Abelian Groups, Rings, Modules, and Homological Algebra

Edited by

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Acknowledgment

We would like to thank participants for making the 2004 Abelian Groups, Rings, and Modules Conference a successful and memorable one, and the contributors for submitting first class research papers and for their patience during the refereeing and editing process.

We would also like to thank staff and colleagues in the College of Sciences and Mathematics for their help in organizing the conference. And many thanks to our friends in the Algebra community for encouraging us and working with us to make the conference and the proceedings a reality. We are most grateful for the financial support we received from the College of Sciences and Mathematics and the Department of Mathematics and Statistics.

Finally, we would like to thank Mrs. Rosie Torbert for the outstanding job she has done helping us turn the research papers into a coherent volume, and many thanks to our colleague, Dr. Darrel Hankerson, for providing the necessary technical assistance in the production of this volume.

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The 2004 Abelian Groups Rings and Modules Conference Participants and Contributors
Honor
EDGAR EARLE ENOCHS
For his dedicated mentoring and contributions to Algebra

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Biography of Professor Edgar Enochs

Edgar Earle Enochs was born on September 13, 1932 in Pike County, Mississippi. He obtained his bachelor's degree in 1958 from Louisiana State University and his Ph.D. degree from the University of Notre Dame, also in 1958, under the supervision of Professor Donald John Lewis. His Ph.D. thesis was titled "Infinite Abelian Groups." In the same year, on June 21, 1958 he married Louise Smith of Baton Rouge, Louisiana. He has seven children: Corinne, Mary Jane, Kathryn, Maureen, Madelaine, Anne, and John, and thirteen grandchildren.

Professor Enochs started his academic career as an instructor at the University of Chicago (1958 - 1960). In 1960, he joined the University of South Carolina as an assistant professor. In 1962 he was promoted to associate professor, and became full professor in 1966. In 1967, he moved to the University of Kentucky, where he has remained since.

Professor Enochs has had an illustrious and prolific career. Having started his research in infinite abelian groups, he has expanded his research interest to a wide range of other areas such as group theory, commutative and non-commutative algebra, modules, category theory, algebraic geometry, homological algebra, and representation theory just to name a few. Most of his papers have resulted in creating and growing new areas of research in Algebra. In particular, his 1963 and 1971 papers on "torsion free covering modules" formed a basis of the work on covers (right approximations) that is still being done today. Another paper that has had a major impact is his 1981 paper on "injective and flat covers and resolvents," which is the foundation of the relative homological algebra research being done today by researchers in the Enochs School. This remarkable paper was followed by the 1985 paper that he co-wrote with one of his students on "balanced functors" that formed a basis for what is now known as Gorenstein relative homological algebra. Professor Enochs has traveled all over the world giving lectures and talks and has continuously hosted research visitors at the University of Kentucky to work on the above research topics (and others) and their connections to commutative and non-commutative algebra, representation theory, sheaves, etc. In many cases, he has single handedly jump-started the visitors' research careers.

Professor Enochs has had a profound impact on mathematics education in the U.S., having supervised over 44 Ph.D. theses, including one of the editors of this book. He is an outstanding teacher and is a recipient of the University of Kentucky's teaching excellence awards: Alumni Association Great Teacher Award and the Sturgill Award for Contributions to Graduate Education.

Even with such stellar accomplishments, Professor Enochs is still the nicest, kindest, and most helpful person, and he is a pure joy to meet and work with.

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Publications of Professor Edgar Enochs

1. Gorenstein categories Tate cohomology on projective schemes (with Sergio Estrada and Juan Ramon Garcia Rozas), submitted.
2. The $\aleph_1$-product of DG-injective complexes (with Alina Iacob), to appear in Proc. Edinburgh Math. Soc.
3. The structure of compact co-Galois groups (with Sergio Estrada, Juan Ramon Garcia Rozas and Luis Oyonarte), to appear in Houston J. Math.
4. Gorenstein flat covers and cotorsion envelopes (with Sergio Estrada and Blas Torrecillas), to appear in J. Algebras Represent. Theory.
5. Covers and envelopes by V-Gorenstein modules (with Juan Antonio Lopez Ramos and Overtoun M.G. Jenda), to appear in Comm. Algebra.
6. A non-commutative generalization of Auslander's last theorem (with Overtoun M.G. Jenda and Juan Antonio Lopez Ramos), to appear in the International Journal of Math. and Math. Sciences.
7. Projective representations of quivers (with Sergio Estrada), to appear in Comm. Algebra.
8. Relative homological algebra in the category of quasi-coherent sheaves (with Sergio Estrada), Adv. in Math., 194 (2005), 284-295.
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Book Chapters

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Books

131. Covers, Envelopes and Cotorsion Theories (with Luis Oyonarte), Nova Science Publishers, 113 pages (2002).

132. Gorenstein Flat Modules (with Juan Antonio Lopez Ramos), Nova Science Publishers, 117 pages (2001).

133. Relative Homological Algebra (with Overtoun Jenda), de Gruyter Expositions in Mathematics, Volume 30 (2000).

Ph.D. theses under the direction of Professor Edgar Enochs

1. **B. Hoyte Maddox**, University of South Carolina, 1964, Absolutely Pure Modules

2. **W. W. Leonard**, University of South Carolina, 1964, Superfluous Submodules

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3. **Pelham Thomas**, University of South Carolina, 1966, Maximal Spaces
4. **James Pleasant**, University of South Carolina, 1966, Certain Relations between Objects and Morphisms in a Category
5. **Arthur Van De Water**, University of South Carolina, 1967, A Property of Modules over Rings with a Left Field of Quotients
6. **David R. Stone**, University of South Carolina, 1968, Torsion-Free and Divisible Modules over Matrix Rings
7. **James R. Smith**, University of South Carolina, 1968, Local Domains with Topologically T-nilpotent Radical
8. **Joong Ho Kim**, University of South Carolina, 1968, On Complete Local Rings
9. **Conduff Childress**, University of South Carolina, 1969, Quotients of Hom and Torsionness
10. **C. Bruce Myers**, University of Kentucky, 1970, F-Torsionless and F-Reflexive Modules
11. **Ann F. Bowe**, University of Kentucky, 1970, Some Aspects of Small Modules
12. **James J. Bowe**, University of Kentucky, 1970, Neat Homorphisms
13. **Thomas J. Cheatham**, University of Kentucky, 1971. Finite Dimensional Rings and Torsion Free Covers
14. **Cary H. Webb**, University of Kentucky, 1972 Tensor and Direct Product
15. **Roger D. Warren**, University of Kentucky, 1972, Free A-Rings
16. **Frank D. Cheatham**, University of Kentucky, 1972, F-Absolutely Pure Modules
17. **David D. Berry**, University of Kentucky, 1975, S-Purity
18. **David D. Adams**, University of Kentucky, 1978, Absolutely Pure Modules
19. **James Patterson, III.**, University of Kentucky, 1979, (X,Y)-Divisible Modules Over Commutative Rings
20. **Peter McCoart Joyce**, University of Kentucky, 1979, Dual Numbers and Finite Abelian Groups
21. **Walter P. Gerlach**, University of Kentucky, 1980, Connecting Locally Compact Abelian Groups
22. **Overtoun M.G. Jenda**, University of Kentucky, 1981, On Injective Resolvents
23. **Richard G. Belshoff**, University of Kentucky, 1990, On Matlis Reflexive Modules
24. **Mark A. Goddard**, University of Kentucky, 1990, Minimal Projective Resolutions of Complexes
25. **Frank Branner**, University of Kentucky, 1991, On the Projective Functor
26. **Victor K. A. Akatsa**, University of Kentucky, 1991, Flat Envelopes and Negative Torsion Functors
27. **Sangwon Park**, University of Kentucky, 1991, The Macaulay-Northcott Functor

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28. **Vivian Cyrus**, University of Kentucky, 1994, The Category of Monoids
29. **Clayton Brooks**, University of Kentucky, 1994, Homotopy Theory of Modules
30. **Albert Bronstein**, University of Kentucky, 1995, On the Representation of Quivers
31. **Okyeon Yi**, University of Kentucky, 1996 Local Nilpotence of Envelopes and Universal Enveloping Algebras
32. **Jinzhong Xu**, University of Kentucky, 1997, Flat Covers of Modules
33. **Christopher Anthony Aubuchon**, University of Kentucky, 1997, A Natural Functor from the Category of Complexes of Left R-modules to the Category left R (epsilon)-Modules
34. **William Todd Ashby**, University of Kentucky, 1998, The Characterization of Graded Principal Ideal Domains and Graded Torsion Free Covering Modules
35. **David W. Dempsey**, University of Kentucky, 2000, Functors and the Preservation of Covers and Envelopes
36. **Julia Varbalow**, University of Kentucky, 2000, Injective and Projective Representations of Quivers
37. **Makhmud Sagandykov**, University of Kentucky, 2000, On Homological Structures of Transformation Groups
38. **Stephen T. Aldrich**, University of Kentucky, 2000, Exact and Semisimple Differential Graded Algebras and Modules
39. **Naveed Zaman**, University of Kentucky, 2000, Minimal Generators
40. **Chris Bullock**, University of Kentucky, 2001, On Chain Numbers
41. **Molly D. Wesley**, University of Kentucky, 2005, Torsion Free Covers of Graded and Filtered Modules
42. **Katherine R. Pinzon**, University of Kentucky, 2005, Absolutely Pure Modules
43. **Alina C. Iacob**, University of Kentucky, 2005, Generalized Tate Cohomology
44. **Todorka N. Nedeva**, University of Kentucky, 2005, Series in the Binomial Polynomials

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Genealogy of Professor Edgar Enochs

- **Karl Theodor Wilhelm Weierstrass** (1815-1897), University of Konigsberg, Honorary Doctor's Degree in 1854
- **Ferdinand Georg Frobenius** (1849-1917), Universitat Berlin, 1870
- **Issai Schur** (1875-1941), Universitat Berlin, 1901
- **Richard Dagobert Braeur** (1901-1977), Universitat Berlin, 1925
- **Donald J. Lewis**, Ph.D., University of Michigan, 1950
- **Edgar Earle Enochs**, Ph.D., University of Notre Dame, 1958

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2004 Abelian Groups, Rings, and Modules Conference Participants

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About the Editors

H. PAT GOETERS was born in Houston, Texas, and at age 11 moved with his family to South Bend, Indiana and a year later to New Haven, Connecticut, following his father's academic career. In 1980, Pat finished his undergraduate studies in mathematics and computer science at Southern Connecticut State University in New Haven, and went to University of Connecticut to pursue a Ph.D. which was completed in 1984 under the supervision of William J. Wickless. After spending one year in a post-doctoral position in Middletown under the tutelage of James D. Reid, Goeters was invited for a tenure track position in Auburn by Ulrich F. Albrecht. Soon after, William Ullery and Overtoun Jenda were hired, and so began a lively algebra group.

OVERTOUN M. G. JENDA was born in Malawi and graduated from Chancellor College, University of Malawi with a bachelor's degree in mathematics. Upon graduation, he worked at Chancellor College as an associate lecturer for a year before moving to the U.S. in 1977 for his graduate studies at University of Kentucky. He obtained his Ph.D. in 1981 under the supervision of Professor Edgar Enochs. He then moved back to Chancellor College where he was a lecturer (assistant professor) for three years. In 1984, he moved to University of Botswana for another three year stint as a lecturer before moving back to University of Kentucky as a visiting assistant professor in 1987. In 1988, he joined a lively algebra research group at Auburn University. In addition to traveling within the U.S., he has been to Belgium, Canada, Czech Republic, Iran, Japan, Russia, South Korea, Spain, and several countries in southern Africa visiting mathematics departments and attending conferences. As a result, he has made long-lasting friends from all over the world that have had a great impact on his mathematics career. Overtoun Jenda is married to Claudine and has two children, Emily and Overtoun, Jr.

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Preface

On the occasion of Edgar Earle Enochs' 72nd birthday, many top researchers in algebra gathered at Auburn University on September 9-11, 2004 to honor Ed, exchange ideas, and renew friendships. This book is a collection of refereed papers by the researchers involved in the talks as well as those who were not able to make it to the conference, and represents most of the current research topics in abelian groups, commutative algebra, commutative rings, group theory, homological algebra, lie algebras, and module theory.

We are excited that many of the veteran researchers in algebra took time from their busy schedules to honor Professor Enochs, and present us with their latest research ideas. The book gives the reader access to the current ideas and techniques of leading researchers. We must add that, according to the master of first order, Laszlo Fuchs, the conference was one of the most comfortable he has ever attended; we concur and attribute this to the participants; their devotion to algebra is evident in the articles submitted.

A rarity compared to some proceeding volumes is that due to Edgar Enochs' venerable contributions to a wide range of topics in algebra, we have in this volume a large collection of high-quality papers, as attested by referees' reports, from many high-level algebraists discussing today's hot research topics. Though steeped in veteran techniques, articles in this volume involve topics that are accessible to the beginning mathematician. Also, in many articles, suggestions of problems and programs for future study are made - it is always nice when one can improve on a master's result (or perhaps knock oneself out trying).

This collection of papers is therefore an excellent addition to the literature and will serve as an invaluable handbook for beginning researchers in algebra as well as specialists. This book is indeed a superb way of honoring a legend in algebra, Edgar Enochs.

HPG
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Chapter 1

Generalizing Warfield's Hom and Tensor Relations

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Abstract We survey generalizations of Warfield's 1968 Homomorphisms and Duality paper. Our main focus is in fixing a module A and examining when Warfield's results hold relative to this fixed A.

1.1 Introduction

Some of the most promising tools in the study of torsion-free abelian groups and modules have been the ideas developed in Warfield's paper [49]. Specifically, the Hom/Tensor functors, $Hom(A, -)$ / $- \otimes A$, and the contravariant functor $Hom(-, A)$, referred to as *Warfield Duality*, where A is a subgroup of the rational integers. In this survey, we will look at generalizing Warfield's results for the integers to more general rings. In particular, the generalizations of Warfield's results to domains is considered, and extensions to general modules is examined. We will start this article with the general setting for Warfield's Hom-Tensor relations.

1.2 Self-Small Modules

When studying a right module A over a ring R, a central role is played by the endomorphism ring $E = End_R(A)$. Because A is an E-R-bimodule, there exists an adjoint pair (H_A, T_A) of functors

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between the categories $\mathcal{M}_R$ and $\mathcal{M}_E$ of right R- and right E-modules respectively defined by

$$H_A(M) = \mathrm{Hom}_R(A, M)$$

for all right R-modules M and

$$T_A(N) = N \otimes_E A$$

for all right E-modules N.

The adjointness of H_A and T_A induces natural transformations $\theta : T_A H_A \to 1_{\mathcal{M}_R}$ and $\phi : 1_{\mathcal{M}_E} \to H_A T_A$ which are defined by $\theta_M(\alpha \otimes a) = \alpha(a)$ and $[\phi_N(x)](a) = x \otimes a$ for all $a \in A$, $\alpha \in \mathrm{Hom}_R(A, M)$ and $x \in N$ whenever $M \in \mathcal{M}_R$ and $N \in \mathcal{M}_E$.

Warfield showed that when $R = \mathbb{Z}$,

$$Hom(A, K \otimes_E A) \cong_{nat} K$$

for all rank-1 modules A and all torsion-free $E = End(A)$-modules K of finite rank, and

$$Hom(A, K) \otimes_E A \cong_{nat} K$$

for all rank-1 modules A and all torsion-free, A-generated modules K of finite rank (that is, K is an image of a direct sum of copies of A).

We call an R-module P *A-projective* if it is a direct summand of $\oplus_I A$ or some index-set I. If I can be chosen to be finite, then P is said to be *A-projective of finite A-rank*. Arnold and Lady showed in [15] that H_A and T_A induce an equivalence between the A-projective modules of finite A-rank and the finitely generated projective right E-modules. However, this equivalence does not extend to an equivalence between $\mathcal{M}_R$ and $\mathcal{M}_E$ unless A is a projective generator of $\mathcal{M}_R$ by Morita's theorem. We denote the largest full subcategories of $\mathcal{M}_R$ and $\mathcal{M}_E$ between which H_A and T_A induce an equivalence by $\mathcal{C}_A$ and $\mathcal{M}_A$ respectively. Clearly, $\mathcal{C}_A$ contains the A-projective modules of finite A-rank while $\mathcal{M}_A$ contains the finitely generated projective right R-modules.

The image of θ_M is called the *A-socle of M*, and is the fully invariant submodule of M generated by all images $\phi(A)$ where $\phi \in \mathrm{Hom}_R(A, M)$. The module M is *A-generated* if $M = S_A(M)$, or equivalently if it is an epimorphic image of $\oplus_I A$ for some index-set I. The *finitely A-generated* modules are those for which I can be chosen to be finite. Arnold and Murley observed in [16] that even if P is A-projective, $H_A(P)$ need not be a projective E-module. Therefore, H_A and T_A may not induce an equivalence between the category of A-projective R-modules and the category of projective right E-modules.

This resembles the difficulties encountered in the study of dualities once interest shifts to the investigation of submodules of A^I for infinite index-sets I. In the latter case, the difficulties can be overcome by restricting the discussion to slender R-modules. To achieve the same in the discussion of A-projective modules of infinite A-rank, Arnold and Murley introduced the notion of self-smallness in [16]. An R-module A is *self-small* if, for all index-sets I and all $\alpha \in \mathrm{Hom}_R(A, \oplus_I A)$, there is a finite subset J of I such that $\alpha(A) \subseteq \oplus_J A$. Finitely generated modules are self-small, as are torsion-free modules of finite rank over integral domains.

Theorem 1.2.1 *[16] Let A be a self-small right R-module. Then, H_A and T_A restrict to an equivalence between the full subcategory of $\mathcal{M}_R$ whose objects are A-projective modules and the category of projective right R-modules.*

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1.3 Projectivity Properties

Call an exact sequence $0 \to B \to C \to M \to 0$ of right R-modules *A-balanced* if A is projective with respect to it; i.e., the induced sequence $0 \to H_A(B) \to H_A(C) \to H_A(G) \to 0$ of right E-modules is exact. Although A generates $\mathcal{C}_A$, it need not be a projective generator. In this section, we describe which self-small right R-modules A are projective generators of $\mathcal{C}_A$. For this, we say that A is *fully faithful (faithful)* as a left E-module if $T_A(M) \neq 0$ for all (finitely generated) non-zero right E-modules M. It is easy to see that, in case A is flat as left E-module, A is faithful if and only if A is fully faithful.

Theorem 1.3.1 *[1] The following are equivalent for a self-small right R-module A:*

a) A is fully faithful as a left E-module.

b) Every epimorphism $F \to P$ with P and F A-projective splits.

c) An exact sequence $0 \to B \xrightarrow{\alpha} M \to P \to 0$ with P A-projective splits if and only if $\alpha(B) + S_A(M) = M$.

d) Every sequence $0 \to U \to M \to N \to 0$ in which M and N are in $\mathcal{C}_A$ is A-balanced.

Arnold and Lady showed in [15] that A is faithful as a left E-module if and only if condition c) holds for all A-projective modules of finite A-rank. However, their arguments do not carry over to the general case.

Turning to morphisms α between modules M and N in $\mathcal{C}_A$, neither $\ker\ \alpha$ nor $\alpha(A)$ need to be in $\mathcal{C}_A$. We thus call a class $\mathcal{C}$ of A-generated groups *A-closed* if it satisfies the following conditions:

i) $\mathcal{C}$ is closed with respect to finite direct sums.

ii) If $G \in \mathcal{C}$ and U is an A-generated subgroup of G, then $U \in \mathcal{C}$.

iii) If $M, N \in C$ and $\alpha \in \mathrm{Hom}_R(M, N)$, then $\ker\ \alpha \in \mathcal{C}$.

Addressing the existence question for A-closed classes, we obtain

Theorem 1.3.2 *[5] The following are equivalent or a self-small right R-module A:*

a) A is flat as a right R-module.

b) There exists an A-closed class $\mathcal{C}$ containing A.

c) $\mathcal{C}_A$ is the largest A-closed class containing the A-projective modules.

In particular, one obtains the following characterization of the elements of $\mathcal{C}_A$ in case A is flat.

Corollary 1.3.3 *[5] Let A be a self-small right R-module which is flat as an E-module. The following are equivalent for an A-generated right R-module M:*

a) $M \in \mathcal{C}_A$.

b) Whenever $0 \to U \to P \to M \to 0$ is an exact sequence with P A-projective, then U is A-generated.

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c) There exists an exact sequence

$$\dots \xrightarrow{\phi_{n+1}} P_n \xrightarrow{\phi_n} P_{n-1} \xrightarrow{\phi_{n-1}} \dots P_0 \xrightarrow{\phi_0} M \to 0$$

such that

i) P_n *is A-projective for all* $n < \omega$, *and*

ii) $0 \to im\ \phi_{n+1} \to P_n \xrightarrow{\phi_n} im\ \phi_n \to 0$ *is A-balanced for all* $n < \omega$.

Consequently the elements of $\mathcal{C}_A$ are exactly the modules for which there exists an A-projective resolution. We call these modules *A-solvable*. By Theorem 1.3.1, all A-projective groups are A-solvable if A is self-small, and the A-generated groups are precisely the epimorphic images of the A-solvable groups. The class of A-solvable modules is not closed with respect to epimorphic images in general, though.

Sequences of A-solvable modules need not be A-balanced; in particular there may exist exact sequences $\oplus_I A \to M \to 0$ with $M \in \mathcal{C}_A$ which are not A-balanced. The existence of such sequences makes it very difficult to develop a comprehensive homological algebra for A-solvable modules. We thus call an A-closed class $\mathcal{C}$ *A-balanced* if every exact sequence $0 \to B \to C \to M \to 0$ with $B, C, M \in \mathcal{C}$ is A-balanced.

Theorem 1.3.4 *[5] The following are equivalent for a self-small right R-module A:*

a) A is faithfully flat as a left E-module.

b) There exists an A-balanced, A-closed class containing all of the A-projective modules.

c) $\mathcal{C}_A$ *is the largest A-balanced, A-closed class containing all of the A-projective modules.*

Given a self-small right R-module A which is faithfully flat as a left E-module, every A-solvable module M admits an exact sequence

$$\dots \xrightarrow{\phi_{n+1}} P_n \xrightarrow{\phi_n} P_{n-1} \xrightarrow{\phi_{n-1}} \dots P_0 \xrightarrow{\phi_0} M \to 0$$

where each P_n is A-projective. Moreover, whenever

$$\dots \xrightarrow{\psi_{n+1}} Q_n \xrightarrow{\psi_n} Q_{n-1} \xrightarrow{\psi_{n-1}} \dots Q_0 \xrightarrow{\psi_0} M \to 0$$

is exact with each Q_n A-projective, then the induced sequences

$$0 \to im\ \psi_{n+1} \to Q_n \xrightarrow{\psi_n} im\ \psi_n \to 0$$

are A-balanced. Therefore, it is possible to develop the concept of an A-projective dimension for an A-solvable module M, and show that it coincides with the projective dimension of the right E-module $H_A(M)$. Moreover, one can define extension functors $\mathrm{Ext}^n_{\mathcal{C}_A}(-,-)$ on $\mathcal{C}_A$ which are naturally equivalent to the functors $\mathrm{Ext}_E(H_A(-), H_A(-))$. For details, see [6].

1.4 The Class $\mathcal{M}_A$

While the discussion so far has been concerned with closure properties of $\mathcal{C}_A$, we now turn to $\mathcal{M}_A$. The results in this section only apply to $R = \mathbb{Z}$.

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Lemma 1.4.1 *[7] A self-small abelian group A is a flat E-module if and only if $S_A(\mathrm{Tor}_1^E(M, A)) = 0$ for all right E-modules M.*

Using this lemma, one obtains

Theorem 1.4.2 *[7] The following conditions are equivalent for a self-small torsion-free abelian group A:*

a) A is faithfully flat as a left E-module.

b) $\mathcal{M}_A$ is closed with respect to submodules.

We now turn to the question as to when $\mathcal{M}_A$ is also closed with respect to products. An additive category $\mathcal{C}$ is *complete* (*cocomplete*) if inverse (direct) limits exist in $\mathcal{C}$. It is easy to see that an additive category $\mathcal{C}$ is cocomplete if and only if coproducts exist in $\mathcal{C}$, and all $\mathcal{C}$-morphisms have cokernels. A similar result holds for complete categories. Therefore, a preabelian category $\mathcal{C}$ is complete and cocomplete if and only if products and coproducts exist in $\mathcal{C}$. This section investigates when $\mathcal{C}_A$ is a complete (cocomplete) category. We want to remind the reader that a class $\mathcal{C}$ of modules over a ring R is the *torsion-free class of a torsion-theory* over R if $\mathcal{C}$ is closed with respect to submodules, products and extensions.

Theorem 1.4.3 *[8] The following conditions are equivalent for a self-small abelian group A:*

a) $\mathcal{M}_A$ is the torsion-free class of some torsion-theory of right $E(A)$-modules.

b) i) A is faithfully flat as an $E(A)$-module.
ii) $\mathcal{C}_A$ is a cocomplete category.
iii) $\mathcal{C}_A$ is a complete category with $\varprojlim_{\mathcal{C}_A} \mathcal{F} \cong T_A H_A(\varprojlim_{\mathcal{A}b} \mathcal{F})$ for all functors $\mathcal{F}$ from a small category into $\mathcal{C}_A$.

The last result in particular raises the question, which conditions have to be satisfied by an R-module A to ensure that $S_A(\prod_I M_i)$ is A-solvable for all families of A-solvable modules $\{M_i\}_{i\in I}$? Following [28], we say that a left R-module A satisfies the *Mittag-Loefler-condition (ML) with respect to a class $\mathcal{M}$ of right R-modules* if A is the direct limit of a filtration

$$\{F_i, \mu_i^j : F_i \to F_j | i, j \in I \ with \ i \leq j\}$$

of finitely presented modules satisfying

(*) For every $i \in I$, there is $j \in I$ with $j \geq i$ such that $ker(1_M \otimes \mu_i) \subseteq ker\ (1_M \otimes \mu_i^j)$ for all $M \in \mathcal{M}$.

Theorem 1.4.4 *[8] The following conditions are equivalent for a self-small abelian group A which is faithfully flat as an $E(A)$-module:*

a) A satisfies ML with respect to $\mathcal{M}_A$.

b) i) $\mathcal{M}_A$ is the torsion-free class of some torsion-theory on $\mathcal{M}_{E(A)}$.
ii) If $\{U_i | i \in I\}$ is a family of A-balanced, A-generated submodules of an A-solvable module M, then $\cap_{i\in I} U_i$ is A-generated.

c) $\mathcal{C}_A$ is a cocomplete category; and $\varprojlim_{\mathcal{C}_A} \mathcal{F} = S_A(\varprojlim_{\mathcal{M}_{\mathcal{R}}} \mathcal{F})$ for all functors $\mathcal{F}$ from a small category into $\mathcal{C}_A$.

d) $S_A(\prod_I M_i)$ is A-solvable for all families $\{M_i | i \in I\}$ of A-solvable modules.

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1.5 Domains Which Support Warfield's Results

Given an integral domain R the *rank* of a torsion-free module B, $rank(B)$, is the size of a maximal linearly independent subset of B. It follows that a rank 1 torsion-free module A is any module that is isomorphic to a nonzero submodule of the quotient field Q of R.

In [26] we were concerned with finding *cancellation modules*; a rank-1 module A is a cancellation module for R if for any two submodules B, C of Q, $AB = AC$ implies $B = C$. The image of $G \otimes_E A \to QG$ inside the divisible hull, QG of G, is denoted by AG. The kernel of $G \otimes_E A \to QG$ is just the torsion submodule of $G \otimes_E A$.

Theorem 1.5.1 *(Theorem 2.3 in [26]) Let R be an integral domain, and let A be a rank-1 module whose endomorphism ring is E. The following are equivalent:*

(a) A is a cancellation module for E.

(b) A is locally free over E.

(c) A is faithfully flat over E.

(d) For all torsion-free E-modules G, $Hom(A, AG) \cong_{nat} G$.

When A is flat over E, $G \otimes_E A \cong AG$, and so, any of the conditions mentioned in the last theorem equate to

$$(e) \quad G \cong_{nat} Hom(A, G \otimes_E A),$$

for every torsion-free right E-module G of finite rank; furthermore, in general, any of the conditions $(a), \ldots, (d)$ imply (e). We do not know if (e) implies that A is flat over E, but clearly (e) implies a weak flatness: if T is the torsion submodule of $G \otimes_E A$, then $Hom(A, T) = 0$ (recall that A is flat over E if and only if $T = 0$ for all torsion-free E-modules G). Thus, any assumption on R that insures $Hom(A, T) = 0 \Rightarrow T = 0$ will afford flatness of A and thus force (e) to be equivalent to (d). An assumption on R that will force the implication $Hom(A, T) = 0 \Rightarrow T = 0$ is R being noetherian of Krull dimension 1.

We call R an *(HT) domain* if $Hom(A, K \otimes_E A) \cong_{nat} K$ for all rank-1 modules A and all torsion-free $E = End(A)$-modules K of finite rank, and we call R a *(TH) domain* if $Hom(A, K) \otimes A \cong_{nat} K$ for all rank-1 modules A and all torsion-free, A-generated modules K of finite rank.

The last theorem was used by Olberding in [38] to obtain the following characterization of (HT) domains.

Corollary 1.5.2 *An integral R is an (HT) domain if and only if the natural map $G \to Hom(A, AG)$ is an isomorphism for all rank-1 modules A and $End(A)$-modules G of finite rank (equivalently, each rank-1 module A is locally principal over its endomorphism ring).*

Analogous to the effort in the last theorem, one can determine when $T_A H_A(G) \cong G$ for every A-generated, torsion-free G of finite rank. The rank-1 module A is said to be *a divisor module* for R if for every submodule C of Q, there exists a submodule B of Q such that $AB = C$.

Theorem 1.5.3 *[26] For a rank-1 module A of an integral domain R, A is a divisor module for $E = End_R(A)$ if and only if for every A-generated, torsion-free module G, $Hom(A, G) \otimes_E A \to G$ is an isomorphism.*

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Stable domains have received a great deal of attention in the literature; these are the domains such that every ideal is projective as a module over its endomorphism ring. Clearly, from the last result, a (TH) domain is stable (the existence of a solution X to $IX = E$, where E is the endomorphism ring of I, shows that I is invertible). Olberding established the converse, if R is stable then R is a (TH) domain in [38].

The other aspect of Warfield's paper that we wish to consider is that of duality. Given a rank-1 module A, take $\mathcal{C}^A$ to be the closure under isomorphism of the class of E-submodules of $\oplus_n A$ for some n. Warfield showed, for the integers, that

$$B \in \mathcal{C}^A \Leftrightarrow Hom(Hom(B, A), A) \cong_{nat} B.$$

Bazzoni and Salce coined the phrase, R is a *Warfield domain* if for all rank-1 modules A, $Hom(Hom(B, A), A) \cong_{nat} B$ for all $B \in \mathcal{C}^A$.

Warfield domains have been examined by many authors (see [38]) and a characterization of them is forthcoming (see [39]). As the characterization of Warfield domains is quite involved we will not go into the details here, other than to give the reader a flavor: a noetherian domain R is Warfield if and only if every ideal of R can be generated by 2 elements. Furthermore, the following implications concerning properties of a domain R are valid and cannot be reversed in general:

$$\text{Warfield domain} \Rightarrow \text{(TH)} \Rightarrow \text{(HT)}.$$

However, the properties are confluent when R is noetherian.

In the next section we will determine the context under which Warfield Duality holds; i.e., when is

$$Hom(Hom(B, A), A) \cong_{nat} B \quad \forall \quad B \in \mathcal{C}^A?$$

1.6 Replicating Duality for Domains

Fix $A \leq Q$. In [18] the domain R is called *A-reflexive* when $Hom_R(Hom_R\,(B, A), A) \cong_{nat} B$ for every B in $\mathcal{C}^A$. The R-reflexive domains are simply called *reflexive*, and reflexive domains have played an historically important role in the development of ring theory (see [32] for a discussion of reflexive domains). For the sake of convenience, we will assume that $R = End_R(A)$.

A domain R is called *divisorial* in [30] when each ideal $I \neq 0$ satisfies $(I^{-1})^{-1} = I$, where $I^{-1} = \{t \in Q \mid tI \subseteq R\}$. Following Heinzer's work, Bazzoni and Salce, in [17] and [18], called R, *A-divisorial*, if for each submodule B of Q with $B \in \mathcal{C}^A$, one has $B^{**} = B$, where $C^* = \{t \in Q \mid tC \subseteq A\}$ for any submodule C of Q. Some authors use different terminology to describe an A-divisorial domain R; for example, in [31], when A is an ideal of R, the terminology is that A is an *m*-canonical ideal for R. Silvana Bazzoni provides an extensive study of A-divisorial domains in [17] under the condition that A is locally a fractional ideal.

In this section, we examine A-reflexive domains. One can provide numerous characterizations of general A-reflexive domains; however we are able to be more specific when we know that A is locally a fractional ideal. For example, establishing that a noetherian A-divisorial domain has A_M a fractional ideal of R_M for every maximal ideal M allows one to show that a noetherian domain is A-divisorial if and only if it is A-reflexive.

Observe that if R is A-divisorial and M is a maximal ideal of R, $A = R^*$ is properly contained in M^* and Q/A contains M^*/A. Furthermore, there are no modules between M and R, so by the duality, there are no modules between A and M^*, and

$$M^*/A \cong R/M.$$

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We conclude that Q/A contains a copy of every simple module when R is A-divisorial.

We now summarize some known results regarding Warfield Duality.

Theorem 1.6.1 *The following are equivalent for a domain R and a rank-1 module A such that $End_R(A) = R$:*

(1) R is A-reflexive.

(2) R is A-divisorial, and $Ext^1_R(B, A)$ is torsion-free for every module $B \in \mathcal{C}^A$.

(3) R is A-divisorial, and Q/A is a universal injective module.

(4) R is A-divisorial, and A is injective relative to any pure exact sequence $0 \to B \to C \to G \to 0$ where $B, C, G \in \mathcal{C}_A$.

(5) The modules of the form $Hom_R(B, A)$ for some module B of finite torsion-free rank are precisely the modules in $\mathcal{C}^A$.

(6) Any $B \in \mathcal{C}^A$ is isomorphic to a relatively divisible submodule of a direct product of copies of A.

The proof of this can be found in the literature (see [18], [32], [41], and [24]).

This result prompts many interesting questions. When R is noetherian, the condition $Ext(B, A)$ torsion-free for every $B \in \mathcal{C}^A$ is superfluous. What are some other circumstances for which this condition is redundant? That is, when does R A-reflexive imply R is A-divisorial? What other condition(s) are there which will force A-divisorial domains to be A-reflexive? When is K_A being a universal injective enough to imply that R is A-reflexive?

The observation that modules of the form $Hom_R(B, A)$ when $B \in \mathcal{C}^A$ localize properly over h-local domains was made in [24] (page 245): If R is h-local, then for any maximal ideal M and any $B \in \mathcal{C} \wedge A$,

$$\mathrm{Hom}(B, A)_M \cong \mathrm{Hom}(B_M, A_M).$$

Therefore, we have

Corollary 1.6.2 *If R is h-local, then R is A-divisorial (respectively A-reflexive) if and only if R_M is A_M-divisorial (respectively A_M-reflexive) for every maximal ideal M.*

In Theorem 4.5 in [18], Bazzoni and Salce observed that A-divisorial domains are h-local by showing that Heinzer's proof that divisorial domains are h-local extends to A-divisorial domains. This important result along with its proof is also contained in the readily available text (page 136 in [23]).

Theorem of Bazzoni-Salce If R is A-divisorial, then R is h-local.

The theorem of Bazzoni and Salce combined with Corollary 1.6.2 allow us to reduce the study of Warfield Duality to the local case.

Reduction to the Local Case R is A-reflexive (A-divisorial) if and only if R is h-local and R_M is A_M-reflexive (A-divisorial) for every maximal ideal M of R.

The phrase Q/A is *cocyclic* means that Q/A is an essential extension of a simple module.

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Theorem of Bazzoni The following are equivalent for an ideal A of a local domain R:

(a) R is A-divisorial.

(b) Q/A is cocyclic and for every nonzero ideal I of R and every decreasing chain $\{J_n\}_n$ of ideals, $\cap_n(J_n + I) = \cap_n J_n + I$.

Furthermore, she is able to show in [17] that, in many cases, R is A-divisorial if and only if Q/A is cocyclic. In particular, she proved the following.

Theorem of Bazzoni If R is noetherian, local and A-divisorial, then A is a fractional ideal of R.

The proof of the above result is aimed at showing noetherian, local and A-divisorial domains have Krull dimension 1, since it then follows that A is a fractional ideal. In order to apply Bazzoni's results on A-divisorial domains, we need to have a (locally) fractional ideal A. For this reason, we wish to know, under what circumstances must A necessarily be a (locally) fractional ideal?

A partial answer to the latter question was obtained in [25]. A *Matlis domain* is an integral domain whose quotient field Q has projective dimension $pd_R\ Q = 1$. If R is h-local, then R is a Matlis domain if and only if for each maximal ideal M of R, Q is countably generated as an R_M-module. So any countable h-local domain is a Matlis domain.

