A Simple, Accurate Approximation for the Outage Probability of Equal-Gain Receivers with Cochannel Interference in an $\alpha$-$\mu$ Fading Scenario

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Abstract—Sums of fading envelopes arise in several wireless communications applications, such as equal-gain combining, outage probability, signal detection, etc. In this paper, based on the intricate task of evaluating the exact statistics of such sums, accurate approximations for the outage probability of equal-gain receivers subject to arbitrary independent cochannel interferers are proposed. In our analysis, an $M$-branch, $N$-interferer system subject to an $\alpha$-$\mu$ fading scenario is considered. The results are obtained in terms of a singlefold integral, which replaces the $M \times N$ nested integrals required in the exact solution. The accuracy of our formulation is checked by comparing it with Monte Carlo simulation results. It is shown that approximate and simulated curves are practically indistinguishable from each other.

Index Terms—$\alpha$-$\mu$ fading channels, cochannel interference, equal-gain combining, moment-based estimators, outage probability.

I. INTRODUCTION

In wireless systems, frequency reuse is a rather common and efficient technique used to provide better utilization of the spectrum. If, on the one hand this improves the spectrum efficiency, on the other hand it gives rise to the so called cochannel interference (CCI). Several techniques are available that combat the deleterious effects of CCI in fading channel, diversity combining schemes being one of them. Equal-gain combining (EGC) is certainly of interest for it provides a performance that approaches that of the maximal-ratio combining (MRC) [1]. Note that, in general, the highest output signal-to-interference-plus-noise ratio (SINR) at the receiver is achieved by use of the optimum combining [2], [3], while MRC is optimum under the assumption of independent interference at each receiving antenna.

Although the performance of systems subject to CCI has been extensively explored in the literature [4]–[11], not many works investigate it as far as the EGC technique is concerned. In fact, the exact performance analysis of EGC alone requires the knowledge of the sum statistics of the branch envelopes, which is available in terms of multifold convolution or the inverse transform of product of moment generation functions. The resulting expressions are usually very unfriendly. Now, if CCI combined with EGC is to be investigated, then the untractability of the problem becomes even more apparent, for the performance analysis requires the knowledge of the statistics of the ratio random variables in which both numerator and denominator appear a sum of random variables. In such a case, simple, accurate approximations are of interest.

In [12], a Nakagami-$m$ approximation to the sum of independent identically distributed (i.i.d.) Nakagami-$m$ RVs is proposed. Anchored on that idea, the parameters of the approximate Nakagami-$m$ distribution of the sum of two correlated identically distributed Nakagami-$m$ RVs were obtained in [13]. In [14], [15], closed-form approximations to the Rayleigh and Rice sum probability density function (PDF), respectively, were obtained based either on a modification of the small argument approximation (Rayleigh case) or on a modification to the sum distribution of squared Ricean RVs (Rice case). A set of works employing moment-based estimators were presented in [16]–[19] for approximating sums of Weibull, Nakagami-$m$, Rice, and Hoyt RVs. More recently [20], highly accurate closed-form approximations to the PDF and cumulative distribution function (CDF) of the sum of independent $\alpha$-$\mu$ (generalized gamma) RVs were provided. In that case, the approximate and exact solutions showed to be almost indistinguishable.

In this paper, relying upon the idea presented in [20], we propose approximate formulations for the outage probability (OP) of EGC receivers with CCI and undergoing $\alpha$-$\mu$ fading. The results are obtained in terms of a singlefold integral, being therefore easily implemented in the most popular computing softwares. The accuracy of our formulation is checked by comparing it with some Monte Carlo simulation results, and an excellent agreement is observed.

II. THE $\alpha$-$\mu$ DISTRIBUTION

The $\alpha$-$\mu$ fading model [21] considers that the signal is composed by multipath clusters propagating in a non-homogeneous environment. It comprises both Nakagami-$m$ and Weibull as special cases. The PDF $f_R(r)$ of the envelope
$R$ is given by
\[ f_R(r) = \frac{\alpha \mu^\alpha r^{\alpha - 1}}{\Gamma(\mu)} \exp\left(-\frac{r^\alpha}{\tilde{r}^\alpha}\right), \] (1)

