symmetries. The symmetries depend on the grades (p and r) of the multiplicative blades and for inner products also on the number of contractions q (say). For example, $\mathbf{A}_p \wedge \mathbf{A}_r$ is antisymmetric if both are odd, otherwise it is symmetric. Finally, we note that the symmetry of both the inner products and the exterior product between $\mathbf{A}_p$ and $\mathbf{A}_r$ may be represented by the overall sign factor obtained by reversing (the operation *reversion* to be defined in the next section) the two multiplicand blades and the product blade of grade $(r+p-2q)$, given by: $(-1)^{r(r-1)/2}\,(-1)^{p(p-1)/2}\,(-1)^{(r+p-2q)(r+p-2q-1)/2}$. This sign factor also reproduce correctly the symmetry of two extreme cases: (i) with $q = 0$, wedge product of the two blades; and (ii) the symmetric zero (lowest) grade scalar with $p = r = q$. The results can be directly verified with the various products of the higher grade elements, derived explicitly following the multiplication rules of exterior algebra in [3]. The symmetry argument will be discussed again in connection with the geometric product.

Since the wedge product between a k- and an $(n-k)$-blade (in n dimensional space) is a pseudoscalar and the number of basis elements of the two blades are same, they constitute *dual* *form.* Inner product (contractions) with unit pseudoscalar I_n establishes a one-to-one mapping from the k-blade basis space to the $(n-k)$-blade basis space and vice-versa. Grassmann algebra thus introduces the important concept of dual form. Each element of the algebra has its dual. For example the dual of a vector (in n-dimension) is the $(n-1)$-blades – the wedge product of the rest $(n-1)$ vectors that are orthogonal to it. In 3-D, the duals of the three unit vectors $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ are respectively:

$I_3.\hat{e}_1 = \hat{e}_1\wedge\hat{e}_2\wedge\hat{e}_3.\hat{e}_1 = \hat{e}_2\wedge\hat{e}_3$, $I_3.\hat{e}_2 = \hat{e}_1\wedge\hat{e}_2\wedge\hat{e}_3.\hat{e}_2 = \hat{e}_3\wedge\hat{e}_1$ and $I_3.\hat{e}_3 = \hat{e}_1\wedge\hat{e}_2\wedge\hat{e}_3.\hat{e}_3 = \hat{e}_1\wedge\hat{e}_2$ – the three independent unit bivectors. Similarly, a k-blade has an $(n-k)$-blade as its dual. Pseudoscalar, the highest-grade element of the algebra i.e. the n-blade is the dual of the lowest zero-grade element scalar. The study of dual spaces states that the pseudoscalar plays the role that the scalar does in normal space. The dual of a scalar, which has no spatial extent, is the pseudoscalar, which has all the spatial extent (volume of the parallelotope in n-dimensions).

The computation of the square of the pseudoscalar I_n is given by: $I_n^2 = (\hat{e}_1 \wedge \hat{e}_2 \wedge \cdots \wedge \hat{e}_n)(\hat{e}_1 \wedge \hat{e}_2 \wedge \cdots \wedge \hat{e}_n)$, one can either reverse the order of the second group or apply a perfect shuffle, both require $(n-1)n/2$ swaps and yielding the sign factor $(-1)^{n(n-1)/2}$, which is 4-periodic, and combined with $\hat{e}_i \cdot \hat{e}_i$ the square is given by $I_n^2 = \pm 1$. So, I_n is invertible and the inverse $I_n^{-1} = \hat{e}_n \wedge \hat{e}_{n-1} \wedge \cdots \wedge \hat{e}_1 = (-1)^{(n-1)n/2} I_n = \pm I_n$ (the general definition of the inverse of an invertible r-blade will be discussed subsequently).

Physical quantities like angular velocity, angular momentum, torque in a force field and the magnetic induction field **B** at a point due to a current-element (given according to Biot-Savart law by the cross-product of the field producing vectorial current-element with the position vector of the point), usually represented by pseudovectors of classical Gibbs-Heaviside VA, are aptly replaced by bivectors defined in Grassman algebra [2]. Exterior algebra also clarifies the origin of imaginary i of complex numbers with the identification to unit bivector basis $e_i \wedge e_j$ and provides a representation of the Levi-Civita index $\epsilon_{ijk\ldots}$ by unit pseudoscalars of three and higher dimensions. It also introduces the notion of dual space and dual form with proper definition of the pseudoscalar, and above all, *forms the basis of GA.*

1.2 Quaternion algebra of Hamilton: While investigating a higher dimensional generalisation of the complex numbers, Hamilton in 1843 has developed the first noncommutative algebra – the quaternion algebra. In analogy with the imaginary i of two component complex number, he introduced the basic unit triplet q_1, q_2, q_3[2] – all square roots of -1, representing the set (S^3) of unit quaternions, to define the 4-component (a_0, a_1, a_2, a_3) quaternion a as:

$$a = a_0 + q_k a_k,\ k = 1, 2, 3 \tag{7}$$

and enunciated the fundamental equation (multiplication rule) of quaternion algebra:

$$q_k q_l = -\delta_{kl} + \epsilon_{klm} q_m \tag{8}$$

If a_0 is zero, then a is called a 'pure quaternion'.

The conjugate of a is defined as $a^* = a_0 - q_k a_k$ and from this definition we immediately have $(a^*)^* = a_0 - (-q_k a_k) = a$; $a_0 = (a + a^*)/2$, $q_k a_k = (a - a^*)/2$. Also, using the fundamental multiplication rule (eq.8), we get:

$a^* a = (a_0 - q_k a_k)(a_0 + q_l a_l) = a_0^2 + a_1^2 + a_2^2 + a_3^2 = a a^*$. The norm of a quaternion a, is the scalar denoted by $|a| = \sqrt{a^* a}$. A quaternion is called a unit quaternion if its norm is 1. The only quaternion with norm zero is zero, and every nonzero quaternion has a unique inverse. It implies that the quaternions form a division algebra. An algebra $\mathcal{A}$ is a division algebra if given $a, b \in \mathcal{A}$ with $ab = 0$, then either $a = 0$ or $b = 0$.[3] In ref. [2], the quaternion algebra described by eq.(8), is discussed in some details.

[2]This notation relates to traditional Hamilton's notation [7] as $q_1 = i$, $q_2 = j$, $q_3 = k$, where the fundamental quaternion equation reads: $i^2 = j^2 = k^2 = ijk = -1$.

[3]Of the four possible normed division algebras (real, complex, quaternion and octonion), GA provide a way to generalize the first three. The octonions being nonassociative [8, 9], the family of GA appears to diverge at the point of the octonions. Nonassociative algebras are also being used in some recent string theoretic models and in quantum systems with magnetic charges. Nonassociativity of the octonions, is closely related to triality. In mathematics, triality is a relationship among three vector spaces, analogous to the duality relation between dual vector spaces. Proper accommodation of this algebra in physical theory requires introduction of nonassociative star products [10].

Quaternions encode rotations by four real numbers only and in most applications the procedure is found to be more efficient than the conventional rotation matrix. In a sequence of rotations, interpolation with quaternionic representation is far more convenient than that with the familiar Euler angles. Quaternions are frequently used in computer graphics programming. In modern mathematics, as we will see, the algebra of quaternions belongs to a even subalgebra of GA, consisting of scalars, bivectors. A quaternion is easily identified with the geometric product (eq.1) of two 3-D vectors and the Eulerian form of the unit quaternion is called *rotor* in GA. Expressed as the sum of a scalar a_0 (the scalar product part) and the pure quaternion part $q_k a_k$, for which the wedge product (bivector $\mathbf{A}$, say) provides an appropriate representation. In fact, the similar algebraic properties of the pure imaginary i and the 'unit' bivector bases allow an algebraic isomorphism. It can be easily shown that this representation of the quaternion in GA is consistent with the quaternion algebra and correctly reproduce the product of two arbitrary quaternions. All these advantages, importance and limitations of quaternion algebra are also discussed in ref. [2].

