Optimal Keyword Auctions with Shadow Costs\textsuperscript{1}

Jun Li
School of International Trade and Economics, University of International Business and Economics,
Beijing, China, 100029, lijunbnu@yahoo.com.cn

Shulin Liu
School of International Trade and Economics, University of International Business and Economics,
Beijing, China, 100029, sliu@uibe.edu.cn

De Liu
Gatton College of Business and Economics, University of Kentucky, Lexington, KY 40506
de.liu@uky.edu

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Abstract: Search engines incur shadow costs whenever an advertisement negatively affects user experiences, which damages the search engine’s long-run revenue. We characterize the optimal mechanism for keyword advertising in a general framework that accommodates click-through rates and shadow costs that can differ across advertisers and positions. We show that shadow costs significantly affect how advertisers are allocated and the optimal payment rule takes a form of “progressive second price”. Under special cases, the optimal mechanism can be implemented as scoring auctions. Through an analysis of special cases, we obtain insights on how shadow costs impact scoring rules and minimum bid policies and bridge the gap between keyword auction theory and practices.

Keywords: Internet advertising; keyword auction; mechanism design; user experience
1. Introduction

Keyword advertising, known as sponsored search, is a form of advertising that appears on search engine result pages. Known for its effectiveness and cost-per-click pricing, keyword advertising has quickly become the most popular form of Internet advertising. According to a report commissioned by Interactive Advertising Bureau (IAB 2011), in 2010 keyword advertising generated $12 billion and accounted for 46% of the total Internet advertising revenue in the United States. The keyword advertising operation is supported by keyword auction platforms such as Google’s AdWords. Each time a user enters a search query, say “hotels in Shanghai,” the search engine returns a set of organic results along with advertisements that are chosen by a keyword auction. To participate in keyword auctions, advertisers submit their advertisements for a specific keyword and associated cost-per-click bid (e.g., $1 per click on “hotels in Shanghai”). These bids are collected and ranked, and those who win the auction will automatically appear in the search result page.

Not surprisingly, academics have turned much attention to keyword auctions as an underlying mechanism for keyword advertising. To this day, many studies have been conducted on modeling the keyword auction (e.g., Aggarwal et al. 2006; Edelman et al. 2007; Varian 2007) and its optimal design (e.g., Garg and Narahari 2009; Iyengar and Kumar 2006; Liu and Chen 2006). Most existing models recognize that in keyword auctions advertisers only need to pay the minimum price to maintain their current position, a feature known as generalized second price (GSP), and are ranked by product of bids and expected click through rates (CTRs). Existing theoretical work suggests that keyword auctions work reasonably well. For example, several authors (Aggarwal et al. 2006; Edelman et al. 2007; Varian 2007) found that even though GSP auction is not truthful, it is simple and yet works reasonably well in terms of equilibrium revenue.
compared with well-known mechanisms such as the Vickrey–Clarke–Groves (VCG) mechanism. Existing research has also established that it is actually efficient to weigh per-click bids with estimated CTRs (Liu and Chen 2006; Liu et al. 2010).

Although the existing models provide strong theoretical foundation for keyword auctions, they fail to lend adequate support for some realistic problems facing search engines. First, most existing models of keyword auctions assume that it is costless to display an advertisement. Thus they do not tell how many advertising positions search engines should permit or when search engines should stop filling positions. More specifically, should minimum bids be advertiser specific and/or position specific? Second, existing models explain why estimated CTRs should be a measure for advertisement quality. But in reality search engines have moved beyond CTRs and have listed several other factors as their concerns for advertisement quality, including relevance of advertisers’ page to the keyword, landing-page quality, and page-loading speed. Existing theories fail to provide answers as to why and how these key auctions should include these factors optimally.

More important, the existing keyword auction literature has paid little attention to the role of user experience. To our knowledge, Abrams and Schwarz (2008) provided the first call for integration of user experience into keyword auction designs. Athey and Ellison (2009) modeled consumers’ click costs, but their goal was to endogenize consumer clicking behavior rather than keyword auction design. Poor user experiences pose a real threat for search engines because they may cause users to avoid advertisements (Cho and Cheon 2004) or to develop negative associations toward the advertised products when they cannot avoid advertisements (Nam et al. 2010), thus undermining a search engine’s long-run revenue (Abrams and Schwarz 2008). Studies have shown that users learn to tune out advertisements or advertising slots they expect to
be useless, leading to so-called “ad blindness” (Drèze and Husssherr 2003; Müller et al. 2009).

CTRs alone can hardly capture user experience. An obtrusive advertisement may achieve high CTRs but decrease user experience. A user may click on a seemingly useful advertisement only to find the landing page to be low-quality, irrelevant, or even fraudulent. Ultimately, because users have to make split-second decisions with very limited information, clicking tells user interests but little about post-clicking user experiences.

But why would an advertiser, who pays for each incoming visitor, want to inflict bad experiences on users? And why would users patronize advertisers who underinvest in the user experience? First, user experiences may manifest long after they buy the goods/services. Because manifestation is delayed, myopic advertisers, one-off campaign runners, and fraudulent sellers may get away with bad user experience and still make a profit. Second, users incur a cost to examine each web page. So they may be reluctant to switch even after learning the goods/services at the current site are not a perfect fit. Advertisers may capitalize on users’ switching cost and underinvest in user experience. User learning is ineffective in either case because of users’ inability to tell good from bad beforehand. Instead of avoiding a few bad advertisements, users end up avoiding all advertisements and even the search engine as a whole. Thus, user experience with advertisements on a search engine is to some extent public goods, which explains why search engines must take some measures to assure that advertisers provide good user experiences.

To address the externality associated with user experience, search engines must account for it in keyword auctions. To capture the impact of advertisements on user experience and ultimately on search engine long-run profits, we introduce a new element called shadow cost. Shadow cost captures the shadow of the future long-term cost to the search engine because of the
impact of advertisements on user experience. The focus of this paper is not on how to estimate shadow costs\(^2\) but rather on how keyword auctions should be designed after shadow costs are considered.

Our goal is to examine how to account for the shadow costs in the optimal design of keyword auction mechanisms (KAMs). The mechanism design literature (Harris and Raviv 1981; Hurwicz 1973; Myerson 1981) looks at how to allocate resources in a way that maximizes the principal’s objective while inducing agents to be truthful about their private information. Following this literature, we assume that advertisers hold private information about their valuation per click. The search engine’s problem is to design a mechanism to match advertisers with positions such that advertisers have incentives to reveal their true valuation per click, and the search engine’s expected net profit is maximized. While previous papers (Garg and Narahari 2009; Iyengar and Kumar 2006) have also examined the mechanism design problem in keyword advertising, this paper adopts a more general framework and is the first to incorporate shadow costs, which significantly affect the optimal mechanism.

Our work consists of two parts. In the first part, we characterize the optimal KAM in a very general framework where CTRs and shadow costs can depend on both advertisements and positions. We obtain the optimal KAM in terms of the probabilities that each advertiser is assigned to each position and the corresponding expected payment by each advertiser. In the

\(^2\) Estimating shadow costs is inevitably more challenging than estimating click through rates. But we believe shadow costs can be estimated using data on advertisers’ websites and users’ behavior and experimentation. For example, search engines can estimate (1) how likely an advertisement will result in an unsatisfactory user experience based on the length and pattern of search session and on the quality of advertiser’s website, (2) the long run impact of unsatisfactory user experience by monitoring the changes in an user’s propensity to use the search engine and to checkout the advertisements in controlled field experiments.
second part, we look for explicit, point-wise allocation and payment rules that are more practical. In particular, we focus on settings where the optimal KAMs can be implemented as scoring auctions. We derive explicit allocation rules for three special cases: (I) without shadow cost, (II) with advertiser-specific per-click shadow cost, and (III) with both a position-specific per-impression shadow cost and an advertiser-specific per-click shadow cost. We show that in all three cases advertisers with positive “net worths” to the search engine are allocated to positions by the descending order of their scores. The first two cases follow a greedy allocation rule; a high position is always filled before a low position, with the first position being the highest. But in the third case, when position-specific shadow cost per click is higher at high positions, the optimal allocation may not be greedy, and a high position may be unfilled. In such a case, the optimal “reserve prices” are generally different across positions.

Our research makes the following contributions. First, we formalize the notion of shadow cost and explain why it should be included as a distinct factor in keyword auction models. Second, we characterize the optimal KAMs with click-through rates and shadow costs that are both advertiser and position specific. We show that the optimal allocation rule can be solved as a linear programming problem, and the optimal payment rule takes a form of “progressive second price”; advertisers pay progressively higher marginal prices for clicks as they reach higher positions. Our results on optimal KAMs generalize those of Garg and Narahari (2009) and Iyengar and Kumar (2006), who consider click-through rates but not shadow costs. Third, by analyzing the three special cases where the optimal KAMs take the form of scoring auctions, we provide practical guidelines on how the optimal KAMs may be implemented in practice and on how shadow costs will impact scoring rules and minimum bids. Our findings about optimal KAMs help bridge the gap between theory and practice: for example, we show that non-greedy
allocation and CTR-dependent minimum bids may arise because of position-specific shadow costs.

