Stochastic integration with respect to fractional Brownian motion and applications

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Abstract

Fractional Brownian motion (fBm) is a Gaussian stochastic process $B = \{B_t, t \geq 0\}$ with zero mean and covariance function given by $E(B_t B_s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$, where $0 < H < 1$ is the Hurst parameter. This process has stationary increments, self-similarity and long-range dependence properties. These properties make fBm a suitable driving noise in different applications like mathematical finance and network traffic analysis. In order to develop these applications, one needs to construct a stochastic calculus with respect to fBm. In the particular case $H = \frac{1}{2}$, the process B is an ordinary Brownian motion but for $H \neq \frac{1}{2}$ it is not a semimartingale and we cannot use the classical Itô calculus. The objective of these notes is to present some recent advances in the stochastic calculus with respect to fractional Brownian motion (fBm) and their applications.

Keywords: Fractional Brownian motion. Stochastic integral. Malliavin calculus. Itô's formula. Stochastic differential equations driven by fractional Brownian motion.

AMS Subject Classification: 60H05, 60H07, 60H10.

1 Fractional Brownian motion

A centered Gaussian process $B = \{B_t, t \geq 0\}$ is called *fractional Brownian motion* (fBm) of Hurst parameter $H \in (0, 1)$ if it has the covariance function

$$R_H(t, s) = E(B_t B_s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{1.1}$$

This process was first introduced by Kolmogorov [24] and studied by Mandelbrot and Van Ness in [29], where a stochastic integral representation in terms of a standard Brownian motion was established.

Fractional Brownian motion has the following *self-similar* property: For any constant $a > 0$, the processes $\left\{a^{-H} B_{at}, t \geq 0\right\}$ and $\{B_t, t \geq 0\}$ have the same distribution. This property is an immediate consequence of the fact that the covariance function (1.1) is homogeneous of order $2H$.

From (1.1) we can deduce the following expression for the variance of the increment of the process in an interval $[s,t]$:

$$E\left(|B_t - B_s|^2\right) = |t-s|^{2H}. \tag{1.2}$$

This implies that fBm has *stationary increments*.

By Kolmogorov's continuity criterion and (1.2) we deduce that fBm has a version with continuous trajectories. Moreover, by Garsia-Rodemich-Rumsey inequality, we can deduce the following modulus of continuity for the trajectories of fBm: For all $\varepsilon > 0$ and $T > 0$, there exists a nonnegative random variable $G_{\varepsilon,T}$ such that $E\left(|G_{\varepsilon,T}|^p\right) < \infty$ for all $p \geq 1$, and

$$|B_t - B_s| \leq G_{\varepsilon,T}|t-s|^{H-\varepsilon},$$

for all $s,t \in [0,T]$. In other words, the parameter H controls the regularity of the trajectories, which are Hölder continuous of order $H-\varepsilon$, for any $\varepsilon > 0$.

For $H = \frac{1}{2}$, the covariance can be written as $R_{\frac{1}{2}}(t,s) = t \wedge s$, and the process B is an ordinary Brownian motion. In this case the increments of the process in disjoint intervals are independent. However, for $H \neq \frac{1}{2}$, the increments are not independent. The covariance between two increments $B_{t+h} - B_t$ and $B_{s+h} - B_s$, where $s+h \leq t$, and $t-s = nh$ is

$$\begin{aligned}\rho_H(n) &= \frac{1}{2}h^{2H}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right) \\ &\approx h^{2H}H(2H-1)n^{2H-2} \to 0\end{aligned}$$

as n tends to infinity. Therefore:

i) If $H > \frac{1}{2}$, $\rho_H(n) > 0$ and $\sum_{n=1}^{\infty} \rho_H(n) = \infty$.

ii) If $H < \frac{1}{2}$, $\rho_H(n) < 0$ and $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$.

In case i) two increments of the form $B_{t+h} - B_t$ and $B_{t+2h} - B_{t+h}$, are positively correlated and the process presents an aggregation behavior. In case ii) these increments are negatively correlated and we say that there is intermittency.

1.1 Moving average representation

Mandelbrot and Van Ness obtained in [29] the following integral representation of fBm in terms of a Wiener process on the whole real line (see also Samorodnitsky and Taqqu [42]):

$$B_t = \frac{1}{C_1(H)} \int_{\mathbb{R}} \left[\left((t-s)^+\right)^{H-\frac{1}{2}} - \left((-s)^+\right)^{H-\frac{1}{2}}\right] dW_s,$$

where $\{W(A), A \text{ Borel subset of } \mathbb{R}\}$ is a Brownian measure on $\mathbb{R}$ and

$$C_1(H) = \left(\int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right)^2 ds + \frac{1}{2H} \right)^{\frac{1}{2}}.$$

Proof. Set $f_t(s) = ((t-s)^+)^{H-\frac{1}{2}} - ((-s)^+)^{H-\frac{1}{2}}$, $s \in \mathbb{R}$, $t \geq 0$. Notice that $\int_{\mathbb{R}} f_t(s)^2 ds < \infty$. In fact, if $H \neq \frac{1}{2}$, as s tends to $-\infty$, $f_t(s)$ behaves as $(-s)^{H-\frac{3}{2}}$ which is square integrable at infinity. For $t \geq 0$ set

$$X_t = \int_{\mathbb{R}} \left[\left((t-s)^+\right)^{H-\frac{1}{2}} - \left((-s)^+\right)^{H-\frac{1}{2}} \right] dW_s.$$

We have

$$\begin{aligned}
E(X_t^2) &= \int_{\mathbb{R}} \left[\left((t-s)^+\right)^{H-\frac{1}{2}} - \left((-s)^+\right)^{H-\frac{1}{2}} \right]^2 ds \\
&= t^{2H} \int_{\mathbb{R}} \left[\left((1-u)^+\right)^{H-\frac{1}{2}} - \left((-u)^+\right)^{H-\frac{1}{2}} \right]^2 du \\
&= t^{2H} \left(\int_{-\infty}^0 \left[(1-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} \right]^2 du + \int_0^1 (1-u)^{2H-1} du \right) \\
&= C_1(H)^2 t^{2H}. \qquad (1.3)
\end{aligned}$$

Similarly, for any $s < t$ we obtain

$$\begin{aligned}
E(|X_t - X_s|^2) &= \int_{\mathbb{R}} \left[\left((t-u)^+\right)^{H-\frac{1}{2}} - \left((s-u)^+\right)^{H-\frac{1}{2}} \right]^2 du \\
&= \int_{\mathbb{R}} \left[\left((t-s-u)^+\right)^{H-\frac{1}{2}} - \left((-u)^+\right)^{H-\frac{1}{2}} \right]^2 du \\
&= C_1(H)^2 |t-s|^{2H}. \qquad (1.4)
\end{aligned}$$

From (1.3) and (1.4) we deduce that the centered Gaussian process $\{X_t, t \geq 0\}$ has the covariance R_H of a fBm with Hurst parameter H. ∎

Notice that the above integral representation implies that the function R_H defined in (1.1) is a covariance function, that is, it is symmetric and nonnegative definite.

It is also possible to establish the following spectral representation of fBm (see Samorodnitsky and Taqqu [42]):

$$B_t = \frac{1}{C_2(H)} \int_{\mathbb{R}} \frac{e^{its} - 1}{is} |s|^{\frac{1}{2}-H} d\widetilde{W}_s,$$

where $\widetilde{W} = W^1 + iW^2$ is a complex Gaussian measure on $\mathbb{R}$ such that $W^1(A) = W^1(-A)$, $W^2(A) = -W^2(A)$, and $E(W^1(A)^2) = E(W^1(A)^2) = \frac{|A|}{2}$, and

$$C_2(H) = \left(\frac{\pi}{H\Gamma(2H) \sin H\pi} \right)^{\frac{1}{2}}.$$

1.2 Hurst's statistical phenomenon of self-similarity

Hurst developed in [23] a statistical analysis of the yearly water run-offs of Nile river. Suppose that $x_1, \ldots, x_n$ are the values of n successive yearly water run-offs. Denote by $X_n = \sum_{k=1}^n x_k$ the cumulative values. Then, $X_k - \frac{k}{n} X_n$ is the deviation of the cumulative value X_k corresponding to k successive years from the empirical means as calculated using data for n years. Consider the range of the amplitude of this deviation:

$$\mathcal{R}_n = \max_{1 \leq k \leq n} \left(X_k - \frac{k}{n} X_n \right) - \min_{1 \leq k \leq n} \left(X_k - \frac{k}{n} X_n \right)$$

and the empirical mean deviation

$$\mathcal{S}_n = \sqrt{\frac{1}{n} \sum_{k=1}^n \left(x_k - \frac{X_n}{n} \right)^2}.$$

Based on the records of observations of Nile flows in 622-1469, Hurst discovered that $\frac{\mathcal{R}_n}{\mathcal{S}_n}$ behaves as cn^H, where $H = 0.7$. On the other hand, the partial sums $x_1 + \cdots + x_n$ have approximately the same distribution as $n^H x_1$, where again H is a parameter larger than $\frac{1}{2}$.

These facts lead to the conclusion that one cannot assume that $x_1, \ldots, x_n$ are values of a sequence of independent and identically distributed random variables. Some alternative models are required in order to explain the empirical facts. One possibility is to assume that $x_1, \ldots, x_n$ are values of the increments of a fractional Brownian motion. Motivated by these empirical observations, Mandelbrot has given the name of Hurst parameter to the parameter H of fBm.

1.3 Non semimartingale property

We have seen that for $H \neq \frac{1}{2}$ fBm does not have independent increments. In this subsection we will show that for $H \neq \frac{1}{2}$ fBm is not a semimartingale. A proof in the case $H > \frac{1}{2}$ can be found in [25] (see also Example 4.9.2 in Liptser and Shiryaev [26]). We will present here the proof given by Rogers in [38] for any $H \neq \frac{1}{2}$. The main arguments of this proof are as follows. For $p > 0$ set

$$Y_{n,p} = n^{pH-1} \sum_{j=1}^n \left| B_{j/n} - B_{(j-1)/n} \right|^p.$$

By the self-similar property of fBm, the sequence $\{Y_{n,p}, n \geq 1\}$ has the same distribution as $\{\widetilde{Y}_{n,p}, n \geq 1\}$, where

$$\widetilde{Y}_{n,p} = n^{-1} \sum_{j=1}^n \left| B_j - B_{j-1} \right|^p.$$

The stationary sequence $\{B_j - B_{j-1}, j \geq 1\}$ is mixing. Hence, by the Ergodic Theorem $\widetilde{Y}_{n,p}$ converges almost surely and in L^1 to $E\left(|B_1|^p\right)$ as n tends to infinity. As a consequence, $Y_{n,p}$ converges in probability as n tends to infinity to $E\left(|B_1|^p\right)$. Therefore,

$$V_{n,p} = \sum_{j=1}^{n} \left| B_{j/n} - B_{(j-1)/n} \right|^p$$

converges in probability to zero as n tends to infinity if $pH > 1$, and to infinity if $pH < 1$. Consider the following two cases:

i) If $H < \frac{1}{2}$, we can choose $p > 2$ such that $pH < 1$, and we obtain that the p-variation of fBm (defined as the limit in probability $\lim_{n\to\infty} V_{n,p}$) is infinite. Hence, the quadratic variation ($p = 2$) is also infinity.

ii) If $H < \frac{1}{2}$, we can choose p such that $\frac{1}{H} < p < 2$. Then the p-variation is zero, and, as a consequence, the quadratic variation is also zero. On the other hand, if we choose p such that $1 < p < \frac{1}{H}$ we deduce that the total variation is infinite.

Therefore, we have proved that for $H \neq \frac{1}{2}$ fBm cannot be a semimartingale.

In a recent paper [7] Cheridito has introduced the notion of *weak semimartingale* as a stochastic process $\{X_t, t \geq 0\}$ such that for each $T > 0$, the set of random variables

$$\Big\{ \sum_{j=1}^{n} f_j (B_{t_j} - B_{t_{j-1}}), n \geq 1, 0 \leq t_0 < \cdots < t_n \leq T,$$
$$|f_j| \leq 1, f_j \text{ is } \mathcal{F}^X_{t_{j-1}}\text{-measurable} \Big\}$$

is bounded in L^0, where for each $t \geq 0$, $\mathcal{F}^X_t$ is the σ-field generated by the random variables $\{X_s, 0 \leq s \leq t\}$. It is important to remark that this σ-field is not completed with the null sets. Then, in [7] it is proved that fBm is not a weak semimartingale if $H \neq \frac{1}{2}$.

Let us mention the following surprising result also proved in [7]. Suppose that $\{B_t, t \geq 0\}$ is a fBm with Hurst parameter $H \in (0,1)$, and $\{W_t, t \geq 0\}$ is an ordinary Brownian motion. Assume they are independent. Set

$$M_t^H = B_t + W_t.$$

Then $\{M_t, t \geq 0\}$ is not a weak semimartingale if $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}]$, and it is a semimartingale, equivalent in law to Brownian motion on any finite time interval $[0,T]$, if $H \in (\frac{3}{4}, 1)$.

1.4 Fractional integrals and derivatives

In this subsection we will recall the basic definitions and properties of the fractional calculus. For a detailed presentation of these notions we refer to [41].

Let $a, b \in \mathbb{R}$, $a < b$. Let $f \in L^1(a, b)$ and $\alpha > 0$. The left and right-sided *fractional integrals* of f of order α are defined for almost all $x \in (a, b)$ by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\, dy \tag{1.5}$$

and

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)\, dy, \tag{1.6}$$

respectively. Let $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$) the image of $L^p(a, b)$ by the operator I_{a+}^{α} (resp. I_{b-}^{α}).

