The spherical retina a conformal geometric algebra model for human like vision

Eduardo Bayro-Corrochano, Member, IEEE, and David Gonzalez-Aguirre

Abstract—In this paper we propose a conformal model for human like vision. In our model the two views are fused in an extended 3D horopter concept. The inverse transformation of the horopter spheres and Poncelet points leads remarkably to a 3D log-polar representation of the visual space. This representation is quite effective for processing the visual space information. We believe that this model is very suitable for building a powerful binocular head for humanoids.

I. INTRODUCTION

This paper presents a novel approach for building stereoscopic perception systems. The traditional 2D horopter is reformulated now as a 3D horopter using conformal geometric algebra. In this framework the visual space is represented as a family of horopter spheres which together with their Poncelet points lead remarkably to a 3D log-polar representation of the visual space. There is quite abundant research activity on image processing using 2D log-polar schemes with either monocular or stereo systems [8],[3]. However this kind of work is representing basically the Cartesian world using polar coordinates, and it fails to fuse the data of the two cameras of the stereoscopic vision system in a single framework. We believe that our human-like computational scheme looks promising for the processing of visual space data. The structure of this paper comprises the following sections. Section two gives a brief introduction to conformal geometric algebra. Section three describes the conformal model for stereoscopic perception and provides insights in the implementation details. The section four presents applications using real images. The conclusions are given in section five.

II. GEOMETRIC ALGEBRA: AN OUTLINE

Let \( G_n \) denote the geometric algebra of \( n \)-dimensions – this is a graded linear space. As well as vector addition and scalar multiplication we have a non-commutative product which is associative and distributive over addition – this is the geometric or Clifford product. A further distinguishing feature of the algebra is that any vector squares to give a scalar. The geometric product of two vectors \( a \) and \( b \) is written \( a\cdot b \) and can be expressed as a sum of its symmetric and antisymmetric parts

\[
ab = a \cdot b + a \wedge b, \tag{1}
\]

where the inner product \( a \cdot b \) and the outer product \( a \wedge b \) are defined by

\[
a \cdot b = \frac{1}{2} (a b + b a) \tag{2}
\]

\[
a \wedge b = \frac{1}{2} (a b - b a). \tag{3}
\]

The inner product of two vectors is the standard scalar or dot product and produces a scalar. The outer or wedge product of two vectors is a new quantity which we call a bivector. We think of a bivector as a oriented area in the plane containing \( a \) and \( b \), formed by sweeping \( a \) along \( b \) – see Figure 1.a.

Thus, \( b \wedge a \) will have the opposite orientation making the wedge product anti-commutative as given in equation 3.

The outer product is immediately generalizable to higher dimensions – for example, \( (a \wedge b) \wedge c \), a trivector, is interpreted as the oriented volume formed by sweeping the area \( a \wedge b \) along vector \( c \). The outer product of \( k \) vectors is a \( k \)-vector or \( k \)-blade, and such a quantity is said to have grade \( k \), see Figure 1.b. A multivector (linear combination of objects of different type) is homogeneous if it contains terms of only a single grade. The geometric algebra provides a means of manipulating multivectors which allows us to keep track of different grade objects simultaneously – much as one does with complex number operations.

In a space of 3 dimensions we can construct a trivector \( a \wedge b \wedge c \), but no 4-vectors exist since there is no possibility of sweeping the volume element \( a \wedge b \wedge c \) over a 4th dimension. The highest grade element in a space is called the pseudoscalar. The unit pseudoscalar is denoted by \( i \) and is crucial when discussing duality.

A. The Geometric algebra of \( n \)-D space

In an \( n \)-dimensional space we can introduce an orthonormal basis of vectors \( \{\sigma_i\} \), \( i = 1, \ldots, n \), such that \( \sigma_i \cdot \sigma_j = \delta_{ij} \). This leads to a basis for the entire algebra:

\[
1, \ \{\sigma_i\}, \ \{\sigma_i \wedge \sigma_j\}, \ \{\sigma_i \wedge \sigma_j \wedge \sigma_k\}, \ \ldots \ \sigma_1 \wedge \sigma_2 \wedge \ldots \wedge \sigma_n. \tag{4}
\]