Theorem 1.6.3 *[25] If R is a local Matlis domain, and R is A-divisorial, then A is a fractional ideal of R.*

In the next section we examine Warfield Duality in a more general context.

1.7 Duality and Infinite Products

Given R-modules A and M, let $M^* = \mathrm{Hom}_R(M, A)$. The assignment $M \mapsto M^*$ defines a contravariant functor from the category of right R-modules to the category of left E-modules. In the same way, setting $N^* = \mathrm{Hom}_E(N, A)$ for all left E-modules N defines a contravariant functor going the other way. For all right R-modules M, there is a natural map $\psi_M : M \to M^{**}$, whose kernel is denoted by $R_A(M)$, and called the *A-radical of M*. Note, $R_A(M) = 0$ if and only if M is a submodule of A^I for some index-set I.

The R-module M is called *A-reflexive* if ψ_M is an isomorphism. If A is slender, then direct summands of A^I are A-reflexive as long as I has non-measurable cardinality. In general, A-reflexive modules have a zero A-radical.

An exact sequence $0 \to B \to C \to M \to 0$ of right R-modules is *A-cobalanced* if the induced sequence $0 \to M^* \to C^* \to B^* \to 0$ of left E-modules is exact.

Proposition 1.7.1 ([4]) *Let A be a slender R-module of non-measurable cardinality. The following are equivalent for a right R-module M of non-measurable cardinality:*

a) M is A-reflexive.

b) There exists an A-cobalanced sequence $0 \to M \to A^I \to N \to 0$ with $R_A(N) = 0$ and $|I|$ non-measurable.

Theorem 1.7.2 *[4] Let A be a slender right R-module whose endomorphism ring is right hereditary. An exact sequence $0 \to P \to M \to N \to 0$ where P is a direct summand of A^I for some index-set I of non-measurable cardinality and M is A-reflexive splits if and only if $R_A(N) = 0$.*

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Using this result, one obtains:

Theorem 1.7.3 *[4] The following are equivalent for a slender R-module A whose endomorphism ring is right and left noetherian:*

a) Every exact sequence $0 \to U \to A^{\omega} \to V \to 0$ *with* $R_A(V) = 0$ *splits.*

b) A is $\aleph_1$*-projective as a left E-module, and E is left hereditary.*

We conclude this section with a result describing projectivity properties of A-reflexive modules:

Theorem 1.7.4 *[4] Let A be a slender right R-module of non-measurable cardinality. Consider the following conditions on A:*

a) If N is a left E-module of non-measurable cardinality with $\mathrm{Ext}^1_R(N, A) = 0$*, then N is projective.*

b) If M is an A-reflexive right R-module, and I has non-measurable cardinality, then every exact sequence $A^I \to M \to 0$ *splits.*

Then, a) always implies b), and the converse holds if E is left hereditary.

1.8 Mixed Groups

We continue our discussion with an application of the concept of A-solvability to the discussion of mixed abelian groups. The class $\mathcal{G}$ was introduced by Glaz and Wickless in [29] as the class of all mixed abelian groups A such that

i) A_p is finite for all primes p,

ii) A/tA is divisible, and

iii) $\mathrm{Hom}(A, tA)$ is torsion.

Several other characterizations of the elements of $\mathcal{G}$ have been obtained; e.g. they are the self-small mixed abelian groups for which A/tA is divisible [10].

Let $A \in \mathcal{G}$ and consider A-generated abelian groups B and C in $\mathcal{G}$. A map $\alpha \in \mathrm{Hom}(B, C)$ induces an $\overline{E}$-module morphism $\overline{H_A(\alpha)} : \overline{H_A(B)} \to \overline{H_A(C)}$ by $\overline{H_A(\alpha)}(\overline{\sigma}) = \overline{\alpha\sigma}$. Define a map $\Delta_{B,C} : \mathrm{Hom}(B, C) \to$ by $\Delta_{B,C}(\alpha) = \overline{H_A(\alpha)}$. The subscripts for Δ are usually omitted unless this would result in ambiguities. The goal of this section is to determine the class of groups B such that $\Delta_{B,C}$ is onto for all A-solvable groups $C \in \mathcal{G}$. Observe that $\Delta(1_B)(\overline{\sigma}) = \overline{H_A(1_B)}(\overline{\sigma}) = \overline{H_A(1_B)(\sigma)} = \overline{\sigma}$ for all $\sigma \in H_A(B)$. Thus, $\Delta_{1_B} = 1_{\overline{H_A(B)}}$. Similarly, $\Delta(\alpha\beta) = \Delta(\alpha)\Delta(\beta)$ for all $\alpha : C \to D$ and $\beta : B \to C$. Finally, in order to simplify our notation, let $\mathcal{F}_B$ denote the functor $\mathrm{Hom}_{\overline{E}}(\overline{H_A(B)}, \overline{H_A(-)})$.

A sequence $0 \to B \to C \xrightarrow{\beta} G \to 0$ is *almost A-balanced* if $H_A(G)/im\ H_A(\beta)$ is torsion. For $A \in \mathcal{G}$, consider the class $\mathcal{G}_A$ of *finitely A-presented groups* which consists of all groups G for which one can find an almost A-balanced exact sequence $0 \to U \to A^n \to G \to 0$ with $n < \omega$ such that U is finitely A-generated.

Theorem 1.8.1 *[13] Let* $A \in \mathcal{G}$ *and let* $B \in \mathcal{G}$ *be A-solvable. The following are equivalent:*

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a) $B \in \mathcal{G}_A$.

b) *The sequence*

$$0 \to \mathrm{Hom}(B, tC) \to \mathrm{Hom}(B, C) \xrightarrow{\Delta} \mathrm{Hom}_{\overline{E}}(\overline{H_A(B)}, \overline{H_A(C)}) \to 0$$

is exact for all A-solvable groups $C \in \mathcal{G}$.

We conclude with a discussion of some applications of the previous result to the category $WALK$ [14]. For a group $A \in \mathcal{G}$, consider the class $\mathcal{TR}$ of all abelian groups whose torsion subgroup is reduced. By [12, Proposition 2.1], $tH_A(G) = H_A(tG)$ for all $G \in \mathcal{TR}$. The symbol $\mathcal{WTR}$ denotes the full subcategory of $WALK$ whose objects are taken from $\mathcal{TR}$. Consider the functors $W : \mathcal{TR} \to \mathcal{WTR}$ defined by $W(G) = G$ and $W(\alpha) = \alpha + \mathrm{Hom}(B, tC)$ for all $B, C \in \mathcal{TR}$, and $W_A : \mathcal{WTR} \to M_{\overline{E}}$ defined by $W_A(G) = \overline{H_A(G)}$ and $W_A(\phi + \mathrm{Hom}(B, tC)) = \overline{H_A(\phi)}$.

Corollary 1.8.2 *[13] Let A be in* $\mathcal{G}$ *such that the* $WALK$*-endomorphism ring of A is a Quasi-Frobenius ring. Then, A is* $WALK$*-injective with respect to any* $WALK$*-monomorphism* $\alpha : G \to C$ *where* $G \in \mathcal{TR}$ *is finitely A-generated, and* $C \in \mathcal{G}_A$.

Following Beaumont's and Pierce's definition in [20], the class $\mathcal{D}$ of *quotient divisible groups (qd-groups)* traditionally consists of those abelian groups G of finite torsion-free rank which have a reduced torsion subgroup and contain a free subgroup F such that G/F is a divisible torsion group. A slightly more general definition of quotient divisibility allows G/F to be the direct sum of a finite and a divisible torsion group in order to ensure that $\mathcal{D}$ also contains the class $\mathcal{G}$ of mixed abelian groups. Turning to the description of qd-groups in terms of smallness conditions, let the symbol $\mathcal{TR}$ denote the class of abelian groups G for which tG is reduced. Our next result shows that the quotient divisible groups are self-small:

Theorem 1.8.3 *The following are equivalent for a group* $A \in \mathcal{TR}$*:*

a) A is quotient divisible.

b) A is $\mathcal{TR}$*-small.*

c) A is D-small.

In particular, $\mathcal{D}$ is the largest subclass $\mathcal{C}$ of $\mathcal{TR}$ which is $\mathcal{C}$ small and contains $\mathbb{Q}$ and all finite groups. Moreover, every quotient divisible group is self-small.

An abelian group A is *qd-flat* if, for each right E-module M, there is a non-zero integer ℓ such that $\ell\,\mathrm{Tor}_1^E(M, A)$ is divisible. Obviously, A is qd-flat if and only if $\mathrm{Tor}_1^E(M, A) \cong D \oplus T$ for some bounded group T and some divisible group D whenever $M \in \mathcal{M}_E$.

Theorem 1.8.4 *Let A be a qd-flat qd-group and k a positive integer such that* $k\,\mathrm{Tor}_1^E(M, A)$ *is divisible for all right E-modules M. A reduced A-generated torsion group G such that* $G_p = 0$ *if* $p|k$ *or* $A_p = 0$ *is A-solvable.*

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Chapter 2

How Far Is An HFD from A UFD?

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Abstract In this paper, we study an invariant $\overline{\Lambda}(R)$ introduced by Scott Chapman to measure how far an HFD R is from being a UFD. We show that if either R contains a prime element or R is a Krull domain with finite divisor class group, then R is a UFD if and only if $\overline{\Lambda}(R) = 0$. However, we give an example of an atomic integral domain R with $\overline{\Lambda}(R) = 0$ which is not an HFD.

2.1 Introduction

An integral domain R is *atomic* if each nonzero nonunit of R is a product of irreducible elements (atoms) of R. If R satisfies the ascending chain condition on principal ideals (ACCP) (in particular, if R is Noetherian or a Krull domain), then R is atomic (but not conversely [16]). An atomic integral domain R is a *half-factorial domain* (HFD) if whenever $x_1 \cdots x_n = y_1 \cdots y_m$ for irreducible $x_i, y_j \in R$, then $m = n$. A UFD is always an HFD, but not conversely. For example, $R = \mathbb{R} + X\mathbb{C}[X]$ is a one-dimensional Noetherian HFD which is not a UFD since $XX = (iX)(-iX)$ are two nonassociated irreducible factorizations of X^2. An atomic integral domain R is a *finite factorization domain* (FFD) if each nonzero nonunit of R has only a finite number of nonassociated irreducible factorizations.

The name HFD was coined by Zaks in [23]. But the idea goes back to a paper of Carlitz [8], where he proved that the ring of integers R in a number field is an HFD if and only if R has class number at most two. For example, $\mathbb{Z}[\sqrt{-5}]$ is an HFD, but not a UFD. The same proof also shows that a Krull domain R with divisor class group $Cl(R)$ is an HFD if $|Cl(R)| \leq 2$, and if each nonzero divisor class contains a height-one prime ideal, then R is an HFD if and only if $|Cl(R)| \leq 2$. However, whether or not a Krull domain R is an HFD depends more on the distribution of the height-one primes ideals in the divisor classes than on the group $Cl(R)$ itself. For more on HFDs, see the recent survey article [11].

In this paper, we study an invariant $\overline{\Lambda}(R)$ introduced by Scott Chapman (cf. [10], [18]) to measure how far an HFD R is from being a UFD. We first show (Theorem 2.2.2) that if an atomic integral

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domain R contains a prime element, then $\overline{\Lambda}(R) = 0$ if and only if R is a UFD. However, we give an example of an atomic integral domain R with $\overline{\Lambda}(R) = 0$ which is not an HFD. We also show (Theorem 2.2.7) that if R is a Krull domain with finite divisor class group, then $\overline{\Lambda}(R) = 0$ if and only if R is a UFD. We then investigate the relationship between $\overline{\Lambda}(R)$ and $\overline{\Lambda}(R_S)$, where $S \subset R$ is a multiplicative set generated by prime elements of R. Finally, we include several open questions.

Throughout, R will always denote an integral domain with group of units $U(R)$ and nonzero elements R^*, the dimension of a ring always means Krull dimension, and X and Y will be indeterminates. As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}/n\mathbb{Z}$, and $\mathbb{F}_q$ will denote the integers, rational numbers, real numbers, complex numbers, integers modulo n, and the finite field with q elements, respectively. General references for factorization in integral domains include [1], [2], [3], and [5]. For any undefined notation or terminology, see [13] or [14].

2.2 $\overline{\Lambda}(R)$

Let R be an atomic integral domain. Following [10], for a nonzero nonunit $x \in R$ and n a positive integer, we define $l_R(x)$ to be the length of a shortest factorization of x, $\eta_R(x)$ to be the number of nonassociated irreducible factorizations of x, $\gamma_R(n) = \{x \mid x \in R \text{ with } l_R(x) = n\}$, $\mu(R, n) = \{\eta_R(x) \mid x \in \gamma_R(n)\}$, $\Lambda(R, n) = |\mu(R, n)|$, and $\overline{\Lambda}(R) = lim_{n\to\infty} \Lambda(R, n)/n$. (We will usually delete the R subscripts when no confusion can occur.) By convention, $\overline{\Lambda}(R) = \infty$ if some $\eta_R(x) = \infty$ (i.e., if R is not an FFD). Thus we will be mainly interested in the case when R is an FFD. So $\overline{\Lambda}(R)$ measures, in some sense, the asymptotic behavior of "the number of the number" of nonassociated irreducible factorizations. Actually, in [10] these definitions were given just for HFDs (in the context of half-factorial monoids), but they work equally well for arbitrary atomic integral domains. The asymptotic behavior of $\eta_R(x)$ has been studied in [17], and a formula for $\eta_R(x)$ when R has class number two is given in [9].

If R is a UFD, then $\overline{\Lambda}(R) = 0$. This follows since $\eta_R(x) = 1$ for each nonzero nonunit $x \in R$ gives $\Lambda(R, n) = 1$ for each positive integer n, and hence $\overline{\Lambda}(R) = lim_{n\to\infty} \Lambda(R, n)/n = lim_{n\to\infty} 1/n = 0$. However, in general, we have been unable to determine conditions for the existence of this limit, and we have no examples where it does not exist. When we write $\overline{\Lambda}(A) = \overline{\Lambda}(B)$ for two atomic integral domains A and B, we mean only that $lim_{n\to\infty} \Lambda(A, n)/n$ exists if and only if $lim_{n\to\infty} \Lambda(B, n)/n$ exists, and if both limits exist, then they are equal. This would be the case, for example, if $\mu(A, n) = \mu(B, n)$ for each positive integer n. Similarly, $\overline{\Lambda}(R) \neq 0$ just means that if $lim_{n\to\infty} \Lambda(R, n)/n$ exists, then it is not zero.

In our first lemma, we isolate the key fact used in our first theorem which shows that it is crucial whether or not R contains a prime element. If R contains a prime element, then $\mu(R, n) \subseteq \mu(R, n+1)$, and hence $\Lambda(R, n) \leq \Lambda(R, n + 1)$, for each positive integer n.

Lemma 2.2.1 *Let R be an atomic integral domain such that $\mu(R, n) \subseteq \mu(R, n + 1)$ for each positive integer n. Then $\overline{\Lambda}(R) = 0$ if and only if R is a UFD.*

Proof We have already observed that $\overline{\Lambda}(R) = 0$ when R is a UFD. Conversely, suppose that R is not a UFD. We show that $\overline{\Lambda}(R) \neq 0$. We may assume that $\eta(x) < \infty$ for each nonzero nonunit $x \in R$. Since R is atomic, but not a UFD, there are irreducible $x_1, \ldots, x_r, y_1, \ldots, y_s \in R$ such that $z = x_1 \cdots x_r = y_1 \cdots y_s$ and no x_i is an associate of any y_j. Next, we show that $\eta(z^n) < \eta(z^{n+1})$ for each positive integer n. To see this, let $z^n = L_1 = L_2 = \cdots = L_k$ be nonassociated irreducible factorizations of z^n. Then $(x_1 \cdots x_r)L_1, (x_1 \cdots x_r)L_2, \ldots, (x_1 \cdots x_r)L_k, (y_1 \cdots y_s)^{n+1}$ are nonassociated irreducible factorizations of z^{n+1}; so $\eta(z^n) < \eta(z^{n+1})$. Let $l(z) = m \geq 2$. By hypothesis, $\eta(z^k) \in \mu(R, mk)$ since $l(z^k) \leq mk$. Thus $\{\eta(z), \ldots, \eta(z^n)\} \subseteq \mu(R, m) \cup \cdots \cup \mu(R, mn) =$

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$\mu(R, mn)$, and hence $\Lambda(R, mn) \geq n$ for each positive integer n. Thus $\Lambda(R, mn)/mn \geq 1/m$ for each positive integer n, and hence $\overline{\Lambda}(R) \neq 0$. □

Theorem 2.2.2 *Let R be an atomic integral domain which contains a prime element. Then $\overline{\Lambda}(R) = 0$ if and only if R is a UFD.*

Proof Let $p \in R$ be prime. Then it is easy to see that $\eta(px) = \eta(x)$ and $l(px) = l(x) + 1$ for each nonzero nonunit $x \in R$. Thus $\mu(R, n) \subseteq \mu(R, n+1)$ for each positive integer n. The theorem now follows directly from Lemma 2.2.1. □

Let $A \subseteq B$ be an extension of integral domains. Probably the simplest examples of HFDs which are not UFDs are certain integral domains of the form $A + XB[X] = \{ f(X) \in B[X] \mid f(0) \in A \}$ or $A + XB[[X]] = \{ f(X) \in B[[X]] \mid f(0) \in A \}$. These two constructions have been studied extensively (cf. [2], [5], [19], [22]) and many special cases have been given for when they yield HFDs (cf. [2], [5], [11], [12]). For example, for any proper extension $K \subset F$ of fields, $K + XF[X]$ and $K + XF[[X]]$ are always HFDs, but not UFDs [2, Theorem 5.3]. More generally, if K is a subfield of an integral domain B, then $K + XB[[X]]$ is always an HFD [5, Proposition 5.1], and $K + XB[X]$ always satisfies ACCP and is an HFD if and only if B is integrally closed [12, Theorem 2.1]. Conditions on K and F often determine properties of $R = K + XF[X]$ or $K + XF[[X]]$; for example, R is Noetherian if and only if $[F : K] < \infty$ and R is integrally closed if and only if K is algebraically closed in F (cf. [7]). Two major differences between these two constructions are that $K + XF[[X]]$ is quasilocal and has no prime elements, while $K + XF[X]$ is never quasilocal and has many prime elements.

Let $K \subset F$ be a proper extension of fields. We next show that for $R = K + XF[[X]]$, we have $\overline{\Lambda}(R) = 0$ (resp., ∞) if F is finite (resp., infinite).

Theorem 2.2.3 *Let $K \subset F$ be a proper extension of fields, and let $R = K + XF[[X]]$. Then R is an HFD, but not a UFD. If F is finite, then $\overline{\Lambda}(R) = 0$. If F is infinite, then $\overline{\Lambda}(R) = \infty$.*

Proof We have already observed that R is an HFD, but not a UFD. First note that each nonzero nonunit of R has the form $\alpha X^n f$ for some $\alpha \in F^*$, $n \geq 1$, and $f \in U(R)$. Let $\{ a_i \}_{i \in I}$ be a set of coset representatives for F^*/K^*. Then $\{ a_i X \}_{i \in I}$ is, up to associates, the set of all irreducible elements of R. Recall that F^*/K^* is finite if and only if F is finite [6]. Hence R is an FFD if and only if F is finite [1, Proposition 5.2]; thus $\overline{\Lambda}(R) = \infty$ if F is infinite. So suppose that F is finite. Then F^*/K^* is a finite cyclic group; say $F^*/K^* = \langle \alpha K^* \rangle$ has order m. Thus $\mu(R, n) = \{ \eta(\alpha^i X^n) \mid 0 \leq i \leq m-1 \}$, and hence $\Lambda(R, n) \leq m$ for each positive integer n. Thus $\overline{\Lambda}(R) = 0$. □

Theorem 2.2.3 gives an example of an HFD R which is not a UFD, but $\overline{\Lambda}(R) = 0$. This shows that the hypothesis in Theorem 2.2.2 that R conains a prime element is essential.

Example 2.2.4 Let $K \subset F$ be a proper extension of fields.

(a) Let $R = K + XF[[X]]$. Then R is a one-dimensional quasilocal HFD, but not a UFD, R has no prime elements, and $\overline{\Lambda}(R) = 0$ when F is finite by Theorem 2.2.3 In particular, $R = \mathbb{F}_2 + X\mathbb{F}_4[[X]]$ has $\overline{\Lambda}(R) = 0$.

(b) Let $R = K + XF[X]$. Then R is a one-dimensional HFD, but not a UFD. Note that any $f \in R$ with $f(0) \neq 0$ is prime in R if and only if it is prime in $F[X]$. Thus R has many prime elements, and hence $\overline{\Lambda}(R) \neq 0$ by Theorem 2.2.2. Recall that R is an FFD if and only if F is finite [1, Proposition 5.2]. Thus $\overline{\Lambda}(R) = \infty$ when F is infinite. In particular, $R = \mathbb{R} + X\mathbb{C}[X]$ has $\overline{\Lambda}(R) = \infty$.

(c) Let $R = \mathbb{F}_2 + X\mathbb{F}_4[X]$. In [20], it is proved that $\overline{\Lambda}(R) = 4/3$. Explicit formulas are computed for $\eta(\alpha X^n)$ for each $\alpha \in \mathbb{F}_4^*$ and positive integer $n \geq 1$. These are used to compute $\mu(R, n)$, and then to show that $\Lambda(R, n) = (4n - r)/3$, where $r \in \{0, 1, 2\}$ and $n \equiv r(mod 3)$. Thus $\overline{\Lambda}(R) = lim_{n \to \infty} \Lambda(R, n)/n = 4/3$.

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Let $K \subset F$ be a proper extension of finite fields and $T = K + XF[X]$. It is conjectured in [20] that $\overline{\Lambda}(T) = \sigma(n)/n$, where $|F^*/K^*| = n$ and $\sigma(n)$ denotes the sum of the positive integers that divide n.

We have just seen that we may have $\overline{\Lambda}(R) = 0$ for R an HFD, but not a UFD. We next give an example of an atomic integral domain R which is not an HFD, but $\overline{\Lambda}(R) = 0$.

Example 2.2.5 Let $R = \mathbb{F}_2[[X^2, X^3]]$. Then R is a one-dimensional local Noetherian integral domain with no prime elements. Note that R is not an HFD since $X^3X^3 = X^2X^2X^2$ are two nonassociated irreducible factorizations of X^6. We show that $\overline{\Lambda}(R) = 0$. Up to associates, the irreducible elements of R are X^2, $X^2 + X^3$, X^3, and $X^3 + X^4$. Let $f \in R$ with $l(f) = n$. Then one can easily check that $ord(f)$ is $3n$, $3n-1$, or $3n-2$. Thus, up to associates, f is either X^{3n}, $X^{3n} + X^{3n+1}$, X^{3n-1}, $X^{3n-1} + X^{3n}$, X^{3n-2}, or $X^{3n-2} + X^{3n-1}$. Hence $\mu(R, n) = \{\eta(X^{3n}), \eta(X^{3n} + X^{3n+1}), \eta(X^{3n-1}), \eta(X^{3n-1} + X^{3n}), \eta(X^{3n-2}), \eta(X^{3n-2} + X^{3n-1})\}$. Thus $\Lambda(R, n) \leq 6$, and hence $\Lambda(R, n)/n \leq 6/n$, for each positive integer n. Thus $\overline{\Lambda}(R) = 0$.

Let R be a Krull domain with divisor class group $Cl(R)$. We have already noted that if $|Cl(R)| \leq 2$, then R is always an HFD; and if each nonzero divisor class contains a height-one prime ideal, then R is an HFD if and only if $|Cl(R)| \leq 2$. Moreover, it is an open question if for every abelian group G, there is a Krull HFD R with $Cl(R) = G$ (this is known to hold for many classes of abelian groups (see [11] or [24])). Also, recall that a Krull domain is always an FFD [1, page 14].

We next give a second criterion to have $\overline{\Lambda}(R) = 0$ if and only if R is a UFD.

Lemma 2.2.6 *Let R be a Krull domain such that $Cl(R)$ has an element of finite order with infinitely many height-one prime ideals in that divisor class. Then $\overline{\Lambda}(R) = 0$ if and only if R is a UFD.*

Proof We have already observed that $\overline{\Lambda}(R) = 0$ when R is a UFD. Conversely, suppose that R is not a UFD. We show that $\overline{\Lambda}(R) \neq 0$. If R has a nonzero principal prime ideal, then $\overline{\Lambda}(R) \neq 0$ by Theorem 2.2.2. Thus we may assume that some nonzero divisor class g with finite order $k \geq 2$ contains infinitely many height-one prime ideals of R. Choose distinct height-one prime ideals P and $\{P_n \mid 1 \leq n < \infty\}$ in class g. For each positive integer n, define nonzero nonunits $x_{n,1}, x_{n,2}, \ldots, x_{n,n+1} \in R$ by $x_{n,i}R = ((P_1 \cdots P_{(i-1)k})P^{(n-i+1)k})_v$. Then each $x_{n,i} \in \gamma(n)$ and $\eta(x_{n,1}) < \eta(x_{n,2}) < \cdots < \eta(x_{n,n+1})$; so $\Lambda(R, n) \geq n + 1$. Thus $\Lambda(R, n)/n > 1$ for each positive integer n, and hence $\overline{\Lambda}(R) \neq 0$. □

Theorem 2.2.7 *Let R be a Krull domain with $Cl(R)$ finite. Then $\overline{\Lambda}(R) = 0$ if and only if R is a UFD.*

Proof If R has only a finite number of height-one prime ideals, then R is a UFD (in fact, a PID) [13, Corollary 13.4]. Otherwise, some divisor class must contain infinitely many height-one prime ideals since $Cl(R)$ is finite. The theorem now follows directly from Lemma 2.2.6. □

Remark 2.2.8 (a) The proof of Lemma 2.2.6 shows that $\overline{\Lambda}(R) \geq 1$, if the limit exists.

(b) In general, a Krull domain R with $Cl(R)$ torsion need not have have an element in $Cl(R)$ with infinitely many height-one prime ideals in that divisor class. For example, let $G = \bigoplus_{n=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$. Then one can use [15, Theorem 8] to construct a Dedekind domain R with $Cl(R) = G$ and no prime elements such that each nonzero divisor class contains at most one maximal ideal of R.

We end this section with an example of a Dedekind HFD R with $Cl(R) = \mathbb{Z}$ and $\overline{\Lambda}(R) = \infty$. In this case, R is an FFD, so $\eta(x) < \infty$ for each nonzero nonunit $x \in R$; but $\Lambda(R, 2) = \infty$.

Example 2.2.9 Let R be a Dedekind domain with $Cl(R) = \mathbb{Z}$ such that R has no prime elements, for each positive integer n there is a unique prime ideal P_n with $[P_n] = n$, there are infinitely many

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prime ideals $\{ Q_n \mid 1 \le n < \infty \}$ all with $[Q_n] = -1$, and these are the only nonzero prime ideals of R (such a Dedekind domain R exists by [15, Theorem 8]). Each irreducible element $x \in R$ is given by $xR = Q_{i_1} \cdots Q_{i_n} P_n$ for some P_n and Q_j's. Note that R is an HFD since $l(x)$ is just the number of P_i's in the prime ideal factorization of xR. Next, we show that $\Lambda(R, 2) = \infty$. For each positive integer n, consider $x_n R = (Q_1 Q_2 \cdots Q_{2n})(P_n)^2$ with $l(x_n) = 2$. This ideal product can be split in $\alpha_n = (2n)!/2n!n!$ ways as $(Q_{i_1} \cdots Q_{i_n} P_n)(Q_{i_{n+1}} \cdots Q_{i_{2n}} P_n)$, and hence $\eta(x_n) = (2n)!/2n!n!$. Also, note that $\alpha_n < \alpha_{n+1}$; so $\{ \eta(x_n) \mid 1 \le n < \infty \} \subseteq \mu(R, 2)$ is infinite. Similarly, each $\mu(R, k)$ is infinite. (For a fixed integer $k \ge 2$ and all positive integers n, define $x_{k,n} \in R$ by $x_{k,n} R = (Q_1 Q_2 \cdots Q_{nk})(P_n)^k$. Then $x_{k,n} \in \mu(R, k)$ and $\eta(x_{k,n}) = (nk)!/k!(n!)^k < \eta(x_{k,n+1})$.) Thus each $\mu(R, k)$ is infinite, and hence $\overline{\Lambda}(R) = \infty$.

2.3 Localization

We have seen in Theorem 2.2.2 and Examples 2.2.4 and 2.2.5 that it is important whether or not R contains a prime element. It thus seems of interest to investigate how $\overline{\Lambda}(R)$ and $\overline{\Lambda}(R_S)$ compare, where $S \subset R$ is a multiplicative set generated by prime elements of R. Let $\mathcal{P} \subset R$ be a set of prime elements of an atomic integral domain R, and let $S = \langle \mathcal{P} \rangle = \{ u p_1 \cdots p_n \mid u \in U(R), p_i \in \mathcal{P} \}$. First observe that R_S is atomic [3, Corollary 2.2], and that R is an HFD (resp., FFD) if and only if R_S is an HFD (resp., FFD) [3, Corollary 2.5 (resp., 2.2)]. Moreover, if S consists of all the prime elements of R, then R_S has no prime elements [3, Corollaries 1.4 and 1.7].

For two atomic integral domains A and B, we write $\overline{\Lambda}(A) \le \overline{\Lambda}(B)$ to mean only that the inequality holds when both limits exist. For example, this would be the case if $\mu(A, n) \subseteq \mu(B, n)$ for each positive integer n.

Theorem 2.3.1 *Let R be an atomic integral domain and $S \subset R$ a multiplicative set generated by prime elements of R. Then $\overline{\Lambda}(R_S) \le \overline{\Lambda}(R)$. Moreover, if S does not contain all the prime elements of R, then $\overline{\Lambda}(R_S) = \overline{\Lambda}(R)$.*

Proof We show that $\mu(R_S, n) \subseteq \mu(R, n)$ for each positive integer n. Thus $\Lambda(R_S, n) \le \Lambda(R, n)$ for each positive integer n, and hence $\overline{\Lambda}(R_S) \le \overline{\Lambda}(R)$. Let $m \in \mu(R_S, n)$. Then $m = \eta_{R_S}(x)$ for some $x \in R_S$ with $l_{R_S}(x) = n$. Write $x = ux'$, where $x' \in R$, $u \in U(R_S)$, and $(x', t) = 1$ for all $t \in S$. Then $\eta_R(x') = \eta_{R_S}(x') = \eta_{R_S}(x) = m$ and $l_R(x') = l_{R_S}(x') = l_{R_S}(x) = n$ (several of these equalities follow from [3, Corollary 1.4]). Thus $m \in \mu(R, n)$.

Suppose that there is some prime $p \in R \setminus S$. Let $m \in \mu(R, n)$. Then $m = \eta_R(x)$ for some $x \in R$ with $l_R(x) = n$. Write $x = sx'$, where $s \in S$ and $(x', t) = 1$ for all $t \in S$. Let $l_R(s) = k$, and set $z = p^k x'$. Then $\eta_{R_S}(z) = \eta_{R_S}(x') = \eta_R(x') = \eta_R(x) = m$ and $l_{R_S}(z) = k + l_{R_S}(x') = l_R(s) + l_R(x') = l_R(x) = n$ (again, use [3, Corollary 1.4]). Thus $m \in \mu(R_S, n)$. Hence $\mu(R, n) \subseteq \mu(R_S, n)$; so $\mu(R, n) = \mu(R_S, n)$, and thus $\overline{\Lambda}(R_S) = \overline{\Lambda}(R)$. □

Corollary 2.3.2 *Let $R[X]$ be an atomic integral domain. Then $\overline{\Lambda}(R[X, X^{-1}]) = \overline{\Lambda}(R[X])$.*

We next give an example to show that we may have $\overline{\Lambda}(R_S) < \overline{\Lambda}(R)$ when S is generated by all the prime elements of R (in this case, R_S has no prime elements). This is somewhat different than what usually happens; most invariants related to lengths of factorizations are not affected by localizing at all the prime elements of R.

Example 2.3.3 Let $K \subset F$ be a proper extension of finite fields with $F^*/K^* = \langle \alpha K^* \rangle$ cyclic of order m. Let $R = K + XF[X]$, and let S be the multiplicative subset of R generated by all the

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prime elements of R. Note that $S = \{ f \in R \mid f(0) \neq 0 \}$ and $R_S = R_M$, where $M = XF[X] \cap R$ is a maximal ideal of R. Also, note that R and R_S are both HFDs and $\overline{\Lambda}(R) \neq 0$ (see Example 2.2.4(b)). Up to associates, the irreducible elements of R_S are $\{ X, \alpha X, \ldots, \alpha^{m-1}X \}$, and thus $\mu(R_S, n) = \{ \eta_{R_S}(\alpha^i X^n) \mid 0 \leq i \leq m-1 \}$. Hence $\Lambda(R_S, n) \leq m$ for each positive integer n, and thus $\overline{\Lambda}(R_S) = 0$.

Clearly R is atomic if $R[X]$ is atomic, but R atomic does not imply that $R[X]$ is atomic [21]. Since $R[X]$ always contains a prime element, $\overline{\Lambda}(R[X]) = 0$ if and only if R is a UFD by Theorem 2.2.2.

Theorem 2.3.4 *Let $R[X]$ be an atomic integral domain. Then $\overline{\Lambda}(R) \leq \overline{\Lambda}(R[X])$.*

Proof Note that $\mu(R, n) \subseteq \mu(R[X], n)$ for each positive integer n. Thus $\overline{\Lambda}(R) \leq \overline{\Lambda}(R[X])$. □

Our final example shows that the inequality in Theorem 2.3.4 may be strict.

Example 2.3.5 Let R be either $\mathbb{F}_2[[X^2, X^3]]$ or $K + XF[[X]]$, where $K \subset F$ is a proper extension of finite fields. In either case, R is a one-dimensional local Noetherian domain with no prime elements, and hence $R[Y]$ is atomic. Then $\overline{\Lambda}(R) = 0$ by Example 2.2.5 and Theorem 2.2.3, respectively. However, $\overline{\Lambda}(R[Y]) \neq 0$ by Theorem 2.2.2 since R is not a UFD.

2.4 Questions

Let R be an atomic integral domain. We end this paper with several questions about $\overline{\Lambda}(R)$. Let $L_R(x)$ denote the length of a longest factorization of a nonzero nonunit $x \in R$. (Note that R is an HFD if and only if $l_R(x) = L_R(x)$ for all nonzero nonunits $x \in R$.)

Question 2.4.1 *Let R be an atomic integral domain.*

(1) Determine conditions on R so that $\overline{\Lambda}(R) = lim_{n\to\infty} \Lambda(R, n)/n$ exists.
(2) Determine the possible values for $\overline{\Lambda}(R)$.
(3) Let R be a Krull domain. Does $\overline{\Lambda}(R) = 0$ if and only if R is a UFD?
(4) Let R be a Krull domain. Do we always have either $\overline{\Lambda}(R) = 0$ or $\overline{\Lambda}(R) = \infty$?
(5) How does the theory change if we use $L_R(x)$ rather than $l_R(x)$ in defining $\overline{\Lambda}(R)$?

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Chapter 3

A Counter Example for A Question On Pseudo-Valuation Rings

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Abstract In this paper, we give a counter example of the following question which was raised by Anderson, Dobbs, and the author in [3, Question 3.14]: Let G be a strongly prime ideal of a ring D such that $G \subset Z(D)$ and $(G : G) = T(D)$ is a PVR. Then $T(D)$ has maximal ideal $Z(D)_S$, where $S = D \setminus Z(D)$, and $Z(D)$ is a prime ideal of D. Is $Z(D)$ also a strongly prime ideal of D?

3.1 Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. The following notation will be used throughout. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, Nil(R) denotes the set of nilpotent elements of R, $Z(R)$ denotes the set of zerodivisors of R, $S = R \setminus Z(R)$, $dim(R)$ denotes the Krull dimension of R, and if B is an R-module, then $Z(B)$ denotes the set of zerodivisors on B, that is, $Z(B) = \{x \in R \mid xy = 0 \text{ in B for some } y \neq 0 \text{ and } y \in B\}$. If I is an ideal of R, then $(I : I) = \{x \in T(R) \mid xI \subset I\}$. We begin by recalling some background material. As in [20], an integral domain R, with quotient field K, is called a *pseudo-valuation domain (PVD)* in case each prime ideal P of R is *strongly prime*, in the sense that $xy \in P, x \in K, y \in K$ implies that either $x \in P$ or $y \in P$. In [5], Anderson, Dobbs and the author generalized the study of pseudo-valuation domains to the context of arbitrary rings (possibly with nonzero zerodivisors). Recall from [5] that a prime ideal P of R is said to be *strongly prime (in R)* if aP and bR are comparable (under inclusion) for all $a, b \in R$. A ring R is called a *pseudo-valuation ring (PVR)* if each prime ideal of R is strongly prime. A PVR is necessarily quasilocal [5, Lemma 1(b)]; a chained ring is a PVR [[5], Corollary 4]; and an integral domain is a PVR if and only if it is a PVD (cf. [1, Proposition 3.1], [2, Proposition 4.2], and [12, Proposition 3]). Recall from [13] and [17] that a prime ideal P of R is called *divided* if it is comparable (under inclusion) to every ideal of R. A ring R is called a *divided ring* if every prime ideal of R is divided. In [8], the author gives another generalization of PVDs to the context of arbitrary rings (possibly with nonzero zerodivisors). Recall from [8] that for a ring R with total quotient ring $T(R)$ such that $Nil(R)$ is a divided prime ideal of R, let $\phi : T(R) \longrightarrow K := R_{Ni(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into K, and ϕ restricted to R is also a ring homomorphism from

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R into K given by $\phi(x) = x/1$ for every $x \in R$. A prime ideal Q of $\phi(R)$ is called a *K-strongly prime* if $xy \in Q$, $x \in K$, $y \in K$ implies that either $x \in Q$ or $y \in Q$. If each prime ideal of $\phi(R)$ is K-strongly prime, then $\phi(R)$ is called a *K-pseudo-valuation ring (K-PVR).* A prime ideal P of R is called a *ϕ-strongly prime* if $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ-strongly prime, then R is called a *ϕ-pseudo-valuation ring ($\phi - PVR$).* It is shown in [8, Corollary 7(2)] that a ring R is a ϕ-PVR if and only if $Nil(R)$ is a divided prime ideal of R and for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in R or $b \mid ac$ in R for each nonunit $c \in R$. Since a PVR is a ϕ-PVR, it is shown in [9, Theorem 2.6] that for each $n \geq 0$ there is a ϕ-PVR with Krull dimension n which is not a PVR. For other related studies on ϕ-rings, we recommend [10], [11], [6], [7], [14].