If $f \in I_{a+}^{\alpha}(L^p)$ (resp. $f \in I_{b-}^{\alpha}(L^p)$) and $0 < \alpha < 1$ then the left and right-sided *fractional derivatives* are defined by

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right), \tag{1.7}$$

and

$$D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} dy \right) \tag{1.8}$$

for almost all $x \in (a, b)$ (the convergence of the integrals at the singularity $y = x$ holds point-wise for almost all $x \in (a, b)$ if $p = 1$ and moreover in L^p-sense if $1 < p < \infty$).

Recall the following properties of these operators:

- If $\alpha < \frac{1}{p}$ and $q = \frac{p}{1 - \alpha p}$ then
$$I_{a+}^{\alpha}(L^p) = I_{b-}^{\alpha}(L^p) \subset L^q(a, b).$$

- If $\alpha > \frac{1}{p}$ then
$$I_{a+}^{\alpha}(L^p) \cup I_{b-}^{\alpha}(L^p) \subset C^{\alpha - \frac{1}{p}}(a, b),$$
where $C^{\alpha - \frac{1}{p}}(a, b)$ denotes the space of $\left(\alpha - \frac{1}{p}\right)$-Hölder continuous functions of order $\alpha - \frac{1}{p}$ in the interval $[a, b]$.

The following inversion formulas hold:

$$I_{a+}^{\alpha}\left(D_{a+}^{\alpha} f\right) = f$$

for all $f \in I_{a+}^{\alpha}(L^p)$, and

$$D_{a+}^{\alpha}\left(I_{a+}^{\alpha} f\right) = f$$

for all $f \in L^1(a, b)$. Similar inversion formulas hold for the operators I_{b-}^{α} and D_{b-}^{α}.

We will make use of the following *integration by parts formula*:

$$\int_a^b \left(D_{a+}^{\alpha} f\right)(s) g(s) ds = \int_a^b f(s) \left(D_{b-}^{\alpha} g\right)(s) ds, \tag{1.9}$$

for any $f \in I_{a+}^{\alpha}(L^p)$, $g \in I_{b-}^{\alpha}(L^q)$, $\frac{1}{p} + \frac{1}{q} = 1$.

1.5 Representation of fBm on an interval

Fix a time interval $[0,T]$. Consider a fBm $\{B_t, t \in [0,T]\}$ with Hurst parameter $H \in (0,1)$. We denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $\mathbf{1}_{[0,t]} \longrightarrow B_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B)$ associated with B. We will denote this isometry by $\varphi \longrightarrow B(\varphi)$.

In this subsection we will establish the representation of fBm as a Volterra process, following the lines of [4] (case $H > \frac{1}{2}$) and [3] (general case).

1.5.1 Case $H > \frac{1}{2}$

It is easy to see that the covariance of fBm can be written as

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |r-u|^{2H-2} du dr, \tag{1.10}$$

where $\alpha_H = H(2H-1)$. Formula (1.10) implies that

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u du dr \tag{1.11}$$

for any pair of step functions φ and ψ in $\mathcal{E}$.

We can write

$$\begin{aligned} |r-u|^{2H-2} &= \frac{(ru)^{H-\frac{1}{2}}}{\beta(2-2H, H-\frac{1}{2})} \\ &\quad \times \int_0^{r \wedge u} v^{1-2H} (r-v)^{H-\frac{3}{2}} (u-v)^{H-\frac{3}{2}} dv, \end{aligned} \tag{1.12}$$

where β denotes the Beta function. Let us show Equation (1.12). Suppose

$r > u$. By means of the change of variables $z = \frac{r-v}{u-v}$ and $x = \frac{r}{uz}$, we obtain

$$
\begin{aligned}
& \int_0^u v^{1-2H}(r-v)^{H-\frac{3}{2}}(u-v)^{H-\frac{3}{2}}dv \\
= \quad & (r-u)^{2H-2}\int_{\frac{r}{u}}^{\infty}(zu-r)^{1-2H}\, z^{H-\frac{3}{2}}dz \\
= \quad & (ru)^{\frac{1}{2}-H}(r-u)^{2H-2}\int_0^1 (1-x)^{1-2H}\, x^{H-\frac{3}{2}}dx \\
= \quad & \beta(2-2H, H-\frac{1}{2})(ru)^{\frac{1}{2}-H}(r-u)^{2H-2}.
\end{aligned}
$$

Consider the square integrable kernel

$$
K_H(t,s) = c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du, \tag{1.13}
$$

where $c_H = \left[\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}\right]^{1/2}$ and $t > s$.

Taking into account formulas (1.10) and (1.12) we deduce that this kernel verifies

$$
\begin{aligned}
& \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du = c_H^2\int_0^{t\wedge s}\left(\int_u^t (y-u)^{H-\frac{3}{2}}y^{H-\frac{1}{2}}dy\right) \\
& \times\left(\int_u^s (z-u)^{H-\frac{3}{2}}z^{H-\frac{1}{2}}dz\right)u^{1-2H}du \\
= \quad & c_H^2\beta(2-2H, H-\frac{1}{2})\int_0^t\int_0^s |y-z|^{2H-2}dzdy \\
= \quad & R_H(t,s).
\end{aligned} \tag{1.14}
$$

Formula (1.14) implies that the kernel R_H is nonnegative definite and provides an explicit representation for its square root as an operator.

From (1.13) we get

$$
\frac{\partial K_H}{\partial t}(t,s) = c_H\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}. \tag{1.15}
$$

Consider the linear operator K_H^* from $\mathcal{E}$ to $L^2(0,T)$ defined by

$$
(K_H^*\varphi)(s) = \int_s^T \varphi(t)\frac{\partial K_H}{\partial r}(t,s)dt. \tag{1.16}
$$

Notice that

$$
\left(K_H^*\mathbf{1}_{[0,t]}\right)(s) = K_H(t,s)\mathbf{1}_{[0,t]}(s). \tag{1.17}
$$

The operator K_H^* is an isometry between $\mathcal{E}$ and $L^2(0,T)$ that can be extended to the Hilbert space $\mathcal{H}$. In fact, for any $s,t \in [0,T]$ we have using (1.17) and (1.14)

$$\begin{aligned}\left\langle K_H^*\mathbf{1}_{[0,t]}, K_H^*\mathbf{1}_{[0,s]}\right\rangle_{L^2([0,T])} &= \left\langle K_H(t,\ \cdot)\mathbf{1}_{[0,t]}, K_H(s,\ \cdot)\mathbf{1}_{[0,s]}\right\rangle_{L^2(0,T)} \\ &= \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du \\ &= R_H(t,s) = \left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\right\rangle_{\mathcal{H}}.\end{aligned}$$

The operator K_H^* can be expressed in terms of fractional integrals:

$$(K_H^*\varphi)(s) = c_H\Gamma(H-\frac{1}{2})s^{\frac{1}{2}-H}(I_{T-}^{H-\frac{1}{2}}u^{H-\frac{1}{2}}\varphi(u))(s). \tag{1.18}$$

This is an immediate consequence of formulas (1.15), (1.16) and (1.6).

For any $a\in[0,T]$, the indicator function $\mathbf{1}_{[0,a]}$ belongs to the image of K_H^* and applying the rules of the fractional calculus yields

$$(K_H^*)^{-1}(\mathbf{1}_{[0,a]}) = \frac{1}{c_H\Gamma(H-\frac{1}{2})}s^{\frac{1}{2}-H}\left(D_{a-}^{H-\frac{1}{2}}u^{H-\frac{1}{2}}\right)(s)\mathbf{1}_{[0,a]}(s). \tag{1.19}$$

Consider the process $W=\{W_t, t\in[0,T]\}$ defined by

$$W_t = B((K_H^*)^{-1}(\mathbf{1}_{[0,t]})). \tag{1.20}$$

Then W is a Wiener process, and the process B has the integral representation

$$B_t = \int_0^t K_H(t,s)dW_s. \tag{1.21}$$

Indeed, for any $s,t\in[0,T]$ we have

$$\begin{aligned}E(W_tW_s) &= E\left(B((K_H^*)^{-1}(\mathbf{1}_{[0,t]}))B((K_H^*)^{-1}(\mathbf{1}_{[0,s]}))\right) \\ &= \left\langle (K_H^*)^{-1}(\mathbf{1}_{[0,t]}), (K_H^*)^{-1}(\mathbf{1}_{[0,s]})\right\rangle_{\mathcal{H}} \\ &= \left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\right\rangle_{L^2(0,T)} = s\wedge t.\end{aligned}$$

Moreover, for any $\varphi\in\mathcal{H}$ we have

$$B(\varphi) = \int_0^T (K_H^*\varphi)(t)dW_t.$$

Notice that from (1.19), the Wiener process W is adapted to the filtration generated by the fBm B and (1.20) and (1.21) imply that both processes generate the same filtration. Furthermore, the Wiener process W that provides the integral representation (1.21) is unique.

The elements of the Hilbert space $\mathcal{H}$ may not be functions but distributions of negative order (see Pipiras and Taqqu [36], [37]). In fact, from (1.18) it follows that $\mathcal{H}$ coincides with the space of distributions f such that $s^{\frac{1}{2}-H}I_{0+}^{H-\frac{1}{2}}(f(u)u^{H-\frac{1}{2}})(s)$ is a square integrable function.

We can find a linear space of functions contained in $\mathcal{H}$ in the following way. Let $|\mathcal{H}|$ be the linear space of measurable functions φ on $[0,T]$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 = \alpha_H \int_0^T \int_0^T |\varphi_r|\,|\varphi_u|\,|r-u|^{2H-2}\,drdu < \infty. \tag{1.22}$$

It is not difficult to show that $|\mathcal{H}|$ is a Banach space with the norm $\|\cdot\|_{|\mathcal{H}|}$ and $\mathcal{E}$ is dense in $|\mathcal{H}|$. On the other hand, it has been shown in [37] that the space $|\mathcal{H}|$ equipped with the inner product $\langle\varphi,\psi\rangle_{\mathcal{H}}$ is not complete and it is isometric to a subspace of $\mathcal{H}$.

The following estimate has been proved in [30]

$$\|\varphi\|_{|\mathcal{H}|} \le b_H \|\varphi\|_{L^{\frac{1}{H}}([0,T])}, \tag{1.23}$$

for some constant $b_H > 0$.

Proof of (1.23). Using Hölder's inequality with exponent $q = \frac{1}{H}$ in (1.22) we get

$$\|\varphi\|_{|\mathcal{H}|}^2 \le \alpha_H \left(\int_0^T |\varphi_r|^{\frac{1}{H}}\,dr\right)^H \left(\int_0^T \left(\int_0^T |\varphi_u|\,(r-u)^{2H-2}\,du\right)^{\frac{1}{1-H}} dr\right)^{1-H}.$$

The second factor in the above expression, up to a multiplicative constant, it is equal to the $\frac{1}{1-H}$ norm of the left-sided fractional integral $I_{0+}^{2H-1}|\varphi|$. Finally is suffices to apply the Hardy-Littlewood inequality (see [45, Theorem 1, p. 119])

$$\left\|I_{0+}^{\alpha} f\right\|_{L^q(0,\infty)} \le c_{H,p}\,\|f\|_{L^p(0,\infty)}, \tag{1.24}$$

where $0<\alpha<1$, $1<p<q<\infty$ satisfy $\frac{1}{q}=\frac{1}{p}-\alpha$, with the particular values $\alpha = 2H-1$, $q=\frac{1}{1-H}$, and $p=\frac{1}{H}$. ■

As a consequence

$$L^2(0,T) \subset L^{\frac{1}{H}}(0,T) \subset |\mathcal{H}| \subset \mathcal{H}.$$

The inclusion $L^2(0,T)\subset|\mathcal{H}|$ can be proved by a direct argument:

$$\begin{aligned}\int_0^T\int_0^T |\varphi_r|\,|\varphi_u|\,|r-u|^{2H-2}\,drdu &\le \int_0^T\int_0^T |\varphi_u|^2\,|r-u|^{2H-2}\,drdu \\ &\le \frac{T^{2H-1}}{H-\frac{1}{2}}\int_0^T |\varphi_u|^2\,du.\end{aligned}$$

This means that the Wiener-type integral $\int_0^T \varphi(t)dB_t$ (which is equal to $B(\varphi)$, by definition) can be defined for functions $\varphi\in|\mathcal{H}|$, and

$$\int_0^T \varphi(t)dB_t = \int_0^T \left(K_H^*\varphi\right)(t)dW_t. \tag{1.25}$$

1.5.2 Case $H < \frac{1}{2}$

We claim that the kernel

$$K_H(t,s) = c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right]$$

where $c_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H,H+1/2)}}$, satisfies

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u) K_H(s,u) du. \tag{1.26}$$

To verify this relation is not so easy as in the case $H < \frac{1}{2}$. In the references [16] and [37] this property is proved using the analyticity of both members as functions of the parameter H. We will give here a direct proof using the ideas of [31]. Notice first that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H (H-\frac{1}{2}) \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}. \tag{1.27}$$

Proof of (1.26). Consider first the diagonal case $s=t$. Set $\phi(s) = \int_0^s K_H(s,u)^2 du$. We have

$$\begin{aligned} \phi(s) &= c_H^2 \left[\int_0^s (\frac{s}{u})^{2H-1} (s-u)^{2H-1} du \right. \\ &\quad -(2H-1) \int_0^s s^{H-\frac{1}{2}} u^{1-2H} (s-u)^{H-\frac{1}{2}} \left(\int_u^s v^{H-\frac{3}{2}} (v-u)^{H-\frac{1}{2}} dv \right) du \\ &\quad \left. +(H-\frac{1}{2})^2 \int_0^s u^{1-2H} \left(\int_u^s v^{H-\frac{3}{2}} (v-u)^{H-\frac{1}{2}} dv \right)^2 du \right]. \end{aligned}$$