2. Geometric algebra: We have already noted that, Grassmann algebra provides the basic framework for Clifford algebra. The inner and exterior products of two vectors complement each other: while the inner product lowers the grade, the other raises it, one is commutative and the other is anticommutative. However, they are not invertible in general. By defining the combined *geometric product* (eq.1), Clifford [11] has shown that the quaternion algebra is just a special case of Grassmann's "theory of extension" (Ausdehnungslehre). The product is associative and executing both lowering and raising of the grade simultaneously. It renders almost all elements to be invertible and the associativity and almost invertibility of the geometric product makes this algebra a formidable tool of mathematical physics.

Since the geometric product of two vectors, in general, produces a scalar and a bivector, it is not closed over the set of vectors but is closed over the larger graded ring of multivectors. The definitions of the inner, exterior and geometric products of two vectors are seamlessly extended in geometric algebra for all multivectors to form a closed algebra of multivectors. Also to start with, it stipulates that for a scalar s and a vector $\mathbf{v}$, the dot product $s.\mathbf{v} = \mathbf{v}.s = 0$ and the exterior product defining the scaling of a vector with the scalar $s \wedge \mathbf{v} = \mathbf{v} \wedge s = s\mathbf{v}$. So, the geometric product $s\mathbf{v} = \mathbf{v}s$ also produces the same scaling operation.

After a long time, both Pauli in the formulation of quantum mechanical spinor algebra and Dirac in his theory of the relativistic electron, though not being appreciated fully, have rediscovered Clifford algebra. It was Hestenes who demonstrated that both the Pauli and Dirac equations are indeed expressible in the language of Clifford algebra and finally realised its wider significance. Mathematically, a geometric algebra may be defined as the Clifford algebra of a vector space with a quadratic form. For a finite dimensional vector space $\mathcal{V}$ over a field $\mathcal{R}$ (or $\mathcal{C}$, real or complex), a quadratic form $\mathcal{Q}$ is a a homogeneous polynomial of degree two in a number of variables and defines a map $\mathcal{Q} : \mathcal{V} \to \mathcal{R}$, such that: $\mathcal{Q}(\lambda\mathbf{v}) = \lambda^2\mathcal{Q}(\mathbf{v})$; $\lambda \in \mathcal{R}$; $\mathbf{v} \in \mathcal{V}$. The term geometric algebra was used by Artin [12] while discussing algebras in association with a number of geometries, including Clifford algebra with the structure of symplectic and orthogonal groups. Clifford's geometric product, comprehensibly captured geometric relations between various objects, providing efficient solution to geometric problems on one hand and on the other facilitating the use of vectors $\mathbf{v}$ as elementary algebraic quantities – all common functions such as logarithms, trigonometric, exponential of $\mathbf{v}$ are now available! Finally, it was Hestenes who stressed the introduction of the term geometric algebra to highlight the importance of Clifford's contribution to the mathematical framework of physics and defines it in terms of the following axiomatic rules for the geometric product (eq.1) for vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$:

(i) associative : $(\mathbf{uv})\mathbf{w} = \mathbf{u}(\mathbf{vw})$,

(ii) left distributive : $\mathbf{u}(\mathbf{v} + \mathbf{w}) = \mathbf{uv} + \mathbf{uw}$,

(iii) right distributive : $(\mathbf{v} + \mathbf{w})\mathbf{u} = \mathbf{vu} + \mathbf{wu}$,

and the magnitude of $\mathbf{v}$, defined as: $\mathbf{vv} \equiv \mathbf{v}.\mathbf{v} = |\mathbf{v}|^2$, in terms of dot or scalar product, where $|v|$ is a positive scalar called the magnitude of $\mathbf{v}$ and $|v| = 0$ implies that $\mathbf{v} = 0$. Starting with the above postulates, Hestenes has shown the consistency with the inner and exterior products of Grassmann algebra using coordinate free approach [6]. Unlike vector algebra, GA naturally accommodates any number of dimensions and any quadratic form, such as one, compatible with the theory of relativity.

In GA, the basic geometric product has no definite symmetry in general and on reversing the order of multiplication, known as reversion, one gets: $\mathbf{vu} = \text{reverse}(\mathbf{uv}) = \tilde{\mathbf{uv}} = \tilde{\mathrm{C}}$. The inner and exterior products can be derived axiomatically as the symmetric and antisymmetric parts of the geometric product, given by, $\mathbf{u}.\mathbf{v} = \frac{\mathbf{uv}+\mathbf{vu}}{2}$, and $\mathbf{u} \wedge \mathbf{v} = \frac{\mathbf{uv}-\mathbf{vu}}{2}$. From the simple axioms of GA it follows that the symmetries of the two products between a bivector and a vector, on the other hand, are just the opposite: $\mathbf{B}.\mathbf{v} = \frac{\mathbf{Bv}-\mathbf{vB}}{2}$, and $\mathbf{B} \wedge \mathbf{v} = \frac{\mathbf{Bv}+\mathbf{vB}}{2}$. The definitions of the inner and exterior products are extended to the geometric product of a vector with a grade-r multivector blade $\mathbf{A}_r$ as, $\mathbf{vA}_r = \mathbf{v}.\mathbf{A}_r + \mathbf{v} \wedge \mathbf{A}_r$, where the inner product $\mathbf{v}.\mathbf{A}_r =< \mathbf{vA}_r >_{r-1}= \frac{1}{2}(\mathbf{vA}_r + (-1)^{r-1}\mathbf{A}_r\mathbf{v})$ lowers the grade of $\mathbf{A}_r$ by one and the exterior product $\mathbf{v} \wedge \mathbf{A}_r =< \mathbf{vA}_r >_{r+1}= \frac{1}{2}(\mathbf{vA}_r - (-1)^{r+1}\mathbf{A}_r\mathbf{v})$ raises the grade of $\mathbf{A}_r$ by one. The symmetries of the two products alternate for the odd or even grade blade $\mathbf{A}_r$.

On the other hand, appropriate multiple inner (dot) products between two higher grade elements of exterior algebra are possible and consequently, their geometric product contains more terms. In n-dimension, the geometric product of two blades of grade p and r, $r > p$, is defined to contain blades of all grades from $r - p$ to $r + p$ ($\leq n$) in steps of $+2$. For example, with two bivectors $\mathbf{B}$ and $\mathbf{B}'$ we get a three component clif (for $n \geq 4$):

$$\mathbf{BB}' =< \mathbf{BB}' >_0 + < \mathbf{BB}' >_2 + < \mathbf{BB}' >_4$$

Therefore, each of the terms on r.h.s. can not be expressed individually by either of the symmetric or the antisymmetric part of the geometric product. Manipulations are usually made by equating the terms of same grade, using the *grade preserving property* of the geometric product (to be discussed in the next subsection).