The paper is structured as follows: next we discuss the related literature followed by a description of our model framework. In section 4 we characterize the feasible and optimal KAMs under a general setting. In section 5 we derive explicit KAMs for three special model specifications. Finally, we discuss the implications of our findings for theory and practice and conclude with future research directions.

## 2. Related Research

Our research is related to the growing literature on keyword auctions. Edelman et al. (2007) and Varian (2007) independently characterized the equilibrium properties of GSP auctions and found that GSP auctions have no truth-telling equilibrium. Several papers have subsequently examined the other properties of the GSP auction and its relationship with the classic VCG mechanism (Ashlagi et al. 2009; Bu et al. 2008; Edelman and Schwarz 2010; Milgrom 2010). Meanwhile, a few papers have examined the design of keyword auctions in terms of ranking rules (Lahaie and Pennock 2007; Weber and Zheng 2007), share structures (Chen et al. 2009), and minimum bids (Liu et al. 2010; Ostrovsky and Schwarz 2009). Recent work (Liu and Viswanathan 2010; Zhu and Wilbur 2011) shows that “hybrid auctions” that allow both per-click and per-impression bids may be viable or even preferred by search engines and other Internet advertising providers. The issue of click fraud in keyword auctions is raised and discussed in Wilbur and Zhu (2009). The empirical literature on keyword auctions is also growing. Animesh et al. (2010) find that the quality-weighted ranking used by Google can somewhat overcome the adverse selection problem in keyword auctions. Ghose and Yang (2009) demonstrate the
nontrivial relationship between positions of the advertisement, click-through rates, and advertisers’ profit per click. Ostrovsky and Schwarz 2009 find in a large-scale field experiment that introduction of theory-driven reserve prices can substantially enhance search engine’s revenue. Yang and Ghose (2010) and Goldfarb and Tucker (2011b) study how keyword advertising interacts with organic search and offline advertising respectively. Unlike previous papers that focus on a particular keyword auction design, we seek the optimal design of KAMs among all feasible mechanisms.

Our research follows the theory of mechanism design, pioneered by Hurwicz (1973), Myerson (1981), and Harris and Raviv (1981). The literature of mechanism design asks how a principal can optimally allocate goods among agents with private information on the valuation of goods. Myerson’s (1981) seminal work has laid the foundation for the mechanism design approach and shows that a standard auction with a minimum bid is optimal for selling a single object under a range of settings. Many advances have since been made in the mechanism design literature, such as optimal mechanism for a multi-product monopolist with unit demand (Harris and Raviv 1981) and for homogeneous multi objects (Maskin and Riley 1989).

Iyengar and Kumar (2006) provided the first paper to study the optimal mechanism in keyword auction settings. They derived the optimal KAM for keyword advertising with click-through rates that are both advertiser and position specific. Garg and Narahari (2009) obtained optimal KAMs under settings when advertisers have identical click-through rates and compared the optimal KAM with GSP and VCG mechanisms. Feng (2008) viewed keyword auctions as a problem of allocating multiple objects for which bidders have a common ranking and examined the optimal mechanism in such a setting. This paper adds to this literature by incorporating shadow costs in the optimal KAMs.
To our knowledge, only two previous papers have considered notions similar to shadow costs. Gonen and Vassilvitskii (2008) analyzed the existence of a symmetric Nash equilibrium under GSP auctions when there is a position-specific reserve price. They proposed a “bi-ladder” auction for a truthful equilibrium to exist. Abrams and Schwarz (2008) introduced an advertiser-specific “hidden cost” under a GSP auction. They argued that by subtracting the hidden cost from bid, search engines can encourage advertisers to create user experience and maximize efficiency. Those two studies differ from ours: they are interested in the equilibrium and efficiency of a particular auction, whereas we study the revenue-maximizing mechanism among all feasible mechanisms. Moreover, we consider a more general framework where both CTRs and shadow costs differ across advertisers and positions. Our results hold new insights unavailable in previous papers. For example, we show for the first time that shadow cost may explain non-greedy allocation and CTR-dependent minimum bids.

3. The Model Setup

3.1 The Keyword Auction Environment

In the keyword auction environment, \( n \) risk-neutral advertisers (bidders) compete for \( k \) ordered advertising positions at a risk-neutral search engine (auctioneer). Let \( i \in N = \{1, 2, \ldots, n\} \) index advertisers/advertisements and \( j \in K = \{1, 2, \ldots, k\} \) index positions. We follow the convention that a higher position has a smaller index number.

Advertiser \( i \) has an expected valuation \( v_i \) per click (valuation for short). We assume that \( v_i \) is independently drawn from \([\underline{v}, \bar{v}]\) according to distribution \( F_i(\cdot) \) which has a strictly positive and continuously differentiable density \( f_i(\cdot) \). We also assume that the hazard rate of each distribution
function, \(f_i(\cdot)/[1 - F_i(\cdot)]\), is non-decreasing.

An advertisement’s click-through rate (CTR) may depend on both the advertisement and the position. We denote \(a_{ij}\) as the CTR of advertiser \(i\) at position \(j\). We assume that the higher the position, the higher the CTR, \(^3\) i.e.:

\[a_{i1} > a_{i2} > \cdots > a_{ik}, \forall i \in N\]

We denote \(c_{ij}\) as the shadow cost of displaying advertiser \(i\) at position \(j\). The risk-neutral search engine (the auctioneer) incurs a shadow cost by displaying advertisements. As discussed before, shadow cost is interpreted as the loss of long-term profits when advertisements cause unsatisfactory user experiences. Like click-through rates, shadow costs may depend on both the advertisement and the position. Shadow costs can be negative, in which case an advertisement positively affects the search engine’s long-term profits. The per-click shadow cost is calculated as \(c_{ij}/a_{ij}\).

Consistent with the keyword auction practice, we assume that advertisers bid on and are charged by clicks. Let \(b_i\) denote the cost-per-click bid by advertiser \(i\).

Throughout the paper, we use bold-faced letters to denote vectors (e.g., \(\mathbf{x}\)) and subscript \(-i\) to denote “all but the \(i\)-th element” (e.g., \(\mathbf{x}_{-i}\)). So \(\mathbf{v} = (v_1, v_2, \ldots, v_n)^\top\) and \(\mathbf{b} = (b_1, b_2, \ldots, b_n)^\top\) denote the (column) vectors for valuations and bids of all advertisers. \(\mathbf{v}_{-i}\) and \(\mathbf{b}_{-i}\) denote the vectors of valuations and bids of all advertisers except \(i\). We denote \(\mathbf{a}_i = (a_{i1}, a_{i2}, \ldots, a_{ik})\) as the (row) vector of advertiser \(i\)'s CTRs at all positions and \(\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)^\top\) as the vector of all

\(^3\) Our results still hold if we generalize this assumption to \(a_{i1} \geq a_{i2} \geq \cdots \geq a_{ik}, \forall i \in N\). In fact, this assumption can be generalized to the case where the ordering of positions by CRT is the same across all advertisers. If the ordering of advertisement positions by CRT does not coincide with the natural order of positions, we can always renumber the positions.
advertisers’ click-through rate vectors. \( c_i \) and \( c \) are similarly defined.

We make the following informational assumptions. The per-click valuation \( v_i \) is advertiser \( i \)’s private information but the distribution functions \( \{F_i(\cdot)\}_{i \in N} \) are common knowledge. As in Liu and Chen (2006), Iyengar and Kumar (2006), and Liu et al. (2010), an advertiser’s click-through rate vector \( \alpha_i \) is known by the advertiser \( i \) and the search engine, but not by other advertisers. We also assume that only the search engine knows the shadow costs \( c \). Throughout the paper, we assume that valuations, CTRs, and shadow costs are independent.

3.2 The Keyword Auction Mechanism

By the revelation principle (Myerson 1979), each Bayesian Nash equilibrium corresponds to a truth-telling equilibrium in a direct mechanism in which agents are simply asked to report their types (valuations). So in this paper we will focus on the family of direct mechanisms. A keyword auction mechanism (KAM) consists two sets of rules: an allocation rule that determines how positions are allocated among advertisers, and a payment rule that determines how much they must pay.

The allocation rule. Denote \( p_{ij}(\mathbf{b}|\alpha, c) \) (\( p_{ij}(\mathbf{b}) \) as a shorthand) as a function that maps the bid vector \( \mathbf{b} \), the click-through rates \( \alpha \), and the shadow costs \( c \) into the probability of assigning advertiser \( i \) to position \( j \). We further denote

\[
p_i(\mathbf{b}) = \sum_{j=1}^{k} \alpha_j p_{ij}(\mathbf{b})
\]

as advertiser \( i \)'s total click-through rate (across all positions that \( i \) is possibly assigned to).