Making the change of variables $u = sx$ in the first integral and using Fubini's theorem yields

$$\begin{aligned} \phi(s) &= c_H^2 \left[s^{2H} \beta(2-2H, 2H) \right. \\ &\quad -(2H-1) s^{H-\frac{1}{2}} \int_0^s v^{H-\frac{3}{2}} \left(\int_0^v u^{1-2H} (s-u)^{H-\frac{1}{2}} (v-u)^{H-\frac{1}{2}} du \right) dv \\ &\quad +2(H-\frac{1}{2})^2 \int_0^s \int_0^v \int_0^w u^{1-2H} (v-u)^{H-\frac{1}{2}} (w-u)^{H-\frac{1}{2}} \\ &\quad \left. \times w^{H-\frac{3}{2}} v^{H-\frac{3}{2}} du dw dv \right]. \end{aligned}$$

Now we make the change of variable $u = vx$, $v = sy$ for the second term and $u = wx$, $w = vy$ for the third term and we obtain

$$\begin{aligned} \phi(s) &= c_H^2 s^{2H} \left[\beta(2-2H, 2H) \ - (2H-1)(\frac{1}{4H} + \frac{1}{2}) \right. \\ &\quad \left. \times \int_0^1 \int_0^1 x^{1-2H} (1-xy)^{H-\frac{1}{2}} (1-x)^{H-\frac{1}{2}} dx dy \right] \\ &= s^{2H}. \end{aligned}$$

Suppose now that $s < t$. Differentiating Equation (1.26) with respect to t, we are aimed to show that

$$H(t^{2H-1} - (t-s)^{2H-1}) = \int_0^s \frac{\partial K_H}{\partial t}(t,u) K_H(s,u) du. \tag{1.28}$$

Set $\phi(t,s) = \int_0^s \frac{\partial K_H}{\partial t}(t,u) K_H(s,u) du$. Using (1.27) yields

$$\begin{aligned}
\phi(t,s) &= c_H^2 (H - \frac{1}{2}) \int_0^s \left(\frac{t}{u}\right)^{H-\frac{1}{2}} (t-u)^{H-\frac{3}{2}} \left(\frac{s}{u}\right)^{H-\frac{1}{2}} (s-u)^{H-\frac{1}{2}} du \\
&\quad - c_H^2 (H - \frac{1}{2})^2 \int_0^s \left(\frac{t}{u}\right)^{H-\frac{1}{2}} (t-u)^{H-\frac{3}{2}} u^{\frac{1}{2}-H} \\
&\quad \times \left(\int_u^s v^{H-\frac{3}{2}} (v-u)^{H-\frac{1}{2}} dv \right) du.
\end{aligned}$$

Making the change of variables $u = sx$ in the first integral and $u = vx$ in the second one we obtain

$$\begin{aligned}
\phi(t,s) &= c_H^2 (H - \frac{1}{2}) (ts)^{H-\frac{1}{2}} \ \gamma(\frac{t}{s}) \\
&\quad - c_H^2 (H - \frac{1}{2})^2 t^{H-\frac{1}{2}} \int_0^s v^{H-\frac{3}{2}} \gamma(\frac{t}{v}) \ dv,
\end{aligned}$$

where $\gamma(y) = \int_0^1 x^{1-2H} (y-x)^{H-\frac{3}{2}} \ (1-x)^{H-\frac{1}{2}} dx$ for $y > 1$. Then, (1.28) is equivalent to

$$\begin{aligned}
& c_H^2 \left[(H - \frac{1}{2}) s^{H-\frac{1}{2}} \ \gamma(\frac{t}{s}) \ - (H - \frac{1}{2})^2 \int_0^s v^{H-\frac{3}{2}} \gamma(\frac{t}{v}) \ dv \right] \\
=\ & H (t^{H-\frac{1}{2}} - t^{\frac{1}{2}-H} (t-s)^{2H-1}).
\end{aligned} \tag{1.29}$$

Differentiating the left-hand side of equation (1.29) with respect to t yields

$$\begin{aligned}
& c_H^2 (H - \frac{3}{2}) \left[(H - \frac{1}{2}) s^{H-\frac{3}{2}} \ \delta(\frac{t}{s}) \ - (H - \frac{1}{2})^2 \int_0^s v^{H-\frac{5}{2}} \delta(\frac{t}{v}) \ dv \right] \\
: \ & = \mu(t,s),
\end{aligned} \tag{1.30}$$

where, for $y > 1$,

$$\delta(y) = \int_0^1 x^{1-2H} (y-x)^{H-\frac{5}{2}} \ (1-x)^{H-\frac{1}{2}} dx.$$

By means of the change of variables $z = \frac{y(1-x)}{y-x}$ we obtain

$$\delta(y) = \beta(2-2H, H + \frac{1}{2}) y^{-H-\frac{1}{2}} (y-1)^{2H-2}. \tag{1.31}$$

Finally, substituting (1.31) into (1.30) yields

$$\begin{aligned}\mu(t,s) &= c_H^2\beta(2-2H,H+\frac{1}{2})(H-\frac{3}{2})(H-\frac{1}{2}) \\ &\quad \times t^{-H-\frac{1}{2}}s(t-s)^{2H-2}+\frac{1}{2}t^{-H-\frac{1}{2}}((t-s)^{2H-1}-t^{2H-1}) \\ &= H(1-2H)\left(t^{-H-\frac{1}{2}}s(t-s)^{2H-2}+\frac{1}{2}(t-s)^{2H-1}t^{-H-\frac{1}{2}}-\frac{1}{2}t^{H-\frac{3}{2}}\right).\end{aligned}$$

This last expression coincides with the derivative with respect to t of the right-hand side of (1.29). This completes the proof of the equality (1.26). ■

The kernel K_H can also be expressed in terms of fractional derivatives:

$$K_H(t,s)=c_H\Gamma(H+\frac{1}{2})s^{\frac{1}{2}-H}\left(D_{t-}^{\frac{1}{2}-H}u^{H-\frac{1}{2}}\right)(s). \tag{1.32}$$

Consider the linear operator K_H^* from $\mathcal{E}$ to $L^2(0,T)$ defined by

$$(K_H^*\varphi)(s)=K_H(T,s)\varphi(s)+\int_s^T(\varphi(t)-\varphi(s))\frac{\partial K_H}{\partial r}(t,s)dt. \tag{1.33}$$

Notice that

$$\left(K_H^*\mathbf{1}_{[0,t]}\right)(s)=K_H(t,s)\mathbf{1}_{[0,t]}(s). \tag{1.34}$$

From (1.26) and (1.34) we deduce as in the case $H>\frac{1}{2}$ that the operator K_H^* is an isometry between $\mathcal{E}$ and $L^2(0,T)$ that can be extended to the Hilbert space $\mathcal{H}$.

The operator K_H^* can be expressed in terms of fractional derivatives:

$$\left(K_H^*\varphi\right)(s)=d_H\ s^{\frac{1}{2}-H}(D_{T-}^{\frac{1}{2}-H}u^{H-\frac{1}{2}}\varphi(u))(s), \tag{1.35}$$

where $d_H=c_H\Gamma(H+\frac{1}{2})$. This is an immediate consequence of (1.33) and the equality

$$\left(D_{t-}^{\frac{1}{2}-H}u^{H-\frac{1}{2}}\right)(s)\mathbf{1}_{[0,t]}(s)=\left(D_{T-}^{\frac{1}{2}-H}u^{H-\frac{1}{2}}\mathbf{1}_{[0,t]}(u)\right)(s).$$

Using the alternative expression for the kernel K_H given by

$$K_H(t,s)=c_H(t-s)^{H-\frac{1}{2}}+s^{H-\frac{1}{2}}F_1(\frac{t}{s}), \tag{1.36}$$

where

$$F_1(z)=c_H(\frac{1}{2}-H)\int_0^{z-1}\theta^{H-\frac{3}{2}}(1-(\theta+1)^{H-\frac{1}{2}})d\theta,$$

one can show that $\mathcal{H}=I_{T-}^{\frac{1}{2}-H}(L^2)$ (see [16] and Proposition 8 of [3]). Notice that

$$C^\gamma([0,T])\subset\mathcal{H}$$

if $\gamma>\frac{1}{2}-H$.

On the other hand, (1.35) implies that

$$\mathcal{H} = \{f : \exists K_H^* f \in L^2(0,T) : f(s) = d_H^{-1} s^{\frac{1}{2}-H} (I_{T-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} K_H^* f(u))(s)\},$$

with the inner product

$$\langle f, g \rangle_{\mathcal{H}} = \int_0^T K_H^* f(s) K_H^* g(s) ds.$$

Consider process $W = \{W_t, t \in [0,T]\}$ defined by

$$W_t = B((K_H^*)^{-1} (\mathbf{1}_{[0,t]})).$$

As in the case $H > \frac{1}{2}$, we can show that W is a Wiener process, and the process B has the integral representation

$$B_t = \int_0^t K_H(t,s) dW_s.$$

Therefore, in this case the Wiener-type integral $\int_0^T \varphi(t) dB_t$ can be defined for functions $\varphi \in I_{T-}^{\frac{1}{2}-H}(L^2)$, and (1.25) holds.

Define the left and right-sided fractional derivative operators on the whole real line for $0 < \alpha < 1$ by

$$D_-^\alpha f(s) := \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(s) - f(s+u)}{u^{1+\alpha}} du$$

and

$$D_+^\alpha f(s) := \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(s) - f(s-u)}{u^{1+\alpha}} du,$$

$s \in \mathbb{R}$, respectively. Then, the scalar product in $\mathcal{H}$ has the following simple expression

$$\langle f, g \rangle_{\mathcal{H}} = e_H^2 \left\langle D_-^{\frac{1}{2}-H} f, D_+^{\frac{1}{2}-H} g \right\rangle_{L^2(\mathbb{R})}, \tag{1.37}$$

where $e_H = C_1(H)^{-1} \Gamma(H + \frac{1}{2})$, $f, g \in \mathcal{H}$, and by convention $f(s) = g(s) = 0$ if $s \notin [0,T]$.

In [3] these results have been generalized to Gaussian Volterra processes of the form

$$X_t = \int_0^t K(t,s) dW_s,$$

where $\{W_t, t \geq 0\}$ is a Wiener process and $K(t,s)$ is a square integrable kernel. Two different types of kernels can be considered, which correspond to the cases $H < \frac{1}{2}$ and $H > \frac{1}{2}$:

i) *Singular case*: $K(\cdot, s)$ has bounded variation on any interval $(u,T]$, $u > s$, but $\int_s^T |K|(dt,s) = \infty$ for every s.

ii) *Regular case*: The kernel satisfies $\int_s^T |K|((s,T],s)^2 ds < \infty$ for each s.

2 Stochastic calculus of variations with respect to fBm

Let $B = \{B_t, t \in [0,T]\}$ be a fBm with Hurst parameter $H \in (0,1)$. Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B(\phi_1), \ldots, B(\phi_n)), \tag{2.1}$$

where $n \geq 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded), and $\phi_i \in \mathcal{H}$.

The *derivative operator* D of a smooth and cylindrical random variable F of the form (2.1) is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(\phi_1), \ldots, B(\phi_n))\phi_i.$$

The derivative operator D is then a closable operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p \geq 1$. For any integer $k \geq 1$ we denote by D^k the iteration of the derivative operator. For any $p \geq 1$ the Sobolev space $\mathbb{D}^{k,p}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}^p = E(|F|^p) + \sum_{j=1}^k E\left(\left\|D^j F\right\|_{\mathcal{H}^{\otimes j}}^p\right).$$

In a similar way, given a Hilbert space V we denote by $\mathbb{D}^{k,p}(V)$ the corresponding Sobolev space of V-valued random variables.

The *divergence operator* δ is the adjoint of the derivative operator. We say that a random variable in $L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator, denoted by Dom δ, if

$$|E(\langle DF, u\rangle_{\mathcal{H}})| \leq c_u \|F\|_{L^2(\Omega)}$$

for any $F \in \mathcal{S}$. In this case $\delta(u)$ is defined by the duality relationship

$$E(F\delta(u)) = E(\langle DF, u\rangle_{\mathcal{H}}), \tag{2.2}$$

for any $F \in \mathbb{D}^{1,2}$.

The following are two basic properties of the divergence operator:

i) $\mathbb{D}^{1,2}(\mathcal{H}) \subset$Dom δ and for any $u \in \mathbb{D}^{1,2}(\mathcal{H})$

$$E\left(\delta(u)^2\right) = E\left(\|u\|_{\mathcal{H}}^2\right) + E\left(\left\langle Du, (Du)^*\right\rangle_{\mathcal{H}\otimes\mathcal{H}}\right), \tag{2.3}$$

where $(Du)^*$ is the adjoint of (Du) in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

ii) For any F in $\mathbb{D}^{1,2}$ and any u in the domain of δ such that Fu and $F\delta(u) + \langle DF, u\rangle_{\mathcal{H}}$ are square integrable, then Fu is in the domain of δ and

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}. \tag{2.4}$$

2.1 Transfer principle

Recall that the operator K_H^* is an isometry between $\mathcal{H}$ and a closed subspace of $L^2(0,T)$. Moreover, $W_t = B((K_H^*)^{-1}(\mathbf{1}_{[0,t]}))$ is a Wiener process such that

$$B_t = \int_0^t K_H(t,s)dW_s,$$

and for any $\varphi \in \mathcal{H}$ we have $B(\varphi) = W(K_H^*\varphi)$.