The last term of the above equation comes from the exterior (wedge) product of the two bivectors and Hestenes [13] denoted the first (scalar product, grade-0) term by $\mathbf{B}.\mathbf{B}'$ and the second (grade-2) term by the 'commutator product' $\mathbf{B} \times \mathbf{B}'$ – the antisymmetric part of $\mathbf{BB}'$. He also observed that the decomposition of $\mathbf{BB}'$ into terms of homogeneous grade cannot be expressed in terms of inner and exterior products alone without decomposing the bivectors into vectors. However, this is not true and with arbitrary (nonsimple) bivectors, the three terms in the above equation are manifestly denoted by [2, 3]:

$$\mathbf{BB}' =< \mathbf{BB}' >_0 + < \mathbf{BB}' >_2 + < \mathbf{BB}' >_4= \mathbf{B} : \mathbf{B}' + \mathbf{B}.\mathbf{B}' + \mathbf{B} \wedge \mathbf{B}' \tag{9}$$

Thus, using appropriate multiple inner products or contractions between two higher grade (r-blade) elements together with the exterior product, the geometric product can accordingly be expressed in terms of homogeneous grades even for nonsimple (which cannot be expressed as the wedge product of independent vector factors) element(s). Similarly, the geometric product of two trivectors $\mathbf{T}$ and $\mathbf{T}'$, containing four terms (for $n \geq 6$) can be written as:

$$\begin{aligned} \mathbf{TT}' &= < \mathbf{TT}' >_0 + < \mathbf{TT}' >_2 + < \mathbf{TT}' >_4 + < \mathbf{TT}' >_6 \\ &= \mathbf{T} :. \mathbf{T}' + \mathbf{T} : \mathbf{T}' + \mathbf{T}.\mathbf{T}' + \mathbf{T} \wedge \mathbf{T}' \end{aligned} \tag{10}$$

The different products on r.h.s. of equations (9) and (10) are explicitly derived in ref. [3]. In general, the geometric product of two r-blades $\mathbf{A}_r$ and $\mathbf{A}'_r$ contains $r + 1$ terms (in $n \geq 2r$) and is given by:

$$\begin{aligned} \mathbf{A}_r\mathbf{A}'_r &= < \mathbf{A}_r\mathbf{A}'_r >_0 + ... + < \mathbf{A}_r\mathbf{A}'_r >_{2r-4} + < \mathbf{A}_r\mathbf{A}'_r >_{2r-2} + < \mathbf{A}_r\mathbf{A}'_r >_{2r} \\ &= \mathbf{A}_r \,(r \,\text{dots})\, \mathbf{A}'_r + \; ... \; + \mathbf{A}_r : \mathbf{A}'_r + \mathbf{A}_r.\mathbf{A}'_r + \mathbf{A}_r \wedge \mathbf{A}'_r. \end{aligned} \tag{11}$$

With $r = 1, 2$ and 3, we get the results for vectors, bivectors and trivectors (equations (1), (9) and (10) respectively. It is to be noted here that, the geometric product of any simple r-blade (which is expressible as the wedge product of r independent vectors) with itself is given by the first term of eq.(11)) only, i.e. the scalar product representing the square magnitude of the simple r-blade, since all other higher grade terms are identically zero. We have also noted earlier that, the geometric product of $\mathbf{A}_p$ and $\mathbf{A}_r$, ($p \leq r$, $r + p \leq n$) contains ($p + 1$) terms (blades) of all grades from $r - p$ to $r + p$ in steps of $+2$ and may be written as: $\mathbf{A}_p\mathbf{A}_r = \mathbf{A}_p \,(p \,\text{dots})\, \mathbf{A}_r + ... + \mathbf{A}_p : \mathbf{A}_r + \mathbf{A}_p.\mathbf{A}_r + \mathbf{A}_p \wedge \mathbf{A}_r$. The definite symmetry (or antisymmetry) of each grade ($p + r - 2q$, $q = 0, 1, \ldots, p$) is represented by the sign factor $(-1)^{p(p-1)/2}\,(-1)^{r(r-1)/2}\,(-1)^{(p+r-2q)(p+r-2q-1)/2}$.

In fact dispensing with the Descartes coordinate system, GA creates great flexibility and consistently generalises to arbitrary n-dimensional space. The coordinate free formulation renders the equations of physics invariant under coordinate transformations, rather than covariant as in the tensorial form. However, one can still recourse to the coordinate system to advantage, when required.

For example, a term by term equality between the product of two quaternions and its analogue in GA can be easily derived using the component representation. Also, the component representation is often much economic than the lengthy verification of various properties and identities in the coordinate free approach [3]. Moreover, various arbitrary definitions are used in the literature to generalise the inner product of vectors for higher grade multivector blades, which are rather confusing. Also, all these try to capture the idea of full contraction (producing the lowest-grade piece of the geometric product) only. One can, however, properly account for the whole gamut of various (multiple) inner products between two blades of different grades by checking the appropriate symmetries of the relevant exterior and inner products [3]. For nonsimple multivector blades, these expressions appear to be convenient and useful.

One may generate a finite-dimensional GA, by choosing a unit pseudoscalar (I_n). The set of all vectors that satisfy $\mathbf{v} \wedge I_n = 0$, forms the vector space. The geometric product of the vectors in this vector space then defines the GA, of which I_n is a member. Since every finite-dimensional GA has a unique I_n (up to a sign), one can define or characterize the GA by it. A pseudoscalar can be interpreted as the volume of an n-parallelotope in an n-dimensional vector space. From the closure property, it follows that $I_n\mathbf{A}_r \equiv I_n\,(r\text{ dots})\,\mathbf{A}_r$. In spaces of odd dimension, the (unit) pseudoscalar I_n commutes with all vectors and so, with all the multivecor blades $\mathbf{A}_r$. Therefore, in the case of odd dimension, the subalgebras of scalars and pseudoscalars consist of elements of the algebra which commute with every element in it. In even dimensional spaces, on the other hand, I_n commutes with all even grade blades and anti commutes with all vectors and with all odd grade blades. Thus, GA makes a sort of distinction with respect to the dimension of the associated vector space. In general, the result may be expressed as: $I_n\mathbf{A}_r = (-1)^{r(n-1)}\mathbf{A}_r I_n$ and used to interchange the inner and exterior products. For example:

$$\begin{aligned}
\mathbf{v}.(\mathbf{A}_r I_n) &= \frac{1}{2}(\mathbf{v}\mathbf{A}_r I_n + (-1)^{n-r-1}\mathbf{A}_r I_n \mathbf{v}) \\
&= \frac{1}{2}(\mathbf{v}\mathbf{A}_r I_n + (-1)^{n-r-1}(-1)^{n-1}\mathbf{A}_r \mathbf{v} I_n) \\
&= \frac{1}{2}(\mathbf{v}\mathbf{A}_r - (-1)^{r+1}\mathbf{A}_r \mathbf{v}) I_n \\
&= \mathbf{v} \wedge \mathbf{A}_r I_n
\end{aligned} \tag{12}$$

This result (eq.12) indicates a sort of duality between the inner and exterior products. An important application of the result is found in defining $\hat{e}^j = (-1)^{j-1} I_{n,\breve{j}}\, I_n^{-1}$, ($I_{n,\breve{j}}$ means $\hat{e}_j$ is excluded from I_n), as the reciprocal set basis element for the set of n orthonormal basis vectors $\{\hat{e}_i\}$, such that $\hat{e}_i.\hat{e}^j = \hat{e}_i.((-1)^{j-1} I_{n,\breve{j}}\, I_n^{-1}) = (-1)^{j-1}(\hat{e}_i \wedge I_{n,\breve{j}})\, I_n^{-1} = I_n\, I_n^{-1}\delta_i^j = \delta_i^j$. Manipulations using geometric product in intermediate steps brings great simplicity in working out various problems. For example, eq.(12) can be written in the general form for the two blades $\mathbf{A}_p$ and $\mathbf{A}_r$ $p + r \leq n$, interchanging the full contractions to exterior product: $\mathbf{A}_p$ (p dots) $(\mathbf{A}_r I_n) = \mathbf{A}_p \wedge \mathbf{A}_r I_n$ [14].