Note that the basis vectors are not represented by bold symbols. Any multivector can be expressed in terms of this basis. In this paper we will specify a geometric algebra \( G_n \) of the \( n \) dimensional space by \( G_{p,q,r} \), where \( p, q \) and \( r \) stand...
for the number of basis vector which squares to 1, -1 and 0 respectively and fulfill $p+q+r$. Its even subalgebra will be denoted by $G_{p,q,r}$. For example $G_{0,2,0}$ has the basis
$$\{1, \sigma_1, \sigma_2, \sigma_1 \sigma_2\},$$
where $\sigma_1^2 = -1, \sigma_2^2 = -1$. This means $p=0, q=2$ and $r=0$. Thus the dimension of this geometric algebra is $n=p+q+r$.

In the n-D space there are multivectors of grade 0 (scalars), grade 1 (vectors), grade 2 (bivectors), grade 3 (trivectors), etc... up to grade $n$. Any two such multivectors can be multiplied using the geometric product. Consider two multivectors $A_r$ and $B_s$ of grades $r$ and $s$ respectively. The geometric product of $A_r$ and $B_s$ can be written as
$$A_r B_s = \langle AB \rangle_{r+s} + \langle AB \rangle_{r+s-2} + \ldots + \langle AB \rangle_{|r-s|}$$
(6)
where $\langle M \rangle_t$ is used to denote the $t$-grade part of multivector $M$, e.g. consider the geometric product of two vectors $ab = \langle ab \rangle_0 + \langle ab \rangle_2 = a \cdot b + a \wedge b$.

B. The geometric algebra of 3-D space

The basis for the geometric algebra $G_{3,0,0}$ of the the 3-D space has $2^3 = 8$ elements and is given by:
$$\{1, \sigma_1, \sigma_2, \sigma_3, \sigma_1 \sigma_2, \sigma_2 \sigma_3, \sigma_3 \sigma_1, \sigma_1 \sigma_2 \sigma_3\} \equiv I.$$ 
(7)
Since the basis vectors are orthogonal, i.e. $\sigma_1 \sigma_2 = \sigma_1 \cdot \sigma_2 + \sigma_1 \sigma_2 = \sigma_1 \wedge \sigma_2$, we write simply $\sigma_1 \sigma_2$.

It can easily be verified that the trivector or pseudoscalar $\sigma_1 \sigma_2 \sigma_3$ squares to $-1$ and commutes with all multivectors in the 3-D space. We therefore give it the symbol $i$; noting that this is not the uninterpreted commutative scalar imaginary $j$ used in quantum mechanics and engineering.

C. Rotors

Multiplication of the three basis vectors $\sigma_1, \sigma_2$, and $\sigma_3$ by $I$ results in the three basis bivectors $\sigma_1 \sigma_2 = I \sigma_3, \sigma_2 \sigma_3 = I \sigma_1$ and $\sigma_3 \sigma_1 = I \sigma_2$. These simple bivectors rotate vectors in their own plane by $90^\circ$, e.g. $(\sigma_1 \sigma_2) \sigma_2 = \sigma_1, (\sigma_2 \sigma_3) \sigma_2 = -\sigma_3$ etc. Identifying the $i, j, k$ of the quaternion algebra with $I \sigma_1, -I \sigma_2, I \sigma_3$ the famous Hamilton relations $i^2 = j^2 = -1, jk = -1$ can be recovered. Since the $i, j, k$ are bivectors, it comes as no surprise that they represent $90^\circ$ rotations in orthogonal directions and provide a well-suited system for the representation of general 3D rotations.

In geometric algebra a rotor (short name for rotator), $R$, is an even-grade element of the algebra which satisfies $R \overline{R} = 1$, where $\overline{R}$ stands for the conjugate of $R$, i.e. $R = a_0 - a_1 \sigma_2 \sigma_3 - a_2 \sigma_3 \sigma_1 - a_3 \sigma_1 \sigma_2 = a_0 - a$.

The transformation in terms of a rotor $a \mapsto Ra \overline{R} = b$ is a very general way of handling rotations; it works for multivectors of any grade and in spaces of any dimension in contrast to quaternion calculus. Rotors combine in a straightforward manner, i.e. a rotor $R_1$ followed by a rotor $R_2$ is equivalent to a total rotor $R$ where $R = R_2 R_1$.