A similar relation holds for the derivative and divergence operators with respect to the processes B and W. That is:

(i) For any $F \in \mathbb{D}_W^{1,2} = \mathbb{D}^{1,2}$

$$K_H^* DF = D^W F,$$

where D^W denotes the derivative operator with respect to the process W, and $\mathbb{D}_W^{1,2}$ the corresponding Sobolev space.

(ii) $\mathrm{Dom}\delta = (K_H^*)^{-1}(\mathrm{Dom}\delta_W)$, and for any $\mathcal{H}$-valued random variable u in Dom δ we have $\delta(u) = \delta_W(K_H^* u)$, where δ_W denotes the divergence operator with respect to the process W.

Suppose $H > \frac{1}{2}$. We denote by $|\mathcal{H}| \otimes |\mathcal{H}|$ the space of measurable functions φ on $[0,T]^2$ such that

$$\|\varphi\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} = \alpha_H^2 \int_{[0,T]^4} \left|\varphi_{r,\theta}\right| \left|\varphi_{u,\eta}\right| |r-u|^{2H-2} |\theta-\eta|^{2H-2} drdud\theta d\eta < \infty.$$

Then, $|\mathcal{H}| \otimes |\mathcal{H}|$ is a Banach space with respect to the norm $\|\cdot\|_{|\mathcal{H}|\otimes|\mathcal{H}|}$. Furthermore, equipped with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}\otimes\mathcal{H}} = \alpha_H^2 \int_{[0,T]^4} \varphi_{r,\theta}\psi_{u,\eta} |r-u|^{2H-2} |\theta-\eta|^{2H-2} drdud\theta d\eta$$

the space $|\mathcal{H}| \otimes |\mathcal{H}|$ is isometric to a subspace of $\mathcal{H} \otimes \mathcal{H}$. A slight extension of the inequality (1.23) yields

$$\|\varphi\|_{|\mathcal{H}|\otimes|\mathcal{H}|} \leq b_H \|\varphi\|_{L^{\frac{1}{H}}([0,T]^2)}. \tag{2.5}$$

For any $p > 1$ we denote by $\mathbb{D}^{1,p}(|\mathcal{H}|)$ the subspace of $\mathbb{D}^{1,p}(\mathcal{H})$ formed by the elements u such that $u \in |\mathcal{H}|$ a.s., $Du \in |\mathcal{H}| \otimes |\mathcal{H}|$ a.s., and

$$E\left(\|u\|^p_{|\mathcal{H}|}\right) + E\left(\|Du\|^p_{|\mathcal{H}|\otimes|\mathcal{H}|}\right) < \infty.$$

3 Stochastic integrals with respect to fractional Brownian motion

In the case of an ordinary Brownian motion, the adapted processes in $L^2([0,T]\times\Omega)$ belong to the domain of the divergence operator, and on this set the divergence operator coincides with Itô's stochastic integral. Actually, the divergence operator coincides with an extension of Itô's stochastic integral introduced by Skorohod in [44]. In this context a natural question is to ask in which sense the divergence operator with respect to a fractional Brownian motion B can be interpreted as a stochastic integral. Note that the divergence operator provides an isometry between the Hilbert Space $\mathcal{H}$ associated with the fBm B and the Gaussian space $H_1(B)$, and gives rise to a notion of stochastic integral in the space of deterministic functions $|\mathcal{H}|$ included in $\mathcal{H}$.

Different approaches have been used in the literature in order to define stochastic integrals with respect to fBm. Lin [25] and Dai and Heyde [15] have defined a stochastic integral $\int_0^T \phi_s dB_s$ as limit in L^2 of Riemann sums in the case $H > \frac{1}{2}$. This integral does not satisfy the property $E(\int_0^T \phi_s dB_s) = 0$ and it gives rise to change of variable formulae of Stratonovich type. A new type of integral with zero mean defined by means of Wick products was introduced by Duncan, Hu and Pasik-Duncan in [17], assuming $H > \frac{1}{2}$. This integral turns out to coincide with the divergence operator.

A construction of stochastic integrals with respect to fBm with parameter $H \in (0,1)$ by a regularization technique was developed by Carmona and Coutin in [6]. The integral is defined as the limit of approximating integrals with respect to semimartingales obtained by smoothing the singularity of the kernel $K_H(t,s)$. The techniques of Malliavin Calculus are used in order to establish the existence of the integrals. The ideas of Carmona and Coutin were further developed by Alòs, Mazet and Nualart in the case $0 < H < \frac{1}{2}$ in [2].

The interpretation of the divergence operator as a stochastic integral has been first studied by Decreusefont and Üstünel in [16]. A stochastic calculus for the divergence process has been developed by Alòs, Mazet and Nualart in [3].

In this section we will discuss the relation between the divergence operator and the path-wise stochastic integral with respect to fBm with parameter $H \in (0,1)$ defined as the limit of the integrals with respect to a regularization of fBm by the convolution with a constant function. The results will be based on the papers [4] (case $H > \frac{1}{2}$), [1] and [9] (case $H > \frac{1}{2}$).

The following definition of the *symmetric stochastic integral* was introduced by Russo and Vallois in [39]. By convention we will assume that all processes and functions vanish outside the interval $[0,T]$.

Definition 1 *Let $u = \{u_t, t \in [0,T]\}$ be a stochastic process with integrable trajectories. The symmetric integral of u with respect to the fBm B is defined as the limit in probability as ε tends to zero of*

$$(2\varepsilon)^{-1} \int_0^T u_s (B_{s+\varepsilon} - B_{s-\varepsilon}) ds,$$

provided this limit exists, and it is denoted by $\int_0^T u_t dB_t$.

3.1 The divergence integral in the case $H > \frac{1}{2}$

The following proposition gives sufficient conditions for the existence of the symmetric integral, and provides a representation of the divergence operator as a stochastic integral (see [4]).

Proposition 2 *Let* $u = \{u_t, t \in [0,T]\}$ *be a stochastic process in the space* $\mathbb{D}^{1,2}(|\mathcal{H}|)$. *Suppose also that a.s.*

$$\int_0^T \int_0^T |D_s u_t| \, |t-s|^{2H-2} \, dsdt < \infty. \tag{3.1}$$

Then the symmetric integral exists and we have

$$\int_0^T u_t dB_t = \delta(u) + \alpha_H \int_0^T \int_0^T D_s u_t \, |t-s|^{2H-2} \, dsdt. \tag{3.2}$$

Remark 1 Under the assumptions of the Proposition 2 the integral $\int_0^T u_t dB_t$ also coincides with the forward and backward integrals.

Remark 2 A sufficient condition for (3.1) is

$$\int_0^T \left(\int_s^T |D_s u_t|^p \, dt \right)^{1/p} ds < \infty$$

for some $p > \frac{1}{2H-1}$.

Sketch of the proof:. Approximate u by

$$u_t^\varepsilon = (2\varepsilon)^{-1} \int_{t-\varepsilon}^{t+\varepsilon} u_s ds.$$

We have

$$\|u^\varepsilon\|_{\mathbb{D}^{1,2}(|\mathcal{H}|)}^2 \leq d_H \, \|u\|_{\mathbb{D}^{1,2}(|\mathcal{H}|)}^2 \,,$$

for some positive constant d_H. Using (2.4) we obtain

$$(2\varepsilon)^{-1} \int_0^T u_s (B_{s+\varepsilon} - B_{s-\varepsilon}) ds = \delta(u^\varepsilon) + (2\varepsilon)^{-1} \int_0^T \left\langle Du_s, \mathbf{1}_{[s-\varepsilon, s+\varepsilon]} \right\rangle_{\mathcal{H}} ds.$$

Finally, take the limit as ε tends to zero. ■

3.1.1 Estimates for the divergence integral

Suppose that $u = \{u_t, t \in [0,T]\}$ is a stochastic process in the space $\mathbb{D}^{1,2}(|\mathcal{H}|)$ such that condition (3.1) holds. Then, for any $t \in [0,T]$ the process $u\mathbf{1}_{[0,t]}$ also belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$ and satisfies (3.1). Hence, by Proposition 2 we can define the indefinite integral $\int_0^t u_s dB_s = \int_0^T u_s \mathbf{1}_{[0,t]}(s) dB_s$ and the following decomposition holds

$$\int_0^t u_s dB_s = \delta(u\mathbf{1}_{[0,t]}) + \alpha_H \int_0^t \int_0^T D_r u_s \, |s-r|^{2H-2} \, drds.$$

The second summand in this expression is a process with absolutely continuous paths. Therefore, in order to deduce L^p estimates and to study continuity properties of $\int_0^t u_s dB_s$ we can reduce our analysis to the process $\delta(u\mathbf{1}_{[0,t]})$. In this section we will establish L^p maximal estimates for this divergence process. We will make use of the notation

$$\int_0^t u_s \delta B_s = \delta \left(u\mathbf{1}_{[0,t]} \right).$$

By Meyer's inequalities (see for example [32]), if $p > 1$, a process $u \in \mathbb{D}^{1,p}(|\mathcal{H}|)$ belongs to the domain of the divergence in $L^p(\Omega)$, and we have

$$E\left(|\delta(u)|^p\right) \leq C_{H,p} \left(\|E(u)\|_{|\mathcal{H}|}^p + E\left(\|Du\|_{|\mathcal{H}| \otimes |\mathcal{H}|}^p \right) \right).$$

As a consequence, applying (2.5) we obtain

$$E\left(|\delta(u)|^p\right) \leq C_{H,p} \left(\|E(u)\|_{L^{1/H}([0,T])}^p + E\left(\|Du\|_{L^{1/H}([0,T]^2)}^p \right) \right).$$

Let $pH > 1$. Denote by $\mathbb{L}_H^{1,p}$ the space of processes $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ such that

$$\|u\|_{p,1} := \left[\int_0^T E(|u_s|^p) ds + E\left(\int_0^T \left(\int_0^T |D_r u_s|^{\frac{1}{H}} dr \right)^{pH} ds \right) \right]^{\frac{1}{p}} < \infty.$$

Using Meyer's inequality and a convolution argument the following maximal L^p inequality for the divergence integral has been established in [4]:

$$E\left(\sup_{t \in [0,T]} \left| \int_0^t u_s \delta B_s \right|^p \right) \leq C \|u\|_{p,1}^p,$$

where the constant $C > 0$ depends on p, H and T.

Assume $pH > 1$ and suppose that $u \in \mathbb{L}_H^{1,p}$. Set $X_t = \int_0^t u_s \delta B_s$. Then, the process X_t has a version with continuous trajectories and for all $\gamma < H - \frac{1}{p}$ there exists a random variable C_γ such that

$$|X_t - X_s| \leq C_\gamma |t-s|^\gamma.$$

This result is also proved in [4]. As a consequence, for a process $u \in \cap_{p>1} \mathbb{L}_H^{1,p}$, the indefinite integral process $X = \left\{ \int_0^t u_s \delta B_s,\ t \in [0,T] \right\}$ is γ-Hölder continuous for all $\gamma < H$. If we assume also that hypothesis (3.1) holds, we deduce analogous continuity results for the symmetric integral process $\int_0^t u_s dB_s$.

3.1.2 Itô's formula for the divergence integral

Suppose that $f, g : [0,T] \longrightarrow \mathbb{R}$ are Hölder continuous functions of orders α and β with $\alpha + \beta > 1$. Young [46] proved that the Riemann-Stieltjes integral $\int_0^T f dg$ exists. As a consequence, if F is a function of class C^2, and $H > \frac{1}{2}$, the path-wise Riemann-Stieltjes integral $\int_0^t F'(B_s) dB_s$ exists for each $t \in [0,T]$. Moreover the following change of variables formula holds:

$$F(B_t) = F(0) + \int_0^t F'(B_s) dB_s. \tag{3.3}$$

In fact, it suffices to show that the second order term

$$R_\pi := \sum_{i=1}^n F''(X_i)(B_{t_i} - B_{t_{i-1}})^2$$

converges to zero almost surely as the norm of the partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ tends to zero, where X_i is an intermediate value between B_{t_i} and $B_{t_{i-1}}$. This follows immediately from the estimate

$$|R_\pi| \le C_\varepsilon \|F''\|_\infty \sum_{i=1}^n |t_i - t_{i-1}|^{2H-\varepsilon}.$$

Moreover, the Riemann-Stieltjes path-wise integral $\int_0^t F'(B_s) dB_s$ coincides with the symmetric integral in the Russo-Vallois sense introduced in Definition 1.

Suppose that F is a function of class $C^2(\mathbb{R})$ such that

$$\max\{|F(x)|, |F'(x)|, |F''(x)|\} \le c e^{\lambda x^2}, \tag{3.4}$$

where c and λ are positive constants such that $\lambda < \frac{1}{4T^{2H}}$. This condition implies

$$E\left(\sup_{0\le t\le T} |F(B_t)|^p\right) \le c^p E\left(e^{p\lambda \sup_{0\le t\le T} |B_t|^2}\right) < \infty$$

for all $p < \frac{T^{-2H}}{2\lambda}$. In particular, we can take $p = 2$. The same property holds for F' and F''.