In this paper, we give a counter example of the following question that was raised by Anderson, Dobbs, and the author in [3, Question 3.14]: Let G be a strongly prime ideal of a ring D such that $G \subset Z(D)$ and $(G : G) = T(D)$ is a PVR. Then $T(D)$ has maximal ideal $Z(D)_S$, where $S = D \setminus Z(D)$, and $Z(D)$ is a prime ideal of D. Is $Z(D)$ also a strongly prime ideal of D?

Our counter example relies on the the idealization construction $R(+)B$ arising from a ring R and an R-module B as in Huckaba [21, Chapter VI]. We recall this construction. For a ring R, let B be an R-module. Consider $R(+)B = \{(r, b) : r \in R, \text{ and } b \in B\}$, and let (r, b) and (s, c) be two elements of $R(+)B$. Define :

1. $(r, b) = (s, c)$ if $r = s$ and $b = c$.
2. $(r, b) + (s, c) = (r + s, b + c)$.
3. $(r, b)(s, c) = (rs, bs + rc)$.

Under these definitions $R(+)B$ becomes a commutative ring with identity. In the following proposition, we state some basic properties of $R(+)B$.

Proposition 3.1.1 *Let R be a ring, B be an R-module, and $Z(B)$ be the set of zerodivisors on B. Then:*

1. *The ideal J of $R(+)B$ is prime (maximal) if and only if $J = P(+)B$, where P is a prime (maximal) ideal of R. Hence $dim(R) = dim(R(+)B)$ [21, Theorem 25.1].*
2. *$(r, b) \in Z(R(+)B)$ if and only if $r \in Z(R) \cup Z(B)$ [21, Theorem 25.3].*
3. *If P is a prime ideal of R, then $(R(+)B)_{P(+)B}$ is ring-isomorphic to $R_P(+)B_P$ [21, Corollary 25.5(2)].*

3.2 Counter Example

Recall that if B is an R-module, then $Z(B) = \{x \in R \mid xy = 0 \text{ in B for some } y \neq 0 \text{ and } y \in B\}$. Also, recall that if R is an integral domain and B is an R-module, then B is said to be *divisible* if r is a nonzero element of R and $b \in B$, then there exists $f \in B$ such that $rf = b$. We start this section with the following lemma.

Lemma 3.2.1 *Let R be an integral domain with quotient field F, P be a prime ideal of R, and $N = R \setminus P$. Then $B = F/P_N$ is a divisible R-module and $Z(B) = P$.*

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Proof It is clear that B is an R-module and $P \subset Z(B)$. Now, suppose that $x(y + F/P_N) = 0$ in B for some $x \in R \setminus P$. Hence $xy = p/n \in P_N$ for some $p \in P$ and $n \in N$. Thus $y = p/nx \in P_N$. Hence $y + F/P_N = 0$ in B. Thus $x \notin Z(B)$. Hence $Z(B) = P$. Next, we show that B is divisible. Let r be a nonzero element of R and $b = x + F/P_N \in B$. Then choose $f = x/r + F/P_N$. Hence $rf = b$, and thus B is divisible. □

The following three propositions are needed.

Proposition 3.2.2 *Let V be a valuation domain of the form $F + M$, where F is a field and M is the maximal ideal of V, and let $R = D + M$ for some subring D of F.*

1. *([16].) If P is a prime ideal of D, then $R_{P+M} = D_P + M$.*
2. *([18, Proposition 4.9(i)].) R is a PVD if and only if either D is a PVD with quotient field F or D is a field.*

Proposition 3.2.3 *([15, Theorem 3.1].) Let R be a ring and B be an R-module. Set $D = R(+)B$. Then:*

1. *If D is a PVR, then R is a PVR.*
2. *If R is a PVD and B is a divisible R-module, then $D = R(+)B$ is a PVR.*

Recall that an integral domain is called a *valuation* domain if for every $a, b \in R$, either $a \mid b$ in R or $b \mid a$ in R.

Proposition 3.2.4 1. *A valuation domain is a PVD ([20, Proposition 1.1]).*

2. *A PVR is quasilocal ([5, Lemma 1(b)]).*
3. *Let R be a ring. Then R is a PVR if and only if a maximal ideal of R is a strongly prime ideal ([5, Theorem 2]).*

Now, we state our example

Example 3.2.5 Let $\mathcal{Z}$ be the ring of integers with quotient field $\mathcal{Q}$. Let $R = \mathcal{Z} + X\mathcal{Q}[[X]]$, F be the quotient field of R, $P = 3\mathcal{Z} + X\mathcal{Q}[[X]]$ is a maximal ideal of R, $N = R \setminus P$, $B = F/P_N$ is an R-module, and set $D = R(+)B$. Then $Z(D) = P(+)B$ is a maximal ideal of D which is not a strongly prime ideal and $G = X\mathcal{Q}[[X]](+)B$ is a strongly prime ideal of D such that $G \subset Z(D)$ and $(G : G) = T(D)$ is a PVR.

Proof By Lemma 2.1 and Proposition 3.1.1(2), we conclude that $Z(D) = P(+)B$. By Proposition 3.1.1(1), $Z(D) = P(+)B$ is a maximal ideal of D. Since D is not quasilocal and $Z(D)$ is a maximal ideal of D, $Z(R)$ is not a strongly prime ideal of D by Proposition 2.4(2 and 3) . Now, $T(D)$ is ring-isomorphic to $R_P(+)B_P$ by Proposition 3.1.1(3). Since $R_P = \mathcal{Z}_{3\mathcal{Z}} + X\mathcal{Q}[[X]]$ by Proposition 2.2(1) and $B_P = B$ by the construction of B, we conclude that $T(D)$ is ring-isomorphic to $\mathcal{Z}_{3\mathcal{Z}} + X\mathcal{Q}[[X]](+)B$. Since it is well known that $\mathcal{Z}_{3\mathcal{Z}} + X\mathcal{Q}[[X]]$ is a valuation domain and hence is a PVD by Proposition 3.4(1) and B is divisible by Lemma 2.1, we conclude that $\mathcal{Z}_{3\mathcal{Z}} + X\mathcal{Q}[[X]](+)B$ is a PVR by Proposition 2.3(2). Hence, $T(D)$ is a PVR and $G = X\mathcal{Q}[[X]](+)B$ is a strongly prime ideal of D. It is clear that $G \subset Z(D)$. Since $yX\mathcal{Q}[[X]] \subset X\mathcal{Q}[[X]]$ for every $y \in \mathcal{Z}_{3\mathcal{Z}} + X\mathcal{Q}[[X]]$, we have $(G : G) = T(D)$ is a PVR. □

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Let R be a ring. Observe that if $Z(R)$ is a strongly prime ideal of R, then $(Z(R) : Z(R)) = T(R)$ is a PVR with maximal ideal $Z(R)$ by [3, Theorem 3.11(b)]. However, if G is a strongly prime ideal of R which is properly contained in $Z(R)$, then $(G : G) = T(R)$ need not be a PVR as in the following example.

Example 3.2.6 Let $\mathcal{Z}$ be the ring of integers and let $\mathcal{C}$ be the field of complex numbers. Let $R = \mathcal{Z} + X\mathcal{C}[[X]]$, F be the quotient field of R, $P = 3\mathcal{Z} + X\mathcal{C}[[X]]$ is a maximal ideal of R, $N = R \setminus P$, $B = F/P_N$ is an R-module, and set $D = R(+)B$. Then $Z(D) = P(+)B$ is a maximal ideal of D which is not a strongly prime ideal and $G = X\mathcal{C}[[X]](+)B$ is a strongly prime ideal of D such that $G \subset Z(D)$ and $(G : G) = T(D)$ is not a PVR.

Proof By an argument similar to that one just given in the proof of the above Example, we conclude that $Z(D) = P(+)X\mathcal{C}[[X]]$ and $T(D)$ is ring-isomorphic to $L = \mathcal{Z}_{3\mathcal{Z}} + X\mathcal{C}[[X]](+)B$. Since $\mathcal{Z}_{3\mathcal{Z}} + X\mathcal{C}[[X]]$ is not a PVD by Proposition 2.2(2), we conclude that L is not a PVR by Proposition 2.3(1). Thus $T(D)$ is not a PVR. Now, since $T(D)$ is ring-isomorphic to L and $X\mathcal{C}[[X]]$ is a strongly prime ideal of R, G is a strongly prime ideal of D. □

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Chapter 4

Co-Local Subgroups of Abelian Groups

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Abstract Dualizing the notion of a localization of an abelian group, we call a subgroup $K \neq \{0\}$ of the abelian group G a co-local subgroup if the natural map $\sigma : Hom(G, G) \to Hom(G, G/K)$ is an isomorphism, i.e., $Hom(G, K) = 0$ and each $\varphi \in Hom(G, G/K)$ is induced by some (unique) $\varphi' \in Hom(G, G)$. While purely indecomposable abelian groups and torsion groups have no co-local subgroups, many co-purely indecomposable groups do have completely decomposable co-local subgroups. If K is a co-local subgroup of a reduced, torsion-free abelian group A, then K is cotorsion-free and a pure subgroup of A. We show that each cotorsion-free group K is isomorphic to a co-local subgroup of some cotorsion-free group G.

4.1 Introduction

The notion of a localization plays an important role in category theory and was investigated in algebraic settings by several authors in [1], [2], and [7]. Special emphasis to the case of localizations of abelian groups was given in [8], [3], and [4]. Our undefined notions of abelian group theory are standard as in [5]. Recall that a localization of an abelian group A is a homomorphism $\alpha : A \to B$ such that for each $\varphi \in Hom(A, B)$ there is a unique endomorphism $\psi : B \to B$ such that $\varphi = \psi \circ \alpha$. Since each localization induces one where the map α is one-to-one, we may assume that $0 \to A \xrightarrow{\alpha} B$ is exact. In this paper we dualize this notion and arrive at the following: The epimorphism $\beta : B \to A \to 0$ is a co-localization if for each $\varphi : B \to A$ there exists a unique $\psi : B \to B$ such that $\varphi = \beta \circ \psi$. Of course, if $\beta : B \to A$ is a co-localization or not fully depends on how the subgroup $K = \ker(\beta)$ is embedded in B. Therefore, an investigation of co-localizations is really an investigation of K. To this end, we define: The subgroup K of B is a *co-local subgroup*, if $K \neq \{0\}$, and the natural map $Hom(B, B) \to Hom(B, B/K)$ is an isomorphism. In other words, K is a co-local subgroup of B, if $Hom(B, K) = 0$ and each $\varphi : B \to B/K$ is induced by some $\psi \in End(B)$. Co-local subgroups have some surprising properties. For example:

- Co-local abelian subgroups are torsion-free, which implies

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- Torsion abelian groups have no co-local subgroups. Moreover,
- If B is purely indecomposable, i.e., B is a pure subgroup of some p-adic numbers, then B has no co-local subgroups. On the other hand,
- Many co-purely indecomposable groups, i.e., groups finite rank $n+1$ and p-rank n, do have free co-local subgroups. Putting our focus of attention on torsion-free groups, we will show:
- If K is a co-local subgroup of a torsion-free group A with divisible part D, then D has a complement G in A such that $K \subseteq G$, and K is a co-local subgroup of G. This allows us to restrict our investigation of co-local subgroups of torsion-free groups A to the case where A is reduced. We will show:
- If K is a co-local subgroup of a torsion-free group A, then K is cotorsion-free. Moreover,
- If K is a co-local subgroup of a reduced torsion-free group, then K is pure in A and A/K is reduced.

We will put together two Black Boxes, c.f. [6], to prove the following:

Theorem 4.1.1 *Let K be a cotorsion-free group. Then there exist cotorsion-free groups A of arbitrarily large cardinality, such that K is isomorphic to a co-local subgroup of A.*

In our construction we will have that $End(A) = \mathbb{Z} = End(A/K)$. The case of co-local subgroups of mixed abelian groups remains enigmatic.

4.2 Basic Properties

Definition 4.2.1 Let K be a subgroup of the abelian group A. Then K is a *co-local subgroup* of A if the natural map $i^* : Hom(A, A) \to Hom(A, A/K)$ is an isomorphism, i.e., for each $\psi : A \to A/K$ there is a unique $\varphi : A \to A$ such that $\psi(a) = \varphi(a) + K$ for all $a \in A$. To avoid trivialities, we require co-local subgroups to be $\neq \{0\}$.

First we collect some preliminaries in the following

Proposition 4.2.2 *(1) $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ do not have any co-local subgroups.*

(2) If A has a co-local subgroup, then $Hom(A, \mathbb{Z}) = 0$.

(3) K is a co-local subgroup of A if and only if $Hom(A, K) = 0$ and for each $\psi \in Hom(A, A/K)$ there is some $\varphi \in End(A)$ such that $\psi(a) = \varphi(a) + K$ for all $a \in A$.

(4) If K is a co-local subgroup of the abelian group A, then K is torsion-free and reduced.

Proof To show (1), let $K \neq \{0\}$ be a proper subgroup of $\mathbb{Q}$. Then there is some prime p such that $\mathbb{Z}(p^\infty)$ is a direct summand of $\mathbb{Q}/K$ and the ring J_p of all p-adic integers is an uncountable subring of $End(\mathbb{Q}/K)$, but $End(\mathbb{Q})$ is countable. This shows that K is not a co-local subgroup of $\mathbb{Q}$. Let $\mathbb{Z}(p^\infty) = \langle a_n : pa_{n+1} = a_n, pa_1 = 0\rangle$. If K is a proper subgroup of $\mathbb{Z}(p^\infty)$, then there is some m such that $K = \langle a_m \rangle$. Moreover, there is an isomorphism $\psi : \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty)/K$ such that $\psi(a_j) = a_{m+j} + K$. If there is some $\varphi \in End(\mathbb{Z}(p^\infty))$ with $\psi(x) = \varphi(x) + K$ for all $x \in \mathbb{Z}(p^\infty)$, then, since K is fully invariant in $\mathbb{Z}(p^\infty)$, $\varphi(K) \subseteq K$ and thus $K \subseteq \ker(\psi)$. This is a contradiction to ψ being injective.

(2) is trivial since $Hom(A, \mathbb{Z}) \neq 0$ implies $Hom(A, K) \neq 0$ for any subgroup $K \neq \{0\}$. Also, (3) follows immediately from the definition.

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To show (4), let K be a co-local subgroup of A such that $t(K)_p$, the p-primary part of the torsion subgroup $t(K)$ of K, is non-trivial. If $t(A)_p$ is not divisible, then A has a cyclic summand of order p^n for some $n \in \mathbb{N}$. But then $Hom(A, K) \neq 0$. This shows that $t(A)_p$ is divisible and $t(K)_p$ is reduced. Therefore $A = B \oplus C$, with $B \cong \mathbb{Z}(p^\infty)$ such that $L = B \cap K \neq \{0\}$ is finite. Then $A/K \cong B/L \oplus M$ for some subgroup M of A/K. Define $\psi : A \to A/K$ such that $\psi \restriction_B : B \to B/L$ is an isomorphism and $\psi(C) = \{0\}$. Since K is a co-local subgroup, there is some $\varphi \in End(A)$ such that $\psi(x) = \varphi(x) + K$ for all $x \in A$. Moreover, $\varphi(C) \subseteq K$, $\varphi(B) \subseteq B$, and $\psi(b) = \varphi(b) + L$ for all $b \in B$. As seen in the proof of (1), this is not possible.

If K is not reduced, then A is not reduced, which implies $Hom(A, K) \neq 0$. Thus K is reduced. □

Corollary 4.2.3 *Torsion groups do not have co-local subgroups.*

We are now ready for the following:

Theorem 4.2.4 *Let K be a co-local subgroup of A. Then*

(1) If $\mathbb{Q}$ is a subgroup of A/K, then $\mathbb{Q}$ is a subgroup of A.

(2) Assume that A is torsion-free and reduced. Then $\mathbb{Z}(p^\infty)$ is not a subgroup of A/K for any prime p.

Proof To show (1), let $A/K = B/K \oplus C/K$ with $B/K \cong \mathbb{Q}$. Define $\psi_1 : A \to A/K$ to be the map $a \mapsto a + K$ followed by the projection of A/K onto B/K with kernel C/K. Thus $\psi_1(b) = b + K$ for all $b \in B$ and $\psi_1(C) = \{0\}$. For any natural number n, define $\psi_n : A \to A/K$ to denote ψ_1 followed by the multiplication by $\frac{1}{n}$. Thus $n\psi_n = \psi_1$ and there exist $\varphi_n \in End(A)$ with $\psi_n(x) = \varphi_n(x) + K$ for all $x \in A$. Moreover, $n\varphi_n(x) + K = \varphi_1(x) + K$ for all $x \in A$ and thus $n\varphi_n - \varphi_1 \in Hom(A, K) = 0$. This shows $n\varphi_n = \varphi_1$ and $\varphi_n(B) \subseteq B$ for all n, which implies $\{0\} \neq \varphi_1(B) \subseteq \bigcap_{n \in \mathbb{N}} nB$. Since K is torsion-free, B is torsion-free and therefore B contains a copy of $\mathbb{Q}$. To show (2), assume that $B/K \cong \mathbb{Z}(p^\infty)$. Define the map $\psi_1 : A \to B/K$ as above. For any $\pi \in J_p$, define $\psi_\pi : A \to B/K$ to be ψ_1 followed by the multiplication by π. Then there is a unique $\varphi_\pi \in End(A)$ such that φ_π induces ψ_π. It is easy to see that $J = \left\{\varphi_\pi : \pi \in J_p\right\} \subseteq End(A)$ is a subring of $End(A)$. Pick some $b \in B - K$ and consider the map $\eta : J_p \to B$ defined by $\eta(\pi) = \varphi_\pi(b)$ for all $\pi \in J_p$. Let $C = \ker(\eta)$. If $C \neq \{0\}$, then J_p/C is a direct sum of a torsion group and a divisible group. Thus, by our assumptions on A, we infer $C = \{0\}$ and η is injective. This shows that $J_p \cong \eta(J_p)$ is a subgroup of B. Since A is torsion-free and reduced, $A = J \oplus C$ with $J \approx J_p$. Now J has a linearly independent subset X of cardinality $2^{\aleph_0}$ and $1 \in X$. Since $\mathbb{Z}(p^\infty)$ is injective, each function $f : X \to \mathbb{Z}(p^\infty)$ extends to a homomorphism $\varphi : J \to \mathbb{Z}(p^\infty)$. On the other hand, any homomorphism $\gamma : J \to A$ is uniquely determined by $\gamma(1)$. This is a contradiction. □

We have a remarkable

Corollary 4.2.5 *If K is a co-local subgroup of the reduced, torsion-free group A, then A/K is reduced.*

The following proposition shows that co-local subgroups are necessarily cotorsion-free in most cases.

Proposition 4.2.6 *Let K be a co-local subgroup of A such that $A/t(A)$ is reduced. Then K is cotorsion-free.*

Proof By Proposition 4.2.2(4) we know that K is torsion-free. Since $\mathbb{Q}$ is injective, K is reduced. Suppose K has a subgroup $J \cong J_p$. Then $A/t(A)$ is p-divisible, since otherwise $Hom(A, J) \neq 0$. But $A/t(A)$ has a subgroup isomorphic to J, which is q-divisible for all primes $q \neq p$. Thus $A/t(A)$ is not reduced. □

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Now we show that certain subgroups of co-local subgroups are co-local again.

Proposition 4.2.7 *Let K_1 be a subgroup of a co-local subgroup K of A such that $Hom(A, K/K_1) = 0$. Then K_1 is a co-local subgroup of A.*

Proof Since K is co-local, $Hom(A, K_1) = 0$. Let $\psi \in Hom(A, A/K_1)$ and $\pi : A/K_1 \to A/K$ the natural epimorphism with $\ker(\pi) = K/K_1$. Let $\psi_1 = \pi \circ \psi$. Since K is co-local, there is some $\varphi \in End(A)$ such that $\psi_1(a) = \varphi(a) + K$ for all $a \in A$. Define $\psi_2 : A \to A/K_1$ by $\psi_2(a) = \varphi(a)+K_1$ for all $a \in A$. Then $(\pi\circ(\psi_2-\psi))(a) = \pi(\psi_2(a))-\pi(\psi(a)) = \varphi(a)+K-\varphi(a)+K = 0$ and thus $(\psi_2 - \psi)(A) \subseteq \ker(\pi) = K/K_1$ and $\psi_2 - \psi \in Hom(A, K/K_1) = 0$. This shows that $\psi = \psi_2$ is induced by φ. □

Corollary 4.2.8 *(1) If K_1 is a direct summand of a co-local subgroup K of A, then K_1 is a co-local subgroup of A.*

(2) If K_1 is a pure quasi-summand of a co-local subgroup K of the torsion-free group A, then K_1 is a co-local subgroup of A.

Proof For (1), note that K/K_1 is isomorphic to a subgroup of K and therefore $Hom(A, K/K_1) = 0$. To show (2), let $nK \subseteq K_1 \oplus K_2 \subseteq K$ for some $n \in \mathbb{N}$ and $\gamma \in Hom(A, K/K_1)$. Then $n\gamma(A) \subseteq (nK + K_1)/K_1 \subseteq (K_1 \oplus K_2)/K_1 \cong K_2$, a subgroup of K. Thus $n\gamma = 0$ and K/K_1 is torsion-free, which implies $\gamma = 0$. □

The additive group J_p of p-adic integers has an abundance of purely indecomposable, pure subgroups, none of which have co-local subgroups, as we will prove next.

Proposition 4.2.9 *Let $A \neq \{0\}$ be a pure subgroup of J_p. Then A has no co-local subgroup.*

Proof Let $a = p^n b \in A$ such that b is a p-adic unit. Then $A/\langle a\rangle \cong A'/\langle p^n\rangle$ with $A \cong A'$ and A has p-rank 1. Thus any epimorphic image A/K of A is a direct sum of at most one cyclic group and copies of $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$'s. Now assume that K is a co-local subgroup of A. By Theorem 4.2.4, either A/K is finite or A contains a copy of J_p, i.e., $p^n J_p \subseteq A$. The first case cannot occur, since A/K finite implies that K contains an isomorphic copy of A and thus $Hom(A, K) \neq 0$. If $p^n J_p \subseteq A$ then $A = J_p$ since A is pure in J_p. Now J_p contains a free subgroup F of rank $2^{\aleph_0}$ and J_p/K is either finite or not reduced. In the first case, K contains a copy of J_p and thus K is not co-local. In the second case, $Hom(J_p, J_p/K)$ has cardinality at least $2^{(2^{\aleph_0})}$ which is bigger than $2^{\aleph_0}$, which is the cardinality of $End(J_p)$. Therefore, K is not co-local. □

The following is now not surprising.

Proposition 4.2.10 *Let K be a co-local subgroup of A. Then*
$E_K(A) = \{\varphi \in End(A) : \varphi(K) \subseteq K\}$ is isomorphic to $End(A/K)$.

Proof Let $\psi_1 : A \to A/K$ be the natural homomorphism and for any $\sigma \in End(A/K)$ define $\psi_\sigma = \sigma \circ \psi_1$. Then there exists a unique $\varphi_\sigma \in End(A)$ such that $\psi_1 \circ \varphi_\sigma = \psi_\sigma$. Since $\ker(\psi_1) = K \subseteq \ker(\psi_\sigma)$, it follows that $\varphi_\sigma \in E_K(A)$ and $E_K(A) \cong End(A/K)$. □

Now we can show another property of co-local subgroups of cotorsion-free groups.

Corollary 4.2.11 *Let K be a co-local subgroup of the torsion-free, reduced group A. Then K is a pure subgroup of A.*

Proof Suppose K is not pure in A. Then $t(A/K)_p \neq \{0\}$ for at least one prime p. Since A is torsion-free and reduced, $t(A/K)_p$ must be reduced by Theorem 4.2.4(2). Therefore, $t(A/K)_p$ has a direct summand C of finite order p^n, which is also a summand of A/K since C is pure-injective. Thus there is some $\psi \in End(A/K)$ of order p^n which implies that $\varphi_\psi \in E_K(A)$ has order p^n. Since A is torsion-free, this implies $\varphi_\psi = 0$ and thus $\psi = 0$, a contradiction. □

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It is now time to give examples of co-local subgroups. To this end, let $S = \{\frac{z}{k} \in \mathbb{Q} : z \in \mathbb{Z}, k \in \mathbb{N}, \gcd(p, k) = 1\}$ be the ring of integers localized at the prime p. Let $F = \oplus_{i=1}^{n} e_i S$ be a free S-module of rank n. Let a_i be p-adic units such that $L = \{a_i : 1 \leq i \leq n\}$ is algebraically independent. Define $\overrightarrow{a} = \sum_{i=1}^{n} e_i a_i$ and $M_{\overrightarrow{a}} = \langle F + \overrightarrow{a}\mathbb{Z}\rangle_*$ a pure submodule of $\widehat{F}$, the p-adic completion of F. This group $M_{\overrightarrow{a}}$ is co-purely indecomposable of rank $n+1$, has p-rank n, and each subgroup of rank $\leq n$ is free. Let E be a proper subset of $N = \{1, 2, ..., n\}$ and $K_E = \oplus_{i \in E} e_i S$.

Claim 4.2.12 *K_E is a co-local subgroup of $M_{\overrightarrow{a}}$.*

Proof Pick $a_i^{(n)} \in S$ such that $a_i \equiv a_i^{(n)} \bmod p^n$ and set $a^{(n)} = \sum_{i=1}^{n} e_i a_i^n$. Then $M_{\overrightarrow{a}} = \langle F \cup \{a_n : n \in \mathbb{N}\}\rangle$. Since L is algebraically independent, it is easy to see that $End(M_{\overrightarrow{a}}) = id_{M_{\overrightarrow{a}}} S$. Define $\pi_E : \widehat{F} \to \widehat{\oplus_{i \in N-E} e_i S}$ to be the natural epimorphism with $\ker(\pi_E) = \widehat{\oplus_{i \in E} e_i S}$. Then $\ker(\pi_E) \cap M_{\overrightarrow{a}} = \oplus_{i \in E} e_i S$ and $\pi_E(M_{\overrightarrow{a}}) \cong \langle (\oplus_{i \in N-E} e_i S) \cup \{\sum_{i \in N-E} e_i a_i\}\rangle$. Easy computations show the rest. $\square$

Now we consider co-local subgroups of torsion-free groups. The next proposition shows that if K is a co-local subgroup of the torsion-free group A, then, w.o.l.o.g., we may assume that A is reduced.

Proposition 4.2.13 *Let K be a co-local subgroup of the torsion-free group $A = D \oplus G$ such that D is divisible and G is reduced. Then $A = D \oplus G'$ for some summand G' of A and K is a co-local subgroup of G'. On the other hand, if K is a co-local subgroup of the torsion-free, reduced group G, then K is a co-local subgroup of $D \oplus G$ for any divisible group D.*

Proof First, assume $D \cap K \neq \{0\}$. Note that K is reduced since $Hom(D, K) = 0$. There exists a subgroup $Q \approx \mathbb{Q}$ of D such that $Q \cap K \neq \{0\}$ and $A = (Q + K) + C$ with $(Q + K) \cap C = K$. Then $Q + K = Q \oplus L$ for some subgroup L of K: Define $\mathcal{F} = \{X : X$ a subgroup of K, such that $Q \cap X = \{0\}$ and X is pure in $Q + K\}$. By Zorn's Lemma, there is a maximal element L in $\mathcal{F}$. Assume $Q \oplus L \subsetneqq Q + K$. Then there is some $k \in K - (Q \oplus L)$. Define $L' = \langle L + k\mathbb{Z}\rangle_*$. Then $Q \cap L' \neq \{0\}$ and there exists $q \in Q, n \in \mathbb{N}, \ell \in L, z \in \mathbb{Z}$ such that $q = \frac{1}{n}(\ell + kz) \in Q \cap L'$. This implies that $\ell = z(\frac{nq}{z} - k) \in z(Q + K) \cap L = zL$ and $k \in Q \oplus L$ follows. This contradiction implies that $Q + K = Q \oplus L$ with $L \subseteq K$ is reduced. Now consider $\psi \in Hom(A, A/K)$ with $\psi(C) = \{0\}$ and $\psi(Q) \subseteq (Q + K)/K \approx Q/(Q \cap K)$, a non-trivial divisible torsion group. Since K is co-local, there exists a (unique) $\varphi \in End(A)$ such that φ induces ψ. Then $\varphi(C) \subseteq K$ and $\varphi(Q) \subseteq Q + K = Q \oplus L$ with L reduced. This implies that $\varphi(Q) \subseteq Q$. Now there are uncountably many of the ψ's, but only countably many of the φ's. This contradiction shows that $D \cap K = \{0\}$.

Each $k \in K$ can be written uniquely as $k = d_k + g_k$ with $d_k \in D$ and $g_k \in G$ and $k \mapsto g_k$ is an injective homomorphism. Define $\gamma : \{g_k : k \in K\} \to D$ by $\gamma(g_k) = d_k$. Since D is injective, there is some $\gamma' : G \to D$ extending γ. Then $G' = \{\gamma'(g) + g : g \in G\}$ is a complement of D in A with $K \subseteq G'$. It follows easily from the definition that K is a co-local subgroup of G'.

The last statement follows since G/K is reduced by Corollary 4.2.5. $\square$

We collect some of the results in this section in the following:

Theorem 4.2.14 *Let K be a co-local subgroup of the torsion-free group A. Then*

(1) K is cotorsion-free.

(2) If A is reduced, then K is pure and A/K is reduced.

Proof (1) follows from Corollary 4.2.3 and 4.2.5, while (2) follows from Proposition 4.2.2 and 4.2.6. $\square$

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4.3 Cotorsion-free Groups as Co-local Subgroups

The goal of this section is that any cotorsion-free group is isomorphic to a co-local subgroup of some cotorsion-free group. We will utilize a slightly modified Black Box together with a standard Black Box. To this end, we introduce the following notation. We closely follow the presentation of the Strong Black Box in [6].

Notation 4.3.1 Let R be a commutative ring with 1 and $\mathbb{S} = \{s_i : i < \omega\}$ a countable multiplicativly closed subset of R such that $1 \in \mathbb{S}$ is the only unit in $\mathbb{S}$. We assume that R is torsion-free and cotorsion-free with respect to $\mathbb{S}$, i.e., $\cap_{s\in\mathbb{S}} sR = \{0\}$ and $Hom(\widehat{R}, R) = 0$, where $\widehat{R}$ is the completion of R in the $\mathbb{S}$-adic topology. We fix $q_n = s_1 s_2 ... s_n$ for all $n < \omega$. If A is an $\mathbb{S}$-pure submodule of the R-module M, i.e., $sM \cap A = sA$ for all $s \in S$, we write $A \subseteq_* M$. Moreover, we write $A \sqsubseteq M$ if A is a direct summand of M.

Moreover, we fix infinite cardinals κ, μ, λ such that $|R| \leq \kappa$, $\mu^\kappa = \mu$, and $\lambda = \mu^+$ is the successor cardinal of μ. Now pick a free R-module $B_0 = \oplus_{0\leq\alpha<\lambda} \mathrm{Re}_\alpha$. We fix a cotorsion free module K such that $|K| \leq \kappa$ and define $B = Ke_{-1} \oplus B_0$ where e_{-1} is just a place holder for the elements of K. If $g \in \widehat{B}$, then there is a countable subset I of λ such that $g = e_{-1}k + \sum_{i\in I} r_i e_i$ and the sequence $\{r_i\}_{i\in I}$ is an $\mathbb{S}$-adic zero-sequence of elements of R, i.e., for all $n \in \mathbb{N}$ one has that $r_i \in q_n R$ for all but finitely many $i \in I$. We define the support of g to be the set $[g] = \{i \in I : r_i \neq 0\}$ and the norm of g is the ordinal $\|g\| = \sup\{i + 1 : i \in [g]\}$.

We define a natural map $\pi_0 : \widehat{B} \to \widehat{B}$ by $\pi_0(e_{-1}) = 0$ and $\pi_\kappa(e_\alpha) = e_\alpha$ for all $\kappa \leq \alpha < \lambda$, i.e., π_0 is the natural projection of $\widehat{B}$ onto $\widehat{B_0}$ with kernel $\widehat{Ke_{-1}}$. In regard to the combinatorics of the Black Box, the copy of K just plays the role of another copy of R. We will usually identify K and Ke_{-1}.

Our goal is to construct a module G such that $K \subseteq_* G \subseteq_* \widehat{B}$ such that $Hom(G, K) = 0$ and $Hom(G, G/K) = R\sigma_K$ where $\sigma_K(x) = x + K$ for all $x \in G$. First we prove the crucial Step Lemma that is at the core of any Black Box construction.

Lemma 4.3.2 ***(Step Lemma).*** *Let $P = Ke_{-1} \oplus_{\alpha\in I^*} \mathrm{Re}_\alpha$ for some subset I^* of λ with $\kappa \subset I^*$and $I = \{\alpha_n : n < \omega\} \subset I^*$ a sequence of ordinals such that $\kappa < \alpha_n < \alpha_{n+1}$ for all $n < \omega$. Let M be a module such that*

(1) $P \subseteq_* M \subseteq_* \widehat{B}$ and $\pi_0(M) \subseteq_* \widehat{B_0}$.

(2) M and $\pi_\kappa(M)$ are cotorsion-free.

(3) $M \cap \widehat{Ke_{-1}} = Ke_{-1}$.

(4) The set $I \cap [g]$ is finite for all $g \in M$.

Let $\varphi \in Hom(B, M)$ such that $\varphi \notin R\pi_\kappa \restriction_B$. Then there exists an element $y \in \widehat{B}$ such that for $M' = \langle M + Ry\rangle_*$ the following hold:

(1') $P \subseteq_* M' \subseteq_* \widehat{B}$ and $\pi_0(M') \subseteq_* \widehat{B_0}$.

(2') M' and $\pi_0(M')$ are cotorsion-free.

(3') $M' \cap \widehat{Ke_{-1}} = Ke_{-1}$.

(4') $\varphi(y) \notin \pi_0(M')$.

(Let $x = \sum_{n<\omega} q_n e_{\alpha_n}$. Then y can be chosen such that $y = x$ or $y = x + \rho b$ for some $\rho \in \widehat{R}$ and $b \in P$.)

Proof For any y as above, we define $M' = \langle M + Ry\rangle_*$, a pure submodule of $\widehat{B}$. First we show that M' satisfies (1'), (2'), and (3'). Note that $[\pi_0(g)] \subseteq [g]$ for all $g \in \widehat{B}$ and $[x] = [\pi_0(rx)] = I$ for all $0 \neq r \in R$ and $[b]$ is finite for all $b \in P$. This implies that M is pure in M' and $M' \cap \widehat{K} = M \cap \widehat{K} = K$. To show that $\pi_0(M')$ is pure in $\pi_0(\widehat{B}) = \widehat{B_0}$, let $g \in \widehat{B_0}$ and $q_n\pi_0(g) = \frac{1}{q_\ell}\pi_0(m + ry) \in \pi_\kappa(M')$ with $m \in M$ and $r \in R$. Then $q_\ell q_n \pi_0(g) = \pi_0(m + ry)$ and by a support argument using

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(4) it follows that $r = q_\ell q_n \widetilde{r}$ for some $\widetilde{r} \in R$. This implies that $\widetilde{m} = \frac{1}{q_\ell} m \in M$. This shows $q_n\pi_0(g) = \pi_0(\widetilde{m} + q_n\widetilde{r}y)$ and thus $q_n\pi_0(g - \widetilde{r}y) = \pi_0(m) \in \pi_0(M) \cap q_n\widehat{B_0} = q_n\pi_0(M)$. It follows that $\pi_0(g - \widetilde{r}y) \in \pi_0(M)$ and $\pi_0(g) = \pi_0(m' + \widetilde{r}y)$ for some $m' \in M$, i.e. $\pi_0(g) \in \pi_0(M')$.