D. Conformal geometric algebra

In order to explain the conformal geometric algebra firstly we introduce the the Minkowski plane $\mathbb{R}^{1,1}$, which has an orthonormal basis $\{e_+, e_-\}$ with the property $e_+^2 = 1, e_2^2 = -1$. Using this basis two extra null bases can be generated $e_{\infty} = (e_+ + e_-)$ and $e_0 = 1/2(e_+ - e_-)$, so that $e_{\infty} e_0 = 0$ and $e_0 e_{\infty} = 1$.

Given the Euclidean space $\mathbb{R}^n$, the conformal space is obtained via $\mathbb{R}^{n+1,1} = \mathbb{R}^n \oplus \mathbb{R}^{1,1}$. Its associated conformal geometric algebra is $G_{n+1,1}$. Any vector $x \in \mathbb{R}^n$ can be embedded in the conformal space using the following mapping
$$x = F(x) = -(x - e_+) e_\infty (x - e_+)$$
$$= x + \frac{1}{2} x^2 e_\infty + e_0,$$
(8)
so that $x^2 = 0$. Note that the vector in conformal space are written italics. A null vector like $x$ whose component $e_0$ is the unity is called normalized. Give the normalized null-vectors $x$ and $y$ we can compute a Euclidean metric
$$x \cdot y = -\frac{1}{2} (x - y)^2,$$
(9)
which corresponds to the length of a cord joining two points lying on the null cone, see for more details [2].

1) Lines, planes, and spheres of the 3D Euclidean space:

Lines, planes and hyperplanes are represented in the conformal space wedging points of the conformal space with the point at infinity
$$l = e_\infty \wedge x_1 \wedge x_2$$
$$\phi = e_\infty \wedge x_1 \wedge x_2 \wedge x_3$$
$$h_n = e_\infty \wedge x_1 \wedge x_2 \wedge x_3 \ldots \wedge x_n$$
(10)
The equation of a hypersphere can be formulated considering the hypersphere centered at point $x \in \mathbb{R}^n$ with radius $\rho^2 = (x - p)^2$, here $p$ is any point lying on the sphere. However we can express this using equation (9) in terms of homogeneous points
$$x \cdot p = \frac{1}{2} \rho^2.$$ 
(11)
Using $x \cdot e = -1$, we can simplify this equation to $x \cdot s = 0$, where
$$s = p - \frac{1}{2} \rho^2 e_\infty = p + \frac{1}{2} (p^2 - \rho^2) e_\infty + e_0.$$ 
(12)
The vector $s$ has the proprieties $s^2 = \rho^2$ and $e \cdot s = -1$. Note that if $\rho = 0$, equation (12) becomes the equation (8) of a point.

The same result can be obtained firstly computing the volume of the
$$\mathcal{V} = e_\infty \wedge x_1 \wedge x_2 \wedge x_3 \ldots \wedge x_{n+1},$$ 
(13)
and then taking the dual of the latter with the pseudoscalar $I = \sigma_1 \sigma_2 \sigma_3 e_\infty e_+$$
$$s = \mathcal{V} \cdot I.$$ 
(14)
Let us give as illustration the computation of a plane and a sphere of $R^3$, given four points in general position

\[ \phi = e_\infty \wedge x_1 \wedge x_2 \wedge x_3 \]

\[ s = \bar{x} \cdot I = (e_\infty \wedge x_1 \wedge x_2 \wedge x_3 \wedge x_4) \cdot I = p - \frac{1}{2} \rho^2 e_\infty. \]  

(15)