Another instructive example may be found in the Gram-Schmidt orthogonalisation procedure. From an arbitrary set of r linearly independent vectors $\mathbf{u}_i,\ i = 1, 2, ..., r,\ r \leq n$, let us construct a graded sequence of r multivector blades: $\mathbf{A}_0 = 1,\ \mathbf{A}_1 = \mathbf{u}_1,\ \mathbf{A}_2 = \mathbf{u}_1 \wedge \mathbf{u}_2,\ ...\ ,\ \mathbf{A}_r = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge ...\ \mathbf{u}_r$. Next we define a new set of vectors $\mathbf{v}_j,\ j = 1, 2, ..., r$: $\mathbf{v}_j = \tilde{\mathbf{A}}_{j-1}\mathbf{A}_j$, where $\tilde{\mathbf{A}}_{j-1}$ is the reverse of $\mathbf{A}_{j-1}$ (see subsection 2.1). Since, the subspace defined by the $(j-1)$ vectors of $\mathbf{A}_{j-1}$ is, by construction, fully contained in the subspace of the (j) vectors of $\mathbf{A}_j$, the geometric product $\tilde{\mathbf{A}}_{j-1}\,\mathbf{A}_j$ contains only the full contraction term, all other terms being identically zero. The full contraction $(j - (j-1) = 1)$ results in the grade one element $\mathbf{v}_j$, which is contained in the subspace of $\mathbf{A}_j$ and orthogonal to the subspace of $\mathbf{A}_{j-1}$. Therefore, the new set of r vectors is an orthogonal set spanning the r dimensional subspace of the n dimensional vector space. This procedure according to geometric algebra [15], fully corresponds to the conventional Gram-Schmidt orthogonalisation process of linear algebra.

Both free and bound vectors are appropriately described in GA using multivectors. Vectors having both magnitude and direction, are usually represented in Gibbs' vector algebra by directed line segments through the origin of a coordinate system i.e. as bound or position vectors. The generalization to include vectors that are not acting through the origin is achieved in 3-D VA by designating a set of six coordinates (of which only four are independent) – known as Plücker coordinates. The extra coordinates represent the offset vector from the origin. This is more conveniently handled in GA through the use of a vector plus bivector clif: $\mathbf{v} = \mathbf{u} + \mathbf{u} \wedge \mathbf{r}$, where the direction of the free vector is that of a usual position vector $\mathbf{u}$ and $\mathbf{r}$ is the offset of this vector from the origin. If the product $\mathbf{u} \wedge \mathbf{r} = 0$, the vectors are parallel implying that $\mathbf{v}$ passes through the origin and we are reverting to a pure position vector. Hence, this naturally generalizes the concept of vectors.

2.1 Some useful operations, properties and theorems of Geometric Algebra: In addition to the geometric product, we have already noted that GA also defines and uses a several unary operations, properties and theorems which are useful in various evaluations and manipulations. These also reveal the richness of the structure of GA and are appended as follows:

(i) **Reversion and Inversion**: The reverse of an arbitrary blade consists in reversing the order of the factors of the blade. For instance, the reverse of a blade $\mathbf{u} \wedge \mathbf{v}$ is $\mathbf{v} \wedge \mathbf{u}$. In general, the reverse of a blade $\mathbf{A}_r$ of grade r is defined as: $\tilde{\mathbf{A}}_r = (-1)^{r(r-1)/2}\mathbf{A}_r$. The reversion only changes the orientation of a blade according to the change of its sign. The positive scalar magnitude of a blade $\mathbf{A}_r$ is given by: $|\mathbf{A}_r| = \sqrt{|\mathbf{A}_r^2|} = \sqrt{\tilde{\mathbf{A}}_r\mathbf{A}_r} = \sqrt{\mathbf{A}_r\tilde{\mathbf{A}}_r}$. Consequently, the inverse $\mathbf{A}_r^{-1}$ of an invertible blade $\mathbf{A}_r$ ($|\mathbf{A}_r| \neq 0$) is defined as: $\mathbf{A}_r^{-1} = |\mathbf{A}_r|^{-2}\tilde{\mathbf{A}}_r \Rightarrow \mathbf{A}_r^{-1}\mathbf{A}_r = \mathbf{A}_r\mathbf{A}_r^{-1} = 1$.

(ii) **Grade selection and grade preserving property**: For the expression $\mathrm{C} = \mathrm{s} + \mathbf{v} + \mathbf{B} + \mathbf{T}$, where s is a scalar, $\mathbf{v}$ a vector, $\mathbf{B}$ a bivector, and $\mathbf{T}$ a trivector, C is referred to as a clif and the terms on the r.h.s. are referred to as multivectors of definite grade from 0 to 3 respectively. In GA, a clif or a general multivector is defined to be the sum of blades A_r of different grade r , such as: $\mathrm{C} = \Sigma\mathrm{A_r}$. The grade selection rule implies, as indicated in equations from (9) to (11), that $< \mathrm{C} >_0 = \mathrm{s}$, $< \mathrm{C} >_1 = \mathbf{v}$, ... $< \mathrm{C} >_\mathrm{r} = \mathbf{A}_\mathrm{r}$ etc. Consequently, $< s\mathrm{C} >_\mathrm{r} = \mathrm{s} < \mathrm{C} >_\mathrm{r}$ and for two clifs C and C$'$, $\mathrm{C} + \mathrm{C}' (= \Sigma_\mathrm{r} < \mathrm{C} >_\mathrm{r} + < \mathrm{C}' >_\mathrm{r}$, i.e. $< \mathrm{C} + \mathrm{C}' >_\mathrm{r} = < \mathrm{C} >_\mathrm{r} + < \mathrm{C}' >_\mathrm{r})$ contains terms of all grades contained in the two clifs. Obviously an identity between two clifs implies that both contain terms of same grades and $< \mathrm{C} >_\mathrm{r} = < \mathrm{C}' >_\mathrm{r}$ for all r [3]. This important rule allows great maneuvering flexibility in calculations.

(iii) **Grade involution**: The grade involution toggles the orientation of a blade if its grade is odd and is defined as: $^\wedge(\mathbf{A}_r) = (-1)^r\mathbf{A}_r$. Involution occurs when something turns in upon itself. Two successive operations produce identity. The reversion is also often called an antiinvolution: it is anti since it reverses order of the vector factors and it is involution since its two successive operations produce identity.

(iv) **Conjugation**: The conjugate of a generic blade $\mathbf{A}_r$ is defined as: $\mathbf{A}_r^* = {}^\wedge(\tilde{\mathbf{A}}_r) = (-1)^r\tilde{\mathbf{A}}_r = (-1)^{r(r+1)/2}\mathbf{A}_r$. Since both involution and reversion returns the original blade on two successive operations, it is also true for conjugation.

(v) **Duality and dualisation**: The dualisation of a generic blade $\mathbf{A}_r$ consists in taking the orthogonal complement of the blade and is defined as: $\mathbf{A}_r^\dagger = I_n\mathbf{A}_r (\equiv I_n\,(\text{r dots})\,\mathbf{A}_r)$, where I_n is the blade with respect to which the dualisation is performed and is usually the unit pseudoscalar of the space. In fact, $\mathbf{A}_r^\dagger$ is the part of the space that is not contained in $\mathbf{A}_r$ and obviously, $\mathbf{A}_r = \tilde{I}_n\mathbf{A}_r^\dagger$. The *dual of a clif* is simply the geometric product of the clif with the unit pseudoscalar.

(vi) **Meet operator**: GA also introduces the *meet* operator as opposed to the joining of (two) subspaces in exterior or progressive product. It extracts the smallest common subspaces of blades $\mathbf{A}_p$ and $\mathbf{A}_r$ and is denoted by the $\vee$ symbol as follows: $\mathbf{A}_p \vee \mathbf{A}_r = < \mathbf{A}_p^\dagger\mathbf{A}_r >_{r-p}$, i.e. the full contraction of the dual of $\mathbf{A}_p$ with $\mathbf{A}_r$. Quite reasonably, the meet operation is termed as regressive product and denoted by the $\vee$ symbol. For two bivectors, it can be easily verified that the regressive product creates the vector that lies on the intersection of the two bivectors. With the notions of duality and the progressive and regressive products, geometric algebra is ideally suited to the study of projective geometry.