Next we show that $\pi_0(M')$ is cotorsion-free. Let $\psi \in Hom(\widehat{R}, M')$ and we may assume that $\psi(1) = m + ry \in M + Ry$. For any $\rho \in \widehat{R}$ we have that $\psi(\rho) = \rho m + \rho r y$ and because of (4) there is some n_0 such that $\psi(\rho) \restriction_{e_{\alpha_n}} = q_n \rho r e_n$ for all $n \geq n_0$. This implies that $\widehat{R}r \subseteq R$ which implies that $r = 0$ since R is cotorsion-free. A similar argument shows that M' is cotorsion-free as well. Now we will show (4'). First we need to show: If $\varphi \in Hom(P, M) - R\pi_0$ then $q_n\varphi \notin R\pi_0$ for all $n < \omega$ as well. By way of contradiction, assume $q_n\varphi = r\pi_0$ for some $r \in R$. Then $q_n\varphi(e_i) = r\pi_0(e_i) = re_i \in q_n\widehat{B} \cap \mathrm{R}e_i = q_n \mathrm{R}e_i$. This implies that $r = q_n\widetilde{r}$ for some $\widetilde{r} \in R$ and it follows that $\varphi = \widetilde{r}\pi_0$. Now let x be as defined above and assume $\varphi(x) \in \pi_0(M') = \pi_0(\langle M + Rx\rangle_*)$. This means that there are $n < \omega$, $m \in M$, $r \in R$ such that

(*) $q_n\varphi(x) = \pi_0(m + rx) = \pi_0(m) + rx$.

Since $q_n\varphi \neq r\pi_0$, there is some $b \in P$ such that $q_n\varphi(b) - \pi_0(b) \neq 0$. Since M is cotorsion-free, there is some $\rho \in \widehat{R}$ such that $\rho(q_n\varphi(b) - \pi_0(b)) \notin M$. Now define $y = x + \rho b$ and assume that for $M'' = \langle M + Ry\rangle_*$ we have that $\varphi(y) \in M''$. This implies that there is some $\ell < \omega$, $m' \in M$, $r' \in R$ such that

(**) $q_\ell\varphi(x + \rho b) = q_\ell\varphi(y) = \pi_0(m' + r'y) = \pi_0(m' + r'x + r'\rho b)$.

Now we multiply (*) by q_ℓ and (**) by q_n and subtract the two equations. We obtain $q_nq_\ell\varphi(\rho b) = \pi_0(q_nm' - q_\ell m) + (q_nr' - q_\ell r)x + \pi_0(q_nr'\rho b)$, recalling that $x = \pi_0(x)$. Another support argument using (4) shows that $q_nr' - q_\ell r = 0$ and we obtain $q_nq_\ell\varphi(\rho b) = \pi_0(q_nm' - q_\ell m) + \pi_0(q_nr'\rho b)$ and $q_\ell(q_n\varphi(\rho b) - \pi_0(r\rho b)) \in \pi_0(M)$. Since $\pi_0(M)$ is pure in $\widehat{B_0}$, we infer $\rho(q_n\varphi(b) - \pi_0(rb)) \in \pi_0(M)$, a contradiction to the choice of b. □

We will utilize the Strong Black Box as presented in [6] and adhere to the notations as defined above. Moreover, λ^0 denotes the set of all ordinals in λ of countable cofinality. The following is a modified version of the Strong Black Box as presented in [6]. We just combine two black boxes over two disjoint stationary sets.

Theorem 4.3.3 *[6]. With the notations as above, let $E \subset \lambda^o$ be a stationary subset of λ such that $\lambda^o - E$ is stationary as well. Let $E = E^{(0)} \cup E^{(1)}$ be a disjoint union of two stationary subsets of λ. Then there exists a family $\{\varphi_\beta\}_{\beta<\lambda}$ such that:*

(1) $\|\varphi_\beta\| \in E$ for all $\beta < \lambda$.

(2) $\|\varphi_\gamma\| \leq \|\varphi_\beta\|$ for all $\gamma \leq \beta < \lambda$.

(3) $\|[\varphi_\gamma] \cap [\varphi_\beta]\| < \|\varphi_\beta\|$ for all $\gamma < \beta < \lambda$. (Recall that $[\varphi] = [dom(\varphi)]$.)

(4) PREDICTION: For any homomorphism $\psi : B \to \widehat{B}$ and for any subset I of λ with $|I| \leq \kappa$ the set $\{\alpha \in E^{(\varepsilon)} : \exists\beta < \lambda$ such that $\|\varphi_\beta\| = \alpha, \psi \restriction_{dom(\varphi)} = \varphi$, and $I \subseteq [\varphi]$, and $K_{e_{-1}} \subset dom(\varphi_\beta)\}$ is stationary for $\varepsilon = 0, 1$. (We will always assume that $Ke_{-1} \subset dom(\varphi_\beta)$.)

We are now ready for the construction of G and begin by setting $G^0 = B = Ke_{-1} \oplus B_0$ which is obviously cotorsion-free. We partition each stationary set $E_{I,\psi}$ into disjoint stationary subsets $E^{(\varepsilon)}_{I,\psi}$, for $\varepsilon = 0, 1$, and define $E^{(\varepsilon)} = \cup E^{(\varepsilon)}_{I,\psi}$. We will define a smooth chain $\{G^\beta\}_{\beta<\lambda}$ of pure R-submodules of $\widehat{B}$ and for $P_\gamma = \oplus_{\alpha\in[\varphi_\gamma]} e_\alpha R$, $\varphi_\gamma : P_\gamma \to \widehat{P_\gamma}$, we will define elements $y_\gamma \in \widehat{P_\gamma}$ such that, for all $\gamma < \beta < \lambda$, we have

(a) $\|y_\gamma\| = \|P_\gamma\|$ $(= \|\beta_\gamma\|)$,

(b) $G^\beta = \left\langle B + \sum_{\gamma<\beta} y_\gamma R\right\rangle_*$, and

(c) G^β is cotorsion-free, $K = \widehat{K} \cap G^\beta$, and G^β/K is cotorsion-free.

Let $\beta < \lambda$ be a limit ordinal and assume that all G^γ, $\gamma < \beta$, satisfies (a), (b), and (c). Just as in [6], it follows that $G^\beta = \cup_{\gamma<\beta} G^\gamma$ satisfies the same conditions.

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Now assume that G^β has been defined and consider φ_β. We may assume that $B_\kappa \subset dom(\varphi_\beta)$. Since $\|\varphi_\beta\| \in E \subset \lambda^o$ and $dom(\varphi_\beta)$ is canonical, there are $\alpha_n \in dom(\varphi_\beta)$ such that $\alpha_n < \alpha_{n+1}$ for all $n < \omega$ and $\|\varphi_\beta\| = \sup\{\alpha_n : n < \omega\}$. Define $I = \{\alpha_n : n < \omega\}$. Then, for any $g \in G^\beta$, $\|I \cap [g]\| < \|\varphi_\beta\|$ and thus $I \cap [g]$ is finite by (a), (b) and clause (3) in the Black Box. We differentiate several cases:

Case 1: $\|\varphi_\beta\| \in E^{(0)}$ and $\varphi_\beta : P_\beta \to \widehat{P_\beta}$ satisfies $\pi_0 \circ \varphi_\beta = \varphi_\beta$, i.e., $image(\varphi_\beta) \subseteq \widehat{B_0} \supseteq \pi_0(\widehat{B})$.

Case 1.1: $image(\varphi_\beta) \subseteq \pi_0(G^\beta)$ and $\varphi_\beta \notin R\pi_0$.

In this case we apply our Step lemma with G^β in place of M. Then we get an element $y = y_\beta \in \widehat{P_\beta}$ such that for $G^{\beta+1} = M' = \langle G^\beta + Ry_\beta \rangle_*$ we have that $\varphi_\beta(y_\beta) \notin \pi_0(G^{\beta+1})$. Moreover, $y_\beta = \sum_{n<\omega} q_n e_{\alpha_n}$ or $y_\beta = \rho b + \sum_{n<\omega} q_n e_{\alpha_n}$ for some $b \in B$. The step Lemma ensures that $G^{\beta+1}$ has the desired properties.

Case 1.2: $image(\varphi_\beta) \nsubseteq \pi_0(G^\beta)$ or $\varphi_\beta \in R\pi_0$.

Here we set $y_\beta = \sum_{n<\omega} q_n e_{\alpha_n}$ and $G^{\beta+1} = \langle G^\beta + Ry_\beta \rangle_* \subset \widehat{B}$.

Case 2: $\|\varphi_\beta\| \in E^{(1)}$.

Now we simplify our Step Lemma by redoing it with $id_{\widehat{B}}$ in place of π_0.

Case 2.1: $image(\varphi_\beta) \subseteq G^\beta$ and $\varphi_\beta \notin R \cdot id_{dom(\varphi_\beta)}$.

Here we proceed as in Case 1.1, apply our Step Lemma, and find y_β such that for $G^{\beta+1} = \langle G^\beta + Ry_\beta \rangle_*$ we have that $\varphi_\beta(y_\beta) \notin G^{\beta+1}$.

Case2.2: $image(\varphi_\beta) \nsubseteq G^\beta$ or $\varphi_\beta \in R \cdot id_{dom(\varphi_\beta)}$. Do the same as in Case 1.2. Then our chain of G^β 's satisfies (a), (b), and (c).

Now we set $G = \cup_{\beta<\lambda} G^\beta$.

As in [6, Lemma 1.2.4] we have:

Lemma 4.3.4 *(a) $B \oplus \oplus_{\beta<\lambda} Ry_\beta$ is a direct sum and $B_0 \oplus \oplus_{\beta<\lambda} Ry_\beta$ is a free R-module.*
(b) If $g \in G - B$, then there are a finite subset N of λ and $k < \omega$ such that $q_k g \in B \oplus \oplus_{\beta \in N} Ry_\beta$ and $[g] \cap [y_\beta]$ is infinite iff $\beta \in N$. Moreover, if $\|g\|$ is a limit ordinal, then $\|g\| = \|y_{\max(N)}\|$.

Again we quote a result from [6, Lemma 1.2.5].

Lemma 4.3.5 *Let G be defined as above and define $G_\alpha = \{g \in G : \|g\| < \alpha\}$. Then:*
(a) $G \cap \widehat{P_\beta} \subseteq G^{\beta+1}$ for all $\beta < \lambda$.
(b) $\{G_\alpha : \alpha < \lambda\}$ is a λ-filtration of G.
(c) If $\beta < \lambda, \alpha < \lambda$ are ordinals such that $\|\varphi_\beta\| = \alpha$, then $G_\alpha \subseteq G^\beta$.

Now we show that $Hom(G, \pi_0(G)) = R\pi_0$:

Note that $\pi_0(G) \approx G/K$ since $\ker(\pi_0) \cap G = G \cap e_{-1}\widehat{K} = e_{-1}K = K$ by our construction.

Now consider $\psi \in Hom(G, \pi_0(G)) - R\pi_0$. Let $\psi' = \psi_B$. Then $\psi' \notin R\pi_0$ since ψ is uniquely determined by ψ'. Let $I = \{\alpha_n : n < \omega\} \subset \lambda$ be a strictly increasing sequence of ordinals such that $\alpha^* = \sup(I) \notin E$. Then $I \cap [g]$ is finite for all $g \in G$. By our step Lemma, there exists an element $y \in \widehat{B}$ such that $\varphi(y) \notin \pi_0(G')$ where $G' = \langle G + Ry \rangle_*$. By our Black Box we have that the set $E^{(0)\prime} = \{\alpha \in E^{(0)} : \exists \beta < \lambda$ such that $\|\varphi_\beta\| = \alpha, \varphi_\beta = \psi' \restriction_{dom(\varphi_\beta)}, y \in dom(\varphi_\beta), I \subseteq [\varphi_\beta]\}$ is stationary. Note that $y \in \widehat{P_\beta}$. Moreover $C = \{\alpha < \lambda : \psi(G_\alpha) \subseteq \pi_0(G_\alpha)\}$ is a cub because $\{G_\alpha\}_{\alpha<\lambda}$ and $\{\pi_0(G_\alpha)\}_{\alpha<\lambda}$ are λ-filtrations of G and $\pi_0(G)$. Now let $\alpha \in C \cap E^{(0)\prime}$. Then $\psi(G_\alpha) \subseteq \pi_0(G_\alpha)$ and there exists an ordinal $\beta < \lambda$ such that $\|\varphi_\beta\| = \alpha$, $y \in \widehat{P_\beta}$, and $I \subseteq [\varphi_\beta]$. This implies that $G_\alpha \subseteq G^\beta$ and $\varphi_\beta \notin R\pi_0$.

Moreover, $P_\beta \subseteq B$ with $\|P_\beta\| = \alpha$ and thus $P_\beta \subseteq G_\alpha \subseteq G^\beta$ and $\psi(P_\beta) \subseteq \pi_0(G^\beta)$. This implies that $\varphi_\beta : P_\beta \to \pi_0(G^\beta)$ with $\varphi_\beta \notin R\pi_0$. Thus, by our construction, $\varphi_\beta(y_\beta) \notin \pi_0(G^{\beta+1})$. On the other hand, $\varphi_\beta(y_\beta) = \psi(y_\beta) \in G \cap \widehat{P_\beta} \subseteq G^{\beta+1}$. This contradiction shows that $Hom(G, \pi_0(G)) = R\pi_0$. A very similar argument, working with the stationary set $E^{(1)}$, will show that $End(G) = Rid_G$, which, of course, implies that $Hom(G, K) = 0$. We have proved the following:

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Theorem 4.3.6 *Let K be a cotorsion-free R-module. Then there exists some cotorsion-free R-module G, such that K is a submodule of G such that $Hom(G, K) = 0$ and $Hom(G, G/K) = R\pi$, where $\pi : G \to G/K$ is the natural map with $\pi(g) = g + K$ for all $g \in G$.*

Let $R = \mathbb{Z}$ with $\mathbb{S} = \mathbb{N}$. Then we have the following:

Corollary 4.3.7 *Let K be a cotorsion-free abelian group. Then there are arbitrarily large cotorsion-free groups G such that K is isomorphic to a co-local subgroup of G.*

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Chapter 5

Partition Bases and $B^{(1)}$- Groups

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Abstract $B^{(1)}$-groups are a subclass of the class of Butler groups, the torsionfree quotients of completely decomposable groups. We study the partition structure associated to a $B^{(1)}$-group G, a lattice-theoretical feature that is behind the direct sum decompositions of G. We determine some of its properties, and give a contribution to the solution of an open problem, that of deciding when the direct sum of two $B^{(1)}$-groups is a $B^{(1)}$-group.

Keywords: Butler group, $B^{(1)}$-group, tent, partition lattice, finite algorithm. A.M.S. Classification: 20K15, 06F99, 06B99.

5.1 Introduction

A $B^{(1)}$-group of rank $m-1$ is a torsionfree Abelian group that is the sum of m rank 1 groups. The class of regular $B^{(1)}$-groups has been amply studied (for history, see [3], [4]), using, as a basic equivalence, quasi-isomorphism [1] instead of isomorphism; this is also what we do in this paper. One of the many reasons for the quasi-isomorphism choice is that it allows us to study the principal properties of a $B^{(1)}$-group G of rank $m-1$ via its "tent", an m-tuple of types (the isomorphy classes of the rank 1 groups that sum up to G), and hence end up in a lattice-theoretical setting. There is, though, another lattice-theoretical context where crucial information can be gathered, namely that of partitions of the set $I = \{1, ..., m\}$; this became clearer the more the theory advanced, and it comes from attaching to G an m-tuple of partitions called a "partition base"; from it, for instance,

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derives the decomposition of G into strongly indecomposable summands [5]. A tent is essentially an $m \times n$ $(0, \infty)$-table, whose rows are the types; the partition point of view derives information mostly from the columns of the tent, called "primes"; this points to a different kind of ordering for $B^{(1)}$-groups, where different properties depend on increasing or diminishing the number of primes (columns) in a tent (the rows determine the rank).

The first task we undertake here is to determine which m-tuples of partitions are partition bases of a $B^{(1)}$-group (Section 5.3). We proceed to prove that every direct summand of G is (up to quasi-isomorphusm) a "partition subgroup" $G(\mathcal{C})$, determined by a partition $\mathcal{C}$ (Section 5.4). In Section 5.5, given two partitions $\mathcal{C}$, $\mathcal{D}$ that are candidates to yield complementary direct summands of a $B^{(1)}$-group, we determine the most general $B^{(1)}$-group (called domain of the couple $(\mathcal{C}, \mathcal{D})$) that decomposes in the form $G(\mathcal{C}) \oplus G(\mathcal{D})$. Finally in Section 5.6 we show that, if one of the two summands is indecomposable, then the two partitions involved have a special shape; this allows a contribution to the solution of an open problem, that of deciding when the direct sum of two $B^{(1)}$-groups is a $B^{(1)}$-group (a problem solved only for the direct sums of two indecomposables [8]).

As is usual in this field, a complicated theory produces easy practical realizations; Section 5.7 should give an instance of this.

Throughout, "group" will stand for "torsionfree abelian group of finite rank". We will adopt quasi-isomorphism as our basic equivalence, hence write "isomorphic" for quasi-isomorphic, "direct summand" for quasi-direct summand; "indecomposable" for strongly indecomposable, etc. In this setting a Krull-Schmidt type result [4] ensures that direct decompositions into indecomposables are unique up to isomorphism.

5.2 Preliminaries

Let $I = \{1, ..., m\}$. If $E \subseteq I$, we will sometimes denote by E^{-1} the complement $I \backslash E$.

$\mathbb{P}(m)(\wedge, \vee)$ will denote the lattice of partitions of I under the ordering "greater = coarser". For $E \subseteq I$ and $i \in I$, set

$p_E = \{I \backslash E, \{i\} \mid i \in E\}$, the *pointed partition* pointed on E,
$b_E = \{E, I \backslash E\}$, the *bipartition* on E,
$p_i = \{\{i\}, I \backslash \{i\}\}$, the pointed partition pointed on $\{i\}$.

$\mathbb{B}(m)$ will denote the set of all bipartitions of I. Setting $b_E + b_F = b_{E+F}$ (where $E + F$ is the usual Boolean sum of subset of I), $\mathbb{B}(m)(+)$ is a $\mathbb{Z}_2$-vector space of dimension $m - 1$.

$\mathbb{T}(\wedge, \vee)$ will denote the lattice of all types ($=$ isomorphy classes of rank 1 groups) with an added maximum ∞ for the type of the 0 group.

Let $(t_1, ..., t_m)$ be a *regular* m-tuple of types, that is $t_j \geq \wedge\{t_i \mid i \neq j \in I\}$. For each $E \subseteq I$, we set:

$$\tau(E) = \wedge\{t_i \mid i \in E\}$$
$$t_E = \tau(E) \vee \tau(I \backslash E).$$

In the following, G will denote a $B^{(1)}$-*group* of rank $m - 1$, that is a torsionfree Abelian group that is the sum of m rank one (pure) subgroups: $G = \langle g_1 \rangle_* + \cdots + \langle g_m \rangle_*$ for suitable elements $g_i \in G$. Throughout, we will assume G *regular*, that is the only relation between these elements will be $g_1 + \cdots + g_m = 0$. (For nonregular $B^{(1)}$- groups see [8, Sec. 1].) The m-tuple $(g_1, \ldots, g_m)$ is called a *base* of G.

Denote by $t_G(g)$ the type of g in G; set *typeset*$(G) = \{t_G(g) \mid g \in G\}$, and $t_i = t_G(g_i)$ for all $i \in I$. Then the m-tuple $(t_1, \ldots, t_m)$ is regular and is called a *type-base* of G. In our setting *there*

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is no loss of generality in supposing that the types of G consist only of zeros and a finite number of infinities, thus a type-base is described by an $m \times n$ table of 0s and ∞s for some n.

Let $g \in G$, $g = r_1 g_1 + \cdots + r_m g_m$ with $r_1, \ldots, r_m \in \mathbb{Q}$. If $\mathcal{C} = \{C_1, \ldots C_k\} = \mathrm{part}_G(g)$ is the partition of I into equal-coefficient blocks [5], then $t_G(g) = t_{C_1} \wedge \cdots \wedge t_{C_k}$ [2, Sec. 2]. Typeset(G) can then be obtained as the image of the map $t : \mathbb{P}(m) \to \mathbb{T}$, defined by

$$t(\mathcal{C}) = t(\{C_1, \ldots, C_k\}) = t_{C_1} \wedge \cdots \wedge t_{C_k} = \tau(I \backslash C_1) \vee \cdots \vee \tau(I \backslash C_k).$$

t is called a *tent* of base $(t_1, \ldots, t_m)$. A tent is an $\wedge$-morphism, hence typeset$(G) = Im(t)$ is a sub-$\wedge$-semilattice of $\mathbb{T}$ [6].

Observation 5.2.1 *[5, 0.b]. $t(\mathcal{C})$ remains the same if the infimum is taken over all but one of the terms (not so for the supremum).*

Then we have $t(p_i) = t_i, t(b_E) = t_E, t(p_E) = \tau(E)$.

Clearly, there is a one-to-one correspondence between type-bases and tents identified up to permutations of the base types. We will say that a tent *represents* a $B^{(1)}$-group if its base is a type-base of the group.

A tent is not necessarily an injective map. For each $\sigma \in Im(t)$, the *minimum partition* $\mathcal{C}$ such that $t(\mathcal{C}) = \sigma$ it is denoted by $part_t(\sigma)$. The m-tuple $(part_t(t_1), \ldots, part_t(t_m))$ that plays a key part in the decompositions of G is called a *partition base* of G (of t) [5].

Minimum partitions of $\vee$-irreducible elements of $Im(t)$ are pointed partitions [7, Lemma l5.3]. The subset A of I is called a *prime* of t if either $A = \emptyset$, or p_A is the minimum partition of a $\vee$-irreducibile element of $Im(t)$. As has been shown in [5], the primes are just the supports of the columns of the table of 0s and ∞s whose rows are the base types. For instance, in the tent

$$\begin{array}{l} t_1 = \infty\ \ 0\ \ 0\ \ 0 \\ t_2 = \infty\ \infty\ \ 0\ \ 0 \\ t_3 = \infty\ \ 0\ \infty\ \ 0 \\ t_4 = \infty\ \ 0\ \ 0\ \infty \end{array}$$

the primes are $A_1 = \{1, 2, 3, 4\}$, $A_i = \{i\}$ for $i = 2, 3, 4$.

In general, adding primes favors indecomposability: more primes means more zeros, hence more connections, thus bigger blocks in the partition base, that is less splittings [5]. E.g., a tent with only one prime defines a completely decomposable group. By regularity of $(t_1, \ldots, t_m)$, for any prime A we have $|A| \neq m - 1$ [6]. $P(t)$ will denote the set of all primes of t; then the set of $\vee$-irreducible elements of $Im(t)$ is $\{\tau(A) \mid A \in P(t)\}$; in this set, $\tau(A) \leq \tau(B)$ implies $A \supseteq B$ [7].

Lemma 5.2.2 *[7, Lemma 1.8]. For each $\sigma \in Im(t)$,*

$$part_t(\sigma) = \vee\{p_A \mid A \in P(t), \tau(A) \leq \sigma\}.$$

Moreover, for each $i \in I$,

$$part_t(t_i) = \vee\{p_A \mid A \in P(t) \text{ and } i \in A\} \leq p_i.$$

If $P = \{A_1, \ldots, A_n\}$ is a set of subsets (of cardinality $\neq m - 1$) of I, a tent t having P as a set of primes consists of n columns of 0s and ∞s with supports A_1, resp. $A_2, \ldots,$ resp. A_n. Considering a tent as a set of primes (columns) rather than as an m-tuple of types (rows) yields a different ordering (wider / narrower) for tents; in particular, if $P(t') \subseteq P(t)$, we will call t' a *primes-sub-tent* of t.

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5.3 Partition Bases

In this Section we characterize m-tuples of partitions that arise as partition bases for tents.

Definition 5.3.1 An m-tuple $(\mathcal{C}_1, \ldots, \mathcal{C}_m)$ of partitions of I is called *almost-distributive* if $\mathcal{C}_i \leq p_i$ for each $i \in I$, and, for each $i \neq j \in I$,

(#) a block of $\mathcal{C}_i$ not containing j is either equal to a block of $\mathcal{C}_j$ or is contained in the block of $\mathcal{C}_j$ containing i. □

Visualizing:

$$\begin{aligned} \mathcal{C}_i &= \{\{i\}, C_j, Y_1, \ldots, Y_s, Z_1, \ldots, Z_t\} \\ \mathcal{C}_j &= \{\{j\}, C_i, H_1, \ldots, H_q, Z_1, \ldots, Z_t\} \end{aligned}$$

where $C_j \supseteq \{j\} \cup H_1 \cup \cdots \cup H_q$ and $C_i \supseteq \{i\} \cup Y_1 \cup \cdots \cup Y_s$.

Building by hand an almost-distributive m-tuple of partitions becomes quickly very complicated, unless most of the partitions are p_i's.

The following proposition shows that this property is necessary for m-tuples of partitions that are partition bases of tents.

Proposition 5.3.2 *If t is a tent with base $(t_1, \ldots, t_m)$ then the m-tuple $(\mathrm{part}_t(t_1), \ldots\ \mathrm{part}_t(t_m))$ is almost-distributive.*

Proof $\mathrm{Part}_t(t_i) \leq p_i$ for each $i \in I$; moreover, by Lemma 5.2.2, $\mathrm{part}_t(t_i) = \vee\{p_A \mid A \in P(t) \text{ and } i \in A\}$.

For fixed $i \neq j \in I$, setting

$$\begin{aligned} H &= \cap\{A \in P(t) \mid i \in A \text{ and } j \notin A\} \\ K &= \cap\{A \in P(t) \mid j \in A \text{ and } i \notin A\}, \end{aligned}$$

we get

$$\begin{aligned} p_H &= \vee\{p_A \mid A \in P(t) \text{ and } i \in A \text{ and } j \notin A\} \\ p_K &= \vee\{p_A \mid A \in P(t) \text{ and } j \in A \text{ and } i \notin A\}. \end{aligned}$$

Therefore, setting $\mathcal{C} = \vee\{p_A \mid A \in P(t) \text{ and } i, j \in A\}$ we have

$$\begin{aligned} \mathrm{part}_t(t_i) &= \mathcal{C} \vee p_H \\ \mathrm{part}_t(t_j) &= \mathcal{C} \vee p_K; \end{aligned}$$

hence each block of $\mathrm{part}_t(t_i)$ not containing j is a block of $\mathcal{C}$, thus is either equal to a block of $\mathrm{part}_t(t_j)$ or is contained in the block of $\mathrm{part}_t(t_j)$ containing $I \backslash K$. But $i \in I \backslash K$, therefore condition (#) holds. □

Theorem 5.3.3 *The almost-distributive property is necessary and sufficient for an m-tuple of partitions to be the partition base of a tent.*

Proof We are only left with proving sufficiency. Let then $(\mathcal{C}_1, \ldots, \mathcal{C}_m)$ be an almost-distributive m-tuple of partitions of I, and set $P = \{A \subseteq I \mid p_A \leq \mathcal{C}_i\ \forall i \in A\}$ (see Section 5 b) for an algorithmic procedure). Then, if t is the tent having $P = P(t)$ as its set of primes, we have $\mathrm{part}_t(t_i) = \mathcal{C}_i$ for each $i \in I$. The definition of P ensures that $\mathrm{part}_t(t_i) \leq \mathcal{C}_i$ for all $i \in I$. Let $C_{i1}, \ldots C_{is_i}$ be the blocks of $\mathcal{C}_i$ different from $\{i\}$. Since $(\mathcal{C}_1, \ldots, \mathcal{C}_m)$ is almost-distributive, it is easy to prove that $C_{i1}^{-1}, \ldots, C_{is_i}^{-1}$ belong to P. Thus $\mathrm{part}_t(t_i) = \vee\{p_A \mid i \in A \text{ and } A \in P\} \geq p_{C_{i_1}^{-1}} \vee \cdots \vee p_{C_{i_k}^{-1}} = \mathcal{C}_i$, therefore $\mathrm{part}_t(t_i) = \mathcal{C}_i$. □

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If $\mathcal{P}$ is an m-tuple of partitions satisfying almost-distributivity, the tent t consisting of all primes compatible with $\mathcal{P}$ is called the *domain* of $\mathcal{P}$, $t = dom(\mathcal{P})$; we may also write $G = dom(\mathcal{P})$ for the $B^{(1)}$-group G with tent t. Which relation ties $G = \mathrm{dom}(\mathcal{P})$ with other groups H having $\mathcal{P}$ as a partition base? If H has tent t' of base $(t'_1, \ldots, t'_m)$ such that $\mathrm{part}_t(t'_i) \leq \mathcal{C}_i$ for each $i \in I$, then the primes of t' are primes of t as well, that is t' is a primes-sub-tent of t. In the tent t of G label each column with a different prime number. Let R be the subgroup of $\mathbb{Q}$ with type ∞ on all primes of t but not of t'. Then $G \otimes R$ has the same tent as H.

5.4 Direct Summands

In the $B^{(1)}$-group G, for $\mathcal{C} = \{C_1, \ldots, C_k\} \in \mathbb{P}(m)$ define the *partition subgroup*

$$G(\mathcal{C}) = \langle g_{C_1} \rangle_* + \cdots + \langle g_{C_k} \rangle_*, \text{ and set}$$
$$G_E = G(p_E).$$

$G(\mathcal{C})$ is clearly a $B^{(1)}$-group. In [5] it has been proved that all indecomposable direct summands of G are (up to ismorphism) partition subgroups. In this Section we show that this holds in general.

If $\mathcal{D} = \{D_1, \ldots, D_s\}$ and $\mathcal{C} = \{C_1, \ldots, C_k\}$ are partitions of I such that $\mathcal{D} \geq \mathcal{C}$, then each block of $\mathcal{D}$ is a union of blocks of $\mathcal{C}$, thus $\mathcal{D}$ can be regarded as a partition of $\mathcal{C}$. Then, setting $D^*_j = \{i \in \{1, \ldots, k\} \mid C_i \subseteq D_j\}$ for each $j \in \{1, \ldots, s\}$ and $\mathcal{D}^* = \{D^*_j \mid j \in \{1, \ldots, s\}\}$, we get:

Lemma 5.4.1 *Let $\mathcal{C}$, $\mathcal{D} \in P(m)$. If $\mathcal{D} \geq \mathcal{C}$, then $G(\mathcal{C})(\mathcal{D}^*) = G(\mathcal{D})$.*

Proof By definition: $G(\mathcal{C})(\mathcal{D}^*) = \langle \sum\{g_{C_i} \mid i \in D^*_1\}\rangle_* + \cdots + \langle \sum\{b_{C_i} \mid i \in D^*_s\}\rangle_* = \langle g_{D_1}\rangle_* + \cdots + \langle g_{D_s}\rangle_* = G(\mathcal{D})$. □

Proposition 5.4.2 *Let $G = G(\mathcal{C}_1) \oplus \cdots \oplus G(\mathcal{C}_k)$ be a decomposition of G into indecomposables. If $J \subseteq \{I, \ldots, k\}$ then $\oplus\{G(\mathcal{C}_i) \mid i \in J\} = G(\wedge\{\mathcal{C}_i \mid i \in J\})$.*

Proof Proceed by induction on m. If $m \leq 2$ then G is indecomposable and the statement is trivially true. Let then $m \geq 3$ with G decomposable: by [2] we have $G = G_E \oplus G_F$ for some tripartition $\{\{i\}, E, F\}$ of I. Then, for each $i \in J$, $G(\mathcal{C}_i)$ is a direct summand either of G_E or of G_F, therefore either $\mathcal{C}_i \geq p_E$ or $\mathcal{C}_i \geq p_F$. Setting $I' = \{i \in \{1, \ldots, k\} \mid \mathcal{C}_i \geq p_E\}$ and $I'' = \{i \in \{1, \ldots, k\} \mid \mathcal{C}_i \geq p_F\}$, we have that $\oplus\{G(\mathcal{C}_i) \mid i \in I'\}$ and $\oplus\{G(\mathcal{C}_i) \mid i \in I''\}$ are decompositions of G_E resp. G_F into indecomposables.

By Lemma 5.4.1 $G(\mathcal{C}_1) = G_E(\mathcal{C}^*_i)$ for each $i \in I'$ and $G(\mathcal{C}_i) = G_F(\mathcal{C}^*_i)$ for each $i \in I''$. Therefore $G_E = \oplus\{G_E(\mathcal{C}^*_i) \mid i \in I'\}$ and $G_{F\cdot} = \oplus\{G_F(\mathcal{C}^*_i)) \mid i \in I''\}$.

Setting $J' = J \cap I'$ and $J'' = J \cap I''$, by induction we get: $\oplus\{G(\mathcal{C}_i) \mid i \in J\} = (\oplus\{G_E(\mathcal{C}^*_i) \mid i \in J'\}) \oplus (\oplus\{G_F(\mathcal{C}^*_i) \mid i \in J''\}) = G_E\left(\wedge\{\mathcal{C}^*_i \mid i \in J'\}\right) \oplus G_F\left(\wedge\{\mathcal{C}^*_i \mid i \in J''\}\right) = G(\wedge\{\mathcal{C}_i \mid i \in J'\}) \oplus G(\wedge\{\mathcal{C}_i \mid i \in J''\}) \leq G(\wedge\{\mathcal{C}_i \mid i \in J\})$. Now, since $(\wedge\{\mathcal{C}_i \mid i \in J'\} \geq p_E$ and $(\wedge\{\mathcal{C}_i \mid i \in J''\} \geq p_F$, it is not hard to verify that $G(\wedge\{\mathcal{C}_i \mid i \in J'\}) \oplus G(\wedge\{\mathcal{C}_i \mid i \in J''\})$ and $G(\wedge\{\mathcal{C}_i \mid i \in J\})$ have the same rank; but the first is a direct summand of G and hence of $G(\wedge\{\mathcal{C}_i \mid i \in J\})$, therefore they coincide. □

Uniqueness of decomposition yields:

Theorem 5.4.3 *If H is a direct summand of G then there exists a partition $\mathcal{C}$ of I such that $H \cong G(\mathcal{C})$.*

Proof Let $H_1 \oplus \cdots \oplus H_r$ be a decomposition of H into indecomposables. For each $i \in \{1, \ldots, r\}$ H_i is an indecomposable of G as well, hence there exists a $j_i \in \{1, \ldots, k\}$ such that $H_i \cong G(\mathcal{C}_{j_i})$. Therefore $H \cong G(\mathcal{C}_{j_1}) \oplus \cdots \oplus G(\mathcal{C}_{j_r}) = G(\wedge\{\mathcal{C}_{j_i} \mid i = 1, \ldots, r\})$, as desired. □

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Corollary 5.4.4 *Let G_1 and G_2 be direct summands of G s.t. $G_1 \cap G_2 = \{0\}$. If $G_1 = G(\mathcal{C})$ and $G_2 = G(\mathcal{D})$ then $G_1 \oplus G_2 = G(\mathcal{C} \wedge \mathcal{D})$.*

If, now, $G(\mathcal{C}) \oplus G(\mathcal{D}) = G(\mathcal{C} \wedge \mathcal{D})$ then $G(\mathcal{C} \vee \mathcal{D}) = \{0\}$, thus $\mathcal{C} \vee \mathcal{D} = \max \mathbb{P}(m)$; moreover, $|\mathcal{C}| + |\mathcal{D}| = |\mathcal{C} \wedge \mathcal{D}| + 1$.

5.5 The Domain of $(\mathcal{C}, \mathcal{D})$

In this Section we build the widest (prime-wise) tent of a $B^{(1)}$-group G splitting into two summands $G(\mathcal{C})$, $G(\mathcal{D})$ with (suitably) given partitions $\mathcal{C}$, $\mathcal{D}$.

Setting, for each $\mathcal{C} = \{C_1, \dots, C_k\} \in \mathbb{P}(m)$, $V(\mathcal{C}) = \{b_E \in \mathbb{B}(m) \mid b_E \geq \mathcal{C}\}$, we have:

a) $V(\mathcal{C})$ is a subspace of $\mathbb{B}(m)$ of dimension $k - 1$.

b) $V(\mathcal{C}) = \langle b_{C_1}, \dots, b_{C_k} \rangle$,

c) $V(\mathcal{C}) \cap V(\mathcal{D}) = V(\mathcal{C} \vee \mathcal{D})$ for each $\mathcal{C}, \mathcal{D} \in \mathbb{P}(m)$,

d) $V(\mathcal{C}) + V(\mathcal{D}) \leq V(\mathcal{C} \wedge \mathcal{D})$ for each $\mathcal{C}, \mathcal{D} \in \mathbb{P}(m)$.

More details and proofs can be found in [6, Sections 3, 4].

Proposition 5.5.1 *For $G(\mathcal{C})$ and $G(\mathcal{D})$ direct summands of G, the following are equivalent:*

(i) $G = G(\mathcal{C}) \oplus G(\mathcal{D})$;

(ii) $\mathcal{C} \vee \mathcal{D} = \max \mathbb{P}(m)$ *and* $|\mathcal{C}| + |\mathcal{D}| = m + 1$;

(iii) $\mathbb{B}(m) = V(\mathcal{C}) \oplus V(\mathcal{D})$.