E. Geometric identities, duals and incidence algebra operations

A circle $z$ can be regarded as the intersection of two spheres $s_1$ and $s_2$. This means that for each point on the circle $x_c \in z$ they lie on both spheres as well $x_c \in s_1$ and $x_c \in s_2$. Assuming that $s_1$ and $s_2$ are linearly independent, we can write for $x_c \in z$

\[(x_c \cdot s_1) s_2 - (x_c \cdot s_2) s_1 = x_c \cdot (s_1 \wedge s_2) = x_c \cdot z = 0, \tag{16}\]

this result tells us that since $x_c$ lies on both spheres $z = (s_1 \wedge s_1)$ should be the intersection of the spheres or a circle. It is easy to see that the intersection with a third sphere leads to a point pair. We have derived algebraically that the wedge of two linearly independent spheres yields to their intersecting circle (see Figure 2), this topological relation between two spheres can be also conveniently described using the dual of the meet operation, namely

\[ z = (z^*)^* = (s_1^* \vee s_2^*)^* = s_1 \wedge s_2, \tag{17}\]

this new equation says that the dual of a circle can be computed via the meet of two spheres in their dual form. This equation confirms geometrically our previous algebraic computation of equation (16).

The dual form of the circle (in 3D) can be expressed by three points lying on it as

\[ z^* = x_{c_1} \wedge x_{c_2} \wedge x_{c_3}, \tag{18}\]

see Figure 2.a.

Fig. 2. a) Circle computed using three points, note its stereographic projection. b) Circle computed using the meet of two spheres.

Similar to the case of planes, lines can be defined by circles passing through the point at infinity as

\[ l^* = x_{c_1} \wedge x_{c_2} \wedge e_\infty. \tag{19}\]

This can be demonstrated by developing the wedge products as in the case of the planes to yield

\[ x_{c_1} \wedge x_{c_2} \wedge e_\infty = x_{c_1} \wedge x_{c_2} \wedge e_\infty + (x_{c_2} - x_{c_1}) \wedge E, \tag{20}\]

from where it is evident that the expression $x_{c_1} \wedge x_{c_2}$ is a bivector representing the plane where the line is contained and $(x_{c_2} - x_{c_1})$ is the direction of the line.

The dual of a point $p$ is a sphere $s$. The intersection of four spheres yields a point. The dual relationships between a point and its dual, the sphere, are:

\[ s^* = p_1 \wedge p_2 \wedge p_3 \wedge p_4 \leftrightarrow p^* = s_1 \wedge s_2 \wedge s_3 \wedge s_4, \tag{21}\]

where the points are denoted as $p_i$ and the spheres $s_i$ for $i = 1, 2, 3, 4$.

A summary of the basic geometric entities and their duals is presented in Table 1.

<table>
<thead>
<tr>
<th>Entity</th>
<th>Representation</th>
<th>Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere</td>
<td>$s = p + \frac{1}{2}(p^* - p^0) e_\infty + e_0$</td>
<td>1</td>
</tr>
<tr>
<td>Point</td>
<td>$x = x + \frac{1}{2}x^* e_\infty + e_0$</td>
<td>1</td>
</tr>
<tr>
<td>Plane</td>
<td>$n = (a - b) / (c - a)$, $d = (a \wedge b \wedge c) I_E$</td>
<td>1</td>
</tr>
<tr>
<td>Line</td>
<td>$L = n_1 \wedge n_2 - x_\infty n_1 n_2 I_E$</td>
<td>2</td>
</tr>
<tr>
<td>Circle</td>
<td>$z = s_1 \wedge s_2$</td>
<td>2</td>
</tr>
<tr>
<td>Point Pair PAIR</td>
<td>$p^* = s_1 \wedge s_2 \wedge s_3 \wedge s_4$</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 1. Entities in conformal geometric algebra

This result tells us that since $x_c$ lies on both spheres $z = (s_1 \wedge s_1)$ should be the intersection of the spheres or a circle. It is easy to see that the intersection with a third sphere leads to a point pair. We have derived algebraically that the wedge of two linearly independent spheres yields to their intersecting circle (see Figure 2), this topological relation between two spheres can be also conveniently described using the dual of the meet operation, namely

\[ z = (z^*)^* = (s_1^* \vee s_2^*)^* = s_1 \wedge s_2, \tag{17}\]

this new equation says that the dual of a circle can be computed via the meet of two spheres in their dual form. This equation confirms geometrically our previous algebraic computation of equation (16).

The dual form of the circle (in 3D) can be expressed by three points lying on it as

\[ z^* = x_{c_1} \wedge x_{c_2} \wedge x_{c_3}, \tag{18}\]

see Figure 2.a.

Fig. 2. a) Circle computed using three points, note its stereographic projection. b) Circle computed using the meet of two spheres.