The allocation rule \( \mathbf{p} \) is a set of probability assignment functions (hereafter we will omit the

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\(^4\)Our main results still hold if advertisers know their shadow costs, if they do not know their CTR vector, or if CTRs and shadow costs are public information. The main assumption here is that the search engine can estimate the CTRs and shadow costs.
domain for all set notations “{}”)

\[ p = \{ p_j(b) \}_{i \in N, j \in K} \]

that satisfies the following conditions,

\[ \sum_{i=1}^{n} p_j(b) \leq 1, \; \forall j \in K \] (2)

\[ \sum_{j=1}^{k} p_i(b) \leq 1, \; \forall i \in N \] (3)

Conditions (2) and (3) require that, respectively, an advertiser can appear in at most one position at a time, and a position can be assigned to at most one advertiser at a time.

**The Payment Rule.** Denote \( m_j(b|\alpha, c) \) \((m_j(b) \text{ as a shorthand})\) as a function that maps the bid vector \( b \), the CTRs \( \alpha \), and the shadow costs \( c \) into *the cost per click to be paid by advertiser i at position j*. The payment rule \( m \) is a set of such payment functions:

\[ m = \{ m_j(b) \} \]

We further denote

\[ m_i(b) = \sum_{j=1}^{k} \alpha_j p_j(b) m_j(b) \] (4)

as advertiser i’s *total payment* (across all positions that advertiser i is possibly assigned to).

Because advertisers are risk neutral, they care only about the total CTR and total payment. Therefore, we may also represent \( p \) and \( m \) by vectors \( \{ p_i(b) \} \) and \( \{ m_i(b) \} \).

The timeline of the game is as follows. First, the search engine announces the mechanism rules \((p, m)\). Next the valuations \( v \), shadow costs \( c \), and CTRs \( \alpha \) are realized. Each advertiser i learns \( v_i \) and \( \alpha_i \), and the search engine learns \( c \) and \( \alpha \). Then, each advertiser i submits a bid \( b_i \) based on the advertiser’s valuation \( v_i \) and CTR \( \alpha_i \). The search engine allocates the positions to advertisers based on the bid vector \( b \), CTRs \( \alpha \), and shadow costs \( c \), and determines the total
payment of each advertiser according to the preannounced rules \((p, m)\).

3.3 Advertiser’s Expected Payoff

Assuming that all advertisers but \(i\) bid their true valuations, \(b_{-i} = v_{-i}\), then we can write advertiser \(i\)’s payoff as

\[
u_i(b_i | v) = \sum_{j=1}^{k} \alpha_j p_j(b_i, v_{-i}) \left[ v_i - m_j(b_i, v_{-i}) \right] = v_i \sum_{j=1}^{k} \alpha_j p_j(b_i, v_{-i}) - \sum_{j=1}^{k} \alpha_j p_j(b_i, v_{-i}) m_j(b_i, v_{-i}) \]

(5)

where \(p_i(b_i, v_{-i})\) and \(m_i(b_i, v_{-i})\), as defined in (1) and (4), are, respectively, advertiser \(i\)’s total CTR and payment under bid vector \((b_i, v_{-i})\), CTRs \(\alpha\), and shadow costs \(c\).

Since advertiser \(i\) only knows \(i\)’s valuation and CTRs, advertiser \(i\)’s expected payoff is

\[
\pi_i(b_i | v_i) = E_E E_c E_{\alpha_{-i}} [u_i(b_i | v)] = E_E E_c E_{\alpha_{-i}} [v_i p_i(b_i, v_{-i}) - m_i(b_i, v_{-i})] = v_i \bar{p}_i(b_i) - \bar{m}_i(b_i)
\]

(6)

where

\[
\bar{p}_i(b_i) = E_c E_{\alpha_{-i}} E_{v_{-i}} [p_i(b_i, v_{-i})] \]

(7)

and

\[
\bar{m}_i(b_i) = E_c E_{\alpha_{-i}} E_{v_{-i}} [m_i(b_i, v_{-i})]
\]

(8)

are interpreted as advertiser \(i\)’s expected click-through rate and expected payment when advertiser \(i\)’s bid is \(b_i\). Note that \(E_{v_{-i}} (\cdot)\) is an expectation with respect to valuations of advertisers other than \(i\), \(E_c\) is an expectation with respect to shadow costs, and \(E_{\alpha_{-i}}\) is an expectation with respect to CTRs of advertisers other than \(i\).

3.4 The Search Engine’s Expected Profit

Because the search engine chooses the mechanism that maximizes expected profit under all possible realizations of \(v, c, \) and \(\alpha\). For any given realization, when all advertisers report their
true valuations, the search engine’s total revenue is $\sum_{i=1}^{n} m_i(v)$ and total shadow cost is $\sum_{j=1}^{k} \sum_{i=1}^{n} p_{ij}(v) c_{ij}$. So for all possible realizations the search engine’s total expected revenue is $R = E_\epsilon E_a E_v \left[ \sum_{i=1}^{n} m_i(v) \right] = \sum_{i=1}^{k} E_{v_i} E_a [\bar{m}_i(v_i)]$ and the total expected shadow cost is $C = E_\epsilon E_a E_v \left[ \sum_{j=1}^{k} \sum_{i=1}^{n} p_{ij}(v) c_{ij} \right]$. The search engine’s total expected profit is the total expected revenue minus the total expected shadow cost:

$$\pi_0 = R - C = \sum_{i=1}^{n} E_{v_i} E_a [\bar{m}_i(v_i)] - E_\epsilon E_a E_v \left[ \sum_{j=1}^{k} \sum_{i=1}^{n} p_{ij}(v) c_{ij} \right]$$

(9)

4. The Optimal Keyword Auction Mechanism

4.1 The Feasible KAM

A feasible KAM must be *individually rational*: any participating advertiser must have nonnegative expected payoff, and be *incentive compatible*; all advertisers must report their true valuations in the direct mechanism. The two requirements can be formally defined as follows:

**Definition 1 (Individual Rationality):** A KAM is individually rational if and only if

$$\pi_i(v_i \mid v_i) \geq 0, \ \forall \ v_i \in [v, \bar{v}]$$

(10)

**Definition 2 (Incentive Compatibility):** A KAM is incentive compatible if and only if

$$\pi_i(v_i \mid v_i) \geq \pi_i(b_i \mid v_i), \ \forall \ b_i \in [v, \bar{v}]$$

(11)

**Definition 3 (Feasible KAM):** A KAM $\langle p, m \rangle$ is feasible if it is individually rational, incentive compatible, and meets the allocation feasibility constraints (2) and (3).

**Lemma 1** A KAM is feasible if and only if the following conditions hold:
(i) \( \pi_i(v_i | v_i) = \pi_i(v | v) + \int_{v_i}^{v} \bar{p}_i(t) dt, \)

(ii) \( \pi_i(v | v) \geq 0, \)

(iii) \( \bar{p}_i(t) \) is non-decreasing in \( t, \)

(iv) Conditions (2) and (3).

All the proofs of lemmas are deferred in the appendix.

By (6) and Lemma 1(i), under any feasible mechanism, advertiser \( i \)'s expected payment is

\[
\bar{m}_i(v_i) = v_i \bar{p}_i(v_i) - \pi_i(v_i | v_i) = v_i \bar{p}_i(v_i) - \pi_i(v | v) = \int_{v_i}^{v} \bar{p}_i(t) dt \quad (12)
\]

We can then calculate the search engine’s expected profit.

**Lemma 2** The search engine’s expected profit under a feasible KAM is

\[
\pi_0 = E_c E_a E_v \left[ \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_j p_j(v) \left[ v_i - \frac{1 - F_j(v_i)}{f_i(v_j)} \frac{c_i}{\alpha_j} \right] \right] - \sum_{i=1}^{n} \pi_i(v | v) \quad (13)
\]

The result in Lemma 2 suggests that the expected revenue of the search engine is also completely determined by the allocation rule \( p \) and \( \{\pi_i(v | v)\} \). In other words, if two different payment rules can lead to the same allocation, they should generate identical expected revenues for the search engine. This result parallels the revenue equivalence theorem obtained for single-object auctions (Myerson 1981). The intuition for this result can be seen from (12): an advertiser’s expected payment is completely determined by the allocation rule \( p \) and the expected payoffs of the lowest valuation advertisers, \( \{\pi_i(v | v)\} \). Therefore, the expected revenue of the search engine, which is the sum of expected payment from advertisers, is also completely determined by \( p \) and \( \{\pi_i(v | v)\} \).

4.2 The Optimal KAM
Lemma 2 implies that a feasible KAM \((p, m)\) is optimal if \(\{p_i(\cdot)\}\) and \(\{\pi_i(v \mid v)\}\) maximizes (13). To meet the individual rationality condition, we must have \(\pi_i(v \mid v) \geq 0\). Because the choice of \(\{\pi_i(v \mid v)\}\) does not affect the optimal choice of \(\{p_i(\cdot)\}\), the optimal design should satisfy \(\pi_i(v \mid v) = 0\) and \(p\) is a solution to the following problem:

\[
\max_p \pi_0 = E_c E_a E_v \left\{ \sum_{j=1}^{k} \sum_{i=1}^{n} p_{ij}(v) \left[ \alpha_{ij} \left( v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right) - c_{ij} \right] \right\} \text{ s.t. (iii) and (iv)} \quad (\text{OP})
\]

**Corollary 1 (Characteristics of the Optimal KAM)** A feasible KAM \((p, m)\) is an optimal mechanism if \(\pi_i(v \mid v) = 0\) and \(p\) solves (OP).