Then, if F satisfies the growth condition (3.4), the process $F'(B_t)$ belongs to the space $\mathbb{D}^{1,2}(|\mathcal{H}|)$ and (3.1) holds. As a consequence, from Proposition 2 we obtain

$$\begin{aligned}
\int_0^t F'(B_s)dB_s &= \int_0^t F'(B_s)\delta B_s + H(2H-1)\int_0^t\int_0^s F''(B_s)(s-r)^{2H-2}drds \\
&= \int_0^t F'(B_s)\delta B_s + H\int_0^t F''(B_s)s^{2H-1}ds. \qquad (3.5)
\end{aligned}$$

Therefore, putting together (3.3) and (3.5) we deduce the following Itô's formula for the divergence process

$$F(B_t) = F(0) + \int_0^t F'(B_s)\delta B_s + H\int_0^t F''(B_s)s^{2H-1}ds. \qquad (3.6)$$

The divergence operator has the following local property:

Lemma 3 *Let u be an element of $\mathbb{D}^{1,2}(\mathcal{H})$. If $u = 0$ a.s. on a set $A \in \mathcal{F}$, then $\delta(u) = 0$ a.s. on A.*

Given a set L of $\mathcal{H}$-valued random variables we will denote by L_{loc} the set of $\mathcal{H}$-valued random variables u such that there exists a sequence $\{(\Omega^n, u^n)\}, n \geq 1\} \subset \mathcal{F} \times L$ with the following properties:

i) $\Omega^n \uparrow \Omega$ a.s.

ii) $u = u^n$, a.e. on $[0,T] \times \Omega_n$.

We then say that $\{(\Omega^n, u^n)\}$ localizes u in L. If $u \in \mathbb{D}^{1,2}_{loc}(\mathcal{H})$ by Lemma 3 we can define without ambiguity $\delta(u)$ by setting

$$\delta(u)|_{\Omega^n} = \delta(u^n)|_{\Omega^n}$$

for each $n \geq 1$, where $\{(\Omega^n, u^n)\}$ is a localizing sequence for u in L.

We state the following general version of Itô's formula proved in [4]:

Theorem 4 *Let F be a function of class $C^2(\mathbb{R})$. Assume that $u = \{u_t, t \in [0,T]\}$ is a process in the space $\mathbb{D}^{2,2}_{loc}(|\mathcal{H}|)$ such that the indefinite integral $X_t = \int_0^t u_s\delta B_s$ is a.s. continuous. Assume that $\|u\|_2$ belongs to $\mathcal{H}$. Then for each $t \in [0,T]$ the following formula holds*

$$\begin{aligned}
F(X_t) = F(0) &+ \int_0^t F'(X_s)u_s\delta B_s \\
&+ \alpha_H\int_0^t F''(X_s)\, u_s\left(\int_0^T |s-\sigma|^{2H-2}\left(\int_0^s D_\sigma u_\theta \delta B_\theta\right)d\sigma\right)ds \\
&+ \alpha_H\int_0^t F''(X_s)u_s\left(\int_0^s u_\theta\,(s-\theta)^{2H-2}\,d\theta\right)ds. \qquad (3.7)
\end{aligned}$$

Remark 1 If the process u is adapted, then the third summand in the right-hand side of (3.7) can be written as

$$\alpha_H \int_0^t F''(X_s)\ u_s \left(\int_0^s \left(\int_0^\theta |s-\sigma|^{2H-2} D_\sigma u_\theta d\sigma \right) \delta B_\theta \right) ds.$$

Remark 2 $\frac{2H-1}{s^{2H-1}}(s-\theta)^{2H-2}\mathbf{1}_{[0,s]}(\theta)$ is an approximation of the identity as H tends to $\frac{1}{2}$. Therefore, taking the limit as H converges to $\frac{1}{2}$ in Equation (3.7) we recover the usual Itô's formula for the the Skorohod integral proved by Nualart and Pardoux [33].

3.2 Stochastic integration with respect to fBm in the case $H < \frac{1}{2}$

The extension of the previous results to the case $H < \frac{1}{2}$ is not trivial and new difficulties appear. In order to illustrate these difficulties, let us first remark that the forward integral $\int_0^T B_t dB_t$ in the sense of Russo and Vallois (with the convergence in L^2) does not exists. In fact, a simple argument shows that, if $t_i = \frac{iT}{n}$, the expectation of the Riemann sums

$$\sum_{i=1}^n B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})$$

diverges:

$$\begin{aligned} \sum_{i=1}^n E\left(B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})\right) &= \frac{1}{2}\sum_{i=1}^n \left[t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H}\right] \\ &= \frac{1}{2}T^{2H}\ \left(1 - n^{1-2H}\right) \to -\infty, \end{aligned}$$

as n tends to infinity. Notice, however, that the expectation of symmetric Riemann sums is constant:

$$\frac{1}{2}\sum_{i=1}^n E\left((B_{t_i} + B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})\right) = \frac{1}{2}\sum_{i=1}^n \left[t_i^{2H} - t_{i-1}^{2H}\right] = \frac{T^{2H}}{2}.$$

We recall that for $H < \frac{1}{2}$ the operator K_H^* given by (1.35) is an isometry between the Hilbert space $\mathcal{H}$ and $L^2(0,T)$. We have the estimate :

$$\left|\frac{\partial K}{\partial t}(t,s)\right| \leq c_H(\frac{1}{2} - H)\ (t-s)^{H-\frac{3}{2}}. \tag{3.8}$$

Consider the following seminorm on the set $\mathcal{E}$ of step functions on $[0,T]$:

$$\begin{aligned} \|\varphi\|_K^2 &= \int_0^T \varphi^2(s) K(T,s)^2 ds \\ &\quad + \int_0^T \left(\int_s^T |\varphi(t) - \varphi(s)|\,(t-s)^{H-\frac{3}{2}} dt \right)^2 ds. \end{aligned}$$

We denote by $\mathcal{H}_K$ the completion of $\mathcal{E}$ with respect to this seminorm. The space $\mathcal{H}_K$ is continuously embedded in $\mathcal{H}$.

The following result is the counterpart of Proposition 2 in the case $H < \frac{1}{2}$ and its has been proved in [1]:

Proposition 5 *Let $u = \{u_t, t \in [0,T]\}$ be a stochastic process in the space $\mathbb{D}^{1,2}(\mathcal{H}_K)$. Suppose that the trace defined as the limit in probability*

$$\mathrm{Tr}Du := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^T \left\langle Du_s, \mathbf{1}_{[s-\varepsilon, s+\varepsilon]} \right\rangle_{\mathcal{H}} ds$$

exists and

$$\begin{aligned} E\left(\int_0^T u_s^2 \left(s^{2H-1} + (T-s)^{2H-1} \right) ds \right) &< \infty, \\ E\left(\int_0^T \int_0^T (D_r u_s)^2 \left(s^{2H-1} + (T-s)^{2H-1} \right) ds dr \right) &< \infty. \end{aligned}$$

Then the symmetric stochastic integral of u with respect to fBm in the sense of Definition 1 exists and

$$\int_0^T u_t dB_t = \delta(u) + \mathrm{Tr}Du.$$

Consider the particular case of the process $u_t = F(B_t)$, where F is a continuously differentiable function satisfying the growth condition

$$\max\left\{ |F(x)|, |F'(x)|, |F''(x)| \right\} \leq ce^{\lambda x^2},$$

where c and λ are positive constants such that $\lambda < \frac{1}{4T^{2H}}$. If $H > \frac{1}{4}$, the process $F(B_t)$ the process belongs to$\mathbb{D}^{1,2}(\mathcal{H}_K)$. Moreover, $\mathrm{Tr}Du$ exists and

$$\mathrm{Tr}Du = H \int_0^T F'(B_t) t^{2H-1} dt.$$

As a consequence we obtain

$$\int_0^T F(B_t) dB_t = \int_0^T F(B_t) \delta B_t + H \int_0^T F'(B_t) t^{2H-1} dt.$$

3.2.1 Itô's formulas for the divergence integral in the case $H < \frac{1}{2}$

An Itô's formula similar to (3.6) was proved in [3] for general Gaussian processes of Volterra-type of the form $B_t = \int_0^t K(t,s) dW_s$, where $K(t,s)$ is a singular kernel. In particular, the process B_t can be a fBm with Hurst parameter $\frac{1}{4} < H < \frac{1}{2}$. Moreover, in this paper, an Itô's formula for the indefinite divergence process $X_t = \int_0^t u_s \delta B_s$ similar to (3.7) was also proved.

On the other hand, in the case of the fractional Brownian motion with Hurst parameter $\frac{1}{4} < H < \frac{1}{2}$, an Itô's formula for the indefinite symmetric integral $X_t = \int_0^t u_s dB_s$ has been proved in [1] assuming again $\frac{1}{4} < H < \frac{1}{2}$.

Let us explain the reason for the restriction $\frac{1}{4} < H$. In order to define the divergence integral $\int_0^T F'(B_s)\delta B_s$, we need the process $F'(B_s)$ to belong to $L^2(\Omega; \mathcal{H})$. This is clearly true, provided F satisfies the growth condition (3.4), because $F'(B_s)$ is Hölder continuous of order $H - \varepsilon > \frac{1}{2} - H$ if $\varepsilon < 2H - \frac{1}{2}$. If $H \leq \frac{1}{4}$, one can show (see [9]) that

$$P(B \in \mathcal{H}) = 0,$$

and the space $\mathbb{D}^{1,2}(\mathcal{H})$ is too small to contains processes of the form $F'(B_t)$.

Following the approach of [9] we are going to extend the domain of the divergence operator to processes whose trajectories are not necessarily in the space $\mathcal{H}$.

Using (1.35) and applying the integration by parts formula for the fractional calculus (1.9) we obtain for any $f, g \in \mathcal{H}$

$$\begin{aligned} \langle f, g \rangle_{\mathcal{H}} &= \langle K_H^* f, K_H^* g \rangle_{L^2(0,T)} \\ &= d_H^2 \left\langle s^{\frac{1}{2}-H} D_{T-}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} f, s^{\frac{1}{2}-H} D_{T-}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} g \right\rangle_{L^2(0,T)} \\ &= d_H^2 \left\langle f, s^{H-\frac{1}{2}} s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{1-2H} D_{T-}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} g \right\rangle_{L^2(0,T)}. \end{aligned}$$

This implies that the adjoint of the operator K_H^* in $L^2(0,T)$ is

$$\left(K_H^{*,a} f \right)(s) = d_H s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{1-2H} D_{T-}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} f.$$

Set $\mathcal{H}_2 = (K_H^*)^{-1} \left(K_H^{*,a} \right)^{-1} (L^2(0,T))$.Denote by $\mathcal{S}_{\mathcal{H}}$ the space of smooth and cylindrical random variables of the form

$$F = f(B(\phi_1), \ldots, B(\phi_n)), \tag{3.9}$$

where $n \geq 1$, $f \in C_b^{\infty}(\mathbb{R}^n)$ (f and all its partial derivatives are bounded), and $\phi_i \in \mathcal{H}_2$.

Definition 6 *Let* $u = \{u_t, t \in [0,T]\}$ *be a measurable process such that*

$$E\left(\int_0^T u_t^2 dt \right) < \infty.$$

We say that $u \in \mathrm{Dom}^* \delta$ *if there exists a random variable* $\delta(u) \in L^2(\Omega)$ *such that for all* $F \in \mathcal{S}_{\mathcal{H}}$ *we have*

$$\int_{\mathbb{R}} E(u_t K_H^{*,a} K_H^* D_t F) dt = E(\delta(u) F).$$

This extended domain of the divergence operator satisfies the following elementary properties:

1. Dom$\delta \subset$ Dom$^*\delta$, and δ restricted to Domδ coincides with the divergence operator.
2. If $u \in$ Dom$^*\delta$ then $E(u)$ belongs to $\mathcal{H}$.
3. If u is a deterministic process, then $u \in$ Dom$^*\delta$ if and only if $u \in \mathcal{H}$.

This extended domain of the divergence operator leads to the following version of Itô's formula for the divergence process, established by Cheridito and Nualart in [9].

Theorem 7 *Suppose that F is a function of class $C^2(\mathbb{R})$ satisfying the growth condition (3.4). Then for all $t \in [0,T]$, the process $\{F'(B_s)\mathbf{1}_{[0,t]}(s)\}$ belongs to* Dom$^*\delta$ *and we have*

$$F(B_t) = F(0) + \int_0^t F'(B_s)\delta B_s + H \int_0^t F''(B_s)s^{2H-1}ds. \tag{3.10}$$

Sketch of the proof. $F'(B_s)1_{[0,t]}(s) \in L^2(\Omega \times [0,T])$ and

$$F(B_t) - F(0) - H\int_0^t F''(B_s^H)s^{2H-1}ds \in L^2(\Omega)\,.$$

Hence, it suffices to show that for any $G \in \mathcal{S}_{\mathcal{H}}$ we have

$$\begin{aligned} & E\left(\left\langle F'(B.)1_{[0,t]}, D.G\right\rangle_{\mathcal{H}}\right) \qquad (3.11)\\ = \; & E\left[G\left(F(B_t) - F(0) - H\int_0^t F''(B_s^H)s^{2H-1}ds\right)\right]. \end{aligned}$$

Equality (3.11) is proved by choosing smooth and cylindrical random variables of the form $G = H_n(B(\varphi))$, where H_n denotes the nth Hermite polynomial, and applying an integration by parts formula. ■

3.2.2 Local time and Tanaka's formula for fBm

Berman proved in [5] that that fractional Brownian motion $B = \{B_t, t \geq 0\}$ has a local time l_t^a continuous in $(a,t) \in \mathbb{R} \times [0,\infty)$ which satisfies the occupation formula

$$\int_0^t g(B_s)ds = \int_{\mathbb{R}} g(a)l_t^a da. \tag{3.12}$$

for every continuous and bounded function g on $\mathbb{R}$. Moreover, l_t^a is increasing in the time variable. Set

$$L_t^a = 2H\int_0^t s^{2H-1}l^a(ds).$$

It follows from (3.12) that

$$2H\int_0^t g(B_s)s^{2H-1}ds = \int_{\mathbb{R}} g(a)L_t^a da.$$

This means that $a \to L_t^a$ is the density of the occupation measure

$$\mu(C) = 2H\int_0^t \mathbf{1}_C(B_s)s^{2H-1}ds,$$

where C is a Borel subset of $\mathbb{R}$. Furthermore, the continuity property of l_t^a implies that L_t^a is continuous in $(a,t) \in \mathbb{R}\times[0,\infty)$.