Similar to the case of planes, lines can be defined by circles passing through the point at infinity as

\[ l^* = x_{c_1} \wedge x_{c_2} \wedge e_\infty, \tag{19}\]

see Figure 3.b.
L. Schwartz shows the developments concerning the image mapping of visual information to the neocortex. He claims that "to simulate the image properties of the human visual system (and perhaps other sensory systems) conformal image mapping is a necessary technique [6]. The mapping function

\[ w = k \log(z + a) \]  

is a widely accepted approximation to the topographic structure of primate V1 foveal and parafoveal regions. An extension of it by simply adding an additional parameter captures the full field topographic map in terms of the dipole map function

\[ w = k \log \left( \frac{z + a}{z + b} \right). \]

However these models are still unsatisfactory, as they can not describe topographic shear due to the fact that they are both explicitly complex-analytic or conformal. Balasubramanian et al [1] suggested a very simple procedure for topographic shear in V1, V2 and V3 assuming that cortical topographic shear is rotational (a compression along iso-eccentricity contours). The authors model the constant rotational with a quasiconformal mapping called the wedge mapping. This mapping using five independent parameters yields an approximation to the V1, V2 and V3 topographic structure unifying these three areas into a single V1-V2-V3 complex as follows: firstly we represent any point in the visual hemi-field with the complex variable \( z = r e^{i\theta} \), where \( r \) and \( \theta \) stand for eccentricity and polar angle respectively. The wedge map for the three visual areas \( V_k, k = 1, 2, 3 \) is the map

\[ \eta_k(r e^{i\theta}) = r, e^{i\theta_k(\theta)}, \]

where the respective functions for V1, V2 and V3 are given by

\[
\begin{align*}
\Theta_1(\theta) &= \alpha_1 \theta, \\
\Theta_2(\theta) &= \left\{ \begin{array}{ll}
-\alpha_2 (\theta - \frac{\pi}{2}) + \Theta_1(\frac{\pi}{2}) & \text{if} \ 0^\circ \leq \theta \leq \frac{\pi}{2}, \\
-\alpha_2 (\theta + \frac{\pi}{2}) + \Theta_1(-\frac{\pi}{2}) & \text{if} \ -\frac{\pi}{2} \leq \theta \leq 0^\circ,
\end{array} \right. \\
\Theta_3(\theta) &= \left\{ \begin{array}{ll}
\alpha_3 \theta + \Theta_3(0^+) & \text{if} \ 0^\circ \leq \theta \leq \frac{\pi}{2}, \\
\alpha_3 \theta + \Theta_3(0^-) & \text{if} \ -\frac{\pi}{2} \leq \theta \leq 0^\circ.
\end{array} \right.
\end{align*}
\]

The wedge warps three copies of V1, V2 and V3 of the visual hemi-field and localize them into a pie form, where each one is compressed by an amount of \( \alpha_k \) in the azimuthal direction, thus this results in a rotational shear in each of the wedges. Finally the wedge map is further modified via a dipole map using equation (28). The result is the full wedge–dipole model depicted semi-qualitatively in Figure 4.

Many visual cortex architectures of the primates and the human have an important feature responsible for the procedure of mixing the visual data of the left and right eyes. It has been shown that ocular dominance columns represent thin strips (5-10 minutes of arc) alternating left and right eye input to the brain. According Yeshurun and Schwartz [11], such architectures, when operated upon with a cepstral filter,
provides a strong cue for binocular stereopsis. An hand this visual cue the creature can sense depth.

In our work we have a different motivation, we extend the 2D horopter concept [7] to 3D horopter sphere for fusing the left and right stereoscopic images in a sphere, see Figure 5.c. This is basically a 3D representation using polar coordinates of the 3D visual space. This representation is after the procedure of computing stereopsis has been carried out. While one traverses the spheres outwards one becomes the sense of directed depth. In conformal geometric algebra the directed depth is a vector pointing outwards from the ego-center of our horopter. The magnitude of it is the scalar value of the depth.