We denote

\[J_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\]

as advertiser \(i\)'s virtual valuation and

\[w_{ij}(v_i) = \alpha_{ij}J_i(v_i) - c_{ij}\]

as the net worth of advertiser \(i\) at position \(j\) to the search engine. By the non-decreasing hazard rate assumption, \(J_i(v_i)\) and \(w_{ij}(v_i)\) are strictly increasing in \(v_i\).

Corollary 1 gives sufficient conditions for a feasible KAM to be optimal. It implies that the optimal KAM maximizes total expected net worth from all advertisers and positions. Recall that the optimal mechanism for single-object auctions maximizes the total expected virtual valuation minus seller’s reservation utility (see, for instance, Myerson 1981). In comparison, the net worth extends virtual valuation by incorporating click-through rates as a multiplier and shadow costs as a reduction. More important, virtual valuation differs only across advertisers; but net worth differs across both advertisers and positions.
Note when \( c = (0, 0, \ldots, 0) \) and click-through rates depend only on positions, Corollary 1 is reduced to the result of Iyengar and Kumar (2006).

Built on Corollary 1, we now characterize the optimal allocation and payment rules.

**Theorem 1** If the allocation rule \( p \) solves (OP) and the payment rule \( m \) satisfies

\[
m_i(b_j, b_{-j}) = b_j p_j(b_j, b_{-j}) - \int_t^b p_j(t, b_{-j})dt, \quad \forall b, \forall i \in N
\]

then \((p, m)\) is an optimal KAM.

Proof: First we will show that bidding truthfully is a Bayesian Nash equilibrium. In fact, if all advertisers but \( i \) bid their true values (i.e., \( b_{-i} = v_{-i} \)), by (14), advertiser \( i \)'s expected payment is \( \bar{m}_i(b_i) = b_i \bar{p}_i(b_i) - \int_t^b \bar{p}(t)dt \). Thus, by (6), we have

\[
\pi_i(v_i, v_{-i}) = v_i \bar{p}(v_i) - \bar{m}_i(v_i) = v_i \bar{p}(v_i) - [v_i \bar{p}_i(v_i) - \int_t^b \bar{p}(t)dt] = \int_t^b \bar{p}(t)dt
\]

Then, advertiser \( i \)'s expected payoff is

\[
\pi_i(b_i, v_i) = v_i \bar{p}_i(b_i) - \bar{m}_i(b_i)
= v_i \bar{p}(b_i) - [b_i \bar{p}(b_i) - \int_t^b \bar{p}(t)dt]
= (v_i - b_i) \bar{p}(b_i) + \int_t^b \bar{p}(t)dt
= \int_t^b \bar{p}(b_i)dt + \int_t^b \bar{p}(t)dt
= \int_t^b \bar{p}(t)dt + \int_t^b [\bar{p}(b_i) - \bar{p}(t)]dt
\leq \int_t^b \bar{p}(t)dt = \pi_i(v_i, v_i)
\]

and the equality holds if and only if \( b_i = v_i \) by condition (iii). Therefore, advertiser \( i \) will bid \( i \)'s true value \( v_i \) and \( i \)'s expected payment is \( \bar{m}_i(v_i) = v_i \bar{p}(v_i) - \int_t^b \bar{p}(t)dt \). Since

\[
\pi_i(v_i, v_{-i}) = v_i \bar{p}(v_i) - \bar{m}_i(v_i) = \int_t^b \bar{p}(t)dt
\]

at the truthful equilibrium, thus \( \pi_i(v_i | v_{-i}) = 0 \) and
Therefore, this KAM meets the conditions (i) and (ii) in Lemma 1. Because this KAM also satisfies conditions (iii) and (iv) in Lemma 1 since $p$ solves (OP), it is a feasible KAM. Finally, this KAM satisfies the conditions of Corollary 1, so it is an optimal KAM. ⊓⊔

4.3 Pointwise Allocation and Payment Rules

Theorem 1 characterizes the optimal mechanism in expected terms: allocation rules that maximize expected revenue and payment rules that yield a certain total expected payment. Notice that we do not claim the uniqueness of allocation and payment rules. In fact, many allocation and payment rules may meet the conditions of Theorem 1 as long as they produce the same expected total click-through rates and payments for advertisers. But in practice, we need an allocation rule that optimizes revenue for each specific realization of $v$ (or $b$), and a corresponding payment rule that specifies what advertisers must pay in each allocation. Next we derive a pair of pointwise optimal allocation and payment rules.

The following result shows that the optimization problem (OP) can be solved pointwise, and the optimal allocation guarantees an advertiser’s total CTRs to be non-decreasing in the advertiser’s valuation.

**Lemma 3** The allocation rule $p$ solves (OP) if it solves

$$
\max_p \sum_{j=1}^k \sum_{i=1}^n p_{ij}(v)\left[\alpha_{ij}J_i(v) - c_{ij}\right] s.t. (iv), \text{ for any } c, \alpha, v
$$

Throughout the rest of the paper, we use valuation vector $v$ to replace bid vector $b$ because advertisers will bid truthfully in any feasible direct mechanism. Without loss of generality, we will focus on deterministic allocation rules.\(^5\)

\(^5\) It is straightforward to show that the solution to (OP') can always be reached at a corner of its
We denote \( \varphi(i) \in \{1, 2, \ldots, n + k\} \) as the position assigned to advertiser \( i \) under the optimal allocation in the Lemma 3. Virtual positions \( k + 1 \) to \( k + n \) are added for notational convenience. We define \( \alpha_{ij} = c_{ij} = 0 \) for any virtual position \( j > k \). We say that an advertiser is assigned if \( \varphi(i) \leq k \) and otherwise, unassigned.

For any \( j \geq \varphi(i) \), let \( b_{ij} = \min\{b_i \mid \varphi(i) \neq \varphi(j) \} \) be the minimal bid advertiser \( i \) must submit to win position \( j \) or higher, given that all other advertisers bid their true valuations. Please note that \( i \) may not obtain position \( j \) at any price (e.g., when \( c_{ij} \) is very large). In such a case, by definition, \( b_{ij} \) is recursively calculated as minimum bid to win position \( j-1 \) (so \( b_{ij} = b_{i,j-1} \)). Clearly \( b_{ij} \) is non-decreasing as position \( j \) gets higher (\( j \) decreases).

**Lemma 4** The optimal payment for the optimal KAM in Theorem 1 is given by:

\[
m_i(v) = \begin{cases} \sum_{x=\varphi(i)}^{k} (\alpha_{ix} - \alpha_{i,x+1}) b_{ix}, & \text{if } i \text{ is assigned} \\ 0, & \text{otherwise} \end{cases} \tag{15}
\]

where \( \alpha_{i,k+1} = 0 \).

We call the optimal payment rule in (15) a **progressive second price** rule. We graphically illustrate this payment rule in Figure 1. The total valuation created by assigning advertiser \( i \) to position \( \varphi(i) \) is represented by area OABCO. The staircase curve (DEF) represents the marginal price paid by advertiser \( i \) as a function of \( i \)'s total CTR. The advertiser pays the area below DEF and retains the area above.

Intuitively, as advertiser \( i \) raises the bid from 0 to \( b_{ik} \), \( i \) obtains position \( k \),\(^6\) and \( i \)'s total clicks increase from 0 to \( \alpha_{ik} \). For the additional clicks (\( \alpha_{ik} \)) received, advertiser \( i \) pays \( b_{ik} \) per click.

---

\(^6\) If advertiser \( i \) cannot obtain position \( k \), we can just skip to the next position \( i \) can get.
Similarly, as advertiser $i$ raises the bid to $b_{i,k-1}$, $i$ gains additional clicks ($\alpha_{i,k-1} - \alpha_{ik}$) and pays $b_{i,k-1}$ for these additional clicks. As $i$ reaches higher and higher positions, $i$ pays progressively higher marginal prices for additional clicks $i$ receives. That is why we call this payment rule a *progressive* second price rule.

Under the GSP payment rule, advertiser $i$’s total payment would be $\alpha_{i,\varphi(i)} b_{i,\varphi(i)}$. So if we fix $\{b_{ik}\}$, an advertiser’s progressive second price payment is lower than the GSP payment. This does not imply, however, that a GSP mechanism generates higher revenue than a progressive second price mechanism. This is because the GSP payment rule does not induce truthful bidding and advertisers generally bid lower than their true valuations under GSP (Edelman et al. 2007). Earlier research has shown that GSP auction is a simplification of the generalized Vickery auction and is helpful in eliminating lowest revenue equilibrium while retaining high revenue ones (Milgrom 2010; Varian 2007).