Let $\mathcal{C}$ and $\mathcal{D}$ be partitions of I such that $\mathbb{B}(m) = V(\mathcal{C}) \oplus V(\mathcal{D})$. For each $i \in I$, denote by b_{E_i} resp. b_{F_i} the projection of p_i on $V(\mathcal{C})$ resp. $V(\mathcal{D})$. It is not hard to show

Proposition 5.5.2 *If $V(\mathcal{C}) \oplus V(\mathcal{D}) = \mathbb{B}(m)$ then $V(\mathcal{C}) = \langle b_{E_1}, \dots, b_{E_m} \rangle$ and $V(\mathcal{D}) = \langle b_{F_1}, \dots, b_{F_m} \rangle$. Moreover, $b_{E_1} + \cdots + b_{E_m} = 0$ and $b_{F_1} + \cdots + b_{F_m} = 0$.*

If $\{i\}$ is, say, a block of $\mathcal{C}$, then $p_i = b_{E_i}$. For each $i \in I$ with $\{i\}$ not a block of $\mathcal{C}$ or $\mathcal{D}$, $b_{E_i} \wedge b_{F_i}$ is a tripartition: supposing w.l.o.g. $i \in E_i^{-1} \cap F_i^{-1}$, that is $E_i^{-1} = \{i\} \cup F_i$ and $F_i^{-1} = \{i\} \cup E_i$, we have

$$b_{E_i} \wedge b_{F_i} = \{\{i\}, E_i, F_i\}.$$

Proposition 5.5.3 *The m-tuple of partitions $(b_{E_1} \wedge b_{F_1}, \dots, b_{E_m} \wedge b_{F_m})$ is almost distributive.*

Proof It is clear that $b_{E_i} \wedge b_{F_i} \leq p_i$ for each $i \in I$; then to prove the statement it is enough to show that for each $i \in I$ condition (#) holds. If $b_{E_i} \wedge b_{F_i} = p_i$, the statement is obvious. Let $\mathcal{C}_i = b_{E_i} \wedge b_{F_i} = \{\{i\}, E_i, F_i\}$ and $\mathcal{C}_j = b_{E_j} \wedge b_{F_j} = \{\{j\}, E_j, F_j\}$. We have four possibilities: $j \in F_i$ and $i \in E_j$; $j \in F_i$ and $i \in F_j$; $j \in E_i$ and $i \in E_j$; $j \in E_i$ and $i \in F_j$. Let us consider the first. Here condition (#) holds if E_i, the block of $\mathcal{C}_i$ not containing j, is contained in E_j, the block of $\mathcal{C}_j$ containing i. Since $b_{E_i}, b_{E_j} \in V(\mathcal{C})$; $b_{F_i}, b_{F_j} \in V(\mathcal{D})$; and $E_i^{-1} = \{i\} \cup F_i$, $F_i^{-1} = \{i\} \cup E_i$, $E_j^{-1} = \{j\} \cup F_j$ and $F_j^{-1} = \{j\} \cup E_j$, it is easy to check that

$$\mathcal{C} \leq b_{E_i} \wedge b_{E_j} = \{E_i \cap E_j, \{i\} \cup (F_i \cap E_j), E_i \cap F_j, \{j\} \cup (F_i \cap F_j)\} \leq b_{E_i \cap F_j}$$
$$\mathcal{D} \leq b_{F_i} \wedge b_{F_j} = \{F_i \cap F_j, \{j\} \cup (F_i \cap E_j), E_i \cap F_j, \{i\} \cup (E_i \cap E_j)\} \leq b_{E_i \cap F_j}$$

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Thus $b_{E_i \cap F_j} \in V(\mathcal{C}) \cap V(\mathcal{D})$. Therefore $E_i \cap F_j = \emptyset$ because $V(\mathcal{C}) \cap V(\mathcal{D}) = \{0\}$, hence $E_i \subseteq E_j$ as desired.

A similar argument works in the remaining cases. □

Definition 5.5.4 Let $(\mathcal{C}, \mathcal{D})$ be a couple of partitions of I such that $\mathbb{B}(m) = V(\mathcal{C}) \oplus V(\mathcal{D})$. We will call *domain* of $(\mathcal{C}, \mathcal{D})$ the tent having $\{A \subseteq I \,|\, p_A \leq b_{Ei} \wedge b_{Fi} \ \forall i \in A\}$ as set of primes, and we will denote it by $dom(\mathcal{C}, \mathcal{D})$. □

Since we just proved that $(b_{E_1} \wedge b_{F_1}, \ldots, b_{E_m} \wedge b_{F_m})$ is almost-distributive, from Proposition 5.5.3 we get

Proposition 5.5.5 *If the tent t is the domain of $(\mathcal{C}, \mathcal{D})$ then $part_t(t_i) = b_{E_i} \wedge b_{F_i}$ for each $i \in I$.*

Proposition 5.5.6 *Let $(\mathcal{C}, \mathcal{D})$ be a couple of partitions of I such that $\mathbb{B}(m) = V(\mathcal{C}) \oplus V(\mathcal{D})$, and let G be represented by the tent t. The following are equivalent:*

(i) $G = G(\mathcal{C}) \oplus G(\mathcal{D})$;

(ii) $part_t(t_i) \leq b_{E_i} \wedge b_{F_i} \forall i \in I$;

(iii) *t is a primes-sub-tent of* dom$(\mathcal{C}, \mathcal{D})$.

Proof Observe that $g_i = -g_{E_i} - g_{F_i}$, where $g_{E_i} \in G(\mathcal{C})$ and $g_{F_i} \in G(\mathcal{D})$, for each $i \in I$.
(i) ⇒ (ii) Let $i \in I$. $G = G(\mathcal{C}) \oplus G(\mathcal{D})$ yields $t_i = t_G(g_i) = t_G(g_{E_i}) \wedge t_G(g_{F_i}) = t(b_{E_i}) \wedge t(b_{F_i}) = t(b_{E_i} \wedge b_{F_i})$, hence $\mathrm{part}_t(t_i) \leq b_{E_i} \wedge b_{F_i}$.
(ii) ⇒ (i) To prove that $G = G(\mathcal{C}) \oplus G(\mathcal{D})$ it is enough to show that $t_G(g_i) = t_{G(\mathcal{C}) \oplus G(\mathcal{D})}(g_i)$, that is $t_i = t(b_{E_i} \wedge b_{F_i}))$, for each $i \in I$. Now $t_i \leq t(b_{E_i} \wedge b_{F_i})$ because $\mathrm{part}_t(t_i) \leq b_{E_i} \wedge b_{F_i}$, hence $t_i = t(b_{E_i} \wedge b_{F_i})$ (the opposite inequality always holds).
(ii) ⇔ (iii) As we saw in the previous Section, dom$(\mathcal{C}, \mathcal{D})$ is the widest tent for which (ii) holds. Then (ii) holds for a tent t if and only if t is a primes-sub-tent of dom$(\mathcal{C}, \mathcal{D})$. □

The following theorem characterizes the primes of dom$(\mathcal{C}, \mathcal{D})$.

Theorem 5.5.7 *Let $V(\mathcal{C}) \oplus V(\mathcal{D}) = \mathbb{B}(m)$ and $A \subseteq I$ s.t. $|A| \neq m - 1$. The following are equivalent:*

(i) *A is a prime of* dom$(\mathcal{C}, \mathcal{D})$;

(ii) $V(P_A) = (V(p_A) \cap V(\mathcal{C})) \oplus (V(p_A) \cap V(\mathcal{D})) = (V(p_A \vee \mathcal{C})) \oplus (V(p_A \vee \mathcal{D}))$;

(iii) $|A| = |\{C \in \mathcal{C} | C \subseteq A\}| + |\{D \in \mathcal{D} | D \subseteq A\}|$.

Proof (i) ⇒ (ii) If A is a prime of dom$(\mathcal{C}, \mathcal{D})$ then $p_A \leq b_{E_i} \wedge b_{F_i}$ for each $i \in A$; thus $V(p_A) \geq V(b_{E_i} \wedge b_{F_i}) = \langle b_{E_i}, b_{F_i} \rangle$, therefore $V(p_A) = \langle p_i | i \in A \rangle = \langle b_{E_i} + b_{F_i} | i \in A \rangle = (V(p_A) \cap V(\mathcal{C})) \oplus (V(p_A) \cap V(\mathcal{D})) = (V(p_A \vee \mathcal{C})) \oplus (V(p_A \vee \mathcal{D}))$.
(ii) ⇒ (i) Let $i \in A$. Since $V(p_A) = (V(p_A) \cap V(\mathcal{C})) \oplus (V(p_A) \cap V(\mathcal{D}))$ we have $b_{E_i} \in V(p_A) \cap V(\mathcal{C})$ and $b_{F_i} \in V(p_A) \cap V(\mathcal{D})$. Therefore $b_{E_i} \wedge b_{F_i} \geq p_A$ for each $i \in A$, hence A is a prime of dom$(\mathcal{C}, \mathcal{D})$.
(ii) ⇔ (iii) is straightforward if we note that $\dim V(p_A \vee \mathcal{C}) = |p_A \vee \mathcal{C}| - 1 = |\{C \in \mathcal{C} | C \subseteq A\}|$ for each partition $\mathcal{C}$ of I. □

Corollary 5.5.8 *Let $V(\mathcal{C}) \oplus V(\mathcal{D}) = \mathbb{B}(m)$. If A is a prime of* dom$(\mathcal{C}, \mathcal{D})$ *then $p_A = (p_A \vee \mathcal{C}) \wedge (p_A \vee \mathcal{D})$.*

Proof Since $V(p_A) = (V(p_A \vee \mathcal{C})) \oplus (V(p_A \vee \mathcal{D})) \leq V((p_A \vee \mathcal{C}) \wedge (p_A \vee \mathcal{D}))$ we have $(p_A \vee \mathcal{C}) \wedge (p_A \vee \mathcal{D}) \leq p_A$, hence $(p_A \vee \mathcal{C}) \wedge (p_A \vee \mathcal{D}) = p_A$. □

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Remark The previous condition is not sufficient. E.g. set $\mathcal{C} = \{\{1, 2\}, \{3, 4\}, \{5\}\}$, $\mathcal{D} = \{\{1, 3\}, \{2, 5\}, \{4\}\}$ and $A = \{1, 2, 3\}$: it is easy to check that $p_A = (p_A \vee \mathcal{C}) \wedge (p_A \vee \mathcal{D})$ but A is not a prime of $\mathrm{dom}(\mathcal{C}, \mathcal{D})$, for p_A is not $\leq b_{Ei} \wedge b_{Fi}$ for any $i \in A$.

5.6 Indecomposable Summands

As we saw before, if G is a $B^{(1)}$-group represented by a tent t then each direct summand of G is a $G(\mathcal{C})$ for some $\mathcal{C} \in \mathbb{P}(m)$, and if $G(\mathcal{D})$ is its complement in G then $V(\mathcal{C}) \oplus V(\mathcal{D}) = \mathbb{B}(m)$. In this section, we will show that if $G(\mathcal{C})$ is indecomposable then $\mathcal{D}$ is a pointed partition; the condition is also sufficient if G is represented by the tent $\mathrm{dom}(\mathcal{C}, \mathcal{D})$.

Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be a partition of I, and p_D a pointed partition of I such that $V(p_D) \oplus V(\mathcal{C}) = \mathbb{B}(m)$; then $|C_j \cap D^{-1}| = 1$ for each $j \in J = \{1, \ldots, k\}$. Thus, setting $r = m - k$, we may assume w.l.o.g. $D = \{1, 2 \ldots, r\}$, $C_j \cap D^{-1} = \{r + j\}$ for each $j \in J = \{1, \ldots, k\}$. Thus for every $j \in J$ we get

$$C_j = X_j \cup \{r + j\}, \text{ with } X_j = C_j \backslash \{r + j\} \subseteq D.$$

Note that X_j may be empty. Setting $t = \mathrm{dom}(\mathcal{C}, p_D)$, it is easy to check that

$$\begin{aligned}
&\mathrm{part}_t(t_i) = b_{E_i} \wedge b_{F_i} = p_i \text{ for all } i \in D, \text{ and}\\
&\mathrm{part}_t(t_{r+j}) = b_{E_{r+j}} \wedge b_{F_{r+j}} = b_{C_j} \wedge b_{X_j} = \{\{r + j\}, X_j, I\backslash C_j\} \text{ for all } j \in J.
\end{aligned}$$

Note that, if $X_j = \emptyset$, then $\mathrm{part}_t(t_{r+j}) = p_{r+j}$.

Lemma 5.6.1 *Let $t = \mathrm{dom}(p_D, \mathcal{C})$. We have:*

(i) all subsets of D are primes of t;

(ii) if A is a prime of t containing $r + j$ for some $j \in J$ then either $C_j \subseteq A$ or $I\backslash C_j \subseteq A$.

Proof (i) is straightforward.
(ii) If A is a prime of t containing $r + j$ for some $j \in J$ then A^{-1} is included in a block of $\mathrm{part}_t(t_{r+j}) = \{\{r + j\}, I\backslash C_j, X_j\}$, hence either $C_j = \{r + j\} \cup X_j \subseteq A$ or $\{r + j\} \cup (I\backslash C_j) \subseteq A$. □

Definition 5.6.2 With the above notation relative to the couple of partitions $(p_D, \mathcal{C})$, we introduce two families of subsets of I (they can be checked on Example c) in Section 5.7):

$$\begin{aligned}
&P(t)_1 = \{\cup\{C_j | j \in J'\} \cup S \mid J' \subset J, S \subseteq \cup\{X_j | j \in J\backslash J'\} \text{ and}\\
&\quad |\cup\{X_j | j \in J\backslash J'\}\backslash S| \geq 2 - |J\backslash J'|\},\\
&P(t)_2 = D^{-1} \cup (\cup\{X_j | j \in J\backslash\{j'\}) \cup S \mid j' \in J, S \subseteq X_{j'}, \text{ and } |X_{j'}\backslash S| \neq 1\}. \square
\end{aligned}$$

Lemma 5.6.3 *The set of nonempty primes of* $\mathrm{dom}(p_D, \mathcal{C})$ *is*

$$P(t)_1 \cup P(t)_2$$

Proof It is not hard to check that if $A \in P(t)_1 \cup P(t)_2$ then A^{-1} is contained in a block of $\mathrm{part}_t(t_i)$ for each $i \in A$, hence A is a prime of t. Let now A be a prime of t. If $A \cap \{r + 1, \ldots, m\} \subset \{r + 1, \ldots, m\}$ then, setting $J' = \{j \in J | r + j \in A\}$, from Lemma 5.6.1 (ii) we get that $C_j \subseteq A$ for each $j \in J'$. Thus, if $S = A\backslash \cup \{C_j | j \in J'\}$ we have $A \in P(t)_1$. Otherwise, if $D^{-1} = \{r + 1, \ldots, m\} \subseteq A$ then, by Lemma 5.6.1(ii), either $C_j \subseteq A$ for each $j \in J$ and hence $A = I$, or there exists a $j \in J$ such that X_j is not contained in A and $I\backslash C_j \subseteq A$, hence $A \in P(t)_2$. □

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Setting, for each $\mathcal{C} = \{C_1, \dots, C_k\} \in \mathbb{P}(m)$ and $A \subseteq I$ with $|A| \neq m - 1$

$$\mathcal{C}[A] = \{j \in J | p_A \leq b_{C_j}\}$$

it is not hard to check

Lemma 5.6.4 *The set of primes of the tent of base* $(t_{C_1}, \dots, t_{C_k})$ *is*

$$\{\mathcal{C}[A] | A \in P(t)\}.$$

Proposition 5.6.5 *If* $G = G(\mathcal{D}) \oplus G(\mathcal{C})$*, with* $G(\mathcal{C})$ *indecomposable, then* $\mathcal{D}$ *is a pointed partition.*

Proof If $\mathcal{D}$ is not a pointed partition then there exist two blocks $D_i, D_j \in \mathcal{D}$ such that $|D_i|$, $|D_j| \geq 2$, and a block C of $\mathcal{C}$ such that $|D_i \cap C| = 1$ and $|D_j \cap C| = 1$. Then, without loss of generality, suppose:

$$D_1 = \{1, 3, \dots\}, D_2 = \{2, 4, \dots\}, C_1 = \{1, 2, \dots\} \text{ and } C_4 = \{4, \dots\}.$$

By Proposition 5.5.5 we have $\text{part}_t(t_1) \leq b_{E_1} \wedge b_{F_1} = \{\{1\}, E_1, F_1\}$ where $b_{E_1} \in V(\mathcal{C})$ and $b_{F_1} \in V(\mathcal{D})$. Thus $\mathcal{C} \leq \{E_1, F_1 \cup \{1\}\}$ and $\mathcal{D} \leq \{F_1, E_1 \cup \{1\}\}$. Therefore $C_1 - \{1\} = \{2, \dots\} \subseteq F_1$, so $D_2 = \{2, 4, \dots\} \subseteq F_1$ and hence $C_4 = \{4, \dots\} \subseteq F_1$. On the other hand $D_1 \subseteq E_1$, hence $E_1 \neq \emptyset$. Thus, setting $E = \{i \in J | C_i \subseteq E_1\}$ and $F = \{i \in J | C_i \subseteq F_1\}$, we have

$$J = \{1\} \cup E \cup F \text{ and } E, F \neq \emptyset.$$

Let t be the tent of base $(t_{C_1}, \dots, t_{C_k})$. By Lemma 5.6.4, for each prime A'' of t'' there exists a prime A of t such that $A'' = \mathcal{C}[A]$. If $A'' = \mathcal{C}[A]$ is a non-trivial prime of t'' then $1 \in A''$ if and only if $1 \in A$. In fact, $1 \in \mathcal{C}[A]$ if and only if $p_A \leq b_{C_1}$ that is either $C_1 \subseteq A$ and so $1 \in A$, or $C_1^{-1} = C_2 \cup \dots \cup C_k \subseteq A$, thus $\mathcal{C}[A] = \{1, \dots, k\}$. Now, if $1 \in A$ then $p_A \leq \{\{1\}, E_1, F_1\}$; therefore either $E_1 \subseteq A$ and hence $E \subseteq \mathcal{C}[A]$, of $F_1 \subseteq A$ and hence $F \subseteq \mathcal{C}[A]$. Thus each prime of t'' containing 1 contains either E or F, therefore $\text{part}_{t''}(t_{C_1}) \leq \{\{1\}, E, F\}$. Hence $G(\mathcal{C})$ is decomposable. □

Proposition 5.6.6 *Let* $t = \text{dom}(p_D, \mathcal{C})$. *If* G *is a* $B^{(1)}$*-group represented by* t *then* $G(\mathcal{C})$ *is indecomposable.*

Moreover, in the above notation for the X_j*, a decomposition of* G_D *into indecomposables is* $G_{X_1} \oplus \dots \oplus G_{X_k}$ *and therefore a decomposition of* G *into indecomposables is*

$$G = G(\mathcal{C}) \oplus G_{X_1} \oplus \dots \oplus G_{X_k}.$$

Proof By Proposition 5.5.6, $G = G_D \oplus G(\mathcal{C})$. $G(\mathcal{C})$ is represented by the tent t'' of base $(t_{C_1}, \dots, t_{C_k})$. By Lemma 5.6.4 we have that $P(t'') = \{\mathcal{C}[A] | A \in P(t)\}$. Then since, by Lemma 5.6.1, $C_1, \dots, C_k$ are primes of t, there follows that $\mathcal{C}[C_1] = \{1\}, \dots, \mathcal{C}[C_k] = \{k\}$ are primes of t''. Therefore $\text{part}_{t''}(t_{C_j}) \geq p_j$ and hence $\text{part}_{t''}(t_{C_j}) = p_j$ for each $j \in J$. Thus, by [6], [8], $G(\mathcal{C})$ is indecomposable.

G_D is represented by the tent t' of base $(t_D, t_i | i \in D)$. By Lemma 5.6.4 we have that $P(t') = \{p_D[A] | A \in P(t)\}$. Now $p_D[A]$ is a subset of $\{1, \dots, r, \{D^{-1}\}\}$. Observe that an $i \in \{1, \dots, r\}$ belongs to $p_D[A]$ if and only if $i \in A$, and that $D^{-1} \in p_D[A]$ if and only if $p_A \leq b_D$, that is either $D^{-1} \subseteq A$ or $D \subseteq A$. Then all proper subsets of D, being primes of t, are also primes of t'. Moreover, Lemma 5.6.4 ensures that $P(t) = P(t)_1 \cup P(t_2)$, hence a prime A of t, not containing $X_1 \cup \dots \cup X_k = D$, contains D^{-1} if and only if $A \in P(t)_2$, thus the set of primes of t' containing $\{D^{-1}\}$ is given by $\{p_D[A] | A \in P(t)_2\} \cup \{D \cup \{D^{-1}\}\}$. Now, if $A \in P(t)_2$ then $A = D^{-1} \cup (\cup\{X_{j'} | j' \in J \backslash \{j\}\}) \cup S$ for some $j \in J$ and $S \subset X_j$, hence setting, for each $B \subseteq \{1, \dots, r\}$,

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$B^* = \{i \in \{1, \ldots, r\} | i \in B\}$, it is easy to check that $p_D[A] = \cup\{X^*_{j'} | j' \neq j\} \cup \{D^{-1}\} \cup S$, therefore we have

$$\{p_D[A] | A \in P(t)_2\} = \{\{\cup X^*_{j'} | j' \neq j\} \cup \{D^{-1}\} \cup S \mid j \in J, S \subset X^*_j\}.$$

Summarizing, all proper subsets of D are primes of t', and

$$\{\{\cup X^*_{j'} | j' \neq j\} \cup \{D^{-1}\} \cup S \mid j \in J, S \subset X^*_j\}\} \cup \{D \cup \{D^{-1}\}\}$$

is the set of primes of t' containing $\{D^{-1}\}$. Therefore we have

$$\text{part}_{t'}(t_i) = p_i \text{ for each } i \in \{1, \ldots, r\} \text{ and}$$
$$\text{part}_{t'}(t_D) = \{\{D^{-1}\}, X^*_1, \ldots, X^*_k\}.$$

Then from ([5, sec. 4]) we get that a decomposition of G_D into indecomposables is $(G_D)_{X^*_1} \oplus \cdots \oplus (G_D)_{X^*_k}$.

To prove the second part of the statement it is enough to observe that $(G_D)_{X^*_j} = G_{Xj}$ for all $j \in J$. □

If G is a $B^{(1)}$-group represented by a primes-subtent of dom$(\mathcal{C}, \mathcal{D})$, then $G(\mathcal{C})$ is not always indecomposable: it is enough to think about a subtent consisting of a single prime. If G is represented by dom$(\mathcal{C}, \mathcal{D})$, we get as a corollary of Proposition 5.6.5 and Proposition 5.6.6

Theorem 5.6.7 *If G is a $B^{(1)}$-group represented by* dom$(\mathcal{C}, \mathcal{D})$ *then $G(\mathcal{C})$ is indecomposable if and only if $\mathcal{D}$ is a pointed partition.*

Proposition 5.6.6 yields a necessary condition for a $B^{(1)}$-group G to equal $G' \oplus G''$ with G'' indecomposable. We will now give a more detailed description of that condition, useful towards deciding when the direct sum of two $B^{(1)}$-groups is a $B^{(1)}$-group, a problem solved up to now only for the direct sum of two indecomposables [8]. Let then in the above notation $G = G(p_D) \oplus G(\mathcal{C})$ with tent t; let t' be the tent of $G' = G(p_D)$, with base $(t_D, t_i | i \in D)$; t'' the tent of $G'' = G(\mathcal{C})$, with base $t_{C_1}, \ldots, t_{C_k}$; $\rho' = \tau(D) = \min(\text{typeset}(G')$; $\rho'' = \tau(\mathcal{C}) = \min(\text{typeset}(G''))$.

Proposition 5.6.8 *In the above setting, the tents t, t', t'' satisfy the following conditions:*

a) $t_D \leq \rho' \vee \rho''$,

b) $t_D = t'(\{\{D^{-1}\}, X_1, \ldots, X_k\})$,

c) $t_{C_j} \leq \tau(X_j) \vee \rho''$, *for all* $j \in J$,

d) $t_{C_j} \leq \tau(X_j) \vee \tau(D \backslash X_j)$, *for all* $j \in J$.

Proof For a) note that $t_D = \tau(D) \vee \tau(D^{-1})$ with $\rho' = \tau(D)$. As for $\tau(D^{-1})$, let us check it prime by prime. Let A be one of its primes, that is $\tau(A) \leq \tau(D^{-1})$; since A is a prime, this yields $A \supseteq D^{-1}$, that is A contains $r+1, r+2, \ldots, m$. Then by Lemma 5.6.1 either $C_j \subseteq A$ for all j, thus $A = I$ and $\tau(A)$ is the minimum of typeset (G); or there is exactly one $j' \in J$ with $I \backslash C_{j'} \leq A$. Then $\tau(A) \leq \tau(I \backslash C_{j'}) \leq t(\mathcal{C}) = \rho''$.
b) $t'(\{\{D^{-1}\}, X_1, \ldots, X_k\}) = t(\{D^{-1}, X_1, \ldots, X_k\})$ can be checked by direct computation.
$t(\{D^{-1}, X_1, \ldots, X_k\}) \leq t_D = \tau(D) \vee \tau(D^{-1})$: to check $\geq$, note that $\tau(D) \leq t(\{D^{-1}, X_1, \ldots, X_k\})$; for $\tau(D^{-1})$ we proceed primewise. Let $\tau(A) \leq \tau(D^{-1})$; we must prove that there is a $j \in J$ such that $\tau(A) \leq \tau(I \backslash X_j)$. As in a), either $C_j \subseteq A$ for all j, and $\tau(A)$ is minimum; or there is exactly one $j' \in J$ with $I \backslash C_{j'} \subseteq A$. But $r + j' \in A$, thus A contains $I \backslash X_{j'}$.
c) follows the same lines as a), once we note that $t_{C_j} = \tau(C_j) \vee \tau(I \backslash C_j)$, and $\tau(C_j) \leq \tau(X_j)$; a check on the primes A of $\tau(I \backslash C_j)$ obtains the result.
For d) observe that $t_{C_j} = \tau(C_j) \vee \tau(I \backslash C_j)$ and $\tau(C_j) \leq \tau(X_j)$, while $\tau(I \backslash C_j) \leq \tau(D \backslash X_j)$. □

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We conclude with an interesting special case:

Proposition 5.6.9 *Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be a partition of I such that $|C_i| \leq 2$ for each $i \in \{1, \ldots, k\}$. If $G(\mathcal{C})$ is a direct summand of G then the complement of $G(\mathcal{C})$ in G is completely decomposable.*

Proof Let $G(\mathcal{D})$ be the complement of $G(\mathcal{C})$ in G, $G(\mathcal{D}_1), \ldots, G(\mathcal{D}_s)$ the indecomposable direct summands of $G(\mathcal{D})$. For each $i \in \{1, \ldots, s\}$, from Corollary 5.4.4 we get $G(\mathcal{C} \wedge \mathcal{D}_i) = G(\mathcal{C}) \oplus G(\mathcal{D}_i)$; since $G(\mathcal{D}_i)$ is indecomposable, $\mathcal{C}$ must then be a pointed partition on the set $\mathcal{C} \wedge \mathcal{D}_i$. Now the nonsingleton block of $\mathcal{C}$ contains exactly two elements, hence $\mathcal{D}_i$ must be a partition on the set $\mathcal{C} \wedge \mathcal{D}_i$, and therefore $G(\mathcal{D}_i)$ must have rank one. Thus each direct indecomposable summand of $G(\mathcal{D})$ has rank one, hence $G(\mathcal{D})$ is completely decomposable. $\square$

5.7 Examples

Say we are given two partitions $\mathcal{C}, \mathcal{D}$ such that $\mathbb{B}(m) = V(\mathcal{C}) \oplus V(\mathcal{D})$, which implies $\mathcal{C} \vee \mathcal{D} = \{I\}$, $\mathcal{C} \wedge \mathcal{D} = \min$. How do we build all $B^{(1)}$-groups G such that $G = G(\mathcal{C}) \oplus G(\mathcal{D})$? Since their tents are primes-subtents of $t = \mathrm{dom}(\mathcal{C}, \mathcal{D})$, it will be enough to build t. We can do this in two steps: a) we get the partition base of t; b) we apply Theorem 5.3.3 to get the primes of t from a partition base. While the procedure is general, we choose as an example the case $\mathcal{D} = p_D$, so that Definition 5.6.2 and Theorem 5.6.7 can be verified.

If $\{i\}$ is a singleton block of $\mathcal{C}$ or of $\mathcal{D}$ then clearly $\mathrm{part}_t(t_i) = p_i$. Let thus w.l.o.g. $i = 1 \in C_1$ (a nonsingleton block of $\mathcal{C}$) and $1 \in D_1$ (a nonsingleton block of $\mathcal{D}$). We will prove that $\mathrm{part}_t(t_1) = \{\{1\}, E_1, F_1\}$ where $p_1 = b_{E_1} + b_{F_1}$ with $b_{E_1} \in V(\mathcal{C})$, that is $b_{E_1} \geq \mathcal{C}$, and $b_{F_1} \in V(\mathcal{D})$, hence $b_{F_1} \geq \mathcal{D}$ (Proposition 5.5.1). There follows

$$\mathcal{C} \leq \{\{1\} \cup F_1, E_1\}, \quad \mathcal{D} \leq \{\{1\} \cup E_1, F_1\},$$

with $C_1 \subseteq \{1\} \cup F_1$, $D_1 \subseteq \{1\} \cup E_1$. Let C_i be another block of $\mathcal{C}$ intersecting D_1; C_i must be contained in E_1. Thus if $C_2, \ldots, C_s$ are the blocks of $\mathcal{C}$ intersecting D_1 and $D_2, \ldots, D_r$ are the blocks of $\mathcal{D}$ intersecting C_1, we have $C_2 \cup \cdots \cup C_s = E_1' \subseteq E_1$, $D_2 \cup \cdots \cup D_r = F_1' \subseteq F_1$ hence

$$\mathrm{part}_t(t_1) \geq \{\{1\}, E_1', F_1', \{j\} | j \in J\} \text{ for } J = I \setminus (\{1\} \cup E_1' \cup F_1'). \tag{5.1}$$

If $J = \emptyset$, our claim is true. If not, there is at least one $j \in J$ such that, calling C_j resp. D_j the block of $\mathcal{C}$ resp. $\mathcal{D}$ containing it, at least one of $C_j \cap F_1'$ or $D_j \cap E_1'$ is nonempty; otherwise $\mathcal{C} \vee \mathcal{D} \leq \{\{1\} \cup E_1' \cup F_1', J\}$, a proper bipartition of I. Moreover, $C_j \cap F_1'$ and $D_j \cap E_1'$ cannot be both nonempty, otherwise $C_j \subseteq F_1'$ and $D_j \subseteq E_1'$ would imply $j \in E_1' \cap F_1'$, a contradiction.

Say then $D_j \cap E_1' \neq \emptyset$; then $j \in E_1$, hence $C_j \subseteq E_1$, $D_j \cap F_1' = \emptyset$, thus

$$\mathrm{part}_t(t_1) \geq \{\{1\}, E_1' \cup C_j, F_1', \{j\} | j \in J' = J \setminus C_j\}$$

and we revert to (5.1) with J' replacing J. The case $C_j \cap F_1' \neq \emptyset$ works analogously; the procedure ends by finite induction.

Here come the examples.

a) Let $\mathcal{C} = \{\{1, 2, 5\}, \{3, 4, 6\}, \{7\}\}$, $\mathcal{D} = p_D = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5, 6, 7\}\}$. We have immediately $\mathrm{part}_t(t_1) = p_1$, $\mathrm{part}_t(t_2) = p_2$, $\mathrm{part}_t(t_3) = p_3$, $\mathrm{part}_t(t_4) = p_4$, $\mathrm{part}_t(t_7) = p_7$. To build $\mathrm{part}_t(t_5)$, let $C_1 = \{1, 2, 5\}$, $D_1 = \{5, 6, 7\}$; then $C_2 = \{3, 4, 6\}$, $C_3 = \{7\}$, $D_2 = \{1\}$, $D_3 = \{2\}$, $E_1' = \{3, 4, 6, 7\}$, $F_1' = \{1, 2\}$. No more steps are needed, $\mathrm{part}_t(t_5) = \{\{5\}, \{3, 4, 6, 7\}, \{1, 2\}\}$. Analogously we get $\mathrm{part}_t(t_6) = \{\{6\}, \{3, 4\}, \{1, 2, 5, 7\}\}$.

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b) We show how to build the primes of a tent given by a partition base $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m)$, that is the primes $A \subseteq I$ such that $p_A \leq \mathcal{C}_i$ for each $i \in A$. Let A^{-1} be a subset of a nonsingleton block of $\mathcal{C}_1$; check wheter A^{-1} is contained in a block of $\mathcal{C}_i$ for each $i \in A$. If so, A is a prime; if not, discard that A^{-1}.

With respect to the partition base built in a), let $A^{-1} = \{2, 7\}$: it is not contained in a block of $\mathrm{part}_t(t_5)$, while $5 \in A$; thus $A = \{1, 3, 4, 5, 6\}$ is not a prime of t. The actual primes of t are listed below.

c) As an application of Lemma 5.6.3 we give here the tent which is the domain of the above couple $(\mathcal{C}, \mathcal{D})$, where $\mathcal{D} = p_D = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5, 6, 7\}\}$, $\mathcal{C} = \{\{1, 2, 5\}, \{3, 4, 6\}, \{7\}\}$. We have $I = \{1, 2, \dots, 7\}$, $D = \{1, 2, 3, 4\}$; $X_1 = \{1, 2\}$, $X_2 = \{3, 4\}$, $X_3 = \emptyset$, $J = \{1, 2, 3\}$; $m = 7$, $r = 4$, $k = 3$. There are 56 primes:

$$
\begin{array}{l}
t_1 = \infty\ \infty\ \infty\ *\ \infty\ *\ *\ 0\ \infty \\
t_2 = \infty\ \infty\ \infty\ *\ \infty\ *\ *\ 0\ \infty \\
t_3 = \infty\ *\ *\ \infty\ *\ \infty\ *\ \infty\ 0 \\
t_4 = \infty\ *\ *\ \infty\ *\ \infty\ *\ \infty\ 0 \\
t_5 = \infty\ 0\ \infty\ 0\ \infty\ 0\ 0\ \infty\ \infty \\
t_6 = \infty\ 0\ 0\ \infty\ 0\ \infty\ 0\ \infty\ \infty \\
t_7 = \infty\ 0\ *\ *\ \infty\ \infty\ \infty\ \infty\ \infty
\end{array}
$$

Here $*$ stands for any choice of 0 or ∞, as long as $|A| \neq m - 1$ is respected (at least two zeros on each column). The last two primes constitute $P(t_2)$; the others constitute $P(t_1)$. The first column summarizes all primes contained in D.

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Chapter 6

Associated Primes of the Local Cohomology Modules

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Abstract In this article we will give some of the ideas we consider important and point out the directions taken by some recent research on the set of associated primes of the local cohomology modules. In addition, we prove the following result.

Let R be a Noetherian ring and $\mathfrak{a}$ be an ideal of R. Let M be an R-module and s be a non-negative integer. Then the following hold:

(a) If $\operatorname{Ext}_R^{s-j}(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^j(M))$ is finitely generated for all $j < s$ and if $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^s(M))$ is a finitely generated R-module, then $\operatorname{Ext}_R^s(R/\mathfrak{a}, M)$ is a finitely generated R-module.

(b) If $\operatorname{Ext}_R^{s+1-j}(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^j(M))$ is finitely generated for all $j < s$ and if $\operatorname{Ext}_R^s(R/\mathfrak{a}, M)$ is a finitely generated R-module, then $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^s(M))$ is a finitely generated R-module.

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6.1 Introduction

Let R be a Noetherian ring and let $\mathfrak{a}$ be an ideal of R and let M be an R-module. An important problem in commutative algebra is to determine when the set of associated primes of the ith local cohomology module $\mathrm{H}_{\mathfrak{a}}^i(M)$ of M with support in $\mathfrak{a}$ is finite. This question, which was raised by Huneke [15], has been studied by many authors. In this survey we review their works and also review the stronger question "is $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^i(M))$ a finitely generated R-module?" which was

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raised by Grothendieck [10]. All the results that we quote were known to the authors except those which have been provided with a proof, i.e., Corollary 6.3.8, Theorem 6.3.9 and Corollary 6.4.4.

6.2 General Case

We start this section with a question due to Huneke [15].
Question A Is the number of associated primes of $\mathrm{H}^i_{\mathfrak{a}}(M)$ finite for all i?

Let $(R, \mathfrak{m})$ be a local ring and let M be a finitely generated R-module. Then for any i the module $\mathrm{H}^i_{\mathfrak{m}}(M)$ is an artinian R-module and so has only finitely many associated primes.

A positive answer to Question A is due to Huneke and Sharp, cf. [14] .

Theorem 6.2.1 *Let $(R, \mathfrak{m})$ be a regular local ring with positive characteristic p. Then $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ is a finite set for any integer $i \geq 0$.*

In [18], Lyubeznik has proved that this result holds if R is of characteristic zero and contains a field. He also showed the same result also holds for unramified regular local rings of mixed characteristic, cf. [19].

On the other hand there is a negative answer to Question A (over a non-local ring) that is due to Singh, cf [23].

Example 6.2.2 Let $R = \mathbb{Z}[u, v, w, x, y, z]/(ux + vy + wz)$. Then $\operatorname{Ass} \mathrm{H}^3_{(x,y,z)}(R)$ is not a finite set.