As an extension of the Itô's formula (3.10), the following result has been proved in [9]:

Theorem 8 *Let $0 < t < \infty$ and $a \in \mathbb{R}$. Then*

$$\mathbf{1}_{\{B_s>a\}}\mathbf{1}_{[0,t]}(s) \in \mathrm{Dom}^*\delta\,,$$

and

$$(B_t-a)^+ = (-a)^+ + \int_0^t \mathbf{1}_{\{B_s>a\}}\delta B_s + \frac{1}{2}L_t^a\,. \tag{3.13}$$

This result can be considered as a version of Tanaka's formula for the fBm. In [12] it is proved that for $H > \frac{1}{3}$, the process $\mathbf{1}_{\{B_s>a\}}\mathbf{1}_{[0,t]}(s)$ belongs to Domδ and (3.13) holds.

The local time λ_t^a has Hölder continuous paths of order $\delta < 1-H$ in time, and of order $\gamma < \frac{1-H}{2H}$ in the space variable, provided $H \geq \frac{1}{3}$(see Table 2 in [21]). Moreover, λ_t^a is absolutely continuous in a if $H < \frac{1}{3}$, it is continuously differentiable if $H < \frac{1}{5}$, and its smoothness in the space variable increases when H decreases.

In a recent paper, Eddahbi, Lacayo, Solé, Tudor and Vives [18] have proved that $l_t^a \in \mathbb{D}^{\alpha,2}$ for all $\alpha < \frac{1-H}{2H}$. That means, the regularity of the local time l_t^a in the sense of Malliavin calculus is the same order as its Hölder continuity in the space variable. This result follows from the Wiener chaos expansion (see [12]):

$$l_t^a = \sum_{n=0}^\infty \int_0^t s^{-nH}p(s^{2H},a)H_n(as^{-H})I_n\left(K_H(s,\cdot)^{\otimes n}\right)ds.$$

In fact, the series

$$\begin{aligned}
&\sum_{n=0}^\infty (1+n)^\alpha E\left[\left(\int_0^t s^{-nH}p(s^{2H},a)H_n(as^{-H})I_n\left(K_H(s,\cdot)^{\otimes n}\right)ds\right)^2\right] \\
=\; &\sum_{n=0}^\infty (1+n)^\alpha n! \int_0^t\int_0^t (sr)^{-nH}p(s^{2H},a)p(r^{2H},a)H_n(as^{-H})H_n(ar^{-H}) \\
&\times \langle K_H(s,\cdot),\ K_H(r,\cdot)\rangle_{\mathcal{H}}\,drds
\end{aligned}$$

is equivalent to

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{-\frac{1}{2}+\alpha} \int_0^t \int_0^t R_H(u,v)(uv)^{-nH-1} du dv \\
= & \sum_{n=0}^{\infty} n^{-\frac{1}{2}+\alpha} \int_0^1 R_H(1,z) z^{-nH-1} dz.
\end{aligned}$$

Then, the result follows from the estimate

$$\left| \int_0^1 R_H(1,z) z^{-nH-1} dz \right| \leq C n^{-\frac{1}{2H}}.$$

4 Stochastic differential equations driven by a fBm

Let $B = \{B_t, t \geq 0\}$ be an m-dimensional fractional Brownian motion of Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$. This means that the components of B are independent fBm with the same Hurst parameter H. Consider the equation on $\mathbb{R}^d$

$$X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds, \quad t \in [0,T], \tag{4.1}$$

where X_0 is a d-dimensional random variable. The integral with respect to B is a path-wise Riemann-Stieltjes integral, and we know that this integral exists provided that the process $\sigma(s, X_s)$ has Hölder continuous trajectories of order larger that $1-H$.

In [27], Lyons considered deterministic integral equations of the form

$$x_t = x_0 + \int_0^t \sigma(x_s) dg_s,$$

$0 \leq t \leq T$, where the $g : [0,T] \to \mathbb{R}^d$ is a continuous functions with bounded p-variation for some $p \in [1,2)$. This equation has a unique solution in the space of continuous functions of bounded p-variation if each component of g has a Hölder continuous derivative of order $\alpha > p-1$. Taking into account that fBm of Hurst parameter H has locally bounded p-variation paths for $p > 1/H$, the result proved in [27] can be applied to Equation (4.1) in the case $\sigma(s,x) = \sigma(x)$, and $b(s,x) = 0$, provided the coefficient σ has a Hölder continuous derivative of order $\alpha > \frac{1}{H} - 1$.

Using the approach based on the notion of p-variation and the general limit theorem proved by Lyons in [28] for differential equations driven by geometric rough paths, Coutin and Qian [14], [13] have established the existence of strong solutions and a Wong-Zakai type approximation limit for stochastic differential equations driven by a fractional Brownian motion with parameter $H > \frac{1}{4}$.

In [40] Ruzmaikina establishes an existence and uniqueness theorem for ordinary differential equations with Hölder continuous forcing. The global solution is constructed, first, in small time intervals where the contraction principle can be applied, provided the Hölder constant is small enough. The main estimates are deduced using Hölder norms.

In [48] the existence and uniqueness of solutions is proved for differential equations driven by a fractional Brownian motion with parameter $H > \frac{1}{2}$, in a small random interval, provided the diffusion coefficient is a contraction in the space $W_{2,\infty}^{\beta}$, where $\frac{1}{2} < \beta < H$. Here $W_{2,\infty}^{\beta}$ denotes the Besov-type space of bounded measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$\int_0^T \int_0^T \frac{(f(t) - f(s))^2}{|t-s|^{2\beta+1}} ds dt < \infty.$$

In [34] Nualart and Rascanu have established the existence and uniqueness of solution for Equation (4.1) using an a priori estimate based on the fractional integration by parts formula, following the approach of Zähle [47]. In this section we will survey the main ideas and results of [34].

4.1 Generalized Stieltjes integrals

Given a function $g : [0, T] \to \mathbb{R}$, set $g_{T-}(s) = g(s) - \lim_{\varepsilon \downarrow 0}(T - \varepsilon)$ provided this limit exists. Take $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and $0 < \alpha < 1$. Suppose that f and g are functions on $[0, T]$ such that $g(T-)$ exists, $f \in I_{0+}^{\alpha}(L^p)$ and $g_{T-} \in I_{T-}^{1-\alpha}(L^q)$. Then the *generalized Stieltjes integral* of f with respect to g is defined by (see [47])

$$\int_0^T f dg = \int_0^T D_{0+}^{\alpha} f_{a+}(s) \, D_{T-}^{1-\alpha} g_{T-}(s) \, ds. \tag{4.2}$$

In [47] it is proved that this integral coincides with the Riemann-Stieltjes integral if f and g are Hölder continuous of orders α and β with $\alpha + \beta > 1$.

Fix $0 < \alpha < \frac{1}{2}$. Denote by $W_0^{\alpha,\infty}(0, T)$ the space of measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$\|f\|_{\alpha,\infty} := \sup_{t \in [0,T]} \left(|f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds \right) < \infty.$$

We have, for all $0 < \varepsilon < \alpha$

$$C^{\alpha+\varepsilon}(0, T) \subset W_0^{\alpha,\infty}(0, T) \subset C^{\alpha-\varepsilon}(0, T).$$

Denote by $W_T^{1-\alpha,\infty}(0, T)$ the space of measurable functions $g : [0, T] \to \mathbb{R}$ such that

$$\|g\|_{1-\alpha,\infty,T} := \sup_{0<s<t<T} \left(\frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y) - g(s)|}{(y-s)^{2-\alpha}} dy \right) < \infty.$$

We have, for all $0 < \varepsilon < \alpha$

$$C^{1-\alpha+\varepsilon}(0,T) \subset W_T^{1-\alpha,\infty}(0,T) \subset C^{1-\alpha}(0,T).$$

For $g \in W_T^{1-\alpha,\infty}(0,T)$ define

$$\begin{aligned}\Lambda_\alpha(g) &:= \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} \left| \left(D_{t-}^{1-\alpha} g_{t-} \right)(s) \right| \\ &\leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha,\infty,T}.\end{aligned}$$

Finally, denote by $W_0^{\alpha,1}(0,T)$ the space of measurable functions f on $[0,T]$ such that

$$\|f\|_{\alpha,1} := \int_0^T \frac{|f(s)|}{s^\alpha} ds + \int_0^T \int_0^s \frac{|f(s)-f(y)|}{(s-y)^{\alpha+1}} dy\ ds < \infty.$$

If f is a function in the space $W_0^{\alpha,1}(0,T)$, and g belongs to $W_T^{1-\alpha,\infty}(0,T)$, then the generalized Stieltjes integral $\int_0^t f dg$ exists for all $t \in [0,T]$ and we have

$$\left| \int_0^t f dg \right| \leq \Lambda_\alpha(g) \|f\|_{\alpha,1}.$$

Indeed,

$$\begin{aligned}\left| \int_0^t f dg \right| &= \left| \int_0^t \left(D_{0+}^\alpha f \right)(s) \left(D_{t-}^{1-\alpha} g_{t-} \right)(s)\, ds \right| \\ &\leq \sup_{0 \leq s \leq t \leq T} \left| \left(D_{t-}^{1-\alpha} g_{t-} \right)(s) \right| \int_0^t \left| \left(D_{0+}^\alpha f \right)(s) \right| ds \\ &\leq \Lambda_\alpha(g) \|f\|_{\alpha,1}.\end{aligned}$$

4.2 Main estimate

Fix $0 < \alpha < \frac{1}{2}$. Given two functions $g \in W_T^{1-\alpha,\infty}(0,T)$ and $f \in W_0^{\alpha,1}(0,T)$ we set

$$h_t = \int_0^t f dg.$$

Then for all $s < t \leq T$ we have

$$\begin{aligned}|h_t| + \int_0^t \frac{|h_t - h_s|}{(t-s)^{\alpha+1}}\, ds \quad &\leq \quad \Lambda_\alpha(g)\, c_{\alpha,T}^{(1)} \int_0^t \left((t-r)^{-2\alpha} + r^{-\alpha} \right) \\ &\qquad \times \left(|f_r| + \int_0^r \frac{|f_r - f_y|}{(r-y)^{\alpha+1}} dy \right) dr, \qquad (4.3)\end{aligned}$$

where $c_{\alpha,T}^{(1)}$ is a constant depending on α and T.

As a consequence of this estimate, if $f \in W_T^{1-\alpha,\infty}(0,T)$ we have

$$\left|\int_s^t fdg\right| \le \Lambda_\alpha(g)\, c_{\alpha,T}^{(2)}\ (t-s)^{1-\alpha}\, \|f\|_{\alpha,\infty}\,,$$

and

$$\left\|\int_0^{\cdot} fdg\right\|_{\alpha,\infty} \le \Lambda_\alpha(g)\, c_{\alpha,T}^{(3)}\quad \|f\|_{\alpha,\infty}\,.$$

Sketch of the proof of (4.3). Using the definition and additivity property of the indefinite integral we obtain

$$\begin{aligned} |h_t - h_s| &= \left|\int_s^t fdg\right| = \left|\int_s^t D_{s+}^{\alpha}(f)(r)\left(D_{t-}^{1-\alpha} g_{t-}\right)(r)\,dr\right| \\ &\le \Lambda_\alpha(g)\left(\int_s^t \frac{|f_r|}{(r-s)^\alpha}dr + \alpha\int_s^t\int_s^r \frac{|f_r - f_y|}{(r-y)^{\alpha+1}}dydr\right). \quad (4.4)\end{aligned}$$

Taking $s=0$ we obtain the desired estimate for $|h_t|$. Multiplying (4.4) by $(t-s)^{-\alpha-1}$ and integrating in s yields

$$\begin{aligned} \int_0^t \frac{|h_t-h_s|}{(t-s)^{\alpha+1}}\ ds &\le \Lambda_\alpha(g)\int_0^t (t-s)^{-\alpha-1} \quad (4.5)\\ &\quad\times\left(\int_s^t \frac{|f_r|}{(r-s)^\alpha}dr + \alpha\int_s^t\int_s^r \frac{|f_r-f_y|}{(r-y)^{\alpha+1}}dydr\right)ds.\end{aligned}$$

By the substitution $s = r-(t-r)y$ we have

$$\int_0^r (t-s)^{-\alpha-1}(r-s)^{-\alpha}ds \le (t-r)^{-2\alpha}\int_0^\infty (1+y)^{-\alpha-1}y^{-\alpha}dy \quad (4.6)$$

and, on the other hand,

$$\int_0^y (t-s)^{-\alpha-1}ds = \alpha^{-1}\left[(t-y)^{-\alpha} - t^{-\alpha}\right] \le \alpha^{-1}(t-y)^{-\alpha}. \quad (4.7)$$

Substituting (4.6) and (4.7) into (4.5) yields

$$\begin{aligned} \int_0^t \frac{|h_t-h_s|}{(t-s)^{\alpha+1}}\ ds &\le \Lambda_\alpha(g)\left[c_\alpha^{(1)}\int_0^t \frac{|f_r|}{(t-r)^{2\alpha}}dr\right.\\ &\quad\left. + \int_0^t\int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}}(t-y)^{-\alpha}dydr\right],\end{aligned}$$

where

$$c_\alpha^{(1)} = \int_0^\infty (1+y)^{-\alpha-1} y^{-\alpha} dy \ = B(2\alpha, 1-\alpha).$$

■

4.3 Deterministic differential equations

Let $0 < \alpha < \frac{1}{2}$ be fixed. Let $g^i \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^m)$, $j = 1, \ldots, m$. Consider the deterministic differential equation on $\mathbb{R}^d$

$$x_t = x_0 + \int_0^t b(s, x_s) ds + \int_0^t \sigma(s, x_s) \, dg_s, \ \ t \in [0,T], \tag{4.8}$$

where $x_0 \in \mathbb{R}^d$.