Figure 1. Advertiser $i$’s payment and payoff
We summarize Lemma 3 and Lemma 4 in the following theorem.

**Theorem 2** If the allocation rule \( p \) solves \((\text{OP}')\) and the payment rule \( m \) is given by (15), then \( (p, m) \) is an optimal KAM.

To summarize, we have characterized the optimal KAM under a general framework. The optimal allocation rule in probabilistic terms is determined by a linear programming in \((\text{OP}')\). Once we obtain the optimal allocation rule, we can implement it using the progressive second price payment rule outlined in Lemma 4. Per result of Lemma 2, the expected profit of the search engine is mainly determined by the optimal allocation rule.

The complexity of the optimal KAM is also mainly determined by that of the optimal allocation problem: the optimal allocation problem \((\text{OP}')\) is a linear-programming (LP) with \( nk \) variables and \( n + k \) constraints. Though polynomial-time algorithms exist for solving problems like \((\text{OP}')\), it is still complex to solve in real time. For instance, with 20 advertisers and 8 positions, the interior point algorithm can take as many as 51.8 million steps to solve \((\text{OP}')\)!

In light of this, we attempt to characterize special cases where the optimal allocation follows some simple and explicit rules.

### 5. The Optimal KAMs under Specific Settings

The purpose of this section is twofold: we hope to derive explicit and deterministic allocation rules and to obtain additional insights about the characteristics of the optimal KAM; we are also interested in the conditions under which the optimal KAM is simple. As suggested by Milgrom (2004), simplicity is highly valued in practical mechanism design. It is even more important for real-time keyword auctions that must be solved in a fraction of a second to avoid
noticeable delays on serving a page.

Throughout this section, we assume that

- (symmetric distribution) \( J(\cdot) = J(\cdot) \)

- (separable CTRs) \( a_{ij} = \alpha_i^a \alpha_j^p \), where \( 1 = \alpha_1^p \geq \alpha_2^p \geq \cdots \geq \alpha_k^p \geq \alpha_{k+1}^p = \cdots = \alpha_{n+1}^p = 0 \)

- (separable shadow cost) \( c_{ij} = c_j^p + \alpha_i^a \alpha_j^p c_i^a \)

The first assumption states that advertisers’ valuations are drawn from the same distribution. This assumption is not essential for deriving explicit optimal allocation rules and payment rules. We make this assumption primarily so that the scoring function is the same for every advertiser. The second assumption, known as the separability condition of the CTRs (Aggarwal et al. 2006; Edelman et al. 2007; Varian 2007), states that CTRs can be separated into an advertiser factor \( \alpha_i^a \) and a position factor \( \alpha_j^p \) (interpreted as the “prominence” of a position). The third assumption states that the shadow cost consists of a position-specific per-impression component \( c_j^p \) and an advertisement-specific per-click component \( c_i^a \). The former arises because position attributes such as location on a page, size, or background color affect the before-click user experience. The latter arises because the quality of the advertiser-specific landing page is the main determinant of the post-click user experience. We can also easily generalize the third assumption by introducing an advertiser-specific per-impression shadow cost in the form of \( \alpha_j^p c_i^a \) (which is proportional to the prominence of a position) and redefine \( c_i^a = c_i^a = \frac{c_i^a}{\alpha_j^p} + c_i^a \) as the new advertiser-specific per-click cost. In such a way, the advertisement-specific component \( c_i^a \) can capture an advertisement’s impact on both before- and post-click user experiences.

**Definition 4 (Scoring Mechanism):** A KAM is a scoring mechanism if there exists a
scoring function $s: N \rightarrow R$\(^7\) such that a high-scored advertiser is allocated before a low-scored advertiser; for any $i, l \in N$, $s(i) > s(l) \Rightarrow \varphi(i) < \varphi(l)$.

**Definition 5 (Greedy Allocation):** An allocation rule is **greedy** if it always fills position $j$ before $j + 1$.

The optimal KAM may not be greedy because if the shadow cost for a position is sufficiently high, it may be left empty while lower positions are filled (see Example 4). The optimal KAM may not be a scoring mechanism either. The following example shows that the highest position may not be filled by advertisers with the highest net worth.

**Example 1:** Suppose there are two positions and two advertisers. If $w_{11}=11$, $w_{12}=7$, $w_{21}=7$, and $w_{22}=5$, it is optimal to assign advertiser 1 to position 1 and advertiser 2 to position 2 with a total net worth of 16. If $w_{11}=8$ instead, the two advertisers are switched under the optimal allocation with a total net worth of 14.

Next we discuss three different cases based on assumptions about shadow costs.

**5.1 Case I. No Shadow Costs ($c_j^p = c_i^a = 0$)**

We use this case as a benchmark for our subsequent results. When shadow costs are absent and CTRs are separable, it is straightforward to show that the optimal KAM is a scoring mechanism with greedy allocation.

**Theorem 3** Under assumption of no shadow costs, the optimal KAM is a greedy scoring mechanism with scoring function

$$s^I(i) = \alpha_i^a J(v_i).$$

Specifically, in the optimal allocation, advertisers with positive $s^I$ scores ($s^I > 0$) are assigned by the descending order of their $s^I$ scores in a greedy fashion.

\(^7\) We require all scoring functions to be in the form of $s(i) = g(v_i, \alpha_i, c_i)$. 
Moreover, if advertisers are indexed by the descending order of their $s^I$ scores, $b_j$ is the solution to

$$\alpha_i^a J(b_j) = \max \{ \alpha_i^a J(v_{j+1}), 0 \}.$$ 

where we define $\alpha_{n+1}^a J(v_{n+1}) = 0$.

By Theorem 3, the optimal allocation in this case involves filling the positions 1 to $k$ in a greedy way by the order of $s^I$ scores from high to low, until there are no more positions or no more advertisers with positive $s^I$ scores. The complexity of the optimal allocation is mainly determined by sorting of advertisers, which can be done in $O(n \log n)$. In the case of 20 advertisers and 8 positions, the sorting may take about 26 steps.

Let $r$ be the solution to $J(v) = 0$. The necessary and sufficient condition for an advertiser to have a positive $s^I$ score is that the advertiser’s valuation must be greater than $r$. So the optimal KAM entails a minimum bid (or reserve price) $r$ for all bidders, regardless the number of advertisers.

**Example 2:** Advertisers 1, 2, 3, and 4 compete for 3 advertisement positions. Let $F(x) = x$, $x \in [0, 1]$, then $J(x) = 2x - 1$. Moreover, $v_1 = 1$, $v_2 = 0.9$, $v_3 = 0.8$, $v_4 = 0.3$, $\alpha_1^p = 1$, $\alpha_2^p = 0.8$, $\alpha_3^p = 0.2$, $\alpha_4^p = 2$, $\alpha_2^a = 4$, $\alpha_3^a = 1$, $\alpha_4^a = 0.5$. Then $s^I(i) = \alpha_i^a J(v_i) = \alpha_i^a [2v_i - 1]$.

<table>
<thead>
<tr>
<th>Advertiser</th>
<th>$v_i$</th>
<th>$J(v_i)$</th>
<th>$\alpha_i^a$</th>
<th>$s^I(i)$</th>
<th>Position</th>
<th>$b_{i1}$</th>
<th>$b_{i2}$</th>
<th>$b_{i3}$</th>
<th>Price</th>
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<td>/</td>
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<td>/</td>
<td>/</td>
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<td>0</td>
</tr>
</tbody>
</table>

5.2 Case II. Advertisement-Specific Per-click Shadow Costs ($c_j^p = 0, c_i^a > 0$)

When there is only an advertiser-specific per-click cost, the shadow cost $c_{ij} = \alpha_i^a \alpha_j^p c_i^a$, i.e., the per-click shadow cost $c_i^a$ depends only on advertisement.

**Theorem 4** Under the assumption $c_{ij} = \alpha_i^a \alpha_j^p c_i^a$, the optimal KAM is a greedy scoring
mechanism with the following scoring function

\[ s^{II}(i) = \alpha^a_i [J(v_i) - c^a_i] \]

Specifically, in the optimal allocation, advertisers with positive \( s^{II} \) scores (\( s^{II} > 0 \)) are assigned by the descending order of their \( s^{II} \) scores in a greedy fashion.

Moreover, if advertisers are indexed by the descending order of their \( s^{II} \) scores, \( b_{ij} \) is the solution to

\[ \alpha^a_i [J(b_{ij}) - c^a_i] = \max \{ \alpha^a_{j+1} [J(v_{j+1}) - c^a_{j+1}], 0 \} \]

where we define \( \alpha^a_{n+1} [J(v_{n+1}) - c^a_{n+1}] = 0 \).

The optimal allocation is similar to that in §5.1: filling the positions 1 to \( k \) sequentially by the descending order of \( s^{II} \) scores, until no more positions or no more advertisers with positive \( s^{II} \) scores. The only difference lies in the scoring rule. Like in §5.1, the optimal KAM is a scoring mechanism with a greedy allocation rule with complexity of \( O(n \log n) \). Given the formula for \( s^{II} \) scores, the minimum bid for any position is a function of shadow cost factor \( c^a_i \). Thus an advertiser with a higher shadow cost must pay a higher minimum bid to be eligible.