(vii) **Periodicity theorem**: An n dimensional real vector space $\mathcal{V}$ with a quadratic form having its signature pair of integers (p, r), where $n = p + r$, is denoted by $\mathcal{V}^{p,r}$ in GA. The algebra on $\mathcal{V}^{p,r}$ is conventionally denoted by $G^{p,r}(\mathcal{V})$. A standard orthonormal basis set $\{e_i\}$ for $\mathcal{V}^{p,r}$ consists of n mutually orthogonal vectors, p of which have norm $+1$ and r of which have norm -1.

Algebras over vector spaces $\mathcal{V}$ (over real or complex fields) are classified according to the dimensions and also by the associated quadratic forms. As an outcome of this classification, every geometric algebra can be decomposed into graded indecomposable factors (algebras). This decomposition is the origin of periodicity theorems and GA exhibit a 8-fold periodicity over the real numbers and an 2-fold periodicity over the complex numbers [16].

2.2 Basic elements of GA and representations of physical objects and operations: The geometric approach considers the basic elements vectors, bivectors etc. as objects with geometric properties, independent of any basis and the algebraic properties of GA are as simple as those of

Euclidean lines, planes and higher dimensional (hyper)surfaces. Although a comparison of elements of different grades is not possible, GA admits, via the geometric product, addition of elements of different grades to form multivector clifs. For example, it can have an element C such that: $\mathrm{C} = \mathrm{s} + \mathbf{v} + \mathbf{B} + \mathbf{T}$, where s is a scalar, $\mathbf{v}$, a vector, $\mathbf{B}$, a bivector and $\mathbf{T}$ is a trivector. This clearly sets GA apart from ordinary algebra. We have already noted that quaternions or spinors are multivector clifs of even subalgebra, formed by the geometric product of two vectors. Hestenes [13] has also shown that the electromagnetic field is best represented by a clif consisting of a vector and a bivector.

Another interesting example may be found in describing the instantaneous particle trajectory in three dimensional space by a multivector clif containing the path length, the instantaneous direction, the curvature and the torsion of the space curve. Denoting the distance along the curve (the path length) by a scalar s and the instantaneous direction by the unit tangent $\hat{\mathbf{u}}$, the curvature (deviation from the straight line) and the torsion (deviation from a plane) may be expressed [17] in terms of a bivector and a trivector formed by $\hat{\mathbf{u}}$, $\hat{\mathbf{n}}$ (unit normal) and $\hat{\mathbf{u}}$, $\hat{\mathbf{n}}$ and unit binormal $\hat{\mathbf{b}}$ (the orthonormal triad defined according to the Frenet-Serret formulae of classical 3-D analytic geometry) respectively. Finally, the trajectory may be represented by the multivector clif: $\mathrm{S} = \mathrm{s} + \hat{\mathbf{u}} + \kappa(\hat{\mathbf{u}} \wedge \hat{\mathbf{n}}) + \kappa^2 \tau(\hat{\mathbf{u}} \wedge \hat{\mathbf{n}} \wedge \hat{\mathbf{b}})$, where κ and τ represent the Frenet-Serret magnitudes of curvature and torsion respectively. With the addition of higher order curvatures, this can be easily extended to higher dimensions.

Elements of GA represent both physical objects and operations and this algebra provides evocative names for its elements (versor, paravector, boomerang etc.). A versor actually refers to a monomial, a geometric product of invertible vectors ($\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$), for example $\mathbf{u}$, $\mathbf{uv}$, $\mathbf{uvw}$ etc. The geometric product of r vectors is called a versor of order r – it is called even (rotor, or spinor) or odd depending upon whether r is even or odd. Versors are operators of GA, which translate, reflect, rotate, dilate, twist and boost other elements and objects of the algebra. In the following, we discuss some of these operations.

Projections, rejections, reflections along vectors, and rotations in planes – all these operations are handled much more efficiently in geometric algebra compared to traditional vector and matrix algebras. Using an invertible vector $\mathbf{u}$ ($\mathbf{u}^{-1} = |\mathbf{u}|^{-2}\,\mathbf{u}$), one can construct the identity $\mathbf{v} = \mathbf{vuu}^{-1}$, and finally write: $\mathbf{v} = (\mathbf{v}.\mathbf{u} + \mathbf{v} \wedge \mathbf{u})\mathbf{u}^{-1} \equiv P_{\mathbf{u}}(\mathbf{v}) + R_{\mathbf{u}}(\mathbf{v})$. Since $P_{\mathbf{u}}(\mathbf{v}) \wedge \mathbf{u} = 0$ and $R_{\mathbf{u}}(\mathbf{v}).\mathbf{u} = 0$, $\mathbf{v}.\mathbf{uu}^{-1}$ can be identified as the orthogonal projection $P_{\mathbf{u}}(\mathbf{v})$ of $\mathbf{v}$ along $\mathbf{u}$ (i.e. its parallel part $\mathbf{v}_{\|\mathbf{u}}$) and $\mathbf{v} \wedge \mathbf{uu}^{-1}$ as the orthogonal rejection $R_{\mathbf{u}}(\mathbf{v})$ of $\mathbf{v}$ from $\mathbf{u}$ (the part $\mathbf{v}_{\perp\mathbf{u}}$, perpendicular to $\mathbf{u}$ and lies in $(\mathbf{u}, \mathbf{v})$ plane). A sort of scaling, with respect to an invertible vector, may be defined using the following combination of projection and rejection: $(\mathbf{v}; \mathbf{u}; \xi) = P_{\mathbf{u}}(\mathbf{v}) + \xi\, R_{\mathbf{u}}(\mathbf{v})$ which scales $\mathbf{v}$ by a factor ξ (a scalar) with respect to $\mathbf{u}$. For $\xi = 1$, it gives $\mathbf{v}$. The above equation can also be written as: $(\mathbf{v}; \mathbf{u}; \xi) = \mathbf{v} + (\xi + 1)(\mathbf{u} \wedge \mathbf{v})\mathbf{u}^{-1} = \xi\, \mathbf{v} + (1 - \xi)(\mathbf{v}.\mathbf{u})\mathbf{u}^{-1}$.