Let us introduce the following assumptions on the coefficients:

H1 $\sigma(t,x)$ is differentiable in x, and there exist some constants $0 < \beta, \delta \leq 1$ and for every $N \geq 0$ there exists $M_N > 0$ such that the following properties hold:

$$\begin{gathered} |\sigma(t,x) - \sigma(t,y)| \leq M_0 |x-y|, \quad \forall\, x \in \mathbb{R}^d, \forall\, t \in [0,T], \\ |\partial_{x_i}\sigma(t,x) - \partial_{x_i}\sigma(t,y)| \leq M_N |x-y|^\delta, \ \forall\, |x|, |y| \leq N, \forall t \in [0,T], \\ |\sigma(t,x) - \sigma(s,x)| + |\partial_{x_i}\sigma(t,x) - \partial_{x_i}\sigma(s,x)| \leq M_0 |t-s|^\beta, \\ \hfill \forall\, x \in \mathbb{R}^d, \forall\, t, s \in [0,T]. \end{gathered}$$

for each $i = 1, \ldots, d$.

H2 The coefficient $b(t,x)$ satisfies for every $N \geq 0$

$$\begin{aligned} |b(t,x) - b(t,y)| &\leq L_N |x-y|, \ \forall\, |x|, |y| \leq N, \forall t \in [0,T], \\ |b(t,x)| &\leq L_0 |x| + b_0(t), \ \forall x \in \mathbb{R}^d, \forall t \in [0,T], \end{aligned}$$

where $b_0 \in L^\rho\left(0,T;\mathbb{R}^d\right)$, with $\rho \geq 2$ and for some constant $L_N > 0$.

Theorem 9 *Suppose that the coefficients σ and b satisfy the assumptions H1 and H2 with $\rho = \frac{1}{\alpha}$, $0 < \beta, \delta \leq 1$ and $0 < \alpha < \alpha_0 = \min\left(\frac{1}{2}, \beta, \frac{\delta}{\delta+1}\right)$. Then Equation (4.8) has a unique continuous solution such that $x^i \in W_0^{\alpha,\infty}(0,T)$ for all $i = 1, \ldots, d$.*

Sketch of the proof. Suppose $d = m = 1$. Fix $\lambda > 1$ and define the seminorm in $W_0^{\alpha,\infty}(0,T)$ by

$$\|f\|_{\alpha,\lambda} = \sup_{t \in [0,T]} \ e^{-\lambda t} \left(|f_t| + \int_0^t \frac{|f_t - f_s|}{(t-s)^{\alpha+1}} \ \ ds \right).$$

Consider the operator $\mathcal{L}$ on defined by

$$(\mathcal{L}f)_t = x_0 + \int_0^t b(s, f_s)ds + \int_0^t \sigma(s, f_s)\, dg_s.$$

There exists λ_0 such that for $\lambda \geq \lambda_0$ we have

$$\|\mathcal{L}f\|_{\alpha,\lambda} \leq |x_0| + 1 + \frac{1}{2}\|f\|_{\alpha,\lambda}.$$

Hence, the operator $\mathcal{L}$ leaves invariant the ball B_0 of radius $2(|x_0|+1)$ in the norm $\|\cdot\|_{\alpha,\lambda_0}$ of the space $W_0^{\alpha,\infty}(0,T)$. Moreover, $\mathcal{L}$ is a contraction operator in $\mathcal{L}(B_0)$ with respect to a different norm $\|\cdot\|_{\alpha,\lambda}$ for a suitable value of $\lambda > 1$. A basic ingredient in the proof of this fact is the estimate

$$\begin{aligned}
&|\sigma(r, f_r) - \sigma(s, f_s) - \sigma(r, h_r) + \sigma(s, h_s)| \\
&\leq M_0|f_r - f_s - h_r + h_s| + M_0|f_r - h_r|(r-s)^\beta \\
&+ M_N|f_r - h_r|\left(|f_r - f_s|^\delta + |h_r - h_s|^\delta\right),
\end{aligned}$$

which is an immediate consequence of the properties of the function σ. This implies the existence of a solution by a fixed point argument. The uniqueness is proved again using the main estimate (4.3). ■

4.3.1 Estimates of the solution

Suppose that the coefficient σ satisfies the assumptions of the Theorem 9 and

$$|\sigma(t,x)| \leq K_0(1 + |x|^\gamma), \tag{4.9}$$

where $0 \leq \gamma \leq 1$. Then, the solution f of Equation (4.8) satisfies

$$\|f\|_{\alpha,\infty} \leq C_1 \exp(C_2 \Lambda_\alpha(g)^\kappa), \tag{4.10}$$

where

$$\kappa = \begin{cases} \frac{1}{1-2\alpha} & \text{if} \quad \gamma = 1 \\ > \frac{\gamma}{1-2\alpha} & \text{if} \quad \frac{1-2\alpha}{1-\alpha} \leq \gamma < 1 \\ \frac{1}{1-\alpha} & \text{if} \quad 0 \leq \gamma < \frac{1-2\alpha}{1-\alpha} \end{cases}$$

and the constants C_1 and C_2 depend on T, α, and the constants that appear in conditions H1, H2 and (4.9).

The proof of (4.10) is based on the following version of Gronwall lemma:

Lemma 10 *Fix $0 \leq \alpha < 1$, $a, b \geq 0$. Let $x : [0,\infty) \to [0,\infty)$ be a continuous function such that for each t*

$$x_t \leq a + bt^\alpha \int_0^t (t-s)^{-\alpha} s^{-\alpha} x_s ds. \tag{4.11}$$

Then

$$x_t \le a + a \sum_{n=1}^{\infty} b^n \frac{\Gamma(1-\alpha)^{n+1} t^{n(1-\alpha)}}{\Gamma[(n+1)(1-\alpha)]}.$$

$$\le a d_\alpha \exp\left[c_\alpha t b^{1/(1-\alpha)}\right], \tag{4.12}$$

where c_a and d_α are positive constants depending only on α (as an example, one can set $c_\alpha = 2\left(\Gamma(1-\alpha)\right)^{1/(1-\alpha)}$ and $d_\alpha = 4e^2 \frac{\Gamma(1-\alpha)}{1-\alpha}$).

This implies that there exists a constants $c_\alpha, d_\alpha > 0$ such that

$$x_t \le a d_\alpha \exp\left[c_\alpha t b^{1/(1-\alpha)}\right].$$

4.4 Stochastic differential equations with respect to fBm

Fix a parameter $\frac{1}{2} < H < 1$. Let $B = \{B_t, t \in [0,T]\}$ be a fractional Brownian motion with parameter H. Choose α such that $1 - H < \alpha < \frac{1}{2}$. By Fernique's theorem, for any $0 < \delta < 2$ we have

$$E\left(\exp\left(\Lambda_\alpha(B)^\delta\right)\right) < \infty.$$

As a consequence, if $u = \{u_t,\ t \in [0,T]\}$ is a stochastic process whose trajectories belong to the space $W_T^{\alpha,1}(0,T)$, almost surely, the path-wise generalized Stieltjes integral integral $\int_0^T u_s dB_s$ exists and we have the estimate

$$\left|\int_0^T u_s dB_s\right| \le G \left\|u\right\|_{\alpha,1}.$$

Moreover, if the trajectories of the process u belong to the space $W_0^{\alpha,\infty}(0,T)$, then the indefinite integral $U_t = \int_0^t u_s dB_s$ is Hölder continuous of order $1-\alpha$, and its trajectories also belong to the space $W_0^{\alpha,\infty}(0,T)$.

Consider the stochastic differential equation (4.1) on $\mathbb{R}^d$ where the process B is an m-dimensional fBm with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ and X_0 is a d-dimensional random variable. Suppose that the coefficients $\sigma^{i,j}, b^i : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ are measurable functions satisfying conditions H1 and H2, where the constants M_N and L_N may depend on $\omega \in \Omega$, and $\beta > 1 - H$, $\delta > \frac{1}{H} - 1$. Fix α such that

$$1 - H < \alpha < \alpha_0 = \min\left(\frac{1}{2}, \beta, \frac{\delta}{\delta+1}\right)$$

and $\alpha \le \frac{1}{\rho}$. Then the stochastic equation (4.3) has a unique continuous solution such that $X^i \in W_0^{\alpha,\infty}(0,T)$ for all $i = 1, \ldots, d$. Moreover the solution is Hölder continuous of order $1-\alpha$.

Assume that X_0 is bounded and the constants do not depend on ω. Suppose that

$$|\sigma(t,x)| \le K_0 \left(1 + |x|^\gamma\right),$$

where $0 \leq \gamma \leq 1$. Then,

$$\|X\|_{\alpha,\infty} \leq C_1 \exp\left(C_2 \Lambda_\alpha(B)^\kappa\right).$$

Hence,for all $p \geq 1$

$$E\left(\|X\|_{\alpha,\infty}^p\right) \leq C_1^p E\left(\exp\left(pC_2\Lambda_\alpha(B)^\kappa\right)\right) < \infty$$

provided $\kappa < 2$, that is,

$$\frac{\gamma}{4} + \frac{1}{2} \leq H$$

and

$$1 - H < \alpha < \frac{1}{2} - \frac{\gamma}{4}.$$

- If $\gamma = 1$ this means $\alpha < \frac{1}{4}$ and $H \leq \frac{3}{4}$
- If $\gamma < 2 - \frac{1}{H}$ we can take any α such that $1 - H < \alpha < \frac{1}{2}$.

5 Applications

In this section we will describe some applications of the stochastic calculus with respect to fBm.

5.1 Vortex filaments based on fBm

The observations of three-dimensional turbulent fluids indicate that the vorticity field of the fluid is concentrated along thin structures called vortex filaments. In his book Chorin [10] suggests probabilistic descriptions of vortex filaments by trajectories of self-avoiding walks on a lattice. Flandoli [19] introduced a model of vortex filaments based on a three-dimensional Brownian motion. A basic problem in these models is the computation of the kynetic energy of a given configuration.

Denote by $u(x)$ the velocity field of the fluid at point $x \in \mathbb{R}^3$, and let $\xi = \mathrm{curl} u$ be the associated vorticity field. The kynetic energy of the field will be

$$\mathbb{H} = \frac{1}{2}\int_{\mathbb{R}^3} |u(x)|^2 dx = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\xi(x)\cdot\xi(y)}{|x-y|}\, dxdy. \tag{5.1}$$

We will assume that the vorticity field is concentrated along a thin tube centered in a curve $\gamma = \{\gamma_t, 0 \leq t \leq T\}$. Moreover, we will choose a random model and consider this curve as the trajectory of a stochastic process three-dimensional fractional Brownian motion $B = \{B_t, 0 \leq t \leq T\}$. This can be formally expressed as

$$\xi(x) = \Gamma \int_{\mathbb{R}^3}\left(\int_0^T \delta(x - y - B_s)\dot{B}_s ds\right)\rho(dy), \tag{5.2}$$

where Γ is a parameter called the circuitation, and ρ is a probability measure on $\mathbb{R}^3$ with compact support.

Substituting (5.2) into (5.1) we derive the following formal expression for the kynetic energy:

$$\mathbb{H} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathbb{H}_{xy} \rho(dx)\rho(dy), \tag{5.3}$$

where the so-called interaction energy $\mathbb{H}_{xy}$ is given by the double integral

$$\mathbb{H}_{xy} = \frac{\Gamma^2}{8\pi} \sum_{i=1}^{3} \int_0^T \int_0^T \frac{1}{|x + B_t - y - B_s|} dB_s^i dB_t^i. \tag{5.4}$$

We are interested in the following problems: Is $\mathbb{H}$ a well defined random variable? Does it have moments of all orders and even exponential moments?

In order to give a rigorous meaning to the double integral (5.4) let us introduce the regularization of the function $|\cdot|^{-1}$:

$$\sigma_n = |\cdot|^{-1} * p_{1/n}, \tag{5.5}$$

where $p_{1/n}$ is the Gaussian kernel with variance $\frac{1}{n}$. Then, the smoothed interaction energy

$$\mathbb{H}_{xy}^n = \frac{\Gamma^2}{8\pi} \sum_{i=1}^{3} \int_0^T \left(\int_0^T \sigma_n(x + B_t - y - B_s)\ dB_s^i \right) dB_t^i, \tag{5.6}$$

is well defined, where the integrals are path-wise Riemann-Stieltjes integrals. Set

$$\mathbb{H}^n = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathbb{H}_{xy}^n \rho(dx)\rho(dy). \tag{5.7}$$

The following result has been proved in [35]:

Theorem 11 *Suppose that the measure ρ satisfies*

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |x - y|^{1-\frac{1}{H}} \rho(dx)\rho(dy) < \infty. \tag{5.8}$$

Let $\mathbb{H}_{xy}^n$ be the smoothed interaction energy defined by (5.6). Then $\mathbb{H}^n$ defined in (5.7) converges, for all $k \geq 1$, in $L^k(\Omega)$ to a random variable $\mathbb{H} \geq 0$ that we call the energy associated with the vorticity field (5.2).