Also the next example gives a negative answer to Question A for a local ring with positive characteristic p, cf. [16].

Example 6.2.3 Let K be a field and $R = K[s, t, u, v, x, y]/(su^2x^2 - (s + t)uxvy + tv^2y^2)$. Then $\operatorname{Ass} \mathrm{H}^2_{(x,y)}(R)$ is not finite.

Recently, Singh and Swanson published a nice paper in this subject, cf. [24]. In their paper we can find many number of counter examples related to Question A. In [20] Marley studied the question for modules over rings of small dimension. He showed that the set $\{\mathfrak{p} \in \operatorname{Supp} \mathrm{H}^i_{\mathfrak{a}}(M) | \operatorname{ht}(\mathfrak{p}) = i\}$ is always a finite set. By using this easy but interesting fact he could prove the following Theorem:

Theorem 6.2.4 *Let $(R, \mathfrak{m})$ be a local ring of dimension d and let M be a finitely generated R-module. Then the set of associated primes of the local cohomology module $H^i_{\mathfrak{a}}(M)$ is finite for all $i \geq 0$ in the following cases:*

(1) $d \leq 3$;

(2) $d = 4$ and R is regular on the punctured spectrum;

(3) $d = 5$ and R is an unramified regular local ring, and M is torsion-free.

In [12] Hellus considered Question A for Cohen-Macaulay local rings. He showed that Question A is true if and only if $\mathrm{H}^2_{(x,y)}(R)$ and $\mathrm{H}^3_{(u,v,z)}(R)$ have finitely many associated primes for all $x, y, z \in R$ and for all regular sequences u, v in R. As a consequence Question A is settled in the affirmative when R is at most 3-dimensional. Later, Zamani improved Hellus's result, cf. [29].

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6.3 Special Case

In this section we focus on the following question.
Question B. Let t be a non-negative integer. Does the module $\mathrm{H}^t_{\mathfrak{a}}(M)$ have only finitely many associated primes?

If M is finitely generated then it is well known that $\operatorname{Ass} \mathrm{H}^i_{\mathfrak{a}}(M)$ is a finite set for $i = 0$. If M is a non-zero finitely generated R-module of finite dimension n, then $\mathrm{H}^{\dim M}_{\mathfrak{a}}(M)$ is an Artinian R-module and so $|\operatorname{Ass} \mathrm{H}^{\dim M}_{\mathfrak{a}}(M)| < \infty$.

The fact that $\mathrm{H}^{\dim R}_{\mathfrak{a}}(M)$ has only finitely many associated primes (in the non-local case) has been observed in [5, Remark 3.11]. In [20], Marley showed that over a local ring R, the module $\mathrm{H}^{\dim R-1}_{\mathfrak{a}}(M)$ has finitely many associated primes.

There are several attempts to give some partial answers to the Question B. In [17] Khashyarmanesh and Salarian used the notions of $\mathfrak{a}$-filter regular sequence and unconditioned strong d-sequence to show the following result.

Theorem 6.3.1 *Let R be a Noetherian ring and let M be a finitely generated R-module. Let t be a non-negative integer. Then $\operatorname{Ass} H^t_{\mathfrak{a}}(M)$ is a finite set if one of the following holds:*

(a) for any $i < t$ the set $\operatorname{Supp} H^i_{\mathfrak{a}}(M)$ is finite.

(b) For any $i < t$ the module $H^i_{\mathfrak{a}}(M)$ is finitely generated.

Let t be a non-negative integer. Assume that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is a finitely generated R-module for all $i < t$. In [26] Tajarod and Zakeri showed that the set $\operatorname{Ass} \mathrm{H}^t_{\mathfrak{a}}(M)$ has an explicit presentation by using an unconditioned $\mathfrak{a}$-filter regular M-sequence.

In [5] Brodmann, Rotthaus and Sharp gave a simple proof (without using the notions of $\mathfrak{a}$-filter regular sequence and unconditioned strong d-sequence) for the following result (note that this result is a consequence of Theorem 6.3.1(b)).

Theorem 6.3.2 *Let R be a Noetherian ring and let M be a finitely generated R-module. Then the following hold:*

(a) If $M \neq \mathfrak{a}M$ then $\operatorname{Ass} H^{grade\,(\mathfrak{a},M)}_{\mathfrak{a}}(M)$ is a finite set.

(b) The set $\operatorname{Ass} H^1_{\mathfrak{a}}(M)$ is finite.

(c) If $H^1_{\mathfrak{a}}(M)$ is finitely generated then $\operatorname{Ass} H^2_{\mathfrak{a}}(M)$ is a finite set.

Finally, Brodmann and Lashgari proved a generalization of Theorem 6.3.1(b) and Theorem 6.3.2 with a nice and simple proof (without using filter regular sequences), cf. [4]:

Theorem 6.3.3 *Let M be a finitely generated R-module. Let t be a non-negative integer such that for each $i < t$, $H^i_{\mathfrak{a}}(M)$ is a finitely generated R-module. Then for any finitely generated submodule N of $H^t_{\mathfrak{a}}(M)$, the set $\operatorname{Ass}(H^t_{\mathfrak{a}}(M)/N)$ has finitely many elements.*

Note that when $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is finitely generated then the set

$$\operatorname{Ass}_R \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)) = \operatorname{Ass}_R \mathrm{H}^t_{\mathfrak{a}}(M)$$

is finite. Thus to give an answer to Question B, we can consider the following conjecture that is proposed by Grothendieck ([10]; Exposé XIII, 1.1]).

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"Let M be a finitely generated R-module and let $\mathfrak{a}$ be an ideal of R. Then $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ is finitely generated R-module for all $j \geq 0$."
Although this conjecture is not true in general, cf. [11]; Example 1, we have the following result that is a generalization of Theorem 6.3.1(b), cf. [2].

Theorem 6.3.4 *Let M be a finitely generated R-module. Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M)$ is a finitely generated R-module for all $i < t$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is finitely generated.*

In [11] Hartshorne introduced the notion of a module cofinite with respect to an ideal $\mathfrak{a}$.

Definition 6.3.5 The R-module M is called $\mathfrak{a}$-cofinite if $\operatorname{Supp}(M) \subseteq \mathrm{V}(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is a finitely generated R-module for each i.

Note that any finitely generated R-module is $\mathfrak{a}$-cofinite module for arbitrary ideal $\mathfrak{a}$. Also if $(R, \mathfrak{m})$ is a local ring then an R-module M is $\mathfrak{m}$-cofinite if and only if M is Artinian.

Recently Divaani-Aazar and Mafi introduced *weakly Laskerian* modules. An R-module M is called weakly Laskerian if for any submodule N of M, the set $\operatorname{Ass}(M/N)$ is finite. It is easy to see that finitely generated modules, Artinian modules and the modules with finite support are weakly Laskerian. By using the technique of spectral sequences they proved the following result, cf. [8]. This result is a generalization of 6.3.1 and 6.3.4.

Theorem 6.3.6 *Let M be a weakly Laskerian module. Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < t$. Then the set of associated primes of $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ and $\operatorname{Ext}^1_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ are finite.*

On the other hand Dibaei and Yassemi proved (without using spectral sequences) the following result that is a generalization of 6.3.4, cf. [9], 6.3.1.

Theorem 6.3.7 *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Let s be a non-negative integer. Let M be an R-module such that $\operatorname{Ext}^s_R(R/\mathfrak{a}, M)$ is a finitely generated R-module. If $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < s$, then the module $\operatorname{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M))$ is finitely generated.*

The following result is a generalization of 6.3.3.

Corollary 6.3.8 *Let $\mathfrak{a}$ be an ideal of R. Let s be a non-negative integer. Let M be an R-module such that $\operatorname{Ext}^s_R(R/\mathfrak{a}, M)$ is a finitely generated R-module. If $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < s$, then for any submodule N of $H^s_{\mathfrak{a}}(M)$ such that $\operatorname{Ext}^1_R(R/\mathfrak{a}, N)$ is finitely generated (for example N might be finitely generated), the module $\operatorname{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M)/N)$ is finitely generated. In particular, $H^s_{\mathfrak{a}}(M)/N$ has finitely many associated primes.*

Proof Let N be a submodule of $\mathrm{H}^s_{\mathfrak{a}}(M)$ such that $\operatorname{Ext}^1_R(R/\mathfrak{a}, N)$ is finitely generated. The short exact sequence

$$0 \to N \to \mathrm{H}^s_{\mathfrak{a}}(M) \to \mathrm{H}^s_{\mathfrak{a}}(M)/N \to 0$$

induces the following exact sequence

$$\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M)) \to \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M)/N) \to \operatorname{Ext}^1_R(R/\mathfrak{a}, N).$$

Since the left hand (by Theorem 6.3.7) and the right hand are finitely generated, we have that $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M)/N)$ is finitely generated. On the other hand

$$\operatorname{Supp} \mathrm{H}^s_{\mathfrak{a}}(M)/N \subseteq \operatorname{Supp} \mathrm{H}^s_{\mathfrak{a}}(M) \subseteq \mathrm{V}(\mathfrak{a}).$$

Therefore $\operatorname{Ass} \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M)/N) = \operatorname{Ass}(\mathrm{H}^s_{\mathfrak{a}}(M)/N)$ is a finite set. □

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In 6.3.7, to ensure that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ is finitely generated, one needs to have infinitely many finiteness conditions on $\mathrm{H}^i_{\mathfrak{a}}(M)$ for $i < s$, that is finiteness of all modules $\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M))$, $i < s$, $j = 0, 1, 2, \cdots$. Here, by a refinement of the proof of [9], 6.3.1, we show that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ is finitely generated provided some certain finitely many conditions are satisfied on the local cohomologies $\mathrm{H}^i_{\mathfrak{a}}(M)$, $i < s$.

Theorem 6.3.9 *Let $\mathfrak{a}$ be an ideal of R and let M be an R-module. Let s be a non-negative integer. Consider the following statements.*

(a) $\mathrm{Ext}^s_R(R/\mathfrak{a}, M)$ is a finitely generated R-module.

(b) $\mathrm{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M))$ is a finitely generated R-module.

Then the following hold:

(i) If $\mathrm{Ext}^{s-j}_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ is finitely generated for all $j < s$ then (b)⇒(a).

(ii) If $\mathrm{Ext}^{s+1-j}_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ is finitely generated for all $j < s$ then (a)⇒(b).

In particular, if $\mathrm{Ext}^{t-j}_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ is finitely generated for $t = s, s+1$ and for all $j < s$, then (a)⇔(b).

Proof (i) We prove it by induction on s. For $s = 0$, the result follows from the equality $\mathrm{Hom}_R(R/\mathfrak{a}, M) = \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^0_{\mathfrak{a}}(M))$. Assume $s > 0$ and $s - 1$ is settled. Assume that E is an injective hull of $M/\Gamma_{\mathfrak{a}}(M)$, and set $N = E/(M/\Gamma_{\mathfrak{a}}(M))$. For all $i \geq 0$, as $\mathrm{H}^i_{\mathfrak{a}}(E) = 0 = \mathrm{Ext}^i_R(R/\mathfrak{a}, E)$, we get the isomorphisms $\mathrm{Ext}^i_R(R/\mathfrak{a}, N) \cong \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ and

$$\mathrm{H}^i_{\mathfrak{a}}(N) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(M).$$

Therefore

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{s-1}_{\mathfrak{a}}(N)) \cong \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$$

is a finitely generated R-module. In addition, for all $j < s - 1$ the modules

$$\mathrm{Ext}^{s-1-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(N)) \cong \mathrm{Ext}^{s-1-j}_R(R/\mathfrak{a}, \mathrm{H}^{j+1}_{\mathfrak{a}}(M))$$

are finitely generated. Now, by induction hypothesis, $\mathrm{Ext}^{s-1}_R(R/\mathfrak{a}, N)$ is finitely generated. Thus $\mathrm{Ext}^s_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is finitely generated too. Consider the following exact sequence

$$\mathrm{Ext}^s_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \to \mathrm{Ext}^s_R(R/\mathfrak{a}, M) \to \mathrm{Ext}^s_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)).$$

Since $\mathrm{Ext}^s_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is finitely generated, we have that $\mathrm{Ext}^s_R(R/\mathfrak{a}, M)$ is finitely generated.

(ii) We prove this by induction on s. For $s = 0$, the result is clear. Let $s > 0$ and $s - 1$ is settled. Assume that E and N are as in the proof of (i). For any $0 \leq j < s - 1$, we have

$$\mathrm{Ext}^{s-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(N)) \cong \mathrm{Ext}^{s-j}_R(R/\mathfrak{a}, \mathrm{H}^{j+1}_{\mathfrak{a}}(M)).$$

Consider the following exact sequence

$$\mathrm{Ext}^s_R(R/\mathfrak{a}, M) \to \mathrm{Ext}^s_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \to \mathrm{Ext}^{s+1}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)).$$

Since $\mathrm{Ext}^s_R(R/\mathfrak{a}, M)$ is finitely generated (by assumption) and $\mathrm{Ext}^{s+1}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is finitely generated (by hypothesis), we have $\mathrm{Ext}^s_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is finitely generated and hence Ext^{s-1}_R $(R/\mathfrak{a}, N)$ is finitely generated. This shows that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{s-1}_{\mathfrak{a}}(N))$ is finitely generated by induction hypothesis. Now the assertion holds. □

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Remark 6.3.10 Theorem 6.3.9 indicates that if M is an R-module such that the modules Ext^i_R $(R/\mathfrak{a}, \mathrm{H}^{s-i}_{\mathfrak{a}}(M))$ for $i = 1, 2, \cdots, s$, and $\mathrm{Ext}^s_R(R/\mathfrak{a}, M)$ are finitely generated, then Hom $(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ is finitely generated. It would be interesting if one can find an example satisfying the above hypothesis such that some (or all) of the modules $\mathrm{H}^j_{\mathfrak{a}}(M)$, $j < s$, are not $\mathfrak{a}$-cofinite.

Corollary 6.3.11 *Let $\mathfrak{a}$ be an ideal of R and let M be an R-module. Let s be a non-negative integer. If $Ext^{s-j}_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ is finitely generated for all $j < s$ and Ass $(H^s_{\mathfrak{a}}(M))$ is a finite set, then Ass $(Ext^s_R(R/\mathfrak{a}, M))$ is finite.*

Proof The proof is the same as Theorem 6.3.9. □

Corollary 6.3.12 *(see [22]) If $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < s$, then $Ext^i_R(R/\mathfrak{a}, M)$ is finitely generated for all $i < s$. In particular, if $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all i, then $Ext^i_R(R/\mathfrak{a}, M)$ is finitely generated for all i.*

6.4 Generalized Local Cohomology

Let M and N be finitely generated R-modules and let $\mathfrak{a}$ be an ideal of R. Then the generalized local cohomology module

$$\mathrm{H}^i_{\mathfrak{a}}(M, N) = \varinjlim_n \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N)$$

was introduced by Herzog in [13] and studied further in [25] and [27]. Note that $\mathrm{H}^i_{\mathfrak{a}}(R, N) = \mathrm{H}^i_{\mathfrak{a}}(N)$. In [28], the set of associated primes of the generalized local cohomology modules is studied. Actually they showed the following simple but useful result.

Lemma 6.4.1 *If M is a finitely generated R-module and N is an $\mathfrak{a}$-torsion R-module then $H^i_{\mathfrak{a}}(M, N) \cong Ext^i_R(M, N)$.*

By using Lemma 3.1 they succeeded in proving the next result that is a generalization of Theorem 6.3.3, cf. [28], Theorem 6.3.1.

Theorem 6.4.2 *Let M and N be finitely generated R-modules and $\mathfrak{a}$ be an ideal of R. Let $t \in \mathbb{N}_0$ be such that $H^i_{\mathfrak{a}}(M, N)$ is finitely generated for all $i < t$ and let K be a finitely generated submodule of $H^t_{\mathfrak{a}}(M, N)$. Then the set Ass $_R H^t_{\mathfrak{a}}(M, N)/K$ is finite.*

Also we have a generalization of Theorem 6.3.4, cf. [1], Theorem 6.2.4.

Theorem 6.4.3 *Let M and N be finitely generated R-modules and $\mathfrak{a}$ be an ideal of R. Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M, N)$ is finitely generated for all $i < t$. Then Hom $_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}$ $(M, N))$ is a finitely generated R-module.*

The following result is a generalization of Theorems 6.3.1(b), 6.3.2, 6.3.3, and 6.3.1.

Corollary 6.4.4 *Let M and N be finitely generated R-modules and $\mathfrak{a}$ be an ideal of R. Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M, N)$ is finitely generated for all $i < t$. Then for any submodule L of $H^t_{\mathfrak{a}}(M, N)$ such that $Ext^1_R(R/\mathfrak{a}, L)$ is a finitely generated R-module (for example L might be a finitely generated R-module), the module $H^t_{\mathfrak{a}}(M, N)/L$ has finitely many associated primes.*

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Proof Let L be a finitely generated submodule of $\mathrm{H}^t_{\mathfrak{a}}(M, N)$. The short exact sequence

$$0 \to L \to \mathrm{H}^t_{\mathfrak{a}}(M, N) \to \mathrm{H}^t_{\mathfrak{a}}(M, N)/L \to 0$$

induces the following exact sequence

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M, N)) \to \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M, N)/L) \to \mathrm{Ext}^1_R(R/\mathfrak{a}, L).$$

Since the left hand (by Theorem 3.3) and the right hand are finitely generated, we have that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M, N)/L)$ is finitely generated. On the other hand

$$\mathrm{Supp}\, \mathrm{H}^t_{\mathfrak{a}}(M, N)/L \subseteq \mathrm{Supp}\, \mathrm{H}^t_{\mathfrak{a}}(M, N) \subseteq \mathrm{V}(\mathfrak{a}).$$

Therefore $\mathrm{Ass}\,(\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M, N)/L)) = \mathrm{Ass}\,(\mathrm{H}^t_{\mathfrak{a}}(M, N)/L)$ is a finite set. □

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Chapter 7

On Inverse Limits of Bézout Domains

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Abstract An example shows that if $A = \varprojlim A_n$ is the inverse limit of an inverse system $\{\varphi_{mn} : A_m \to A_n \mid m \geq n\}$ of Bézout (hence Prüfer) domains A_n, then A need not be a Prüfer (or a Bézout) domain. If, however, each transition map φ_{mn} is surjective, the question whether A must be a Prüfer domain is more subtle. A partial result is given for this context. Enhancement of this result is considered by means of associated inverse systems of CPI-extensions, with applications to Prüfer domains, Bézout domains and locally divided domains.

7.1 Introduction

This note is a sequel to the work initiated on inverse limits of integral domains in [5]. Because much of [5] had to do with applications to certain infinite-dimensional integral domains called $P^{\infty}VD$s, it was natural to restrict attention in [5] to inverse limits of some special types of inverse systems indexed by $\mathbb{N}$, the set of positive integers. The contexts of several other applications in [5] were motivated by the work in [6] on direct limits of integral domains. As a central result in [6] stated that any direct limit (over a directed index set) of Prüfer domains is a Prüfer domain, it was natural to ask in [5] whether the class of Prüfer domains is stable under inverse limit. In the quasilocal case, there is a complete answer [5, Theorem 2.1 (g)]: the inverse limit of any inverse system of valuation domains (indexed by $\mathbb{N}$) is a valuation domain. For the special type of inverse system emphasized in [5], it was established in [5, Theorem 2.21] that the class of Prüfer domains is stable under inverse limit *for that type* of inverse system. The general question of whether the class of Prüfer domains is stable under inverse limits of *arbitrary* inverse systems indexed by $\mathbb{N}$ was left open in [5]. In this note, we resolve that question.

Sadly, the answer is negative, as Example 7.2.1 presents an inverse system of Prüfer domains whose index set is $\mathbb{N}$ and whose inverse limit is not a Prüfer domain. From the point of view of category theory, this fact is somewhat palatable, since a nontrivial product of rings is an inverse limit (granted *not* over a directed index set) and is never an integral domain (Prüfer or otherwise). Nevertheless, and more to the point, we notice that the inverse system $\{\varphi_{mn} : A_m \to A_n \mid m \geq n\}$

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in Example 7.2.1 lacks one important ingredient; namely, its transition maps φ_{mn} are not surjective. We thus come to a sharpening of the question: Is the class of Prüfer domains stable under inverse limits of inverse systems which are indexed by $\mathbb{N}$ and which have surjective transition maps? The bulk of this paper studies this question.

The prime ideals P of $A := \varprojlim A_n$ include those of the A_n (assuming surjective φ_{mn}) but we do not know if that is essentially the complete story, as it was in the earlier context [5, Theorem 2.5 (a)]. (A related problem is that if $B := \varprojlim B_n$ is another inverse limit such that $\mathrm{Spec}(A_n) \cong \mathrm{Spec}(B_n)$ as partially ordered sets for each n, then it need not be the case that $\mathrm{Spec}(A) \cong \mathrm{Spec}(B)$ [11, page 354, lines 1–14; Propositions 2.1 and 3.1]; for a positive partial result in this regard, see [11, Theorem 5.7].) Our methods consider only $P \in \cup\, \mathrm{Spec}(A_n)$ as we seek to determine if A_P is a valuation domain. Theorem 7.2.3 and Corollary 7.2.4 provide a positive answer if each A_n is a Bézout domain and each φ_{mn} is surjective when restricted to unit groups. Proposition 7.2.6 (b) shows that two canonical valuation domains containing A_P are isomorphic and hence, in a sense, equally approximate A_P. One of these canonical extensions of A_P is studied via an associated inverse system in which each A_n is replaced with a suitable CPI-extension (in the sense of [1]) so that each transition map in the new inverse system has kernel a divided prime ideal (in the sense of [2]). The latter inverse system falls under the rubric of [5], thus permitting use of results such as the above-mentioned [5, Theorem 2.21]. For the sake of clarity, some of the "Prüferian" applications in Proposition 7.2.6 (a) are couched in the more general context of locally divided domains (in the sense of [2], [3]). Finally, Remark 7.2.7 explains that if the A_n are merely (commutative) rings rather than integral domains, then even in the presence of surjective transition maps, $\mathrm{Spec}(\varprojlim A_n)$ may be much larger than $\cup\, \mathrm{Spec}(A_n)$.

In addition to the notational conventions indicated above, we mention the following. All rings considered are commutative with identity. If A is a ring, then $U(A)$ denotes the set of units of A, $\mathrm{Spec}(A)$ denotes the set of prime ideals of A and "dimension" refers to the Krull dimension of A. If A is a domain with quotient field K, then an *overring of* A is any ring B such that $A \subseteq B \subseteq K$. Any unexplained material is standard, as in [9], [10].

7.2 Results

We begin with a negative answer to the naïve question.

Example 7.2.1 There exists an inverse system $\{\varphi_{mn} : A_m \to A_n \mid m \geq n\}$ such that A_n is a Bézout (hence Prüfer) domain for each $n \in \mathbb{N}$ but $A := \varprojlim A_n$ is not a Prüfer domain (and hence is not a Bézout domain).

Proof Suppose, for the moment, that there exists an integrally closed integral domain A such that A is not a Prüfer domain and the set of minimal valuation overrings of A is denumerable, say $\{V_i \mid i \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, put $A_n := \cap_{i=1}^{n} V_i$. By [10, Theorem 107], A_n is a Bézout (and, hence, Prüfer) domain for each $n \in \mathbb{N}$. Moreover, $\cap_{n=1}^{\infty} A_n = \cap_{i=1}^{\infty} V_i = A$ since A is integrally closed [9, page 231]. If $m \geq n$ in $\mathbb{N}$, define $\varphi_{mn} : A_m \to A_n$ to be the inclusion map. Then $\{\varphi_{mn} \mid m \geq n\}$ evidently forms an inverse system, but its inverse limit, $\varprojlim A_n = \cap_{n=1}^{\infty} A_n = A$, is not a Prüfer (or a Bézout) domain.

It remains to construct an integral domain A with the properties supposed above. To this end, let k be a countable field, X an indeterminate over k and $V = k(X) + M$ a valuation domain with maximal ideal $M \neq 0$. Then $A := k + M$ has the desired properties. Indeed, A is integrally closed but not a Prüfer domain, by standard facts about $D + M$ constructions [9, Exercise 11 (2), page 202; Exercise 13 (2), page 286]. Also, the set of minimal valuation overrings of A is in one-to-one

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correspondence with the set of (minimal) valuation domains W of $k(X)$ contained properly between k and $k(X)$: see [9, Exercise 13 (2), page 203]. Since k is countable, the set of monic irreducible polynomials in $k[X]$ (resp., $k[X^{-1}]$) is denumerable (cf. [10, Exercise 8, page 8]). It is well known that such polynomials serve to classify the valuation domains W in question (cf. [12]) and so the set of such W is denumerable. □

We next fix the **riding assumptions and notation** for the rest of the paper. We assume given an $\mathbb{N}$-indexed inverse system of integral domains A_k, $\{\varphi_{mn} : A_m \to A_n \mid m \geq n\}$, which has the property that each of its transition maps φ_{mn} is surjective. Put

$$A := \varprojlim A_n, \;\; \Phi_n : A \to A_n \text{ the canonical map, } Q_n := \ker(\Phi_n)$$

and

$$Q_{mn} := \ker(\varphi_{mn}) \text{ for } m \geq n.$$

The next result collects some useful facts. They may be proved as in the corresponding parts of [5, Theorem 2.1, Lemma 2.2 and Proposition 2.4], although the ambient hypotheses for the cited results were more stringent than our current riding assumptions.

Lemma 7.2.2 *(a)* $A = \{(a_n) \in \prod A_n \mid \varphi_{n+1,n}(a_{n+1}) = a_n$ *for each* $n \in \mathbb{N}\}$.

(b) For each $n \in \mathbb{N}$, Φ_n *is surjective and is the composite of the inclusion map* $A \hookrightarrow \prod A_k$ *and the canonical projection* $\prod A_k \to A_n$.

(c) For each $n \in \mathbb{N}$, $Q_n \in \operatorname{Spec}(A)$ *and* $A/Q_n \cong A_n$.

(d) For each $n \in \mathbb{N}$, $Q_n = \{(a_k) \in A \mid a_k = 0$ *for each* $k \leq n\}$.

(e) $Q_1 \supseteq Q_2 \supseteq Q_3 \supseteq \dots$ *and* $\cap Q_n = 0$.

(f) If $r \geq n \in \mathbb{N}$, *then* $Q_{rn} = \Phi_r(Q_n)$, $\Phi_r^{-1}(Q_{rn}) = Q_n$, $\varphi_{r+1,r}$ *restricts to a surjection* $Q_{r+1,n} \to Q_{rn}$, *and* $\varphi_{r+1,r}^{-1}(Q_{rn}) = Q_{r+1,n}$.

(g) If $r \geq n \in \mathbb{N}$, *then* $\varprojlim\{Q_{rn} \mid r \geq n\} = Q_n$ *canonically.*

We turn now to the main question, namely, whether A_n being a Prüfer domain for each n implies that A is a Prüfer domain; i.e., that A_P is a valuation domain for each (without loss of generality) nonzero $P \in \operatorname{Spec}(A)$. Our proofs require the restriction that P contain some Q_ν, a condition that was automatically satisfied by the pullbacks treated in [5]. (See [5, Theorem 2.5 (a)]. We do not know if the riding assumptions of the present paper ensure the $P \supseteq Q_\nu$ condition. See also Remark 7.2.7.) In view of Example 7.2.1, it seems natural to focus first on the case in which each A_n is a Bézout domain. For this context, Theorem 7.2.3 gives a positive conclusion if $\varphi_{n+1,n}(U(A_{n+1})) = U(A_n)$ for each n. (Notice that, since $\varphi_{n+1,n}$ is surjective for each n, the latter condition holds automatically if A_{n+1} is quasilocal, that is a valuation domain, for each n. However, if A_{n+1} is not quasilocal, it need not be the case that $\varphi_{n+1,n}(U(A_{n+1})) = U(A_n)$.) Note that, in contrast with the methods in [5], Theorem 7.2.3 and Corollary 7.2.4 avoid the assumption that $Q_{n+1,n}$ is a divided prime ideal of A_{n+1} for each n.

Theorem 7.2.3 *For each n, suppose that* A_n *is a Bézout domain and that* $\varphi_{n+1,n}$ *induces a surjection* $U(A_{n+1}) \to U(A_n)$. *If, in addition,* $P \in \operatorname{Spec}(A)$ *is such that* $P \supseteq Q_\nu$ *for some* ν, *then* A_P *is a valuation domain.*

Proof It is enough to show that if $\alpha, \gamma \in A_P$, then either $\alpha \in \gamma A_P$ or $\gamma \in \alpha A_P$. Without loss of generality, we may assume that $\alpha, \gamma \in P$. Write $\alpha = (\alpha_n)$, $\gamma = (\gamma_n) \in \prod A_n$. By restricting attention to the (cofinal) set $\{n \in \mathbb{N} \mid n \geq \nu\}$ and relabeling, we may assume that $P \supseteq Q_1$, and so $\alpha_n, \gamma_n \in P_n := \Phi_n(P)$ for each $n \geq 1$. Without loss of generality, $\alpha_n \neq 0$ and $\gamma_n \neq 0$ for all n.

Since A_n is a Bézout domain, it is a GCD-domain (in the sense of [10, page 32]). Let $d_n := \gcd(\alpha_n, \gamma_n)$; in other words, d_n is *a* greatest common divisor of α_n and γ_n in A_n. Then $\alpha_n = d_n\alpha'_n$

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and $\gamma_n = d_n\gamma_n'$, where $\alpha_n', \gamma_n' \in A_n$ and $\gcd(\alpha_n', \gamma_n') = 1$. Fix n for the moment. Then, with $\varphi := \varphi_{n+1,n}$, we have the equations

$$\alpha_n = \varphi(\alpha_{n+1}) = \varphi(d_{n+1})\varphi(\alpha_{n+1}') = d_n\alpha_n',$$

$$\gamma_n = \varphi(\gamma_{n+1}) = \varphi(d_{n+1})\varphi(\gamma_{n+1}') = d_n\gamma_n'.$$

Since A_{n+1} is a Bézout domain, $1 = \gcd(\alpha_{n+1}', \gamma_{n+1}')$ is an A_{n+1}-linear combination of α_{n+1}' and γ_{n+1}'. Applying φ, we see that 1 is an A_n-linear combination of $\varphi(\alpha_{n+1}')$ and $\varphi(\gamma_{n+1}')$. Thus, $\gcd(\varphi(\alpha_{n+1}'), \varphi(\gamma_{n+1}')) = 1$. It now follows via [10, Theorem 49 (a)] from the above displayed equations that

$$\gcd(\alpha_n, \gamma_n) = \varphi(d_{n+1})\gcd(\varphi(\alpha_{n+1}'), \varphi(\gamma_{n+1}')) = \varphi(d_{n+1}).$$

As any two gcds of α_n and γ_n are associates, there exists $u_n \in U(A_n)$ such that $\varphi_{n+1,n}(d_{n+1}) = u_n d_n$.

Since $U(A_1) = \varphi_{21}(U(A_2))$, we may redefine d_2 (to be an associate of the former d_2) so as to ensure that $\varphi_{21}(d_2) = d_1$. (Specifically, replace d_2 with v_2d_2, where $v_2 \in U(A_2)$ satisfies $\varphi_{21}(v_2) = u_1^{-1}$.) Similarly, we may use the hypotheses to redefine $d_3, d_4, \ldots$ so that $\varphi_{n+1,n}(d_{n+1}) = d_n$ for all $n \geq 1$. By *abus de langage*, we keep the above α_n', γ_n' notation. Then $(\alpha_n') \in A$, since $\varphi := \varphi_{n+1,n}$ satisfies

$$d_n\varphi(\alpha_{n+1}') = \varphi(d_{n+1})\varphi(\alpha_{n+1}') = \varphi(\alpha_{n+1}) = \alpha_n = d_n\alpha_n'$$

and $d_n \neq 0$. Similarly, $(\gamma_n') \in A$. Observe that it suffices to show that $(\alpha_n')A_P$ and $(\gamma_n')A_P$ are comparable under inclusion, for $\delta := (d_n) \in A$ satisfies $\alpha = \delta(\alpha_n')$ and $\gamma = \delta(\gamma_n')$. Thus, we may replace α and γ with (α_n') and (γ_n'), respectively. In other words, we may assume that $\gcd(\alpha_n, \gamma_n) = 1$ for each n.

We next give two ways to complete the proof. First, recall that $\gcd(\alpha_n, \gamma_n) = 1$ for each n. Hence, $\alpha_nA_n + \gamma_nA_n = A_n$ for each n. Then localizing at P_n yields that

$$(A_n)_{P_n} = \alpha_n(A_n)_{P_n} + \gamma_n(A_n)_{P_n} \subseteq P_n(A_n)_{P_n} \subset (A_n)_{P_n},$$

the desired contradiction.

The following is an alternate way to finish the proof. Since inverse limit preserves monomorphisms, we can view $A \subseteq D := \varprojlim (A_n)_{P_n}$. As A_n is a Prüfer domain, $(A_n)_{P_n}$ is a valuation domain for each n, and so by [5, Theorem 2.1 (g)], D is a valuation domain. Thus, without loss of generality, $\alpha\gamma^{-1} \in D$. In particular, $\xi_n := \alpha_n\gamma_n^{-1} \in (A_n)_{P_n}$ for all n. Hence, $\xi_n = b_nz_n^{-1}$ for some $b_n \in A_n$ and $z_n \in A_n \setminus P_n$. As $\alpha_n\gamma_n^{-1}$ is in "lowest terms" and A_n is a GCD-domain, it follows that $\gamma_n | z_n$ in A_n, whence $z_n \in A_n\gamma_n \subseteq P_n$, the desired contradiction, thus completing the alternate proof. □

For an example illustrating Theorem 7.2.3, begin with a valuation domain (V, M) having prime spectrum

$$M = P_1 \supset P_2 \supset \cdots \supset P_n \supset P_{n+1} \supset \cdots \supset 0$$

and consider the inverse system defined by $A_n := V/P_n$, with the transition maps $\varphi_{mn} : V/P_m \to V/P_n$ the canonical surjections if $m \geq n$.

Corollary 7.2.4 *For each n, suppose that A_n is a Bézout domain and that $\varphi_{n+1,n}$ induces a surjection $U(A_{n+1}) \to U(A_n)$. If, in addition,* $\mathrm{Spec}(A) = \cup\{\mathrm{im}(\mathrm{Spec}(A_n) \to \mathrm{Spec}(A)) \mid n \in \mathbb{N}\}$*, then A is a Prüfer domain.*

Proposition 7.2.6 studies further the condition that A_P is a valuation domain. First, recall from [1], [7] that if P is a prime ideal of an integral domain R, the *CPI- extension* of R with respect to P is the integral domain given by the following pullback:

$$R(P) := R_P \times_{R_P/PR_P} R/P = R + PR_P.$$

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We assume familiarity with the material on Spec($R(P)$) in [1], [7]. Note also that PR_P is a divided prime ideal of $R(P)$: cf. [1, Proposition 2.5, Theorem 2.4], [2, Lemma 2.4 (b), (c)].

Suppose that $\{\varphi_{mn} : A_m \to A_n \mid m \geq n\}$ satisfies our riding hypotheses. We proceed to define an inverse system $\{\varphi^*_{mn} : A^*_m \to A^*_n \mid m \geq n \geq 2\}$, called the *associated inverse system of* $\{\varphi_{mn}\}$, which is more tractable. For each $n \geq 2$ in $\mathbb{N}$, let

$$A^*_n := A_n(Q_{n1}) = A_n + Q_{n1}(A_n)_{Q_{n1}}.$$

Define $\varphi^*_{n+1,n} : A^*_{n+1} \to A^*_n$ by

$$\varphi^*_{n+1,n}(a + qz^{-1}) = \varphi_{n+1,n}(a) + \varphi_{n+1,n}(q)\varphi_{n+1,n}(z)^{-1}$$

for all $a \in A_{n+1}$, $q \in Q_{n+1,1}$ and $z \in A_{n+1} \setminus Q_{n+1,1}$. Since Lemma 7.2.2 (f) ensures that $Q_{n+1,1} = \varphi^{-1}_{n+1,n}(Q_{n1})$, an easy calculation verifies that $\varphi^*_{n+1,n}$ is well defined. Then the inverse system $\{\varphi^*_{mn}\}$ is obtained by defining

$$\varphi^*_{mn} := \varphi^*_{n+1,n} \circ \varphi^*_{n+2,n+1} \circ \cdots \circ \varphi^*_{m,m-1} \text{ if } m > n+1 \geq 3.$$

By analogy with the riding notation, we put $A^* := \varprojlim A^*_n$, $Q^*_n := \ker(A^* \to A^*_n)$ and $Q^*_{mn} := \ker(\varphi^*_{mn})$ if $m \geq n \geq 2$.

Lemma 7.2.5 (a) establishes that, apart from rescaling by using all $n \geq 2$, $\{\varphi^*_{mn}\}$ satisfies our riding hypotheses, and Lemma 7.2.5 (b) shows that $\{\varphi^*_{mn}\}$ has a desirable property which was assumed for the inverse systems treated in [5].