In place of $\mathbf{u}$, using an invertible simple bivector $\mathbf{B}$ representing a oriented plane, again from the identity $\mathbf{v} = \mathbf{vBB}^{-1}$, projection and rejection of the vector $\mathbf{v}$, on and from the plane respectively, can be similarly defined as: $\mathbf{v} = \mathbf{v}.\mathbf{BB}^{-1} + \mathbf{v} \wedge \mathbf{BB}^{-1} = P_{\mathbf{B}}(\mathbf{v}) + R_{\mathbf{B}}(\mathbf{v})$. This result can be similarly extended for any invertible multivector blade $\mathbf{A}_r$ in projecting a vector into the multivector subspace. The idea is to express a vector as a sum of two terms, one in the subspace of $\mathbf{A}_r$ and another in its orthogonal complement (dual) subspace of $\mathbf{A}_r^{\dagger}$: $\mathbf{v} = (\mathbf{v}.\mathbf{A}_r + \mathbf{v} \wedge \mathbf{A}_r)\mathbf{A}_r^{-1} = P_{\mathbf{A}_r}(\mathbf{v}) + R_{\mathbf{A}_r}(\mathbf{v})$. Since $P_{\mathbf{A}_r}(\mathbf{v}) \wedge \mathbf{A}_r = 0$ and $R_{\mathbf{A}_r}(\mathbf{v}).\mathbf{A}_r = 0$; $P_{\mathbf{A}_r}(\mathbf{v})$ lies in $\mathbf{A}_r$ and $R_{\mathbf{A}_r}(\mathbf{v})$ is orthogonal to $\mathbf{A}_r$ and lies in the orthogonal complement $\mathbf{A}_r^{\dagger}$ of $\mathbf{A}_r$. It is also interesting to note that:
$R_{\mathbf{A}_r}(\mathbf{v}) = \mathbf{v} \wedge \mathbf{A}_r\mathbf{A}_r^{-1} = \mathbf{v} \wedge \mathbf{A}_r I_n^{-1} I_n \mathbf{A}_r^{-1} = \mathbf{v}.(\mathbf{A}_r I_n^{-1}) I_n \mathbf{A}_r^{-1}$, according to eq.(12);
$= \mathbf{v}.\mathbf{A}_r^{\dagger}\,(\mathbf{A}_r^{\dagger})^{-1} = P_{\mathbf{A}_r^{\dagger}}(\mathbf{v})$, i.e. the orthogonal projection of $\mathbf{v}$ in $\mathbf{A}_r^{\dagger}$ is equal to the orthogonal rejection from $\mathbf{A}_r$. Also, it is obvious that both projection and rejection leave the scalars untouched: $P_{\mathbf{A}_r}(\lambda) = \lambda = R_{\mathbf{A}_r}(\lambda)$. One can similarly project a multivector into a subspace. For any simple p-blade $\mathbf{A}_p$, rewriting the identity $\mathbf{A}_p = \mathbf{A}_p(\mathbf{A}_r\mathbf{A}_r^{-1})$ using the associative property of geometric product, as: $\mathbf{A}_p = (\mathbf{A}_p\mathbf{A}_r)\mathbf{A}_r^{-1}$, where $\mathbf{A}_r$ is an invertible blade ($r > p,\ r + p \leq n$). It can be easily verified that only two terms of the final geometric product survive and we get:
$\mathbf{A}_p = (< \mathbf{A}_p\mathbf{A}_r >_{r-p} + < \mathbf{A}_p\mathbf{A}_r >_{r+p})\mathbf{A}_r^{-1} = (\mathbf{A}_p\ (\text{p dots})\ \mathbf{A}_r + \mathbf{A}_p \wedge \mathbf{A}_r)\mathbf{A}_r^{-1} = P_{\mathbf{A}_r}(\mathbf{A}_p) + R_{\mathbf{A}_r}(\mathbf{A}_p)$.

Now, let us consider the vector:
$$\begin{aligned} \mathbf{v}' &= -P_{\mathbf{u}}(\mathbf{v}) + R_{\mathbf{u}}(\mathbf{v}) = -\mathbf{v}_{\|\mathbf{u}} + \mathbf{v}_{\perp\mathbf{u}} \\ &= -(\mathbf{v}.\mathbf{u})\mathbf{u}^{-1} + (\mathbf{v} \wedge \mathbf{u})\mathbf{u}^{-1} = -(\mathbf{u}.\mathbf{v})\mathbf{u}^{-1} - (\mathbf{u} \wedge \mathbf{v})\mathbf{u}^{-1} \\ &= -\mathbf{u}\,\mathbf{v}\,\mathbf{u}^{-1}, \end{aligned} \tag{13}$$

obtained by reverting the component $\mathbf{v}_{\|\mathbf{u}}$ ($P_{\mathbf{u}}(\mathbf{v})$) of the vector $\mathbf{v}$ parallel to $\mathbf{u}$ and leaving $\mathbf{v}_{\perp\mathbf{u}}$ ($R_{\mathbf{u}}(\mathbf{v})$)

unaltered. Since $\mathbf{v}'^2 = \mathbf{u}\mathbf{v}\mathbf{u}^{-1}\mathbf{u}\mathbf{v}\mathbf{u}^{-1} = \mathbf{v}^2$, the operation preserves the magnitude and represents simple reflection of the vector $\mathbf{v}$ along $\mathbf{u}$ (or equivalently in the plane orthogonal to $\mathbf{u}$). Projection and rejection along a direction ($\mathbf{u}$) thus, defines reflection of $\mathbf{v}$ along that direction. Reflection from the plane $\mathbf{B}$, on the other hand, is obtained by reverting $R_{\mathbf{B}}(\mathbf{v})$ (or $\mathbf{v}_{\perp\mathbf{B}}$) and keeping $P_{\mathbf{B}}(\mathbf{v})$ (or $\mathbf{v}_{\|\mathbf{B}}$) unchanged and is expressed as: $\mathbf{v}' = P_{\mathbf{B}}(\mathbf{v}) - R_{\mathbf{B}}(\mathbf{v}) = \mathbf{v}.\mathbf{B}\mathbf{B}^{-1} - \mathbf{v}\wedge\mathbf{B}\mathbf{B}^{-1} = -\mathbf{B}\mathbf{v}\mathbf{B}^{-1}$. The general result for any invertible blade $\mathbf{A}_r$ may be accordingly expressed as: $\mathbf{v}' = \mp\mathbf{A}_r\mathbf{v}\mathbf{A}_r^{-1}$, the sign $\mp$ depends on the grade r of the reflecting blade.

Reflection is generally treated using reflection matrix and matrix algebra. However, the simplicity and power of the approach following GA is amply evident in the description of more general, composite reflections in higher dimensions ($n \geq 4$) [3]. Interestingly, replacing the rays by unit vectors and the vertex-ray interface interaction producing reflection and refraction of the incident ray by rotation-like operators, Sugon et al [18] have reformulated the laws of geometric optics using GA.

Following eq.(13), next consider a successive second reflection of $\mathbf{v}$ along $\mathbf{w}$. This may be expressed as:

$$\begin{aligned}\mathbf{v}'' &= -\mathbf{w}\,\mathbf{v}'\,\mathbf{w}^{-1} = -\mathbf{w}(-\mathbf{u}\,\mathbf{v}\,\mathbf{u}^{-1})\mathbf{w}^{-1} = \mathbf{w}\,\mathbf{u}\,\mathbf{v}\,\mathbf{u}^{-1}\,\mathbf{w}^{-1}\\ &= (\mathbf{w}\,\mathbf{u})\mathbf{v}(\mathbf{w}\,\mathbf{u})^{-1},\ \text{with } \mathbf{w}\mathbf{u} = \mathbf{w}.\mathbf{u} + \mathbf{w}\wedge\mathbf{u}. \qquad (14)\end{aligned}$$

The combined operation also preserves the magnitude of $\mathbf{v}$ ($\mathbf{v}^2 = \mathbf{v}''^2$) and for two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, it can be readily shown that $\mathbf{v}_1.\mathbf{v}_2 = \mathbf{v}_1''.\mathbf{v}_2''$. The transformation thus preserves both length and angle and must be a pure rotation. The geometric product of the two invertible vectors, represents normalisable elements of an even subalgebra (isomorphic with quaternion in 3-D). In fact, the normalized geometric product $\mathrm{R} = \hat{\mathbf{w}}\hat{\mathbf{u}}$, called *rotor*, is the generator of pure rotation and since $\mathrm{R}^{-1} = \tilde{\mathrm{R}} = \hat{\mathbf{u}}\hat{\mathbf{w}}$, eq.(14) may be rewritten as: $\mathbf{v}'' = \mathrm{R}\,\mathbf{v}\,\tilde{\mathrm{R}}$. Also with $\mathrm{R}' = \mathbf{w}\mathbf{u}$, the transformation $\mathrm{R}'\,\mathbf{v}\,\tilde{\mathrm{R}}'$ produces, like quaternions, both rotation and dilation. Magnitude of bivectors in an Euclidean space are characterized by square root of a negative number and using the generalisation of the concept of exponential function of multivectors introduced by Hestenes [13], the rotors can be represented by elliptic functions as:

$$\mathrm{R} = \exp(\frac{\hat{\mathbf{B}}|\theta|}{2}) = \cos(\frac{|\theta|}{2}) + \hat{\mathbf{B}}\,\sin(\frac{|\theta|}{2}), \qquad (15)$$

with, $\cos(\frac{|\theta|}{2}) = \hat{\mathbf{w}}.\hat{\mathbf{u}}$, $\sin(\frac{|\theta|}{2}) = |\hat{\mathbf{w}}\wedge\hat{\mathbf{u}}|$ and $\hat{\mathbf{B}} = \frac{\hat{\mathbf{w}}\wedge\hat{\mathbf{u}}}{|\hat{\mathbf{w}}\wedge\hat{\mathbf{u}}|}$. The rotor expressed as the exponential of the bivector $\mathbf{B}$, generates a rotation through the bilinear expression of eq.(14). The bivector encodes both the the direction (defined by the oriented plane of $\hat{\mathbf{B}}$) and the magnitude (the angle of rotation θ – twice the angle between $\hat{\mathbf{w}}$ and $\hat{\mathbf{u}}$ and provides unambiguous specification of rotation in any dimension. The half-angle appears in the expression of the rotor is due to the bilinear form of the operation. Although it is less evident in lower dimensions, the bilinear transformation is much easier to handle than the one-sided rotation matrix. As the number of dimensions increases, the latter becomes very complicated.