Why we believe that this representation is useful?. Let us answer this question firstly showing the advantages of our mathematical system. In the conformal geometric algebra the computational unit is the sphere. One can map all the 3D visual information onto a family of spheres using the 3D horopter concept. As the 3D visual information is mapped on spheres, we can start to use this representation however in the 5D space of the conformal geometric algebra. We can explode then all the computational advantages of this mathematical system like to apply incidence algebra between circles, planes and spheres and to utilize linear transformations like translators, rotors and dilators in terms of spinors. In this way the 3D visual information on the sphere can be treated more efficiently in the conformal geometric algebra framework for various humanoid tasks like recognition, reasoning, planning and conduct autonomous actions.

We can also claim that all the efforts on conformal mapping are restricted to the mapping on the primate and human visual areas, however in our paper we are introducing a mathematical framework in order to have an artificial way how we can fuse in 3D the images of the left and right cameras for depth sensing necessary for recognition, representation, reasoning and planning.

B. Horopter and the conformal model

The horopter is the 3D geometric locus in space where an object has to be placed in order to stimulate exactly two points in correspondence in the left and right retinas of a biologic binocular vision system [10]. In Figure 5.c we see a horopter depending of an azimuth angle $\kappa$. In other words the horopter represents a set of points that cause minimal (almost zero) disparity on the retinas. We draw the horopter tracing an arc through the fixation point and the nodal points of the two retinas, see Figure 5.a. The theoretical horopter is known as the Vieth-Müller circle. Note that each fixation distance has its own Vieth-Müller circle. According to this theoretical view, the following assumptions can be made: each retina may be a perfect circle, both retinas are of the same size, corresponding points are perfectly matched in their retina locations, and points in correspondence are evenly spaced across the nasal and temporal retinas of the right and left eyes. If an object is located in either side of the horopter, a small amount of disparity is caused by the eyes. The brain analyzes this disparity and computes the relative distance of the object with respect to the horopter. In a narrow range near of the horopter the stereopsis does not exist. That is due to very small disparities which are not enough to stimulate stereopsis. Empirical horopter measurements (even done using the Nonius method) do not agree with the Vieth-Müller circle. There are two obvious reasons for this inconsistence either due to irregularities in the distribution of visual directions in the two eyes or a result of the optical distortion in the retinal image. There are various physiological reasons why the horopter can be distorted. Other cause of distortion is the asymmetric distribution of oculocentric visual distributions. In addition to a regional asymmetry in local signs in one eye, the distribution between the two eyes may not be congruent (correspondence problem), this may be another cause of horopter distortion. Asymmetric mapping from retina to the neocortex in both eyes also causes a deviation of the horopter from the Vieth-Müller circle. In this work we attempt to build an artificial binocular system. Deviations of the horopter in this artificial system will depend if the digital cameras fulfill the above summarized requirements for obtaining the Vieth-Müller circles. In the next subsections we explain our measures taken in order to avoid a distortion of the horopters. One important contribution of this work is to show that the horopter is naturally embedded...
in the unit sphere of the conformal geometric algebra. The simple configuration of the horopter shown in Figure 5.a is nothing than a very naive geometric representation using polar coordinates of the geometric locus of the visual space. In contrast using the tools of conformal geometric algebra we can claim that binocular vision can be reformulated advantageously using an spherical retina. Now we show how we find the horopter in the sphere of conformal geometry. Actually we are dealing with a bunch of spheres intersecting the centers of the cameras $L_C$ and $R_C$, see Figure 5.b. This is the pencil of spheres in the projective space of spheres. Note that the $L_C$ and $R_C$ cameras centers are Poncelet points. Since a stereo system only sees in front of it, we consider the spheres emerging towards the front. When the space locus of objects expands, the centers of the spheres move along the bisector line of the stereo rig, this is when the depth $\delta$ grows, see Figure 5.a. From now onwards we will use the term horopter sphere rather than the horopter circle, because when we change the azimuth of the horopter circle we are simply selecting a different circle of a particular horopter sphere $s_i$, see Figure 5.b. As a result we can consider that all the points of the visual space are lying on the pencil of the horopter spheres. Let us translate this description in terms of equations of the conformal geometric algebra. We call the unit horopter sphere $s_0$ the one which has as center the egocenter of the stereo rig, see Figure 5.c and using the sphere equation, $s_0 = p - \frac{1}{2} \rho_0^2 e_\infty = e_0 + \frac{1}{2} (c_0^2 - \rho_0^2)e_\infty + e_0$, where its center (egocenter) is attached to the true origin of space of the conformal geometric algebra and the radius is the half of stereo rig length $\rho_0 = \frac{1}{2} |L_C - R_C|$. The center $c_i$ of any horopter sphere $s_i$ moves towards the point at infinity as $c_i = c_0 + \frac{1}{2} (c_0^2 - \rho_0^2)e_\infty + e_0$, where $c_0$ is the Euclidean 3D vector. Thus we can write the equation of the sphere $s_i$ as $s_i = c_i + \frac{1}{2} (c_i^2 - \rho_0^2)e_\infty + e_0$, where the radius is computed in terms of the stereopsis depth $\rho_i = \frac{1}{2}(1 + \delta)$. Consider the figure of the model for visual human system in Figure 5.c, we see that the horopter circles laye on a pencil of planes $\pi_i$. We can obtain the same circles $z_i$ simply by intersecting in our conformal model such pencil of planes with the pencil of tangent spheres as is depicted in Figure 5.c. The intersection is computed using the meet operation of the duals of the plane and sphere and taking the dual of the result as $z_i = \pi_i \wedge s_i$. Now, taking the meet of any couple of horopter spheres we gain a circle which lies on the front parallel plane with respect to the digital cameras common plane $z = s_i \wedge s_j$. Later on taking the meet of this circle with the unit horopter we regain the Poncelet points $L_C$ and $R_C$ which in our terms is called point pair $PP_{LR} = z = s_0 = L \wedge (s_0^*)$, note that the second part of the equation computes the point pair wedging the dual of sphere $s_0^*$ with the line crossing the camera centers $L_C$ and $R_C$. If we further consider the Figure 5.b., the intersecting plane $\pi_i$ cuts the horopter spheres generating geometric locus on the plane as depicted in Figure 5.b-c. These horopter circles fulfill an interesting property. If one takes an inversion of all horopter circles with centers on the line $l$ we get the radial lines of the polar diagram, see at the right in Figure 6.a. Now, since the plane $\pi_i$ (by varying angle $\kappa$) intersects the family of horopter spheres producing the horopter circles of Figure 6.a whose inverse is a 2D log-polar diagram, we can conclude that the inverse of the arrangement of horopter spheres and Poncelet points is equivalent a 3D log-polar diagram, as depicted in Figure 6.b. To understand this better, lets us take any radial line of the 2D log-polar diagram and express it in conformal geometric algebra $L = X \wedge Y \wedge e_\infty$. Now, applying an inversion to this line we get a circle, i.e. $z = e_4 L e_4$. Note that this inversion is implemented as a reflection on $e_4$. The 3D log-polar diagram is an extraordinary result, because contrary to the general belief that conformal image processing takes places in 2D log-polar diagram, we can consider that the visual processing rather takes place in a 3D log-polar diagram. This claim is novel and it is promising, because this framework can be used for 3D flow estimation as opposite to the use of one view or even an arrangement of two log-polar cameras. In this paper we use our conformal model in a real binocular head which will be used for an humanoid, or as an intuitive vision-based man-machine interface and also for advanced 3D automated visual inspection.