Lemma 7.2.5 *Let $\{\varphi_{mn} : A_m \to A_n \mid m \geq n\}$ be an $\mathbb{N}$-indexed inverse system of locally divided integral domains for which φ_{mn} is surjective for each $m \geq n$ in $\mathbb{N}$. Let $\{\varphi^*_{mn} : A^*_m \to A^*_n \mid m \geq n\}$ be the associated inverse system (using the notation introduced above). Then:*

(a) φ^*_{mn} is surjective for each $m \geq n \geq 2$ in $\mathbb{N}$.

(b) $Q^*_{n+1,n}$ is a divided prime ideal of A^*_{n+1} for each $n \geq 2$.

Proof (a) Without loss of generality, $m = n + 1$. Then it is easy to verify the assertion by using the explicit construction of $\varphi^*_{n+1,n}$ given above, since Lemma 7.2.2 (f) ensures that $\varphi_{n+1,n}$ sends $Q_{n+1,1}$ onto Q_{n1} and $A_{n+1} \setminus Q_{n+1,1}$ onto $A_n \setminus Q_{n1}$.

(b) Since $Q_{n+1,n} \subseteq Q_{n+1,1}$, a direct calculation using the above explicit construction of $\varphi^*_{n+1,n}$ shows that

$$Q^*_{n+1,n} := \ker(\varphi^*_{n+1,n}) = Q_{n+1,n}(A_{n+1})_{Q_{n+1,1}}.$$

The assertion is a consequence of the following useful fact: if $P \subseteq Q$ are prime ideals of an integral domain R such that R_Q is a divided domain, then PR_Q is a divided prime ideal of $R(Q) := R + QR_Q$. (Apply this fact to $R = A_{n+1}$, $P = Q_{n+1,n}$, and $Q = Q_{n+1,1}$.) To prove the above "useful fact", note by an easy calculation that one has to show that $PR_P = PR_Q$, and so an appeal to the proof of a characterization of locally divided domains [3, Theorem 2.4] completes the argument. □

Proposition 7.2.6 *Let $\{\varphi_{mn} : A_m \to A_n \mid m \geq n\}$ satisfy the riding hypotheses, with $A := \varprojlim A_n$. Let $\{\varphi^*_{mn} : A^*_m \to A^*_n \mid m \geq n\}$ be the associated inverse system, with $A^* := \varprojlim A^*_n$. Then:*

(a) Let $\mathcal{C}$ be a class of integral domains. If $A_n \in \mathcal{C}$ for each $n \in \mathbb{N}$, then $A^* \in \mathcal{C}$ in each of the following cases: $\mathcal{C}$ is the class of all (i) Prüfer domains, (ii) Bézout domains, (iii) divided domains, (iv) locally divided domains.

(b) Suppose that A_n is a locally divided domain for each n (for instance, repeat the hypotheses in (a).) Let $P \in \operatorname{Spec}(A)$ with $P \supseteq Q_1$; take $P_n := \Phi_n(P)$. Put $B := \varprojlim A_n(P_n)$. Then $\mathcal{P} := \varprojlim P_n(A_n)_{P_n} \in \operatorname{Spec}(B)$. Moreover, the canonical injection $A_P \to B_{\mathcal{P}}$ is an isomorphism if and only if the canonical injection $A_P \to \varprojlim (A_n)_{P_n}$ is an isomorphism. Indeed, $B_{\mathcal{P}}$ and $\varprojlim (A_n)_{P_n}$ are isomorphic as A_P-algebras.

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Proof (a) Note that $A_n^* \in \mathcal{C}$ for each $n \geq 2$. Indeed, for (i) and (ii), this holds since each overring of a Prüfer (resp., Bézout) domain is a Prüfer (resp., Bézout) domain [9], while for (iii) and (iv), the proof of [4, Proposition 2.12] combines with [2, Lemma 2.2 (a), (c)] to ensure that the class of divided (resp., locally divided) domains is stable for CPI-extensions. Let $\{\varphi_{mn}^* : A_m^* \to A_n^*\}$ be the associated inverse system, with $A^* := \varprojlim A_n^*$.

The strategy is now to apply appropriate results of [5] to $\{\varphi_{mn}^*\}$. To be able to do so, we must verify that $\{\varphi_{mn}^*\}$ satisfies the riding assumptions of [5]. In view of Lemma 7.2.5, it follows from [5, Remark 2.24] that we need only verify that A_2^* is not a field and $Q_{n+1,n}^* \neq 0$ for all $n \geq 2$.

If A_2^* is a field, then by cofinality, we can delete the index $2 \in \mathbb{N}$. If the concern persists, then by cofinality, we may assume that $A_{n+1}^* = A_{n+1}(Q_{n+1,1})$ is a field for *each* $n \in \mathbb{N}$, whence $Q_{n+1,1} = 0$ and $\varphi_{n+1,1}$ is an isomorphism for each $n \in \mathbb{N}$. In that case, $A \cong A_1 \in \mathcal{C}$ and so, since $A_n^* \cong A_n$ for each n, $A^* \cong \varprojlim A_n = A \in \mathcal{C}$.

Similarly, if passing to cofinal index sets does not remove concerns about $Q_{n+1,n}^*$, then we may assume that $Q_{n+1,n}^* = 0$ for *each* $n \in \mathbb{N}$. By Lemma 7.2.5, it follows that $A^* \cong A_2^* \in \mathcal{C}$.

We may now apply the results of [5] to $\{\varphi_{mn}^*\}$ as follows: for (i), use [5, Theorem 2.21]; for (ii), use [5, Corollary 2.23]; for (iii), use [5, Corollary 2.17 (a)]; and for (iv), use [5, Corollary 2.17 (b)].

(b) As $P \supseteq Q_1 \supseteq Q_n = \ker(\Phi_n)$, we have $P_n \in \mathrm{Spec}(A_n)$ for each n. As $P = \Phi_n^{-1}(P_n)$, we infer a canonical ring homomorphism $\alpha : A_P \to D := \varprojlim (A_n)_{P_n}$. It is straightforward to use the construction of α to verify that α is an injection. We next sketch how to rework the construction of the "associated inverse system" to produce B.

We produce an inverse system $\{\psi_{mn} : B_m \to B_n \mid m \geq n \geq 2\}$ as follows. For each $n \in \mathbb{N}$, consider the CPI-extension $B_n := A_n(P_n) = A_n + P_n(A_n)_{P_n}$. As $\varphi_{n+1,n}^{-1}(P_n) = P_{n+1}$ (as a consequence of Lemma 7.2.2 (f), (g)), we can mimic the construction of $\varphi_{n+1,n}^*$ to produce a surjective ring homomorphism $\psi_{n+1,n} : B_{n+1} \to B_n$ and, hence, the required surjection $\psi_{mn} : B_m \to B_n$ by composition if $m > n + 1 \geq 3$. We show that the methods of [5] apply, more or less, in studying $B := \varprojlim B_n$.

Observe that the kernel of $\psi_{n+1,n}$ is $Q_{n+1,n}(A_{n+1})_{P_{n+1}}$. Since the hypothesis in (b) ensures that $(A_{n+1})_{P_{n+1}}$ is a divided domain, reasoning as in the proof of Lemma 7.2.5 (b) shows that $\ker(\psi_{n+1,n})$ is a divided prime ideal of B_{n+1}. There are two ways that the methods of [5] might not apply: either each such $\psi_{n+1,n}$ is an isomorphism or each B_n is a field. In the first case, all the canonical maps in question are isomorphisms, since A_P, $B_{\mathcal{P}}$ and $\varprojlim (A_n)_{P_n}$ all canonically identify with $(A_1)_{P_1}$ in this case. In the second case, each $P_n = 0$ by the standard theory of CPI-extensions, whence the inverse systems defining A and B are essentially the same, with A_P, $B_{\mathcal{P}}$ and $\varprojlim (A_n)_{P_n}$ all canonically identified with the quotient field of A_1 in this case. Thus, we can assume henceforth that the inverse system $\{\psi_{mn}\}$ satisfies the riding assumptions in [5].

View $\mathcal{P} := \varprojlim P_n(A_n)_{P_n}$ canonically inside $\varprojlim B_n = B$. It is straightforward to use the condition $\varphi_{n+1,n}^{-1}(P_n) = P_{n+1}$ to verify that $\mathcal{P} \in \mathrm{Spec}(B)$. (The same conclusion holds in the two cases noted above, for then $\mathcal{P} \cong P_1(A_1)_{P_1}$ and $B \cong B_1$.) Therefore, by [5, Proposition 2.15 (d)], the canonical ring homomorphism $\beta : B_{\mathcal{P}} \to E := \varprojlim (B_n)_{P_n(A_n)_{P_n}}$ is an isomorphism. Moreover, there is an isomorphism $\gamma : D \to E$ because one has compatible isomorphisms $(A_n)_{P_n} \to (B_n)_{P_n(A_n)_{P_n}}$ at every level. To finish the proof of (b), it suffices to find a ring homomorphism $\delta : A_P \to B_{\mathcal{P}}$ such that $\beta \circ \delta = \gamma \circ \alpha : A_P \to E$.

By composing the inclusions $A \to B$ and $B \to B_{\mathcal{P}}$, one obtains an injection $f : A \to B_{\mathcal{P}}$. We claim that $f(A \setminus P) \subseteq B \setminus \mathcal{P}$. Indeed, if $a = (a_n) \in A \cap \mathcal{P}$, then $a_n \in P_n(A_n)_{P_n} \cap A_n = P_n$ for each n, whence $a \in \varprojlim P_n = P$, thus proving the claim. The universal mapping property of localization produces a unique ring homomorphism $\delta : A_P \to B_{\mathcal{P}}$ that extends f, and a routine calculation verifies that $\beta \circ \delta = \gamma \circ \alpha$, to complete the proof. □

In the context of Proposition 7.2.6 (b), suppose that A_n is a Prüfer (hence, locally divided) domain for each n. Then both $B_{\mathcal{P}}$ and $\varprojlim (A_n)_{P_n}$ are valuation domains, by [5, Theorem 2.21 and Theorem 2.1 (g)]. (In the two degenerate cases noted above, the assertion about $B_{\mathcal{P}}$ follows since $B \cong B_1$ is

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Prüfer in these cases.) Thus, we come to the main point of Proposition 7.2.6 (b): these two standard ways to produce a valuation domain containing A_P are isomorphic, and A_P coincides with the first of these valuation domains if and only if A_P coincides with the second.

Remark 7.2.7 It is well known (cf. [6]) that if $\{B_i\}$ is a *directed* system of (commutative) rings indexed by a directed index set, then $\mathrm{Spec}(\varinjlim B_i) \cong \varprojlim \mathrm{Spec}(B_i)$. Accordingly, it may seem reasonable to speculate that if $\{D_n\}$ is an *inverse* system of rings which is indexed by $\mathbb{N}$ and has surjective transition maps, then there should be a close connection between $\mathrm{Spec}(\varprojlim D_n)$ and $\varinjlim \mathrm{Spec}(D_n)$. If each D_n is an integral domain, this is indeed so for certain natural inverse systems: see [5, Theorem 2.5 (a)]. However, the following example shows that the situation can be more complicated if the D_n are not integral domains. In this example, each D_n is a principal ideal ring.

Let $\{k_i \mid i \in \mathbb{N}\}$ be any sequence of fields. For each $n \in \mathbb{N}$, put $D_n := \prod_{i=1}^{n} k_i$. If $r \geq n$ in $\mathbb{N}$, let $\varphi_{rn} : D_r \to D_n$ denote the canonical projection map; of course, each φ_{rn} is surjective. Moreover, $\varinjlim \mathrm{Spec}(D_n)$ is countable, since it can be viewed as a union of a countable chain of finite sets. However, $\{\varphi_{rn} \mid r \geq n\}$ leads to $D := \varprojlim D_n$ which is such that $\mathrm{Spec}(D)$ is not countable. Indeed, $D \cong \prod_{i=1}^{\infty} k_i$ canonically, and so $\mathrm{Spec}(D)$ is the Stone-Čech compactification of $\mathbb{N}$ when $\mathbb{N}$ is endowed with the discrete topology. (The "Stone-Čech" part of the preceding assertion seems to be folklore. In case $k_i = \mathbb{R}$ for all i, this piece of folklore follows from [8, items 7.10 and 7.11, page 105].) We conclude from this example that care must be taken if one attempts to extend the work in [5] and this note to $\mathbb{N}$-indexed inverse systems having surjective transition maps for arbitrary (commutative) rings.

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Chapter 8

An Elementary Proof of Grothendieck's Theorem

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Abstract We prove a theorem analogous to Smith's theorem but for matrices with Laurent polynomials as entries. Then we show that this result is equivalent to Grothendieck's theorem about vector bundles on the projective line.

8.1 Introduction

Throughout this article k will be an arbitrary field. We will consider matrices of Laurent polynomials, i.e., elements of $k[x^{-1}, x]$. We will consider row operations of one type and column operations of another type on such matrices. The row operations will be those of multiplying a row by a scalar $\alpha \in k$, $\alpha \neq 0$, interchanging two rows and then adding a multiple of one row by an element of $k[x]$ to another distinct row. The column operations are similar except that here we can add a multiple of a column by an element of $k[x^{-1}]$ (instead of $k[x]$) and add it to another distinct column. So we will say two matrices are equivalent if we can obtain one from the other by using a sequence of such row and column operations.

Given a column matrix $C = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$ we define $deg\ C$ to be the maximum of the $deg\ p_i$ where we set $deg\ 0 = -\infty$. In a similar manner we define $ord\ C$ to be the minimum of the $ord\ p_i$ where $ord\ 0 = +\infty$ as usual.

On such a column matrix $C \neq 0$ we will perform our row operations but where the row operations of the third kind are restricted to the following: if i is such that $deg\ p_i$ is minimal among the $deg\ p_i$,

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where $p_i \neq 0$, then if $p_j \neq 0$ $(j \neq i)$ we replace p_j with $p_j + \alpha\, x^s\, p_i$ $(\alpha \in k,\ s \geq 0)$ where

$$deg\ (p_j + \alpha\, x^s\, p_i) < deg\ p_j.$$

Then we observe that if we can get C' from C to be a sequence of such operations then we have

$$ord\ C \leq ord\ C' \leq deg\ C' \leq deg\ C.$$

And then clearly we can get such a C' of the form $C = \begin{pmatrix} \bar{p} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and so

$$ord\ C \leq ord\ \bar{p} \leq deg\ \bar{p} \leq deg\ C.$$

Of course this is just essentially part of the Smith algorithm for putting a matrix with entries in $k[x]$ in a diagonal form.

For a row matrix $R = (p_1\ \cdots\ p_n)$ we have an analogous claim except that here we consider the $p_i \neq 0$ of largest order. Then we replace $p_j \neq 0$ $(j \neq i)$ with $p_j + \alpha x^{-s}$ $(\alpha \in k,\ s \geq 0)$ so that $ord(p_j + \alpha x^{-s} p_i) > ord\ p_j$. And again with $R \neq 0$ we can get R' from R to be a sequence of such operations with

$$ord\ R \leq ord\ R' \leq deg\ R' \leq deg\ R,$$

and finally get $(\bar{p}\ 0 \cdots\ 0)$ with

$$ord\ R \leq ord\ \bar{p} \leq deg\ \bar{p} \leq deg\ R$$

where we define $ord\ R$ and $deg\ R$ as we defined $ord\ C$ and $deg\ C$ for column matrices. These observations will be applied mainly to the rows of 2×2 matrices.

8.2 The Main Theorem

Theorem 8.2.1 *If $P = (p_{ij}) \in \mathcal{M}_n(k[x^{-1}, x])$ and if $det\ P = \alpha x^s$ for some $s \in \mathbb{Z}$ and $\alpha \in k$, $\alpha \neq 0$ then P is equivalent to a unique diagonal matrix of the form $diag(x^{a_1}, x^{a_2}, \cdots, x^{a_n})$ with $a_1 \geq a_2 \geq \cdots \geq a_n$.*

Proof We first observe that using only row operations we can get P equivalent to an upper triangular matrix. And then using the fact that $det\ P = \alpha x^s$ we see that we can assume all the diagonal entries are of the form x^a for $a \in \mathbb{Z}$. □

Now to prove the first part of our theorem we will prove several lemmas about 2×2 matrices.

Lemma 8.2.2 *If $P = \begin{pmatrix} x^a & p \\ 0 & x^b \end{pmatrix}$ where $a \geq b$ then P is equivalent to $\begin{pmatrix} x^a & 0 \\ 0 & x^b \end{pmatrix}$*

Proof Using our row operations of taking multiples of the second row and adding to the first row we see that we can assume that $deg\ p < b$. Using the column operations we see that we can also assume $ord\ p > a$. So since $b \leq a$ this means we can get $p = 0$ by the indicated operations, i.e., we can get $\begin{pmatrix} x^a & 0 \\ 0 & x^b \end{pmatrix}$. □

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Lemma 8.2.3 *If* $P = \begin{pmatrix} x^a & 0 \\ 0 & x^b \end{pmatrix}$ *and if* $b > a$ *then* P *is equivalent to some* $\begin{pmatrix} x^a & \bar{p} \\ 0 & x^b \end{pmatrix}$ *where either* $\bar{p} = 0$ *or* $a < ord\ \bar{p} \leq deg\ \bar{p} < b$.

Proof The argument is essentially that of the proof of Lemma 8.2.2. □

The next two lemmas are crucial to our argument.

Lemma 8.2.4 *If* $P = \begin{pmatrix} x^a & p \\ 0 & x^b \end{pmatrix}$ *where* $p \neq 0$ *and where* $a < ord\ p \leq deg\ p < b$ *then* P *is equivalent to a matrix of the form* $\begin{pmatrix} x^{\bar{b}} & \bar{p} \\ 0 & x^{\bar{a}} \end{pmatrix}$ *where* $a < \bar{a}, \bar{b} < b$ *and where* $a + b = \bar{a} + \bar{b}$.

Proof We consider the row $C = (x^a\ p)$. Using our observations about column matrices above, we see that with one column operation we can get a $\bar{C} = (\bar{p}\ p)$ with $a < ord\ \bar{p}$. But now applying our operations on the column matrix $\bar{C}$ we see we can finally get a matrix $(p'\ 0)$. But it is easy to see that $p'|det\ P$ so in fact we can assume $p' = x^{a'}$. Then by our observations on degrees and orders in the column situations we see that $a < a' < b$. If we apply the same column operations we applied to $(x^a\ p)$ and then to $(\bar{p}\ p)$ to the original matrix $P = \begin{pmatrix} x^a & p \\ 0 & x^b \end{pmatrix}$ we see that we get P equivalent to a matrix of the form $\begin{pmatrix} x^{\bar{a}} & 0 \\ \bar{p} & p' \end{pmatrix}$ since $x^{\bar{a}}\ p' = det\ P = x^{a+b}$ and we see that $p' = x^{\bar{b}}$ where $\bar{a} + \bar{b} = a + b$. Then since $a < \bar{a} < b$ we also get $a < \bar{b} < b$. Then exchanging the rows and exchanging the columns of $\begin{pmatrix} x^{\bar{a}} & 0 \\ \bar{p} & x^{\bar{b}} \end{pmatrix}$ we get the desired matrix. □

Lemma 8.2.5 *If* $\begin{pmatrix} x^a & p \\ 0 & x^b \end{pmatrix}$ *where* $p \neq 0$ *and where* $a < ord\ p < deg\ p < b$ *then* P *is equivalent to a matrix of the form* $\begin{pmatrix} x^{\bar{a}} & 0 \\ 0 & x^{\bar{b}} \end{pmatrix}$ *with* $a < \bar{a}, \bar{b} < b$.

Proof We first apply Lemma 8.2.4. Then we apply Lemma 8.2.2 or Lemma 8.2.3 (whichever is applicable), and then Lemma 8.2.4 again (if applicable) etc. It is clear that our procedure terminates and that then we have the desired matrix. □

Proof of Theorem 8.2.1. We now prove the first claim of our theorem. So we assume P is upper triangular and that it has $x^{a_1}, x^{a_2}, \cdots, x^{a_n}$ as its diagonal entries. If we apply our lemmas 8.2.2 to 8.2.5 to the 2×2 principal submatrix $\begin{pmatrix} x^{a_1} & p_{12} \\ 0 & x^{a_2} \end{pmatrix}$ we see that this matrix is equivalent to a diagonal matrix. But then applying the same operations to P we see that we can assume also that $p_{12} = 0$. Then using the same argument on the submatrix $\begin{pmatrix} x^{a_2} & p_{23} \\ 0 & x^{a_3} \end{pmatrix}$ we see that we can assume $p_{23} = 0$. Continuing we see that we can assume now that P is upper triangular and that the first super diagonal is 0. Now we begin again but with the submatrix $\begin{pmatrix} x^{a_1} & p_{13} \\ 0 & x^{a_3} \end{pmatrix}$ and then the submatrix $\begin{pmatrix} x^{a_2} & p_{24} \\ 0 & x^{a_4} \end{pmatrix}$ and so forth and see that we can also assume the second superdiagonal is 0. Then finally we get P equivalent to a diagonal matrix. Exchanging rows and exchanging columns if necessary we can get the diagonal entries in any desired order. Finally, in the next section we will exhibit the connection with Grothendieck's theorem. The easy argument for uniqueness in Grothendieck's theorem will give us the uniqueness of our diagonal matrix. □

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8.3 Grothendieck's Theorem

Let R be a commutative ring and let us consider the scheme

$$(\mathbf{P}^1(R) = Proj\ R[x_0, x_1], \mathcal{O}).$$

We see that the category of quasi-coherent sheaves over the projective line can be considered in terms of certain representations of the quiver $\bullet \to \bullet \leftarrow \bullet$. For if we take the basic affine open sets $D_+(x_0)$, $D_+(x_1)$ y $D_+(x_0) \cap D_+(x_1) = D_+(x_0x_1)$ covering the projective line, we have the inclusions

$$D_+(x_0) \hookleftarrow D_+(x_0x_1) \hookrightarrow D_+(x_1),$$

so applying the structure sheaf $\mathcal{O}$ associated to $\mathbf{P^1}(R)$ we get

$$\mathcal{O}(D_+(x_0)) \hookrightarrow \mathcal{O}(D_+(x_0x_1)) \hookleftarrow \mathcal{O}(D_+(x_1)),$$

but now, $\mathcal{O}(D_+(x_0)) = R[x_0, x_1]_{(x_0)}$, $\mathcal{O}(D_+(x_0x_1)) = R[x_0, x_1]_{(x_0x_1)}$ and

$$\mathcal{O}(D_+(x_1)) = R[x_0, x_1]_{(x_1)}.$$

So we may identify

$$R[x_0, x_1]_{(x_0)},\ R[x_0, x_1]_{(x_0x_1)},\ \text{and } R[x_0, x_1]_{(x_0)}$$

with the rings $R[\frac{x_1}{x_0}]$, $R[\frac{x_1}{x_0}, \frac{x_0}{x_1}]$, $R[\frac{x_0}{x_1}]$ respectively. So if we call $x = x_1/x_0$ we follow that the scheme $(\mathbf{P^1}(R), \mathcal{O})$ can be seen as a representation of the quiver $\bullet \to \bullet \leftarrow \bullet$, given by

$$R[x] \hookrightarrow R[x, x^{-1}] \hookleftarrow R[x^{-1}].$$

Hence, a sheaf of quasi-coherent modules $\mathcal{F}$ on $\mathbf{P^1}(R)$ is a sheaf of $\mathcal{O}$-modules, that is, a representation of the form

$$M \xrightarrow{f} P \xleftarrow{g} N,$$

with $M \in R[x]$-Mod, $N \in R[x^{-1}]$-Mod and $P \in R[x, x^{-1}]$-Mod, and with f a $R[x]$-linear map and g a $R[x^{-1}]$-linear, satisfying the quasi-coherence property, that is $\mathcal{F}|_{Spec R[x]} \cong \widetilde{M}$, $\mathcal{F}|_{Spec R[x^{-1}]} \cong \widetilde{N}$ and $\mathcal{F}|_{Spec R[x,x^{-1}]} \cong \widetilde{P}$. Since $\widetilde{M}$ and $\widetilde{N}$ are also quasi-coherent, it follows that

$$\widetilde{M}|_{Spec R[x,x^{-1}]} \cong \widetilde{P} \cong \widetilde{N}|_{Spec R[x,x^{-1}]},$$

so

$$P = \Gamma(Spec R[x, x^{-1}], \widetilde{P}) \cong \widetilde{M}(Spec R[x, x^{-1}]) = S^{-1}M$$

and

$$P = \Gamma(Spec R[x, x^{-1}], \widetilde{P}) \cong \widetilde{N}(Spec R[x, x^{-1}]) = T^{-1}N,$$

being $S = \{1, x, x^2, \cdots\}$, $T = \{1, x^{-1}, x^{-2}, \cdots\}$ and the isomorphism are just $S^{-1}f$ and $T^{-1}g$.

Considering the category in this way we are able to give a short and elementary proof of Grothendieck's theorem (Theorem 8.3.6).

We present some well-known results concerning quasi-coherent sheaves over $\mathbf{P^1}(R)$ that are easy to prove in terms of our representations. We shall use these later in proving Grothendieck's Theorem. Some of the results presented now are included in [1, 2].

We begin by classifying all representations of the form $R[x] \xrightarrow{f} R[x, x^{-1}] \xleftarrow{g} R[x^{-1}]$.

Proposition 8.3.1 *Each representation of the form* $R[x] \xrightarrow{f} R[x, x^{-1}] \xleftarrow{g} R[x^{-1}]$ *is isomorphic to some* $R[x] \hookrightarrow R[x, x^{-1}] \xleftarrow{x^n} R[x^{-1}]$, *with* $n \in \mathbb{Z}$.

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Proof We may define (see [1]) a pair of adjoint functors (D, H) between the categories of $R[x]$-modules and $\mathfrak{Qco}(\mathbf{P^1}(R))$ in the following way, $D : R[x]\text{-Mod} \to \mathfrak{Qco}(\mathbf{P^1}(R))$ defined by $D(L) = L \xrightarrow{i} S^{-1}L \xleftarrow{id} S^{-1}L$ is a right adjoint of $H : \mathfrak{Qco}(\mathbf{P^1}(R)) \to R[x]\text{-Mod}$, given by $H(M \to P \leftarrow N) = M$. Then, by using this, we have

$$\begin{tikzcd} R[x] \arrow[r, "f"] \arrow[d, "id"] & R[x, x^{-1}] \arrow[d, "h^{-1}"] & R[x^{-1}] \arrow[l, "g"'] \arrow[d, "h^{-1} \circ g"] \\ R[x] \arrow[r, hook] & R[x, x^{-1}] \arrow[r, equal] & R[x, x^{-1}] \end{tikzcd}$$

where $h = (S^{-1}f)^{-1}$, and from this it follows

$$\begin{tikzcd} R[x] \arrow[r, hook] \arrow[d, "id"] & R[x, x^{-1}] \arrow[d, "h"] & R[x^{-1}] \arrow[l, "d"'] \arrow[d, "id"] \\ R[x] \arrow[r, "f"] & R[x, x^{-1}] & R[x^{-1}] \arrow[l, "g"'] \end{tikzcd}$$

(where $d = h^{-1} \circ g = (S^{-1}f) \circ g$). Then, since columns are isomorphisms we deduce that

$$(R[x] \xrightarrow{f} R[x, x^{-1}] \xleftarrow{g} R[x^{-1}]) \cong (R[x] \hookrightarrow R[x, x^{-1}] \xleftarrow{d} R[x^{-1}])$$

(notice that $R[x] \hookrightarrow R[x, x^{-1}] \xleftarrow{d} R[x^{-1}]$ is in $\mathfrak{Qco}(\mathbf{P^1}(R))$ because $T^{-1}d = S^{-1}f \circ T^{-1}g$ is an isomorphism). But if $T^{-1}d : R[x, x^{-1}] \to R[x, x^{-1}]$ is an isomorphism, $T^{-1}d(1)$ must be a unit of $R[x, x^{-1}]$, so $d = u \cdot x^n$, with $u \in R$ and $n \in \mathbb{Z}$; in fact we can suppose $d = x_n$ because $R[x] \hookrightarrow R[x, x^{-1}] \xleftarrow{ux^n} R[x^{-1}]$ and $R[x] \hookrightarrow R[x, x^{-1}] \xleftarrow{x^n} R[x^{-1}]$ are obviously isomorphic. Finally, we see that x^n and x^m give isomorphic representations if, and only if, $n = m$. If $R[x] \hookrightarrow R[x, x^{-1}] \xleftarrow{x^n} R[x^{-1}]$ and $R[x] \hookrightarrow R[x, x^{-1}] \xleftarrow{x^m} R[x^{-1}]$ are isomorphic, we have a diagram

$$\begin{tikzcd} R[x] \arrow[r, hook] \arrow[d, "\alpha"] & R[x, x^{-1}] \arrow[d, "\beta"] & R[x^{-1}] \arrow[l, "x^n"'] \arrow[d, "\gamma"] \\ R[x] \arrow[r, hook] & R[x, x^{-1}] & R[x^{-1}] \arrow[l, "x^m"'] \end{tikzcd}$$

with commutative squares. But it is clear that $\alpha = \cdot z$, $\beta = k \cdot x^l$ and $\gamma = \cdot z'$, for some $0 \neq k, z, z' \in R$, $l \in \mathbb{Z}$, and then, by the commutativity of the first square, it follows $k \cdot x^l = z'$, so $k = z'$ and $l = 0$, and from the second square, $z'x^n = zx^m$, so $n = m$. $\square$

In terms of quasi-coherent sheaves, a representation $R[x] \hookrightarrow R[x, x^{-1}] \xleftarrow{x^n} R[x^{-1}]$, with $n \in \mathbb{Z}$, corresponds to the (unique) line bundles of degree n over $\mathbf{P}^1$, which is denoted by $\mathcal{O}(n)$. So this justifies the following definition

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Definition 8.3.2 A representation $R[x] \hookrightarrow R[x, x^{-1}] \xleftarrow{x^n} R[x^{-1}]$, $n \in \mathbb{Z}$ is denoted by $\mathcal{O}(n)$.

Proposition 8.3.3 $\mathcal{O}(n) \otimes \mathcal{O}(m) \cong \mathcal{O}(n+m)$.

Proof This is obvious because, in general, if A, B are R-modules, $T^{-1}A \otimes_{T^{-1}R} T^{-1}B \cong T^{-1}(A \otimes_R B)$, for any multiplicatively closed set T, and this isomorphism is precisely $a/t \otimes b/t' \to (a \otimes b)/tt'$ (notice that $S^{-1}R[x] = R[x, x^{-1}]$). □

Another well-known result which is easy to prove under our notation is the following.

Proposition 8.3.4 *Let $m, n \in \mathbb{Z}$ be two integers. Then* $\mathrm{Hom}(\mathcal{O}(m), \mathcal{O}(n))$ *is trivial if $m > n$ and is equal to the space of polynomials of degree $n - m$ whenever $m \leq n$.*

Proof Let (f, g, h) be a morphism between $\mathcal{O}(m)$ and $\mathcal{O}(n)$, so f is an $R[x]$-morphism, g is an $R[x^{-1}, x]$-morphism and h is an $R[x^{-1}]$-morphism. Then we must have $x^m g(1) = x^n h(1)$, and $f(1) = g(1) \in R[x]$, so $x^{m-n} g(1) = h(1) \in R[x^{-1}]$, hence $m - n \leq 0$ and $g(1) = f(1)$ is a polynomial of degree less than or equal to $n - m$, which determines uniquely the morphism $\mathcal{O}(m) \to \mathcal{O}(n)$. □

Corollary 8.3.5 *The space of 0-cohomologies of $\mathcal{O}(n)$ is trivial if $n < 0$ and is the space of polynomials of degree less than or equal to n whenever $n \geq 0$.*

Proof This is obvious, by noticing that $H^0(\mathcal{O}(n)) = \mathrm{Hom}(\mathcal{O}(0), \mathcal{O}(n))$, and applying Proposition 8.3.4. □

It is very well known [4] that vector bundles over the projective line, $\mathbf{P^1}$, are direct sums of line bundles in an essentially unique way (Grothendieck's theorem). The representations of $\mathfrak{Qco}(\mathbf{P^1}(k))$ which correspond to vectors bundles are $M \to P \leftarrow N$, with M, N finitely generated and free (for example $k[x] \hookrightarrow k[x^{-1}, x] \xleftarrow{x} k[x^{-1}]$). In this section, we are going to prove this theorem, in terms of representations of the quiver $\bullet \to \bullet \leftarrow \bullet$.

Theorem 8.3.6 *(Grothendieck) Each representation of $\mathfrak{Qco}(\mathbf{P^1}(k))$ of the form $M \to P \leftarrow N$, with M, N finitely generated and free, is a direct sum of*

$$\mathcal{O}(j_i) \equiv k[x] \hookrightarrow k[x^{-1}, x] \xleftarrow{x^{j_i}} k[x^{-1}],$$

$j_i \in \mathbb{Z}$ $i = 1, \cdots, n$ with $j_1 \leq j_2 \leq \cdots \leq j_n$. Moreover the integers $\{j_1, \cdots, j_n\}$ are uniquely determined.

Proof First of all note we can suppose $M \xrightarrow{f} P \xleftarrow{g} N$, with $M = k[x]^n$, $P = k[x^{-1}, x]^n$, $N = k[x^{-1}]^n$, is of the form $M \hookrightarrow P \xleftarrow{h} N$, by using the right adjoint functor. Let P be the $n \times n$ matrix associated to h, $P = (p_{ij})$, $p_{ij} \in k[x^{-1}, x]$. We know the $k[x]$-linear map h has a unique extension to a $k[x^{-1}, x]$-isomorphism between $k[x^{-1}, x]^n$, so $det(P)$ is a unit of $k[x^{-1}, x]$, that is, $det(P) = ux^l$, $l \in \mathbb{Z}$, $0 \neq u \in R$ and, in fact, we can suppose $det(P) = x^l$, $l \in \mathbb{Z}$. Changing a base of M (over $k[x]$) just amounts to performing our row operations on P. Likewise changing a base of N corresponds to our column operations on P. So we can assume P is a diagonal matrix. This proves that each of our representations is a direct sum as desired

To get uniqueness we follow an argument given by Grauert and Remmert in [3]. Let us suppose we have two decompositions

$$\mathcal{O}(j_1) \oplus \cdots \oplus \mathcal{O}(j_n) \cong \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)$$

with $j_1 \leq \cdots \leq j_n$ and $k_1 \leq \cdots \leq k_n$. Let i be the first index for which $j_i \neq k_i$ and suppose $j_i < k_i$. By Proposition 8.3.3, we have

$$\mathcal{O}(j_i - j_1) \oplus \cdots \oplus \mathcal{O} \oplus \mathcal{O}(j_i - j_{i+1}) \oplus \cdots \oplus \mathcal{O}(j_i - j_n) \cong$$

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$$\mathcal{O}(j_i - k_1) \oplus \cdots \oplus \mathcal{O}(j_i - k_i) \oplus \mathcal{O}(j_i - k_{i+1}) \oplus \cdots \oplus \mathcal{O}(k_n)$$

then the number of $\mathcal{O}(t)$'s with $t \geq 0$ is different in both sides, which leads to a contradiction, by Corollary 8.3.5, with the dimension of 0-cohomologies in both sides. □

Remark 8.3.7 Straightforward modifications of the proof of Theorem 8.3.6 allow to prove the analogous result for a noncommutative case, that is, for the decomposition of a "noncommutative" vector bundle of the form

$$k[x;\sigma] \xrightarrow{f} k[x, x^{-1};\sigma] \xleftarrow{g} k[x^{-1};\sigma],$$

where $\sigma : k \to k$ is an automorphism.

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Chapter 9

Gorenstein Homological Algebra

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9.1 Introduction

This is a survey article concerning the variety of relative homological algebra that is now called Gorenstein homological algebra. We will give a brief history of the subject, point out some connections with other areas, and finally give some support to:

Metatheorem (Henrik Holm): Every result in classical homological algebra has a counterpart in Gorenstein homological algebra.

Daniel Gorenstein wrote his thesis [2] under Zariski at Harvard. In it he studied certain singularities of plane algebraic curves. These would now be called Gorenstein singularities and their

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associated local rings would be called Gorenstein local rings. His work soon motivated the notion of a Gorenstein local ring of arbitrary Krull dimension. Then Bass (in [1]) wrote his famous "On the ubiquity of Gorenstein rings". It seems that Bass intended the title to be a historical comment. It was a prophetic one. Gorenstein rings and related Gorenstein topics have surfaced in commutative algebra, in algebraic combinatorics [5], in the repair of the proof of Fermat's last theorem [6], and in the active area of Gorenstein liaison in algebraic geometry ([3] and [4]).

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9.2 Tate Homology and Cohomology

One of the basic ideas of Gorenstein homological algebra is that of a Gorenstein projective module. These are modules which admit a certain complete projective resolution. The idea for these resolutions predates Gorenstein's work and seems to have first occurred in the work (unpublished) of John Tate.

We recall the facts. Let G be a finite group and consider the group ring ZG and the ZG-module Z with the trivial action ($gn = n$ for all $g \in G,\ n \in Z$). Then Z is reflexive ($Z \cong Z^{**}$ naturally where $Z^* = \mathrm{Hom}_{ZG}(Z, ZG)$ is the algebraic dual). Every left (right) ZG-module is also a right (left) ZG-module by using the antiautomorphism of ZG corresponding to the function $g \mapsto g^{-1}$ ($g \in G$). So then in fact $Z \cong Z^*$.