One can also apply a rotor on a subspace. This will produce rotations of all the individual vectors of the corresponding multivector blade by the rotor. For example, if we take a r-blade $\mathbf{A}_r = \mathbf{v}_1\wedge\mathbf{v}_2\wedge ...\wedge\mathbf{v}_r$, then $(\mathbf{v}_1\wedge\mathbf{v}_2\wedge\ ...\ \wedge\mathbf{v}_r)'' = \mathrm{R}\,\mathbf{v}_1\wedge\mathbf{v}_2\wedge\ ...\ \wedge\mathbf{v}_\mathrm{r}\,\tilde{\mathrm{R}} =<\mathrm{R}\,\mathbf{v}_1\,\mathbf{v}_2\,...\,\mathbf{v}_\mathrm{r}\,\tilde{\mathrm{R}}>_\mathrm{r}$ $=<\mathrm{R}\,\mathbf{v}_1\,\tilde{\mathrm{R}}\mathrm{R}\,\mathbf{v}_2\,\tilde{\mathrm{R}}\mathrm{R}\,...\,\tilde{\mathrm{R}}\mathrm{R}\,\mathbf{v}_\mathrm{r}\,\tilde{\mathrm{R}}>_\mathrm{r} = \mathrm{R}\,\mathbf{v}_1\,\tilde{\mathrm{R}}\wedge\mathrm{R}\,\mathbf{v}_2\,\tilde{\mathrm{R}}\wedge\ ...\ \wedge\mathrm{R}\mathbf{v}_\mathrm{r}\,\tilde{\mathrm{R}} = \mathbf{v}_1''\wedge\mathbf{v}_2''...\wedge\mathbf{v}_\mathrm{r}''$, since the bilinear operation with the rotor is grade preserving.

Composition of rotations simply corresponds to multiplication of rotors. For example, if we rotate a cube in yz-plane through $\pi/2$ first and subsequently in zx-plane, again through $\pi/2$, applying rotor the equivalent single rotation can be found easily. Let $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ be the orthonormal Cartesian basis vectors and the product rotor for the combined rotation is given by:

$$\begin{aligned}\mathrm{R} &= \mathrm{R}_1\mathrm{R}_2 = \exp(\hat{\mathrm{e}}_3\wedge\hat{\mathrm{e}}_1\frac{\pi}{4})\exp(\hat{\mathrm{e}}_2\wedge\hat{\mathrm{e}}_3\frac{\pi}{4})\\ &= \{\cos(\frac{\pi}{4}) + \hat{e}_3\wedge\hat{e}_1\,\sin(\frac{\pi}{4})\}\{\cos(\frac{\pi}{4}) + \hat{e}_2\wedge\hat{e}_3\,\sin(\frac{\pi}{4})\}\\ &= \frac{1}{2}\{(1+\hat{e}_3\wedge\hat{e}_1)(1+\hat{e}_2\wedge\hat{e}_3)\} = \frac{1}{2} + \frac{1}{2}(\hat{e}_2\wedge\hat{e}_3 + \hat{e}_3\wedge\hat{e}_1 + \hat{e}_1\wedge\hat{e}_2)\\ &= \cos(\frac{\pi}{3}) + \frac{1}{\sqrt{3}}(\hat{e}_2\wedge\hat{e}_3 + \hat{e}_3\wedge\hat{e}_1 + \hat{e}_1\wedge\hat{e}_2)\sin(\frac{\pi}{3})\end{aligned}$$

Therefore, the combined rotation is in the (1, 1, 1) plane of the Cartesian system and the angle of rotation is $\frac{2\pi}{3}$.

For the product rotor $\mathrm{R} = \mathrm{R}_2\mathrm{R}_1$, $\mathrm{R}\tilde{\mathrm{R}} = \mathrm{R}_2\mathrm{R}_1\tilde{\mathrm{R}}_1\tilde{\mathrm{R}}_2 = 1$. Thus, rotors form a group and multiple rotors compose single-sidedly. Also for any vector $\mathbf{v}$, if we make a rotation of 2π, $\mathbf{v}'$ returns to $\mathbf{v}$. What happens to R is surprising; using (15) above, we see that $\mathrm{R} \to -\mathrm{R}$. Nothing quantum-mechanical is invoked here and it indicates that spinors share identical algebra with rotors. It turns out that it is possible to represent a Pauli spinor by rotors or more specifically by an arbitrary even element in the geometric algebra of 3-space. In fact, spinors may be regarded as nonnormalised rotors in GA – more importantly GA provides an explicit construction and description of spinors. It is also important to note that rotors can handle much more complex rotations and in the non-Euclidean space, the bivectors may possess a positive square and rotors are no longer elliptic but hyperbolic. For example, the corresponding rotors for the bivectors containing a time-like component as in $x-t$, $y-t$ and $z-t$ space-time surfaces of a 4-D space-time continuum, produce Lorentz boost in addition to the usual rotations on three orthogonal spatial planes $x-y$, $x-z$ and $y-z$). It turns out that the Lorentz boost is a sort of generalized rotation obtained from the same rotor prescription of GA. Since spinors allow a more general treatment of the notion of invariance under rotation and Lorentz boosts, from the above discussion it appears that they can be used without reference to relativity, but arise naturally in the discussions of Lorentz group.

Translations or linear shifts in space are usually defined by the addition with a vector representing the shift. This, however, works only for vectors representing locations in space and fails in the case free vectors and other geometric objects. These problems are removed in the projective geometric algebra by adding an extra dimension to reality and representing the geometric objects of the target space (under study) with linear subspaces of the augmented model space. In this setting both the rotor and the translator can be obtained via two consecutive reflections along $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$. If the two unit vectors are intersecting, $\hat{\mathbf{w}}\,\hat{\mathbf{u}}$ is a rotor R producing rotation by twice the angle between them. But If they are parallel, $\hat{\mathbf{w}}\,\hat{\mathbf{u}}$ represents a translator T and produces translation in the direction perpendicular to $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$ by twice the distance between them. Also a continuous shift or drifting of the plane of rotation may be produced by combining a translator (T) with a rotor to form the special motor $\mathrm{M} = \mathrm{T}\,\mathrm{R}$ – the generator of *twist*.

From a discussion of spin groups, spinors can be defined in any number of dimensions as elements of the even subalgebra of some real geometric algebra [19]. The physical properties of Pauli and Dirac spinors are also discussed and finally evaluate the relationship among Dirac, Lorentz, Weyl, and Majorana spinors. Lounesto [16], on the other hand, has used a special type of Dirac self adjoint clif (consisting of five multivectors – two scalars, two vectors and a bivector), termed *boomerang*, for the reconstruction of the Weyl and Majorana spinors.[4] Another mathematical representation, *Twistor*, introduced by Penrose in 1967 which attracts recent attention of string theorists, also requires an understanding of spinors and may be viewed as an extension of spinor algebra. Arcaute et al [20] have proposed reinterpretation of twistors within the framework of GA, in terms of 4-d spinors.