IV. BUILDING THE CONFORMAL HOROPTER USING A BINOCULAR HEAD

A half-body humanoid is being built in order to research models and algorithms for perception action systems based on conformal geometric algebra, this system includes a robotic arm, hand, neck (pan-tilt unit) and stereoscopic vision l(see Figure 7.a-b). The front parallel cameras system provides an excellent platform to implement the conformal horopter (see Figure 6.b). In order to support the assumptions
made upon the model, we use effective techniques to adjust focal distances, manage calibration and shooting control of both cameras. Since both cameras are the same model and have the same lenses mounted, we could therefore assume equal image resolution; in terms of human vision parameters this would be taken as equal retina size on both eye balls. This is a very important issue to assure so that the image and geometric process will be done in a non degenerated horopter. So priori standard camera calibration, the focal distance is set as follows. First we capture a gray scale image \( I_l(x, y, t_0) \) at time \( t_0 \) on the left camera, then we map the image into its Fourier representation \( F_l(u, v, t_0) = \mathcal{F}\{I_l(x, y, t_0)\} \). Hence, the spectral density \( P_l(u, v, t_0) = |F_l(u, v, t_0)|^2 \) is computed in order to determine the energy distribution in the bidimensional signal. The high frequency region could be thought as the zone which extends beyond certain cut-off radius \( R_c \), i.e. where the frequency variable \( u \) or \( v \) (or perhaps both) represents the high frequency components of the image, see Figure 7.c.