where $\tilde{r} = \sqrt[\alpha]{E(R^\alpha)}$, $\alpha > 0$ (which is related to the non-linearity of the medium), and $\mu > 0$ (which is associated to the number of multipath clusters). $\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t)dt$ is the gamma function, and $E(\cdot)$ denotes expectation. The CDF of $R$ can be found
\[ F_R(r) = \frac{\Gamma(\mu, \frac{r^\alpha}{\tilde{r}^\alpha})}{\Gamma(\mu)}, \] (2)

where $\Gamma(z, y) = \int_y^\infty t^{z-1} \exp(-t)dt$ is the incomplete gamma function. The $k$-th moment $E(R^k)$ can be expressed as
\[ E(R^k) = \tilde{r}^k \frac{\Gamma(\mu + k/\alpha)}{\mu^{k/\alpha} \Gamma(\mu)}. \] (3)

For $\alpha = 2$, (1) particularizes to the Nakagami-$m$ PDF, whereas for $\mu = 1$, (1) reduces to the Weibull one.

### III. System Model

In our analysis, we consider an EGC receiver composed by $M$ antennas conveniently spaced so that the signals arriving at them are independent. In EGC, the received signals are cophased, equally weighted, and added to give the resultant desired signal. We assume that there are $N$ cochannel interferers and that the desired signals are coherently summed, whereas the interfering signals are incoherently summed [22]. In such a case, the signal-to-interference power ratio (SIR) of these systems are modeled as
\[ Z = \left(\frac{X}{Y}\right)^2, \] (4)

where
\[ X = \sum_{i=1}^M X_i, \] (5)

represents the sum of the desired signals at the diversity branches and
\[ Y^2 = \sum_{j=1}^N \sum_{i=1}^M Y_{i,j}^2, \] (6)

stands for the sum of the powers of the interference signals at the diversity branches. Here, $X_i$ is the desired signal envelope at the $i$-th diversity branch and $Y_{i,j}$ denotes the envelope of the interference signal caused by the $j$-th interfering carrier at the $i$-th diversity branch. In the calculations used here, $X_i$ and $Y_{i,j}$ are assumed to be $\alpha$-$\mu$ distributed with parameters $(\alpha_i, \mu_i, \tilde{x}_i)$ and $(\alpha_{i,j}, \mu_{i,j}, \tilde{y}_{i,j})$, respectively. In this case, $\tilde{x}_i = \alpha_i \sqrt{E(X_i^\alpha)}$ and $\tilde{y}_{i,j} = \alpha_{i,j} \sqrt{E(Y_{i,j}^\alpha)}$.

### IV. Outage Probability - Exact Formulation

The OP is defined as the probability that the received signal goes beneath a given threshold, $z_{th}$, and it can be expressed as
\[ P_{out} = Pr[Z < z_{th}], \] (7)

which is the CDF of $Z$, namely $F_Z(z)$, evaluated at $Z = z_{th}$. In [23], an unified approach for computing the OP in wireless systems is proposed, which is given by
\[ F_Z(z_{th}) = \int_0^\infty F_X(y \sqrt{z_{th}}) f_Y(y) dy, \] (8)

From (8), note that a tractable exact expression for the OP is very difficult to attain, if not impossible. This is because neither the PDF of $Y$, $f_Y(y)$, nor the CDF of $X$, $F_X(x)$, can be obtained in a simple manner for the general case. One of the possible exact solutions involves multifold integrals or integral of the product of moment generating functions, certainly non attractive approaches as the number of diversity branches and interfering signals increases. For instance, for $M$ branches and $N$ interferers the number of nested integrals necessary to solve this problem is $M \times N$, which turns out to be intractable for practical applications (e.g., $M = 2$ and $N = 6$, then 12 integrals are necessary). In order to circumvent this, we propose an approximate formulation for the OP given in (8). Besides being very simple, it is highly accurate, as shall be seen from the numerical examples given in Section VI.

### V. Outage Probability - Approximate Formulation

With the aim at arriving at a tractable approximate expression, we propose initially to approximate $f_X(y)$ and $F_X(y \sqrt{z_{th}})$ by the PDF and CDF of an $\alpha$-$\mu$ variate, as shown in (1) and (2), respectively. The motivation for this comes from the fact that the sum of independent $\alpha$-$\mu$ variates can be well approximated by a single $\alpha$-$\mu$ variate [20]. The procedure is detailed next.