It may be noted that specification of axis of rotation is not possible for dimensions other than three. Representing the plane of rotation by a unit bivector $\hat{\mathbf{B}}$, GA appropriately generalises the description. On the other hand, a rotation with only one plane of rotation is a simple rotation and all rotations in two and three dimensions are simple. In an earlier paper [3] we have described the rotational transformation eq.(14) in terms of simple bivector $\mathbf{B}$. However, in four and higher dimensions, complex rotations involving multiple planes of rotation may appear in addition to simple rotations which can be described adequately in GA. For example, rotations can be decomposed either into one or two planar rotations in 4-D and are called simple and double rotations respectively. If the angles θ_1 and θ_2 of the two independent planar rotations of a double rotation have the same magnitude, then it is said to be isoclinic. More generally, in an even n-dimensional space, a rotation is isoclinic if all its $\frac{n}{2}$ angles are equal (up to the sign).

2.3 Applications and advantages of geometric algebra: Hestenes has paved the way for an extensive application of GA, for a variety of convenient and new formulations of modern physics. He has formulated standard problems on particle and rigid body dynamics using GA and developed the *spinor theory* of rotations and rotational dynamics [6]. Calculations with spinors are demonstrably more efficient compared to the conventional matrix theory and provides new insights into

[4] A spinor is determined up to a phase factor by its bilinear covariant (the special clif in the present case), which is in turn determined by its spinor – representing a boomerang, which returns back [16].

the treatment of the topics discussed. Specially in rotational dynamics and celestial mechanics, this unique treatment has both practical as well as theoretical advantages. Hestenes [21] has also given an invariant formulation of the Hamiltonian mechanics in terms of *geometric calculus* [15] – a generalization of the calculus of differential forms according to GA.

The structures of the Pauli's σ- and Dirac's γ-matrices correspond to the structures of geometric algebra [22], rendering an implicit geometric interpretation for quantum mechanics. Consistent formulations of both classical and quantum mechanics (QM) with GA facilitates the introduction of spin as a physical observable and thereby removes the first conceptual hurdle for quantum-classical unification. Moreover, Hestenes [23] has finally developed GA on four-dimensional Minkowski metric, christened as the Spacetime Algebra (STA)[5]. In fact, with the introduction of the geometric product, GA removes much of the mathematical divide among classical, quantum, and relativistic physics.

From a reformulation of the Dirac theory in terms of STA, The spin is revealed in this formulation as a dynamical property of the electron motion, associated with a local circulatory motion *zitterbewegung* (zbw) – first proposed by Schrödinger. Hestenes has further argued that the zbw idea offers a natural interpretation of the theory and is characterized by the complex phase factor of the wave function – a main feature, which it shares with its nonrelativistic limit. Consequently, the barrier penetration in nonrelativistic quantum mechanics can also be interpreted as manifestations of the zbw. The quantum phase has a general interpretation in terms of the Pancharatnam-Berry geometric phase [24]. According to Hestenes, the zbw argument provides an even more literal geometrical interpretation. Zitterbewegung has recently been detected in varied experimental situations [25].

Developing the nonrelativistic QM as a statistical theory over phase space, the Weyl-Wigner-Moyal (WWM) formalism [26] represents the observables in terms of the corresponding phase space functions (c-number) instead of the Hilbert space operators of standard QM. Replacing the conventional product of functions, the star product regime (also called the Moyal product or Weyl-Groenewold product) [27] used in this formalism, produce noncommutative composition of the phase space functions (the so-called deformation quantization). The star product encodes the quantum mechanical action and the formalism thus accommodates the uncertainty principle in systematic analogy with the noncommuting Hilbert space operators.

The strength of GA may also be argued using 'Occam's Razor' as it provides a simpler and economic mathematical model for the description of physical theories, naturally extending from one to two, to higher dimensions. The effectiveness of this algebra is amply demonstrated as it encapsulates the usual four Maxwell's equations of electromagnetism describing the electromagnetic field for the charge density ρ and current density $\mathbf{j}$ as sources in a *single, compact equation* [28]:

$$(c^{-1}\frac{\partial}{\partial t} + \nabla)\mathrm{F} = \rho - \frac{\mathbf{j}}{c}, \tag{16}$$

where c is the velocity of light. The electromagnetic field F is described as the sum of the electric field vector and the magnetic field bivector, i.e. represented by a clif and $\nabla\mathrm{F} = \nabla.\mathrm{F} + \nabla\wedge\mathrm{F}$, defined similarly like the geometric product. The four geometrically distinct parts of eq.(15) – its scalar, vector, bivector, and pseudoscalar parts, are respectively equivalent to the standard set of four equations. It should be noted here that the unification of the separate equations for divergence and curl in electromagnetism in a single equation is nontrivial – the unified equation can be inverted directly to determine the field. The formulation reveals that electromagnetism has no chirality and offers resolution to the Pierres puzzle in this context [29]. The apparent chirality turns out to be actually an artifact of the standard electromagnetic theory. It allows descriptions of electrodynamics and special relativity by extending the algebra of space to the algebra of spacetime (e.g. GA on Minkowski space, with replacement of the Euclidean metric by the Minkowski metric) [14, 30].

Recently, Daviau and Bertrand [31] have argued that three geometric algebras are sufficient to describe all interactions of modern physics and discussed the advantages of the approach. The algebra of the usual space, algebra of space-time and a third with only two more dimensions of space are sufficient to describe all aspects of electromagnetism, including the quantum wave of the electron, gravitation, the electro-weak interactions and the gauge group of the standard model, with

[5]In GA, paravectors (simplest clif represented by scalar plus vector) provide an alternative approach to the STA, called algebra of physical space (APS).

electro-weak and strong interactions.

In summary we note that, the geometric algebra provides an unified and comprehensive mathematical language for physics as it provides the appropriate representation for both physical Variables and operations by the elements of the algebra. The geometric products of two 2-D and 3-D vectors represent respectively ordinary complex numbers and quaternions and thereby generalizes complex analysis with even subalgebra. GA reduces rotations and Lorentz transformations to algebraic multiplication, and more generally it allows computational geometry without matrices or tensors and via *Conformal geometric algebra* (the conformal model projects onto a surrounding sphere, instead of a plane as in the case of projective geometry) gives a whole new language for doing geometry on the computer and currently being exploited in computer graphics. GA is also being used to investigate fractal geometry [32] and rotors applied to analyze and study low dimensional systems like quantum ring, monolayer and bilayer graphene [33].

GA offers great notational economy that simplifies many mathematical expressions. It formulates classical physics in an efficient spinorial formulation with tools, that are closely related to ones familiar in quantum theory, such as spinors and projectors. Thereby, it unites Newtonian mechanics, relativity, quantum theory, and more in a single formalism and language that is as simple as the algebra of the Pauli spin matrices. Spinors cannot be represented by tensors and besides the arduous language barrier of the tensor algebra, it requires a time consuming exercise to prove the covariance of physical quantities and equations for the coordinate-based tensor formulation. In this sense GA is more general than tensor algebra. From a study of electrically anisotropic medium, Matos *et al* [34] have also claimed that geometric algebra is the most natural setting and provide a better mathematical framework than tensors and dyadics in the formulation of anisotropy. However, pure grade multivectors can represent only antisymmetric tensors of rank equal to or less than the dimension of the base vector space. Unlike the grade of a multivector, the rank of a tensor is not restricted by the dimension of the vector space. Hestenes has proposed to introduce tensor as multilinear functions defined on geometric algebras [30].

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