**Example 3:** Continue with Example 2. Let \( c^a_1 = 0.2, c^a_2 = 0.5, c^a_3 = 0.7, c^a_4 = 0.3 \). Then \( s^{II}(i) = \alpha^a_i [J(v_i) - c^a_i] = \alpha^a_i [2v_i - 1 - c^a_i] \).

<table>
<thead>
<tr>
<th>Advertiser</th>
<th>( v_i )</th>
<th>( J(v_i) )</th>
<th>( \alpha^a_i )</th>
<th>( c^a_i )</th>
<th>( s^{II}(i) )</th>
<th>Position</th>
<th>( b_{i1} )</th>
<th>( b_{i2} )</th>
<th>( b_{i3} )</th>
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<td>/</td>
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<td>/</td>
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</table>

**5.3 Case III. Advertisement- and Position-Specific Shadow Costs** \( (c^p_j > 0, c^a_i > 0) \)

When there are both an advertiser-specific per-click cost and a position-specific per-impression cost, the shadow cost \( c^*_j = c^p_j + \alpha^a_i \alpha^p_j c^a_i \).
Theorem 5 Under the assumption $c_{ij} = c_j^p + \alpha_i^p \alpha_j^p c_i^a$ and
\[
\frac{c_1^p}{\alpha_1^p} \leq \frac{c_2^p}{\alpha_2^p} \leq \cdots \leq \frac{c_k^p}{\alpha_k^p}
\] (16)
the optimal KAM is a greedy scoring mechanism with the following scoring function
\[
s^{\text{III}}(i) = \alpha_i^a [J(v_i) - c_i^a]
\]
Specifically, in the optimal allocation, advertisers with $s^{\text{III}}(i) \geq \frac{c_j^p}{\alpha_j^p}$ are assigned by the descending order of their $s^{\text{III}}$ scores in a greedy fashion.

Moreover, if advertisers are numbered in the descending order of $s^{\text{III}}$ scores, $b_{ij}$ is solution to
\[
\alpha_i^a [J(b_{ij}) - c_i^a] = \max\{ \alpha_{j+1}^a [J(v_{j+1}) - c_{j+1}^a] \frac{c_j^p}{\alpha_j^p} \}
\]
where we define $\alpha_{n+1}^a J(v_{n+1}) = 0$.

The optimal KAM in Theorem 5 also fills positions sequentially by the descending order of $s^{\text{III}}$ scores, until there are no more positions or no more advertisers with an $s^{\text{III}}$ score that exceeds the advertiser’s shadow cost. Similar to §5.1, the optimal allocation rule has a complexity of $O(n \log n)$. Given the formula for $s^{\text{III}}$ scores and the minimize score requirement $\{ \frac{c_j^p}{\alpha_j^p} \}$, the minimum bid to qualify for a position is different across advertisers and positions. An advertiser with a higher click-through rate $\alpha_i^a$ and a lower shadow cost $c_i^a$ pays a lower minimum bid, whereas a position with a higher shadow cost $c_j^p / \alpha_j^p$ requires a higher minimum bid.

The condition (16) maintains that for any given advertiser, the per-click shadow cost ($= c_j^p / \alpha_j^p$) does not decrease with position. As a result, if an advertiser does not generate a positive net worth at position $j$, neither does the advertiser at position $j+1$. Condition (16) is not
needed for the optimal KAM to be a scoring mechanism but is needed for it to be greedy, as shown by the following example.

**Example 4:** Continue with example 3. Let $\frac{c_i^p}{\alpha_i^p} = 0.2, \frac{c_i^p}{\alpha_i^p} = 1.3, \frac{c_i^p}{\alpha_i^p} = 0.4$. Then $s_{\text{III}}(i) = \alpha_i^p [J(v_i) - c_i^p] = \alpha_i^p \{2v_i - 1 - c_i^p\}$.

<table>
<thead>
<tr>
<th>Advertiser</th>
<th>$v_i$</th>
<th>$J(v_i)$</th>
<th>$\alpha_i^a$</th>
<th>$c_i^a$</th>
<th>$s_{\text{III}}(i)$</th>
<th>$s_{\text{III}}(i) - c_i^p / \alpha_i^p$</th>
<th>Position</th>
<th>$b_{i1}$</th>
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To summarize, all three special cases involve a scoring mechanism, but with different scoring rules and minimum bids. In the first case (no shadow costs), the advertisers are ranked by virtual valuation weighted by CTRs, and the minimum bid $r$ is the same across all advertisers and positions. In the second case (only an advertiser-specific per-click shadow cost), advertisers’ rankings are affected by both their CTRs and shadow costs, and the minimum bids are now a function of advertisers’ shadow costs factor $c_i^a$ (Theorem 4). In the third case (an advertiser-specific per-click shadow cost and a position-specific per-impression shadow cost), advertisers are ranked the same way as the second case if (16) holds but the minimum bids become dependent on advertiser-specific shadow cost and position-specific shadow cost – see Theorem 5. While the first two cases involve greedy allocation, the third case may not. These results highlight the broad impact of shadow costs on optimal keyword auction design.

6. Discussion and Conclusion

Search engines incur shadow costs when an advertisement negatively affects user...
experiences. We formulate keyword auction design as a mechanism design problem in which a search engine faces advertisers with private valuation per click, and CTRs and shadow costs differ across advertisers as well as positions. Our analysis on the optimal KAMs for the general setting and three special cases yield the following implications of shadow costs for keyword auctions.

**Implications of shadow costs for minimum bids.** In the optimal auction literature, minimum bid exists to exclude bidders who have negative virtual valuation. In previous research on optimal KAMs (Garg and Narahari 2009; Iyengar and Kumar 2006), the optimal minimum bid exists purely for revenue considerations and depends only on the distribution of valuations. Our findings for the case I (no shadow cost) are consistent with the previous papers: the optimal minimum bid is independent of advertiser’s CTRs and exists only for revenue considerations. But with shadow costs (cases II and III), optimal minimum bids exist not only for revenue considerations but also for allocation efficiency – an efficient allocation would exclude advertisers whose valuations are less than the shadow costs they create. Furthermore, when shadow costs have a position-specific per-impression component, minimum bids are no longer independent of CTRs: given the per impression shadow cost, advertisers who have high CTRs should have a smaller minimum bid. This finding provides a theoretical justification for using CTR-dependent minimum bids observed in search engine practices.

**Implications of shadow costs for number of positions and when to fill a position.** From the perspective of optimal KAMs, some positions may be unfilled because other positions get priority and none of the remaining advertisers can generate a positive net worth at these positions.

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8 Minimum bids are used to increase auction revenue by forcing bidders to pay more. Minimum bids decrease efficiency because they exclude bidders who may have a positive valuation for the auctioned object.
When shadow costs consist of only an advertiser-specific component, this usually means that low positions (i.e., ones with low CTRs $\alpha_j^n$) are not filled. However, high positions may be unfilled while low positions are filled, when shadow cost consists of a position-specific component and high positions have higher shadow costs on a per-click basis ($c_j^n / \alpha_j^n$). This finding provides theoretical justifications for the observation that search engines sometimes avoid showing advertisements at the top of the search results; rather they feature advertisements on the right side because top positions incur higher per-click shadow costs than do right-side positions. Our result resonates with the empirical observation by Goldfarb and Tucker (2011a) that it may be undesirable to advertise on the most intrusive positions in targeted Internet advertising.

**Implications on how to incorporate shadow costs.** At the beginning of the paper, we made a point that user experience factors, such as relevance of advertisements to search terms and landing-page quality, should be considered in keyword auctions. But how? Our results show that shadow costs enter the keyword auction through two main constructs: scoring rules and minimum bids. Results in Case II suggest that the advertiser-specific component enters both the scoring rule and minimum bids. Specifically, in the calculation of scores and minimum bids, we can simply subtract the advertiser-specific shadow cost from the advertiser’s virtual valuation. In contrast, the position-specific component enters only the minimum bid calculations. It is worth noting that shadow costs do not enter keyword auctions as a weighting factor as CTR does. Our results thus suggest an alternative to the practice that stuffs CTRs and shadow-cost-related factors into a single weighting factor (often called “quality score”).

As a nascent mechanism for a new market, the design of keyword auctions is still evolving. As the keyword advertising business thrives, advertisers have learned how to do well in keyword auctions including driving up their CTRs sometimes at the cost of user experiences. It is
therefore all the more important to incorporate shadow costs into the next generation of keyword auction designs. As the first researchers to study shadow costs in a mechanism design framework, we hope to both highlight the issues created by shadow costs and to bridge the gap between keyword auction theory and practice.