If $H = \frac{1}{2}$, fBm B is a classical three-dimensional Brownian motion. In this case condition (5.8) would be $\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |x - y|^{-1} \rho(dx)\rho(dy) < \infty$, which is the assumption made by Flandoli [19] and Flandoli and Gubinelli [20]. In this last paper, using Fourier approach and Itô's stochastic calculus, the authors show that $Ee^{-\beta\mathbb{H}} < \infty$ for sufficiently small negative β.

The proof of Theorem 11 is based on the stochastic calculus of variations with respect to fBm and the application of Fourier transform.

Sketch of the proof of Theorem 11. The proof will be done in two steps:

Step 1(Fourier transform) Using

$$\frac{1}{|z|} = \int_{\mathbb{R}^3} (2\pi)^3 \frac{e^{-i\langle \xi, z\rangle}}{|\xi|^2} d\xi$$

we get

$$\sigma_n(x) = \int_{\mathbb{R}^3} |\xi|^{-2} e^{i\langle \xi, x\rangle - |\xi|^2/2n} \, d\xi.$$

Substituting this expression in (5.6), we obtain the following formula for the smoothed interaction energy

$$\begin{aligned}
\mathbb{H}^n_{xy} &= \frac{\Gamma^2}{8\pi} \sum_{j=1}^{3} \int_0^T \int_0^T \left(\int_{\mathbb{R}^3} e^{i\langle \xi, x+B_t-y-B_s\rangle} \, \frac{e^{-|\xi|^2/2n}}{|\xi|^2} \right) dB^j_s dB^j_t \\
&= \frac{\Gamma^2}{8\pi} \int_{\mathbb{R}^3} |\xi|^{-2} e^{i\langle \xi, x-y\rangle - |\xi|^2/2n} \, \|Y_\xi\|^2_{\mathbb{C}} \, d\xi, \qquad (5.9)
\end{aligned}$$

where

$$Y_\xi = \int_0^T e^{i\langle \xi, B_t\rangle} dB_t$$

and $\|Y_\xi\|^2_{\mathbb{C}} = \sum_{i=1}^3 Y^i_\xi \overline{Y_\xi}^i$. Integrating with respect to ρ yields

$$\mathbb{H}^n = \frac{\Gamma^2}{8\pi} \int_{\mathbb{R}^3} \|Y_\xi\|^2_{\mathbb{C}} \, |\xi|^{-2} \, |\widehat{\rho}(\xi)|^2 \, e^{-|\xi|^2/2n} d\xi \geq 0. \qquad (5.10)$$

From Fourier analysis and condition (5.8) we know that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |x-y|^{1-\frac{1}{H}} \rho(dx)\rho(dy) = C_H \int_{\mathbb{R}^3} |\widehat{\rho}(\xi)|^2 \, |\xi|^{\frac{1}{H}-4} d\xi < \infty. \qquad (5.11)$$

Then, taking into account (5.11) and (5.10), in order to show the convergence in $L^k(\Omega)$ of $\mathbb{H}^n$ to a random variable $\mathbb{H} \geq 0$ it suffices to check that

$$E\left(\|Y_\xi\|^{2k}_{\mathbb{C}} \right) \leq C_k \left(1 \wedge |\xi|^{k\left(\frac{1}{H}-2\right)} \right). \qquad (5.12)$$

Step 2 (Stochastic calculus) We will present the main arguments for the proof of the estimate (5.12) for $k=1$. Relation (3.2) applied to the process $u_t = e^{i\langle \xi, B_t\rangle}$ allows us to decompose the path-wise integral $Y_\xi = \int_0^T e^{i\langle \xi, B_t\rangle} dB_t$ into the sum of a divergence plus a trace term:

$$Y_\xi = \int_0^T e^{i\langle \xi, B_t\rangle} \delta B_t + H \int_0^T i\xi e^{i\langle \xi, B_t\rangle} t^{2H-1} dt. \qquad (5.13)$$

On the other hand, applying the three dimensional version of Itô's formula (3.6) we obtain

$$e^{i\langle \xi, B_T\rangle} = 1 + \sum_{j=1}^{3} \int_0^T i\xi_j e^{i\langle \xi, B_t\rangle} \delta B_t^j - H \int_0^T t^{2H-1} |\xi|^2 e^{i\langle \xi, B_t\rangle} dt. \qquad (5.14)$$

Multiplying both members of (5.14) by $i\xi|\xi|^{-2}$ and adding the result to (5.13) yields

$$Y_\xi = p_\xi \left(\int_0^T e^{i\langle \xi, B_t\rangle} \delta B_t \right) - \frac{i\xi}{|\xi|^2} \left(e^{i\langle \xi, B_T\rangle} - 1 \right) := Y_\xi^{(1)} + Y_\xi^{(2)},$$

where $p_\xi(v) = v - \frac{\xi}{|\xi|^2} \langle \xi, v\rangle$ is the orthogonal projection of v on $\langle \xi\rangle^\perp$. It suffices to derive the estimate (5.12) for the term $Y_\xi^{(1)}$. Using the duality relationship (2.2) for each $j = 1, 2, 3$ we can write

$$E\left(Y_\xi^{(1),j} \overline{Y}_\xi^{(1),j} \right) = E\left(\left\langle e^{i\langle \xi, B_\cdot\rangle}, p_\xi^j D_\cdot \left(p_\xi^j \int_0^T e^{-i\langle \xi, B_t\rangle} \delta B_t \right) \right\rangle_{\mathcal{H}} \right). \qquad (5.15)$$

The commutation relation $\langle D(\delta(u)), h\rangle_{\mathcal{H}} = \langle u, h\rangle_{\mathcal{H}} + \delta(\langle Du, h\rangle_{\mathcal{H}})$ implies

$$D_r^k \left(\int_0^T e^{-i\langle \xi, B_t\rangle} \delta B_t^j \right) = e^{-i\left\langle \xi, B_r^k\right\rangle} \delta_{k,j} + (-i\xi^k) \int_0^T \mathbf{1}_{[0,t]}(r) e^{-i\langle \xi, B_t\rangle} \delta B_t^j.$$

Applying the projection operators yields

$$\begin{aligned} p_\xi^j D_r \left(p_\xi^j \int_0^T e^{-i\langle \xi, B_t\rangle} \delta B_t \right) &= e^{-i\langle \xi, B_r\rangle} \left(I - \frac{\xi^* \xi}{|\xi|^2} \right)_{j,j} \\ &= e^{-i\langle \xi, B_r\rangle} \left(1 - \frac{\left(\xi^j\right)^2}{|\xi|^2} \right) \end{aligned}$$

Notice that the term involving derivatives in the expectation (5.15) vanishes. This cancellation is similar to what happens in the computation of tha variance of the divergence of an adapted process, in the case of the Brownian motion. Hence,

$$\begin{aligned} \sum_{j=1}^{3} E\left(Y_\xi^{(1),j} \overline{Y}_\xi^{(1),j} \right) &= 2\ E\left(\left\langle\ e^{-i\langle \xi, B_\cdot\rangle}, e^{-i\langle \xi, B_\cdot\rangle}\ \right\rangle_{\mathcal{H}} \right) \\ &= 2\alpha_H \int_0^T \int_0^T E\left(e^{i\langle \xi, B_s - B_r\rangle} \right)\ |s-r|^{2H-2}\ ds dr \\ &= 2\alpha_H \int_0^T \int_0^T e^{-\frac{|s-r|^{2H}}{2} |\xi|^2}\ |s-r|^{2H-2}\ ds dr, \end{aligned}$$

which behaves as $|\xi|^{\frac{1}{H}-2}$ as $|\xi|$ tends to infinity. This completes the proof of the desired estimate for $k=1$. ■

In the general case $k \geq 2$ the proof makes use of the *local nondeterminism property* of fBm:

$$\mathrm{Var}\left(\sum_i (B_{t_i} - B_{s_i})\right) \geq k_H \sum_i (t_i - s_i)^{2H}.$$

5.1.1 Decomposition of the interaction energy

Assume $\frac{1}{2} < H < \frac{2}{3}$. For any $x \neq y$, set

$$\widehat{\mathbb{H}_{xy}} = \sum_{i=1}^{3} \int_0^T \left(\int_0^t \frac{1}{|x + B_t - y - B_s|} \, dB_s^i \right) dB_t^i. \tag{5.16}$$

Then $\widehat{\mathbb{H}_{xy}}$ exists as the limit in $L^2(\Omega)$ of the sequence $\widehat{\mathbb{H}_{xy}^n}$ defined using the approximation $\sigma_n(x)$ of $|x|^{-1}$ introduced in (5.5) and the following decomposition holds

$$\begin{aligned}
\widehat{\mathbb{H}_{xy}} &= \sum_{i=1}^{3} \int_0^T \int_0^t \frac{1}{|x - y + B_t - B_r|} \delta B_r^i \delta B_t^i. \\
&\quad -H^2 \int_0^T \int_0^t \delta_0(x - y + B_t - B_r)(t-r)^{2(2H-1)} dr dt. \\
&\quad +H(2H-1) \int_0^T \left(\int_0^t \frac{1}{|x - y + B_t - B_r|} (t-r)^{2H-2} dr \right) dt \\
&\quad +H \int_0^T \left(\frac{1}{|x - y + B_T - B_r|} (T-r)^{2H-2} + \frac{1}{|x - y + B_r|} r^{2H-1} \right) dr.
\end{aligned}$$

Notice that in comparison with $\mathbb{H}_{xy}$, in the definition of $\widehat{\mathbb{H}_{xy}}$ we chose to deal with the half integral over the domain

$$\{0 \leq s \leq t \leq T\},$$

and to simplify the notation we have omitted the constant $\frac{\Gamma^2}{8\pi}$. Nevertheless, it holds that $\mathbb{H}_{xy} = \frac{\Gamma^2}{8\pi}\left(\widehat{\mathbb{H}_{xy}} + \widehat{\mathbb{H}_{yx}}\right)$, and we have proved using Fourier analysis that $\mathbb{H}_{xy}$ has moments of any order.

The following results have been proved in [35]:

1. All the terms in the above decomposition of $\widehat{\mathbb{H}_{xy}}$ exists in $L^2(\Omega)$ for $x \neq y$.
2. If $|x-y| \to 0$, then the terms behave as $|x-y|^{\frac{1}{H}-1}$, so they can be integrated with respect to $\rho(dx)\rho(dy)$.
3. The bound $H < \frac{2}{3}$ is sharp: For $H = \frac{2}{3}$ the weighted self-intersection local time diverges.

5.2 Application to financial mathematics

Fractional Brownian motion has been applied to describe the behavior to prices of assets and volatilities in stock markets. The long-range dependence self-similarity properties make this process a suitable model to describe these quantities.

5.2.1 Fractional Black Scholes model

Assume the price of a stock is modelled as

$$S_t = S_0 e^{\mu t + \sigma B_t},$$

where B_t is a fBm with Hurst parameter H and μ and $\sigma > 0$ are constants. If $H \neq \frac{1}{2}$ this model admits arbitrage (see [38], Shiryaev [43], Cheridito [8]). In the case $H > \frac{1}{2}$, one can construct an arbitrage in the following way. Suppose μ coincides with the interest rate r, and define the strategy (α_t, β_t), where α_t is the number of bonds and β_t is the number of assets, by

$$\begin{aligned}
\alpha_t &= 1 - e^{2B_t}, \\
\beta_t &= 1(e^{B_t} - 1).
\end{aligned}$$

The value of this strategy at time t is

$$V_t = \alpha_t e^{rt} + \beta_t S_t = e^{rt} \left(e^{B_t} - 1\right)^2.$$

This strategy is self-financing because

$$\begin{aligned}
dV_t &= re^{rt} \left(e^{B_t} - 1\right)^2 dt + 2\sigma e^{rt + \sigma B_t} \left(e^{B_t} - 1\right) dB_t \\
&= r\alpha_t e^{rt} + \beta_t dS_t,
\end{aligned}$$

however, $V_0 = 0$ and $V_t > 0$ for all $t > 0$. So, this strategy is an arbitrage.

5.2.2 Stochastic volatility models

In Comte and Renault [11], and Hu [22], the following model with stochastic volatility is considered. The price of an asset S_t is given by

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t,$$

where $\sigma_t = f(Y_t)$ and

$$dY_t = \alpha(m - Y_t)dt + \beta_t dB_t.$$

The process W_t is an ordinary Brownian motion and B_t is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, independent of W. Notice that Y_t is a fractional Ornstein-Uhlenbeck process. Examples of functions f are $f(x) = e^x$ and $f(x) = |x|$. Let us mention the following results on this model that are proved in [22]:

1) The market is incomplete and martingale measures are not unique.

2) Set $\gamma_t = (r - \mu)/\sigma_t$ and

$$\frac{dQ}{dP} = \exp\left(\int_0^T \gamma_t dW_t - \frac{1}{2} \int_0^T |\gamma_t|^2 dt \right).$$

Then, Q is the *minimal martingale measure* associated with P.

3) The risk minimizing-hedging price of an European call option is given by

$$v = e^{-rT} E_Q \left[(S_T - K)^+ \right].$$

If $\mathcal{G}_t$ denotes the filtration generated by fBm, it holds that

$$\begin{aligned} v &= e^{-rT} E_Q \left[E_Q \left((S_T - K)^+ |\mathcal{G}_T \right) \right] \\ &= e^{-rT} E_Q \left[C_{BS}(S_0, \sigma) \right], \end{aligned}$$

where $\sigma = \sqrt{\int_0^T \sigma_s^2 ds}$.

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