The contribution of a certain component \( P_l(u, v, t) \) to the zone is determined (weighed to avoid rippling) by \( B(i, j, R_c) = 1 - e^{-\sigma (u^2 + v^2)/R_c^2} \). This zone implies all the components of the signal that constitute the corners or borders in space domain, features found in a well-focused camera. The energy amount in this region at certain time \( E_R(t) \) is the sharpness of the image we calculate it as discrete weighed integration \( E_R(t_0) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} B(i, j, R_c) P(i, j, t_0) \). While these calculations take place we physically adjust the focal length of the lens until the maximal \( E_{R_{Max}}(t_n) \) is reached see Figure 7.d. Once the left camera has gone through this stage we do the same with the right one. Finally since we do not only wish to get the best possible focus for each camera, but also to make the system evenly focused, the strategy consists in establishing a subwindow

\[
W_l(x, y, R_s) = \{(x - R_s) \leq x \leq (x + R_s) \times \{(y - R_s) \leq y \leq (y + R_s)\} \}
\]

in the left image (such that its content appears in the 3D field of view, so it can be found in both images), and apply this subwindow as correlation kernel with the right image in order to find the best match. This Kernel was improved for our purposes with an edge detector filter (3x3 pixels).

Later on, we calibrated the stereo rig with a well-known method from [9]. Uniform temporal distribution of features on images is an important restriction which is been tackled by computational thread schemes. Spatial distribution of features is assured with a full camera calibration including lenses distortion (radial-tangential) and projective compensations. The disparities found in images are handled as head-centric iso-disparities surfaces. These are the meshes that supports stereo matching, and more over scene reconstruction. Since off-line calibration procedure has been finished, the next stage consists in computing those iso-disparities surfaces using a multiresolution algorithm, this method is inspired on census mapping for correspondence estimation, however our algorithm uses a central pixel difference. This process has been accelerated by a set of constraints (ordering, uniqueness, disparity limit, disparity continuity, etc.) [9]. When dense mesh reconstruction is wished a median-like filter must be applied to distinguish those outliers which were not taken by the disparity limit constraint. This median-like filter considers only those pixels with non zero disparity and the value set to every filtered pixel will depend on the mean and median of the neighborhood. The variation of azimuth angle \( \kappa \) generates a pencil of planes which could be seen as the epipolar plane whose intersections with the images planes are the scan lines of the images (see figure 6.b), this is possible because the epipolar calibration has been done during the wrapping phase (where lens distortion and center of the cameras are compensated) of the on-line 3D scene mesh generation. Finally the 3D mesh of the field of view is formed by two sets of vertices (conformal 5D points).

Due to the geometric nature of the approach and the features provided by the incidence algebra [2] the usage of geometric primitives (as planes, lines, spheres etc.) provide a powerful tool-box for further applications, however until this point the content in the visual space has been encoded with conformal points. The idea is to find these geometric primitives from the mesh in such a manner that effects of noise and quantization induced by the digital images could be diminished. In this way we address an intuitive geometric approach for visual space perception on humanoids, among other applications. The conformal model has been used to build an application where a scene reconstruction and modeling (done with the geometric space rectification) has been done, this data is used to segment elements in the field of view bases on its depth (in a multiresolution manner). More information (color skin, shape, etc.) is used to estimate the user pose or gesture. Once this pose is know we can define assets in the 3D virtual world, in such a way that if the user touches this asset some command is dispatched see Figure 8.

V. CONCLUSIONS

In this work we propose a novel conformal model for human like vision. In this work the old concept of horopter
circles is extended to a horopter spheres which leads to a useful 3D log-polar representation of the visual space. This model has certain biological plausibility and looks promising for making stereoscopic vision of humanoids more effective for real time tasks.

REFERENCES