To navigate the complexity of introducing shadow costs, we simplified assumptions. For example, we assumed the risk neutrality of advertisers and the search engine and the independence between valuations, CTRs, and shadow costs. It would be interesting to examine whether relaxing these assumptions would lead to significant new insights. In the analysis of special cases, we make additional assumptions such as separable CTRs. While empirical evidence refutes the separability assumption, our interaction with keyword advertising professionals suggests that the discrepancy may be tolerable in pursuit of theoretical results.

As a first step toward understanding shadow costs, additional research could extend this paper in several ways. It is clearly desirable to examine shadow costs from the social welfare point of view by incorporating users. It may also be interesting to examine how shadow costs affect the competition between advertisers in a more detailed model. Another potentially interesting extension is to examine how shadow costs interact with payment schemes, including cost-per-click, CPM (cost-per mille-impressions), and cost-per-sale, in the optimal keyword auction designs. Our research also raises many interesting empirical questions. For example, how large are the shadow costs? How do shadow costs differ across advertisers and positions? How well do search engines account for shadow costs when they assign advertisers to positions? How do different keyword auction designs impact user experience and search engine long-run revenue? Finally, while we use search engine keyword advertising as a context for our model and analysis, our results clearly have implications for future research to consider other advertising formats and
platforms, including display advertising and mobile advertising.

References


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**Appendix**

**Proof of Lemma 1:**

First, by (6)

\[
\pi_i (b_i | v_i) = E_i E_{a_{-i}} E_{v_{-i}} \left[ v_i p_i (b_i, v_{-i}) - m_i (b_i, v_{-i}) \right]
\]

\[
= E_i E_{a_{-i}} E_{v_{-i}} \left[ b_i p_i (b_i, v_{-i}) - m_i (b_i, v_{-i}) + (v_i - b_i) p_i (b_i, v_{-i}) \right]
\]

\[
= \pi_i (b_i | b_i) + (v_i - b_i) \overline{p}_i (b_i)
\]

(if part) Suppose conditions (i)–(iv) hold, we show the mechanism is feasible. Of course, the condition (iv) implies conditions (2) and (3) in the definition of feasibility. By conditions (i) and (ii), we have

\[
\pi_i (v_i | v_i) = \pi_i (v | v) + \int_{\xi} \overline{p}_i (t) dt \geq \pi_i (v | v) \geq 0
\]
So the individual rationality condition (10) is met.

From (A1), conditions (i) and (iii),

\[ \pi_i(v_i | v_i) - \pi_i(b_j | v_i) = \pi_i(v_i | v_i) - \pi_i(b_j | b_j) - (v_i - b_j) \bar{p}_i(b_j) \]

\[ = \int_0^t \bar{p}_i(t) dt - (v_i - b_j) \bar{p}_i(b_j) \]

\[ = \int_0^t [\bar{p}_i(t) - \bar{p}_i(b_j)] dt \geq 0 \]

where the first equality is due to (A1), the second is due to condition (i), and the inequality is due to condition (iii). Thus the incentive compatibility condition (11) is met. We conclude that the mechanism that meets conditions (i) ~ (iv) is feasible.

(only if part) Suppose a KAM is feasible: we now show that it must satisfy conditions (i)~(iv). Obviously, when the individual rationality condition (10) and allocation constraints (2) and (3) hold, the conditions (ii) and (iv) must also hold.

Applying the incentive compatibility condition (11) and equality (A1), we have

\[ \pi_i(v_i | v_i) \geq \pi_i(b_j | v_i) = \pi_i(b_j | b_j) + (v_i - b_j) \bar{p}_i(b_j) \]

i.e.,

\[ \pi_i(v_i | v_i) - \pi_i(b_j | b_j) \geq (v_i - b_j) \bar{p}_i(b_j) \]

Similarly, by an exchange of \( b_i \) and \( v_i \), \( \pi_i(b_j | b_j) - \pi_i(v_i | v_i) \geq (b_j - v_i) \bar{p}_i(v_i) \), i.e.,

\[ \pi_i(v_i | v_i) - \pi_i(b_j | b_j) \leq (v_i - b_j) \bar{p}_i(v_i) \]

Therefore, \( (v_i - b_j) \bar{p}_i(b_j) \leq \pi_i(v_i | v_i) - \pi_i(b_j | b_j) \leq (v_i - b_j) \bar{p}_i(v_i) \). Hence the condition (iii) must hold. Furthermore, \( \pi_i(v_i | v_i) \) is non-decreasing and continuous, thus \( \pi_i(v_i | v_i) \) is absolutely continuous and \( \frac{d \pi_i(v_i | v_i)}{d v_i} = \bar{p}_i(v_i) \), a.e. We conclude that condition (i) also holds. \( \square \)

**Proof of Lemma 2:**
By (12), (7) and (1), we have

\[
R = \sum_{i=1}^{n} E_{a_i} E_{v_i}[\bar{m}_i(v_i)] = \sum_{i=1}^{n} E_{a_i} \left\{ \int_0^\infty v_i \bar{p}_i(v_i) - \pi_i(v_i) - \int_0^\infty \bar{p}_i(t)dt \right\} f_i(v_i) dv_i = \sum_{i=1}^{n} E_{a_i} \left\{ \int_0^\infty v_i \bar{p}_i(v_i) f_i(v_i) dv_i - \int_0^\infty \bar{p}_i(t)dt f_i(v_i) dv_i \right\} = \sum_{i=1}^{n} E_{a_i} \left\{ \int_0^\infty v_i \bar{p}_i(v_i) f_i(v_i) dv_i - \int_0^\infty \left[1 - F_i(v_i)\right] \bar{p}_i(v_i) dv_i \right\} = \sum_{i=1}^{n} E_{a_i} \left\{ \int_0^\infty v_i \bar{p}_i(v_i) f_i(v_i) dv_i - \int_0^\infty \left[1 - F_i(v_i)\right] \bar{p}_i(v_i) dv_i \right\} - \sum_{i=1}^{n} \pi_i(v_i) = \sum_{i=1}^{n} E_{a_i} \left\{ \int_0^\infty v_i \bar{p}_i(v_i) f_i(v_i) dv_i \right\} - \sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} E_{a_i} \left\{ \int_0^\infty v_i \bar{p}_i(v_i) f_i(v_i) dv_i \right\} - \sum_{i=1}^{n} \pi_i(v_i) = E_c E_{a_v} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{k} \alpha_{ij} p_{ij}(v) \left[ v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] \right\} - \sum_{i=1}^{n} \pi_i(v_i) \]

where the second equality is due to (12), the fourth equality is an application of integration by parts, the sixth equality is due to (7), and the seventh equality is due to (1).

Substituting (9), we can obtain the search engine’s expected profit

\[
\pi_0 = R - C = \sum_{i=1}^{n} E_{a_i} E_{v_i}[\bar{m}_i(v_i)] - E_c E_{a_v} \left[ \sum_{i=1}^{n} \sum_{j=1}^{k} \alpha_{ij} p_{ij}(v) c_{ij} \right] = E_c E_{a_v} \left\{ \sum_{j=1}^{k} \sum_{i=1}^{n} \alpha_{ij} p_{ij}(v) \left[ v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] \right\} - \sum_{i=1}^{n} \pi_i(v_i) = E_c E_{a_v} \left\{ \sum_{j=1}^{k} \sum_{i=1}^{n} \alpha_{ij} p_{ij}(v) \left[ v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} - \frac{c_{ij}}{\alpha_{ij}} \right] \right\} - \sum_{i=1}^{n} \pi_i(v_i).
\]

□

Proof of Lemma 3:
We will show \( p \) satisfies condition (iii) and thus \( p \) solves (OP). Consider valuation vectors \( \mathbf{v} = (v_1, v_2, ..., v_n)^T \) and \( \mathbf{v}' = (v_1, v_2, ..., v_{i-1}, v_i + \varepsilon, ..., v_n)^T \), for any \( \varepsilon > 0 \). Let \( \mathbf{P} = \{ p_j(v) \} \) and \( \mathbf{P}' = \{ p_j(v') \} \) be optimal allocation under \( \mathbf{v} \) and \( \mathbf{v}' \) respectively. Let

\[
    w_i(\mathbf{v}, \mathbf{P}) = \sum_{j=1}^{n} p_j(\mathbf{v})[\alpha_j J_i(v_j) - c_j]
\]

is the total net worth of advertiser \( i \) and \( w_i(\mathbf{v}, \mathbf{P}') \) represent the total net worth of advertisers other than \( i \). \( w_i(\mathbf{v}, \mathbf{P}') \) and \( w_i(\mathbf{v}, \mathbf{P}) \) are similar for allocation \( \mathbf{P}' \). Since \( \mathbf{P} \) and \( \mathbf{P}' \) are optimal under \( \mathbf{v} \) and \( \mathbf{v}' \) respectively, we have

\[
    w_{-i}(\mathbf{v}, \mathbf{P}) + w_i(\mathbf{v}, \mathbf{P}) \geq w_{-i}(\mathbf{v}, \mathbf{P}') + w_i(\mathbf{v}, \mathbf{P}')
\]

\[
    w_{-i}(\mathbf{v}', \mathbf{P}') + w_i(\mathbf{v}', \mathbf{P}') \geq w_{-i}(\mathbf{v}', \mathbf{P}) + w_i(\mathbf{v}', \mathbf{P})
\]