Firstly, we investigate the approximation of $F_X(y \sqrt{z_{th}})$ by the CDF of an $\alpha$-$\mu$ variate (2), i.e.,
\[ F_X(y \sqrt{z_{th}}) \approx \frac{\Gamma(\mu_S, \mu_S (y \sqrt{z_{th}})^\alpha / \tilde{x}_S^{\alpha_S})}{\Gamma(\mu_S)}, \] (9)

In order to render (9) a good approximation, we use moment-based estimators to calculate $\mu_S$, $\mu_S$, and $\tilde{x}_S = \sqrt{E(\tilde{X}^{\alpha_S})}$ from the exact moments of $X$. Assume, for the moment, the knowledge of $E(X)$, $E(X^2)$, and $E(X^4)$. Then, moment-based estimators for $\alpha_S$, $\mu_S$, and $\tilde{x}_S$ can be written as [21]
\[ \Gamma^2(\mu_S + 1/\alpha_S) \] (10)

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\[ \Gamma^2(\mu_S + 1/\alpha_S) \Gamma(\mu_S + 2/\alpha_S) - \Gamma^2(\mu_S + 1/\alpha_S) = E(X^2) - E^2(X), \] (10)

The system of transcendental equations (10) and (11) must be numerically solved for $\alpha_S$ and $\mu_S$. To this end, built-in routines available in most popular computing softwares can be used in an efficient and straightforward manner. In MATHEMATICA, for instance, the required function is FindRoot. Having obtained $\alpha_S$ and $\mu_S$, then $\tilde{x}_S$ is estimated as in (12). It remains to find the exact moments $E(X)$, $E(X^2)$, and $E(X^4)$ required into (10), (11), and (12). By using the
multinomial expansion, these moments are obtained in terms of the individual moments of the \( \alpha-\mu \) summands as [20]

\[
E(X^n) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \cdots \sum_{n_{M-1}=0}^{n_{M-2}} \binom{n}{n_1} \binom{n_1}{n_2} \cdots \binom{n_{M-2}}{n_{M-1}} \sum_{n_M=0}^{n_{M-1}} \binom{n_{M-1}}{n_{M-1}}
\]

Note that these individual moments are given in (3) for the respective parameters \((\alpha, \mu, \hat{z}_t)\).

Now, let us analyze the approximation of \( f_Y(y) \) by the PDF of an \( \alpha-\mu \) variate (1), i.e.,

\[
f_Y(y) \approx \frac{\alpha I_{\mu,\mu}I_{\nu,\nu}^{-1}}{y^\nu \Gamma(\mu) \Gamma(\nu)} \exp \left( -\frac{\mu y^{\alpha}}{\nu} \right),
\]

where \( \hat{r} = \sqrt{E(R^2)} \) as in (1). The \( m \)-th moment of (15) is given by

\[
E(Y^m) = \frac{\hat{r}^{2m}}{\mu^{2m} \Gamma(m)} \prod_{i=1}^{m} \Gamma(\mu_i)
\]

which is the same as (3), except that \( k = 2m \). Having observed that, moment-based estimators for \( \alpha_i, \mu_i \), and \( \hat{y}_t \) can be written as [21]

\[
\hat{\alpha}_i = \frac{\Gamma(\mu_i + 2/\alpha_i)}{\Gamma(\mu_i + 4/\alpha_i) - \Gamma(\mu_i + 2/\alpha_i)} \quad \text{and} \quad \hat{\mu}_i = \frac{E(Y^2)}{E(Y^4) - E(Y^2)^2}
\]

\[
\hat{\mu}_i = \frac{\Gamma(\mu_i + 4/\alpha_i)}{\Gamma(\mu_i + 8/\alpha_i) - \Gamma(\mu_i + 4/\alpha_i)} \quad \text{and} \quad \hat{\mu}_i = \frac{E(Y^2)}{E(Y^4) - E(Y^2)^2}
\]

\[
\hat{y}_t = \frac{\mu_i^{\alpha_i} \Gamma(\mu_i) E(Y^2)}{\Gamma(\mu_i + 2/\alpha_i)}
\]

Note that as (6) consists of the sum of squared envelopes, the estimators are modified so that the contour conditions are satisfied. The exact moments required in (17), (18) and (19) can be evaluated as

\[
E(Y^{2n}) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \cdots \sum_{n_{M-1}=0}^{n_{M-2}} \binom{n}{n_1} \binom{n_1}{n_2} \cdots \binom{n_{M-2}}{n_{M-1}} \sum_{n_{M-1}}^{n_{M-2}} \binom{n_{M-1}}{n_{M-1}}
\]

\[
E(Y_m^{2n}) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \cdots \sum_{n_{N-1}=0}^{n_{N-2}} \binom{n}{n_1} \binom{n_1}{n_2} \cdots \binom{n_{N-2}}{n_{N-1}} \sum_{n_{N-1}}^{n_{N-2}} \binom{n_{N-1}}{n_{N-1}}
\]

in which

\[
E(Y^{2n}) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \cdots \sum_{n_{M-1}=0}^{n_{M-2}} \binom{n}{n_1} \binom{n_1}{n_2} \cdots \binom{n_{M-2}}{n_{M-1}} \sum_{n_{M-1}}^{n_{M-2}} \binom{n_{M-1}}{n_{M-1}}
\]

where \( m = 1, 2, \ldots, M \) and the individual moments in (21) are given in (3) for the respective parameters \((\alpha_{i,j}, \mu_{i,j}, \hat{y}_{i,j})\). Then, by substituting (9) and (14) in (8), we arrive at an approximate expression for the OP of EGC with CCI in \( \alpha-\mu \) fading channels.

It is noteworthy that the procedure applied here is very simple, with the determination of the appropriate parameters being done straightforwardly. In addition, an excellent match between the approximate and simulation data is obtained.

VI. NUMERICAL RESULTS

In this Section, we compare our approximate expressions with some Monte Carlo simulation results.

Figs. 1 and 2 illustrate the OP versus normalized SIR threshold for \( \mu_i = \mu_{i,j} = 2.5 \), \( \alpha_i = \alpha_{i,j} = 1.5 \) and \( N = 6 \). As expected, an improvement of the performance is noticed as \( \mu \) increases. In addition, an excellent match is also attested when approximate and simulated curves are compared. By varying the number of interferers signals, Fig. 4 sketches the OP for \( M = 3 \) using the same fading parameters of the Fig. 3. As expected, the performance decreases when \( N \) increases. A myriad of other fading conditions have been investigated and, in all of them, an excellent adjustment has observed.

Fig. 5 explores a scenario with distinct interference power distributions. In this case, it is useful to define a ratio of powers in order to examine the distinct interference-power distribution

\[
\lambda(dB) = 10 \log_{10} \frac{\sum_{i=1}^{M} P_i^S}{\sum_{i=1}^{M} \sum_{j=1}^{N} P_i^S q_{i,j}},
\]

where \( P_i^S \) is the desired signal power at the \( i \)-th diversity branch and \( P_i^S \) is the power of the \( i \)-th interfering cochannel at the \( i \)-th diversity branch. Moreover, a vector \( q = [q_{1,1}, q_{1,2}, \ldots, q_{i,j}] \) of normalized interference powers is defined, for which \( \sum_{j=1}^{N} \sum_{i=1}^{M} q_{i,j} = 1 \). It is noteworthy approximate and simulated curves are practically coincident.

VII. CONCLUSIONS

New accurate approximate formulations for the outage probability of equal-gain receivers subject to multiple independent cochannel interferers in \( \alpha-\mu \) fading channels have been provided. The results are obtained in terms of a singlefold integral, which replaces the \( M \times N \) nested integrals required in the exact solution. An excellent match between the approximate and simulation results has been observed.
Fig. 1. Outage probability versus normalized SIR threshold for $\mu_i = \mu_{i,j} = 2.5$, $N = 2$, $M = 3$, and $\alpha_i = \alpha_{i,j}$ varying.

Fig. 2. Outage probability versus normalized SIR threshold for $\mu_i = \mu_{i,j} = 2.5$, $N = 2$, $M = 4$, and $\alpha_i = \alpha_{i,j}$ varying.

Fig. 3. Outage probability versus normalized SIR threshold for $\mu_i = \mu_{i,j} = 2.5$, $\alpha_i = \alpha_{i,j} = 1.5$, $N = 6$, and $M$ varying.

Fig. 4. Outage probability versus normalized SIR threshold for $\mu_i = \mu_{i,j} = 2.5$, $\alpha_i = \alpha_{i,j} = 1.5$, $M = 3$, and $N$ varying.

REFERENCES


Outage probability versus normalized SIR threshold for distinct interference power distributions using $\mu_i = \mu_{i,j} = 2$, $M = 2$, $N = 2$, and $\alpha_i = \alpha_{i,j}$ varying.

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