Noticing that if we hold the allocation constant, the total net worth of advertisers other than \( i \) is the same under \( \mathbf{v} \) and \( \mathbf{v}' \), i.e., \( w_{-i}(\mathbf{v}, \mathbf{P}) = w_{-i}(\mathbf{v}', \mathbf{P}) \) and \( w_{-i}(\mathbf{v}, \mathbf{P}') = w_{-i}(\mathbf{v}', \mathbf{P}') \). So we combine the two inequalities as

\[
    w_i(\mathbf{v}, \mathbf{P}) + w_i(\mathbf{v}', \mathbf{P}') \geq w_i(\mathbf{v}', \mathbf{P}) + w_i(\mathbf{v}, \mathbf{P}')
\]

Substituting

\[
    w_i(\mathbf{v}, \mathbf{P}) - w_i(\mathbf{v}', \mathbf{P}') = [J_i(v_i) - J_i(v_i + \varepsilon)] p_i(\mathbf{v})
\]

and

\[
    w_i(\mathbf{v}, \mathbf{P}') - w_i(\mathbf{v}', \mathbf{P}') = [J_i(v_i) - J_i(v_i + \varepsilon)] p_i(\mathbf{v}')
\]

we have

\[
    [J_i(v_i) - J_i(v_i + \varepsilon)] p_i(\mathbf{v}) \geq [J_i(v_i) - J_i(v_i + \varepsilon)] p_i(\mathbf{v}')
\]

Since \( J_i(v_i) < J_i(v_i + \varepsilon) \), we must have \( p_i(\mathbf{v}) \leq p_i(\mathbf{v}') \), which implies that \( p \) satisfies (iii). Moreover, \( p \) satisfies (iv) by the definition of (OP). So \( p \) solves (OP). □

**Proof of Lemma 4:**

By (15), any unassigned advertiser pays zero. The total payment from assigned advertiser \( i \)
is,

\[ m_i(v) = v_i p_i(v_i, v_{-i}) - \int_{\varphi(i)}^0 p_i(t, v_{-i}) \, dt \]

\[ = v_i \alpha_{i, \varphi(i)} - \int_{\varphi(i)}^0 p_i(t, v_{-i}) \, dt \]

By the definition of \( b_{ij} \), it can be seen that \( b_{ij} \) is non-increasing with position. Moreover, when advertiser \( i \) bids between \([b_{is}, b_{i,s-1}]\), she will obtain position \( s \) for some \( s > \varphi(i) \).\(^9\)

Therefore,

\[ \int_{\varphi(i)}^0 p_i(t, v_{-i}) \, dt = \int_{\varphi(i)}^{b_{i,\varphi(i)}} p_i(t, v_{-i}) \, dt + \int_{b_{i,\varphi(i)}}^{b_{i,\varphi(i)+1}} p_i(t, v_{-i}) \, dt \]

\[ = \alpha_{i, \varphi(i)} (v_i - b_{i, \varphi(i)}) + \int_{\varphi(i)}^{b_{i,\varphi(i)+1}} p_i(t, v_{-i}) \, dt \]

\[ = \alpha_{i, \varphi(i)} (v_i - b_{i, \varphi(i)}) + \alpha_{i, \varphi(i)+1} (b_{i, \varphi(i)} - b_{i, \varphi(i)+1}) + \cdots \]

\[ + \alpha_{i, k-1} (b_{i, k-2} - b_{i, k-1}) + \int_{b_{i, k-1}}^{b_{i, k}} p_i(t, v_{-i}) \, dt + \int_{b_{i, k}}^{b_s} p_i(t, v_{-i}) \, dt \]

\[ = \alpha_{i, \varphi(i)} (v_i - b_{i, \varphi(i)}) + \alpha_{i, \varphi(i)+1} (b_{i, \varphi(i)} - b_{i, \varphi(i)+1}) + \cdots \]

\[ + \alpha_{i, k-1} (b_{i, k-2} - b_{i, k-1}) + \alpha_{i, k} (b_{i, k-1} - b_{i, k}) + 0 \]

\[ = \alpha_{i, \varphi(i)} v_i - \sum_{k=\varphi(i)}^{k_s} (\alpha_{i, k} - \alpha_{i, k+1}) b_{is} \]

where \( \alpha_{i, k+1} = 0 \). Then we get \( m_i(v) = \sum_{k=\varphi(i)}^{k_s} (\alpha_{i, k} - \alpha_{i, k+1}) b_{is} \). \( \square \)

**Proof of Theorem 4:**

Under assumption that \( c_{ij} = \alpha_i^a \alpha_j^p c_i^a \), (OP') becomes

\[ \max_p \sum_{i=1}^n \sum_{j=1}^p p_{ij}(v) \alpha_j^p \left[ J(v_i) - c_i^a \right] \] s.t. (iv), for any \( c, \alpha, v \) \hspace{1cm} (OP'')

\(^9\) When \( b_{is} = b_{i,s-1} \), advertiser \( i \) can’t get position \( s \) but the proof can nevertheless go through.
It is straightforward that the optimal KAM is a scoring mechanism.

To see that the greedy allocation rule is optimal, we note that according to (OP'), if \( s^{\text{II}}(i) < 0 \), it is never optimal to assign \( i \) to any position. On the other hand, it is never optimal to leave position \( j \) empty if there exists some unassigned advertiser \( i \) such that \( s^{\text{II}}(i) > 0 \). Also, it is obviously never optimal to fill position \( j \) (\( \leq k \)) while \( j - 1 \) is empty. Now, suppose \( s^{\text{II}}(i) > s^{\text{II}}(l) \) for some advertisers \( i \) and \( l \). For any positions \( j < s \), we have \( \alpha_j s^{\text{II}}(i) + \alpha_j s^{\text{II}}(l) \geq \alpha_j s^{\text{II}}(i) + \alpha_j s^{\text{II}}(l) \). So it is never optimal to assign advertiser \( l \) before advertiser \( i \). The above arguments suggest that the allocation function stated in Theorem 4 is optimal. □

**Proof of Theorem 5:**

Under the assumption \( c_{ij} = c_{ij}^{\alpha} + \alpha_i^{\alpha} \alpha_j^{\alpha} c_{ij}^{\alpha} \), (OP') becomes

\[
\max_{\mathbf{p}} \sum_{j=1}^{k} \sum_{i=1}^{n} p_{ij}(\mathbf{v}) \alpha_j^{\alpha} \left\{ \alpha_i^{\alpha}\left[j(\mathbf{v}_i) - c_i^{\alpha}\right] - \frac{c_{ij}^{\alpha}}{\alpha_j^{\alpha}} \right\}, \quad \text{s.t. (iv), for any } \mathbf{c}, \mathbf{a}, \mathbf{v} \quad \text{(OP'')}
\]

Since (16) holds, it is easy to know that the optimal KAM is a scoring mechanism.

To see that the greedy allocation rule is optimal, we note that, according to (OP''), if \( s^{\text{III}}(i) - \frac{c_{ij}^{\alpha}}{\alpha_j^{\alpha}} < 0 \), it is never optimal to assign \( i \) to position \( j \). On the other hand, it is never optimal to leave position \( j \) empty if there exists some unassigned advertiser \( i \) such that \( s^{\text{III}}(i) - \frac{c_{ij}^{\alpha}}{\alpha_j^{\alpha}} > 0 \). If there are two advertisers \( i \) and \( l \) with \( s^{\text{III}}(i) > s^{\text{III}}(l) > 0 \), then for any \( s > j \), \( \alpha_j^{\alpha} [s^{\text{III}}(i) - \frac{c_{ij}^{\alpha}}{\alpha_j^{\alpha}}] + \alpha_j^{\alpha} [s^{\text{III}}(l) - \frac{c_{ij}^{\alpha}}{\alpha_j^{\alpha}}] \) (note that \( \alpha_j^{\alpha} \geq \alpha_s^{\alpha} \geq 0 \) and the equality...
only holds when \( \alpha_j^p = \alpha_k^p = 0 \) or \( j > k \). Therefore, like in Theorem 4, it is never optimal to place advertiser \( l \) before advertiser \( i \). Finally, if \( s^{III}(i) - \frac{c_j^p}{\alpha_j^p} > 0 \), it is never optimal to assign advertiser \( i \) to position \( j + 1 \) while \( j \) is empty. This is because \( \alpha_j^p (s^{III}(i) - \frac{c_j^p}{\alpha_j^p}) > \alpha_{j+1}^p (s^{III}(i) - \frac{c_{j+1}^p}{\alpha_{j+1}^p}) \) (note that \( \alpha_j^p > \alpha_{j+1}^p \) and \( \frac{c_j^p}{\alpha_j^p} \leq \frac{c_{j+1}^p}{\alpha_{j+1}^p} \)). The above arguments suggest that the allocation function stated in Theorem 5 is optimal. \( \square \)