It is not hard to establish that $\mathrm{Ext}^n(Z, ZG) = 0 = \mathrm{Ext}^n(Z^*, ZG)$ for $n > 0$. So beginning with a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to Z \to 0$$

of Z as a left module and with each P_n finitely generated and projective (ZG is left and right noetherian) and using the above we see that by taking duals we get an exact sequence

$$0 \to Z^* \to P_0^* \to P_1^* \to P_2^* \to \cdots$$

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with each P_n^* a finitely generated projective right module. But considering these and Z^* as left modules and using the isomorphism $Z \cong Z^*$ to splice the exact sequences

$$0 \to Z^* \to P_0^* \to P_1^* \to P_2^* \to \cdots$$

and

$$\cdots \to P_2 \to P_1 \to P_0 \to Z \to 0$$

we get an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to P_0^* \to P_1^* \to P_2^* \to \cdots$$

of finitely generated projective modules. This exact sequence has the property that $Z = \mathrm{coker}(P_1 \to P_0)$ and is such that its algebraic dual is also exact.

This complex is a homological invariant of Z and so is used to compute Tate homology and cohomology in all degrees. The cohomology is computed by applying $\mathrm{Hom}_{ZG}(-, N)$ to the complex for any left ZG-module and then computing the cohomology. Using $M \otimes_{ZG}$- for a right ZG-module M we get the homology with the analogous procedure.

See Chapter XII of [1] or II.7 of [1] for treatments of Tate homology and cohomology.

In what follows, a *complete projective resolution* of a module M (if such exists) will mean an exact sequence

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to P^1 \to P^2 \to \cdots$$

of projective modules such that $M = \ker(P^0 \to P^1)$ and such that the sequence stays exact if we apply the function $\mathrm{Hom}(-, P)$ for any projective module.

The complex constructed above for $\mathbb{Z}$ over ZG is a complete projective resolution of Z.

A *complete injective resolution* of a module is defined dually.

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9.3 Auslander and Gorenstein Rings

Let R be a commutative and local noetherian ring with maximal ideal $\mathcal{M}$. Elements $x_1, x_2, ..., x_d \in \mathcal{M}$ are said to form an *R-sequence* if

$$R \xrightarrow{x_1} R,\ \frac{R}{(x_1)} \xrightarrow{x_2} \frac{R}{(x_1)}, \cdots, \frac{R}{(x_1, ..., x_{d-1})} \xrightarrow{x_d} \frac{R}{(x_1, ..., x_{d-1})}$$

are injections. If it is possible to find such $x_1, x_2, ..., x_d$ so that $\mathcal{M} = (x_1, x_2, ..., x_d)$ then R is said to be regular local. A weaker condition is that we can find such $x_1, x_2, ..., x_d$ so that $(x_1, x_2, ..., x_d)$

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is large in $\mathcal{M}$ in the sense that $\dfrac{R}{(x_1, x_2, ..., x_d)}$ is artinian. In this case R is said to be *Cohen-Macaulay*. Then since $\dfrac{R}{(x_1, x_2, ..., x_d)} \neq 0$ we have soc $\left(\dfrac{R}{(x_1, x_2, ..., x_d)}\right) \neq 0$. If furthermore dim soc $\left(\dfrac{R}{(x_1, x_2, ..., x_d)}\right) = 1$ (with the dimension over the residue field $k = \frac{R}{\mathcal{M}}$) then R is said to be *Gorenstein*.

Then in 1966-67, Maurice Auslander gave a series of lectures in Pierre Samuel's seminar in Paris [1]. In these notes Auslander considers a Gorenstein R and finitely generated modules M such that for some (all) R-sequences $x_1, ..., x_d \in \mathcal{M}$ with $\dfrac{R}{(x_1, ..., x_d)}$ artinian, $x_1, ..., x_d$ is also an M-sequence (so $M \xrightarrow{x^1} M$ etc. are also injections). Such an M is said to be a *maximal Cohen-Macaulay module*. He argued that the equivalent homological conditions are that $\mathrm{Ext}^n(M, R) = 0 = \mathrm{Ext}^n(M^*, R)$ for $n \geq 1$ and that M is reflexive. Then with a little more work than in the Tate situation (here we don't necessarily have $M \cong M^*$) he argues that these conditions are equivalent to M having a complete projective resolution.

Auslander's ideas were developed further in Auslander-Bridger [2] and then in Auslander- Buchweitz [3]. In the latter work we have the beginnings of Gorenstein homological algebra. They work with a local, commutative noetherian R admitting a dualizing module D. If R is Gorenstein then R is such a ring with $D = R$. So we will state some of their results in this restricted situation. They prove that over such an R, for a finitely generated N there is an exact sequence

$$0 \to L \to C \to N \to 0$$

where C is a finitely generated maximal Cohen-Macaulay R-module such that Hom $(D, C) \to$ Hom $(D, N) \to 0$ is exact. So $C \to N$ is what Xu ([4], pg. 29) calls a *special $\mathcal{C}$-precover* of N where $\mathcal{C}$ is the class of finitely generated maximal Cohen-Macauley modules. Auslander and Buchweitz call these precovers *maximal Cohen-Macaulay approximations*. Having these precovers is a first step toward developing a Gorenstein homological algebra.

The works [1], [2] and [3] are rich in results and ideas. Another fundamental idea introduced by Auslander in [1] is that of the G-dimension of a finitely generated module M.

It can be stated in this manner. We have G-dim $M \leq n$ (G-dim short for *G-dimension*) if for any partial projective resolution

$$0 \to C \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

of M with $P_0, ..., P_{n-1}$ finitely generated and projective, C has a complete projective resolution.

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9.4 The Kaplansky Program

In the 1950s, Irving Kaplansky (then at the University of Chicago) gave impetus to studying rings by considering their categories of modules. Two (among others) of his students, H. Bass and S. Chase, proved the value of this program in their theses ([1] and [2]). Bass gave internal characterizations of those rings such that every module admits a projective cover and Chase showed that right coherence for a ring is equivalent to closure of the class of flat left R-modules under products. Then in 1963 Bass wrote his ubiquity paper. He showed that Gorenstein local rings (and these had an internal definition) are characterized by the equivalent property that R has finite self injective dimension. Since R is noetherian, this property is equivalent to the property that every projective module has finite injective dimension.

In some way Bass' results (and also Matlis duality) were anticipated by Dieudonné [3].

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9.5 Iwanaga-Gorenstein Rings

Y. Iwanaga (in [1]) carried out the Kaplansky program with respect to his generalization of the commutative Gorenstein local rings R. He considered R which are left and right noetherian and such that for some n, $0 \leq n < \infty$, inj $\dim_R R$, inj dim $R_R \leq n$. He calls such an R an *n-Gorenstein* ring. So an *Iwanaga-Gorenstein* ring R is one which is n-Gorenstein for some n.

We have

Theorem 9.5.1 *(Iwanaga [3]). Over an Iwanaga-Gorenstein ring R the following are equivalent for any left or right R-module M*

a) proj. dim. $M < \infty$

b) inj. dim. $M < \infty$

If in fact R is n-Gorenstein, it is easy to get that proj. dim. $M \leq n$ and inj. dim. $M \leq n$ for any M as in the theorem.

If $n = 0$ (so R is self-injective, i.e., R is quasi-Frobenius) we recover the familiar result that an R-module is projective if and only if it is injective.

There are many ways to generate examples of Iwanaga-Gorenstein rings. One of the early examples corresponds to a submonoid $S \subset N$. If k is a field then the subring of $k[[x]]$ consisting of

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all $\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots$ such $\alpha_n \neq 0$ implies $n \in S$ is Iwanaga-Gorenstein if and only if S is symmetric (see [1], pg. 553). Complete intersections are such rings ([1], pg. 541).

In general it seems that when we have a functorial procedure for a change of ring which is such that when R is left and right noetherian so is the new ring, then the procedure also preserves the property of being Iwanaga-Gorenstein. Examples of such procedures are going from R to $R[x]$, to $R[[x]]$, to $M_n(R)$, to $R(G)$ (G a group) and from R to $\hat{R}$ when R is a local ring. In [2] there is information about the group ring case.

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9.6 Gorenstein Homological Algebra

Inspired by the work of Auslander, Auslander-Buchweitz and that of Iwanaga, Enochs and Jenda attempted in [4] to initiate the full study of Gorenstein homological algebra.

As a starting point we have

Definition 9.6.1 A module C is said to be *Gorenstein projective* if and only if it has a complete projective resolution.

Then by duality we have

Definition 9.6.2 A module G is said to be *Gorenstein injective* if and only if it has a complete injective resolution.

By such a resolution we mean an exact sequence

$$\cdots \to E^{-2} \to E^{-1} \to E^0 \to E^1 \to E^2 \to \cdots$$

of injective modules with $M = \ker(E^0 \to E^1)$ and such that $\mathrm{Hom}(E, -)$ leaves the sequence exact when E is an injective module.

The main result in [2] is

Theorem 9.6.3 *If R is n-Gorenstein and $0 \to C \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ is any partial projective resolution of an R-module M, then C is Gorenstein projective. If $0 \to N \to E^0 \to \cdots \to E^{n-1} \to G \to 0$ is a partial injective resolution of an R-module N, then G is Gorenstein injective.*

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So these results just say G- proj. dim. $M \leq n$ and G- injective. dim. $N \leq n$ where *G- proj. dim.* and *G- inj. dim.* denote the Gorenstein projective and injective dimensions of the modules. These notions are generalizations of Auslander's G-dim. (see [2] of section 2).

The main consequence of these theorems is that over an n-Gorenstein ring every module has a Gorenstein projective precover and a Gorenstein injective preenvelope (and in fact an envelope). This allows us to see that every module M over an Iwanaga-Gorenstein ring has a *Gorenstein projective resolution*. By this is meant a complex

$$\cdots \to C_2 \to C_1 \to C_0 \to M \to 0$$

where each C_n is a Gorenstein projective module and which becomes an exact complex when Hom $(C, -)$ is applied to it when C is an arbitrary Gorenstein projective module (since $C = R$ is such a module we see that in fact the original complex is exact).

In an analogous manner we see that every module N has a *Gorenstein injective resolution*

$$0 \to N \to G^0 \to G^1 \to \cdots$$

Here we apply the functors Hom$(-, G)$ with G Gorenstein injective and get exact sequences. Since G can be an arbitrary injective module E, we see that the original complex is also exact in this situation.

These complexes allow one to mimic classical homological algebra and define derived functors $\mathrm{Gext}^n(M, N)$. Balance in this situation means $\mathrm{Gext}^n(M, N)$ can be computed using either the Gorenstein projective resolution of M or the Gorenstein injective resolution on N. We also get long exact sequences associated with short exact sequences $0 \to M^{'} \to M \to M^{''} \to 0$ and $0 \to N^{'} \to N \to N^{''} \to 0$ but only under the additional hypotheses that Hom $(C, -)$ leaves the first exact when C is Gorenstein projective and Hom $(-, G)$ leaves the second exact when G is Gorenstein injective. We note that $G\mathrm{ext}^0(M, N) =$ Hom (M, N) as in the classical situation.

A complete treatment of these results is given in chapters 10, 11 and 12 of [3].

There are various results that suggest the proper situation to use Gorenstein homological algebra is over a ring R admitting a dualizing module or even a dualizing complex (see [5], for example).

For an excellent treatment of some parts of Gorenstein homological algebra see Christensen's "Gorenstein Dimensions" [1]. But also Henrik Holm's thesis and his other work and that of Anders Frankild and Peter Jorgensen are good sources for information on this topic.

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9.7 Generalized Tate Homology and Cohomology

Let R be any ring. Let M be a left R-module such that for some partial projective resolution of M

$$0 \to C \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

C is a Gorenstein projective module. Then C admits a complete projective resolution

$$\cdots \to Q^{-2} \to Q^{-1} \to Q^0 \to Q^1 \to Q^2 \to \cdots = Q$$

Then given any other partial projective resolution

$$0 \to C' \to P'_{n-1} \to \cdots P'_0 \to M \to 0$$

of M, using the generalized Schanuel's lemma we get that C' is also Gorenstein projective. So C' has a complete projective resolution

$$\cdots \to Q'^{-2} \to Q'^{-1} \to Q'^{0} \to Q'^{1} \to Q'^{2} \to \cdots = Q'$$

chasing diagrams and using properties of Gorenstein projective modules we get maps of complexes Q and $Q \to Q'$ and $Q' \to Q$. Then it can be shown that these are homotopy equivalences. This means we can get well defined functors in the usual manner.

For example, if N is any other left R-module, we form the complex Hom (Q, N) and then compute its cohomology. The convention is that the group $H^m(\mathrm{Hom}(Q, N))$ is denoted $\widehat{\mathrm{Ext}}_R^{m+n}(M, N)$. With an analogous convention the complexes $Q \otimes_R N$ (but where we start with M a right R-module and let N be a left R-module) give us groups $\widehat{\mathrm{Tor}}_n^R(M, N)$.

So these are the *generalized Tate cohomology* and *homology groups*. In Tate's situation the module $M = Z$ is already Gorenstein projective (so corresponds to n being 0 in the partial projective resolution). If we assume G is Gorenstein injective when $0 \to N \to E^0 \to \cdots \to E^{n-1} \to G \to 0$ is a partial injective resolution of the left R-module, then using a complete injective resolution of G and then applying the functors Hom $(M, -)$ and then again computing cohomology we again get Tate cohomology modules again denoted $\widehat{\mathrm{Ext}}^n(M, N)$. When both G-proj. dim. $M < \infty$ and G-inj. dim. $N < \infty$ then in [1] it is shown that the two methods given above of computing $\widehat{\mathrm{Ext}}^n(M, N)$ give the same groups, i.e., we have a Tate cohomology version of balance.

It seems likely we also get this sort of balance when computing $\widehat{\mathrm{Tor}}_n(M, N)$ (when G-proj. dim. $M < \infty$ and G-proj. $N < \infty$) with M a right and N a left R-module.

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9.8 The Avramov-Martsinkovsky Program

If a left R-module M has a Gorenstein projective resolution

$$\cdots \to C_2 \to C_1 \to C_0 \to M \to 0$$

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and if $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ is a projective resolution of M, then there is a commutative diagram

$$\begin{array}{ccccccccc} \to & P_2 & \to & P_1 & \to & P_0 & \to & M & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \| & \\ \to & C_2 & \to & C_1 & \to & C_0 & \to & M & \to 0 \end{array}$$

Then since the complexes

$$\to C_2 \to C_1 \to C_0 \to 0 = C$$

and

$$\to P_2 \to P_1 \to P_0 \to 0 = P$$

are used to compute $G\mathrm{ext}^n(M, N)$ and $\mathrm{Ext}^n(M, N)$ we see that we have natural transformations

$$G\mathrm{ext}^n(M, N) \to \mathrm{Ext}^n(M, N)$$

for all $n \geq 0$.

When $n = 0$ these are isomorphisms. If G-proj. dim. $M < \infty$ then we also have the Tate cohomology groups $\widehat{\mathrm{Ext}}^n(M, N)$. There is a little more work involved to argue there are natural transformations

$$\mathrm{Ext}^n(M, N) \to \widehat{\mathrm{Ext}}^n(M, N)$$

So for each n we have the diagram

$$G\mathrm{ext}^n(M, N) \to \mathrm{Ext}^n(M, N) \to \widehat{\mathrm{Ext}}^n(M, N)$$

In [1], Avramov and Marsinkovsky give an argument that not only are each of the diagrams exact sequences but

Theorem 9.8.1 *If M if finitely generated and G-proj. $M = d < \infty$ then there is an exact sequence*

$$0 \to \mathrm{Gext}^1(M, N) \to \mathrm{Ext}^1(M, N) \to \widehat{\mathrm{Ext}}^1(M, N) \to \mathrm{Gext}^2(M, N) \to \cdots$$

$$\cdots \to \mathrm{Gext}^d(M, N) \to \mathrm{Ext}^d(M, N) \to \widehat{\mathrm{Ext}}^d(M, N) \to 0$$

incorporating each of the diagrams above.

Iacob in [2] gives another way to prove their result.

She considers the map $\phi : P \to C$ of complexes we have as above. Associated with this map of complexes we have the mapping cone complex $M(\phi)$ and the exact sequence

$$0 \to C \to M(\phi) \to P[1] \to 0$$

of complexes. This exact sequence splits at the module level, so if we apply the functor $\mathrm{Hom}(-, N)$ to this exact sequence of complexes we still get an exact sequence of complexes. So there is an associated exact sequence of cohomology groups. Those associated with Hom (C, N) and Hom $(P[1], N)$ give Gext and Ext groups. She argues that under the hypothesis the other cohomology groups are Tate cohomology groups $\widehat{\mathrm{Ext}}^n(M, N)$ with n in the range $1 \leq n \leq d$. There are some advantages to her approach. First, there is no need to assume M is finitely generated or that G-proj. dim. on $M < \infty$, but only that M has a Gorenstein projective resolution (however G-proj. dim. $M < \infty$ is the usual hypothesis that is used to guarantee this). But perhaps the biggest advantages to her approach are that her approach dualizes and that it so easily gives the existence of the desired exact sequences.

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9.9 Gorenstein Flat Modules

Having in hand notions of Gorenstein projective and injective modules, it is natural to ask if there is a good notion of a Gorenstein flat module. In [1] it was shown that there are negative torsion functors $\mathrm{Tor}^R_{-n}(M, N)$ for $n \geq 0$ which can be computed using an injective resolution of the right module M or a complex

$$0 \to N \to F^0 \to F^1 \to \cdots$$

of modules where the F^n are flat and such that $E\otimes-$ makes the complex exact when E is an injective module (we need R right coherent to guarantee the existence of such a complex).

This result suggested the (perhaps strange) definition:

Definition 9.9.1 If left R-module N is said to be *Gorenstein flat* if there is an exact complex

$$\cdots \to F^{-1} \to F^0 \to F^1 \to \cdots$$

of flat modules with $N = \ker(F^0 \to F^1)$ and such that $E\otimes-$ leaves the complex exact whenever E is an injective right R-module.

There are several results that suggest that this is the right definition. Perhaps the most significant of these is the next result.

Theorem 9.9.2 *([2, Theorem 2.1]). If R is Iwanaga-Gorenstein then the left R-module N is Gorenstein flat if and only if $N = \varinjlim C_i$ for an inductive system $(C_i)_{i\in I}$ of finitely generated Gorenstein projective modules.*

It is an open question whether modules in general have Gorenstein projective precovers. In the absolute case it is trivial to argue that modules have projective precovers and somewhat more complicated to argue that they have flat covers.

In the Gorenstein situation we have the following result.

Theorem 9.9.3 *([3]). If R is right coherent every left module has a Gorenstein flat cover.*

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9.10 Salce's Cotorsion Theories

In [2] Salce defined cotorsion theories. He used a version of orthogonality with respect to the functor Ext.

Let R be a ring and $\mathcal{F}$ a class of left R-modules. Let $\mathcal{F}^\perp$ consist of all the modules C such that $\mathrm{Ext}^1(F, C) = 0$ for all $F \in \mathcal{F}$. Similarly, for a class $\mathcal{C}$ let ${}^\perp\mathcal{C}$ consist of all the F such that $\mathrm{Ext}^1(F, C) = 0$ for all $C \in \mathcal{C}$. Then the pair $(\mathcal{F}, \mathcal{C})$ of classes is called a *cotorsion theory* (on the category of left R-modules) if $\mathcal{F}^\perp = \mathcal{C}$ and ${}^\perp\mathcal{C} = \mathcal{F}$. The cotorsion theories of most interest are the complete ones.

Definition 9.10.1 A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be *complete* if for each left R-modules M and N there are exact sequences

$$0 \to C \to F \to M \to 0$$

$$0 \to N \to \bar{C} \to \bar{F} \to 0$$

with $F, \bar{F} \in \mathcal{F}$ and $C, \bar{C} \in \mathcal{C}$. We note that in the language of Xu ([4] of section 2), $F \to M$ is a special $\mathcal{F}$-precover and $N \to \bar{C}$ is a special $\mathcal{C}$-preenvelope. An additional property of interest is the so-called hereditary property.

Definition 9.10.2 A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be *hereditary* if it satisfies the conditions:

a) if $0 \to C' \to C \to C'' \to 0$ is exact with $C', C \in \mathcal{C}$ then $C'' \in \mathcal{C}$.

b) if $0 \to F' \to F \to F'' \to 0$ is exact with $F, F'' \in \mathcal{F}$ then $F' \in \mathcal{F}$.

Now let R be an Iwanaga-Gorenstein ring. Let $\mathcal{L}$ be the class of left R-modules of finite projective dimension (equivalently of finite injective dimension). Then ${}^\perp\mathcal{L} = \mathcal{C}$ is the class of Gorenstein projective modules and $\mathcal{L}^\perp = \mathcal{G}$ is the class of Gorenstein injective modules. Furthermore, $(\mathcal{C}, \mathcal{L})$ and $(\mathcal{L}, \mathcal{G})$ are complete and hereditary cotorsion theories.

This result is essentially in [1] (although not stated quite in this form). This is one of the few instances where a class (here the class $\mathcal{L}$) can serve as the class on both sides of cotorsion theories.

The other well-known instance is where M is the class of all left R-modules and we have complete hereditary cotorsion theories $(\mathcal{P}, \mathcal{M})$, $(\mathcal{M}, \mathcal{E})$ with $\mathcal{P}$ and $\mathcal{E}$ the classes of projective and injective modules, respectively.

An important result appears in [3]. They prove that in our situation if $\mathcal{L}' \subset \mathcal{L}$ is the class of finitely generated modules of finite projective dimension, then $(\mathcal{L}')^\perp = \mathcal{L}^\perp = \mathcal{G}$. This result is useful in proving basic properties about the class $\mathcal{G}$.

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9.11 Other Possibilities

It seems likely there is a version of Gorenstein homological algebra in the category of quasi-coherent sheaves over a scheme, at least for a scheme which in an obvious sense is locally Gorenstein. There are techniques developed in [1] which could be useful in developing such a theory.

There are other abelian categories where such a theory might be used. In [2] there is a notion of a *Gorenstein projective complex* in the category of complexes over a ring. This notion is the straight-forward modification of the definition for modules. In [4] there is a completely different approach which uses a weaker kind of projectivity. Hovey in [3] has exhibited connections with Quillen's model category structures.

Given a complete cotorsion theory $(\mathcal{F}, \mathcal{C})$ on the category of left R-modules (or possibly in any abelian category) there is the notion of a complete $\mathcal{F}$-resolution and a complete $\mathcal{C}$-resolution of an object which mimic the complete projective and injective resolutions of a module. This would lead to the notion of Gorenstein projective and injective objects relative to the cotorsion theory $(\mathcal{F}, \mathcal{C})$. Some of what was noted above indicates there are examples where such a study might be fruitful.

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Chapter 10

Modules and Point Set Topological Spaces

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Abstract Begin with a self-small self-slender right R-module G and its ring of R-endomorphisms E. We construct a commutative diagram of categories, functors, and functions that contains elements from abelian groups, the complete category of left modules and the category of right modules over E, elements from algebraic topology, and elements from point-set topology. The diagram is a partial answer to the *Langlands Program* that seeks to find nontrivial connections between areas of mathematics. Applications include a complete set of topological invariants for abelian groups, a unique decomposition of certain topological spaces, and a topological characterization of the flat dimension of the left E-module G.

10.1 The Diagram

We will construct a commutative diagram, Diagram (1), that contains the category of abelian groups, the categories of left modules and right modules over a ring, categories of complexes, and categories of topological spaces.

Let G be a right module over some indeterminant ring R, and write $E = \mathrm{End}_R(G)$. **Mod-**E is the category of right E-modules, and let E**-Mod** be the category of left E-modules. Let **Ab** denote the category of abelian groups, and let **SAb** denote the category of sequences $(\cdots, A_2, A_1)$ of abelian groups A_k. A map f in **SAb** is a sequence

$$f = (\cdots, f_2, f_1) : (\cdots, A_2, A_1) \longrightarrow (\cdots, B_2, B_1)$$

of abelian group maps $f_k : A_k \longrightarrow B_k$ for each integer $k \geq 1$.

Let **Complex** denote the category whose objects are complexes of abelian groups and whose maps are the homotopy equivalence classes $[f]$ of chain maps $f : \mathcal{Q} \longrightarrow \mathcal{Q}'$ between complexes $\mathcal{Q}$ and $\mathcal{Q}'$.

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$$\begin{tikzcd}
 & G\text{-}\mathbf{Plex} \arrow[rr, "\gamma"] \arrow[dl, "\mathrm{h}_G"'] \arrow[dr, "\alpha"] & & \mathbf{Free\ Complex} \arrow[dl, shift left, "D_*"] \\
\mathbf{Mod}\text{-}E \arrow[dr, "\mathrm{Tor}^*_E(\cdot,\, G)"'] & & \mathbf{M\text{-}Spaces} \arrow[ur, shift left, "\beta"] \arrow[dl, shift right, "H^M_*"'] & \\
 & \mathbf{SAb} \arrow[ur, shift right, "C_*"'] & & G\text{-}\mathbf{coPlex} \arrow[uu, "\epsilon"'] \arrow[ul, "\delta"'] \arrow[dl, "\mathrm{h}^G"] \\
 & & E\text{-}\mathbf{Mod} \arrow[ul, "\mathrm{Ext}^*_E(\cdot,\, G)"] &
\end{tikzcd}$$

Diagram (1)

A *G-plex* is a complex

$$\mathcal{Q} = \qquad \cdots \xrightarrow{\ \delta_3\ } Q_2 \xrightarrow{\ \delta_2\ } Q_1 \xrightarrow{\ \delta_1\ } Q_0$$

of abelian groups such that

1. Each Q_k is a direct summand of a direct sum of copies of G and
2. The induced complex $\mathrm{Hom}_R(G, \mathcal{Q})$ is exact, i.e., given an integer $k > 0$ each map $f : G \longrightarrow \ker \delta_k$ lifts to a map $g : G \longrightarrow Q_{k+1}$ such that $\delta_{k+1} \circ g = f$.

Dualizing, a *G-coplex* is a complex

$$\mathcal{W} = \qquad W_0 \xrightarrow{\ \sigma_0\ } W_1 \xrightarrow{\ \sigma_1\ } W_2 \xrightarrow{\ \sigma_2\ } \cdots$$

of abelian groups such that

1. Each W_k is a direct summand of a direct product of copies of G, and
2. The induced complex $\mathrm{Hom}_R(\mathcal{W}, G)$ is exact.

Let G-**Plex** denote the full subcategory of **Complex** whose objects are the G-plexes, and let G-**Coplex** denote the full subcategory of **Complex** whose objects are G-coplexes $\mathcal{W}$.

Let

$$H^C_k(\cdot) : \mathbf{Complex} \longrightarrow \mathbf{Ab}$$

denote the *k-th homology functor*. That is, for integers $k > 0$, given a complex $\mathcal{Q}$ then *the k-th homology group of* $\mathcal{Q}$ is

$$H^C_k(\mathcal{Q}) = \ker \delta_k / \mathrm{image}\ \delta_{k+1}$$

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and the *zero-th homology group* for a complex $\mathcal{Q}$ is

$$H_0^C(\mathcal{Q}) = Q_0/\text{image } \delta_1 = \text{coker } \delta_1.$$

The *homology functor*

$$H_*^C(\cdot) : \mathbf{Complex} \longrightarrow \mathbf{SAb}$$

is defined by

$$H_*^C(\cdot) = (\cdots, H_2(\cdot), H_1(\cdot)).$$

This homology functor will be restricted to full subcategories of **Complex** without a change in notation, e.g., we have the homology functors $H_*^P(\cdot) : G\text{-}\mathbf{Plex} \longrightarrow \mathbf{SAb}$ and

$$H_*^F(\cdot) : \mathbf{Free\ Complex} \longrightarrow \mathbf{SAb}$$

where **Free Complex** is the full subcategory of **Complex** whose objects are complexes whose terms are free abelian groups. This homology functor is used in this paper but for technical reasons does not appear in Diagram (1).

Observe that for each G-plex $\mathcal{Q}$

$$\mathrm{Hom}_R(G, \mathcal{Q}) = \qquad \cdots \xrightarrow{\delta_2^*} \mathrm{Hom}_R(G, Q_1) \xrightarrow{\delta_1^*} \mathrm{Hom}_R(G, Q_0)$$

is a complex. Define the functor

$$\mathrm{h}_G(\cdot) = H_0^C \circ \mathrm{Hom}(G, \cdot) : G\text{-}\mathbf{Plex} \longrightarrow \mathbf{Mod}\text{-}E.$$

Dually, observe that for each G-coplex $\mathcal{W}$

$$\mathrm{Hom}_R(\mathcal{W}, G) = \qquad \cdots \xrightarrow{\sigma_2^*} \mathrm{Hom}_R(W_1, G) \xrightarrow{\sigma_1^*} \mathrm{Hom}_R(W_0, G)$$

is a complex. Define the functor

$$\mathrm{h}^G(\cdot) = H_0^C \circ \mathrm{Hom}_R(\cdot, G) : G\text{-}\mathbf{Coplex} \longrightarrow E\text{-}\mathbf{Mod}.$$

The Torsion and Extension functors are naturally homology functors and they play a fundamental role in the applications of our discussions. Let

$$\mathrm{Tor}_E^*(\cdot, G) : \mathbf{Mod}\text{-}E \longrightarrow \mathbf{SAb}$$

denote the functor defined by

$$\mathrm{Tor}_E^*(\cdot, G) = (\cdots, \mathrm{Tor}_E^2(\cdot, G), \mathrm{Tor}_E^1(\cdot, G))$$

and dually define the functor

$$\mathrm{Ext}_E^*(\cdot, G) : E\text{-}\mathbf{Mod} \longrightarrow \mathbf{SAb}$$

as

$$\mathrm{Ext}_E^*(\cdot, G) = (\cdots, \mathrm{Ext}_E^2(\cdot, G), \mathrm{Ext}_E^1(\cdot, G)).$$

Now let us gravitate to the category **Spaces** of point set topological spaces. More precisely, the category of point set topological spaces is denoted by **Spaces**, and in a departure from the norm,

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the maps in **Spaces** are the homotopy equivalence classes $[f]$ of continuous maps $f : X \longrightarrow Y$ between topological spaces. We will write $X \sim Y$ when two topological spaces X and Y are *homotopic*. To each topological space X we fix a simplicial approximation $\langle X \rangle$ of X and then form the associated complex $\beta(X)$ of free abelian groups from $\langle X \rangle$. Inasmuch as $X \sim \langle X \rangle$, the free complex $\beta(X)$ is unique up to homotopy. Thus we have defined a functor

$$\beta(\cdot) : \textbf{Spaces} \longrightarrow \textbf{Free Complex}.$$

For each integer $k \geq 0$ we define the *k-th homology group of* X to be the k-th homology group of the free complex $\beta(X)$, or symbolically

$$H_k^S(X) = H_k^F(\beta(X)).$$

We can then define the homology functor

$$H_*^S(\cdot) : \textbf{Spaces} \longrightarrow \textbf{SAb}$$

on the category of point set topological spaces by

$$H_*^S(\cdot) = (\cdots, H_2^S(\cdot), H_1^S(\cdot)).$$

Given an indexed set $\{X_i \,\big|\, i \in \mathcal{I}\}$ of topological spaces let

$$\bigvee_{i \in \mathcal{I}} X_i = \text{the one point union of the } X_i.$$

The following is a classic construction due to Moore. Given integers $k > 0, n \geq 0$ let S^k denote the k-sphere and fix a continuous function

$$f_n : S^k \longrightarrow S^k \text{ of degree } n.$$

Such functions exist in abundance. See [15, Example 2.31]. Let D^{k+1} be the $k+1$-disk and note that S^k is the boundary of D^{k+1}. Define

$$C_k(f_n) = \frac{S^k \cup D^{k+1}}{\{x \sim f_n(x) \,\big|\, x \in S^k\}}.$$

For $k \geq 2$ the constructed space is simply connected. However, it is possible by choosing different boundary functions f_n and f_n' that the end results $C_1(f_n)$ and $C_1(f_n')$ are not homotopic spaces. They are certainly not simply connected for $n \neq 1$. See [15, page 368]. Thus our applications will deal almost exclusively with the cases $k \geq 2$.

If A is a finitely generated abelian group then $A = A_1 \oplus \cdots \oplus A_t$ for some indecomposable cyclic groups A_p. Then we define

$$C_k(A) = C_k(A_1) \vee \cdots \vee C_k(A_t).$$

This construction is extended to any abelian group A by taking the direct limit of the finitely generated subgroups of A so that

$$C_k(A) = C_k(\varinjlim A_o) = \varinjlim C_k(A_o)$$

where A_o ranges over the finitely generated subgroups of A. See [15, pages 314]. The space $C_k(A)$ is called an *M-space concentrated in degree k*. For integers $k \geq 2$, $C_k(A)$ is simply connected. For a fixed group A and integer $k \geq 2$, $C_k(A)$ is unique up to homotopy. See [15, page 368]. Let **M-Spaces** denote the full subcategory of **Spaces** whose objects are homotopic to the one point

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unions of Moore k-Spaces $C_k(A)$ ranging over abelian groups A and integers $k > 0$. That is $X \in$ **M-Spaces** iff there is a sequence of abelian groups $S = (\cdots, A_2, A_1)$, and a fixed set of generators and relations for A_1, such that

$$X \sim \bigvee_{k>0} C_k(A_k).$$

There are homology functors

$$H_k^M(\cdot) : \textbf{M-Spaces} \longrightarrow \textbf{SAb}$$

for each integer $k > 0$ so as usual there is a functor

$$H_*^M(\cdot) : \textbf{M-Spaces} \longrightarrow \textbf{SAb}$$

defined by

$$H_*^M(\cdot) = (\cdots, H_2^X(\cdot), H_1^X(\cdot)).$$

The *function*

$$C_* : \textbf{SAb} \longrightarrow \textbf{M-Spaces}$$

sends a sequence of abelian groups $S = (\cdots, A_2, A_1)$ to

$$C_*(S) = \bigvee_{k\geq 2} C_k(A_k).$$

(Notice that the first subscript is k=2, not 1.)

We summarize uniqueness in the following two lemmas.

Lemma 10.1.1 [15, Page 143] *Let A be an abelian group and fix an integer $k > 0$. The M-space $C_k(A)$ satisfies the following group isomorphisms for each integer $p > 0$.*

$$H_p^M(C_k(A)) = \begin{cases} A & \text{if } p = k \\ 0 & \text{if } p \neq k \end{cases} \tag{10.1}$$

Lemma 10.1.2 [15, page 368] *Let $k \geq 2$ be an integer, and let X be an M-space concentrated at k. Then*

$$C_k(H_k^M(X)) \sim X.$$

The maps α, γ, δ, and ϵ are then the unique maps that make the diagram commute. A classic result in topology states that C_* is not a functor, so the maps $\alpha, \gamma, \delta, \epsilon$ are *not functors*.

There is a homology functor $H_*^F(\cdot)$: **Free Complex** $\longrightarrow$ **SAb** so there is a function

$$D_* = C_* \circ H_*^F : \textbf{Free Complex} \longrightarrow \textbf{M-Spaces}$$

on the objects of the categories.

It is worth noting that we have included abelian groups, point set topology, algebraic topology, the category of right modules over a (not necessarily commutative) ring E and its category of left modules, together with the usual derived functors $\mathrm{Tor}_E^*(\cdot, G)$ and $\mathrm{Ext}_E^*(\cdot, G)$ in a nontrivial way.

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10.2 Self-Small and Self-Slender Modules

We assume throughout this section that G is a self-small and a self-slender right R-module. With this assumption we will be able to supplant some functors and functions in Diagram (1) with category equivalences and bijections, as in Diagram (2).

$$
\begin{tikzcd}
& G\text{-}\mathbf{Plex} \arrow[rr, "\gamma"] \arrow[dl, "\mathrm{h}_G"'] \arrow[dd, "H_*^P"] \arrow[dr, "\alpha"] & & \mathbf{Free\ Complex} \arrow[dl, "D_*"] \\
\mathbf{Mod}\text{-}E \arrow[ur, shift right, "\mathrm{t}_G"'] \arrow[dr, "\mathrm{Tor}_E^*(\cdot,\, G)"'] & & \mathbf{M\text{-}Spaces} \arrow[ur, shift left, "\beta"] \arrow[dl, "H_*^M"'] & \\
& \mathbf{SAb} \arrow[ur, shift right, "C_*"'] & & G\text{-}\mathbf{coPlex} \arrow[ll, "H_*^c"] \arrow[ul, "\delta"'] \arrow[uu, "\epsilon"'] \arrow[dl, "\mathrm{h}^G"'] \\
& & E\text{-}\mathbf{Mod} \arrow[ul, "\mathrm{Ext}_E^*(\cdot,\, G)"] \arrow[ur, shift right, "\mathrm{h}^G"'] &
\end{tikzcd}
$$

Diagram (2)

Given a cardinal c, let $G^{(c)}$ denote the direct sum of c copies of G, and let G^c denote the product of c copies of G. The R-module G is said to be *self-small* if for each cardinal c the natural map

$$\mathrm{Hom}_R(G, G)^{(c)} \longrightarrow \mathrm{Hom}_R(G, G^{(c)})$$

is an isomorphism. Dually the R-module G is *self-slender* if for each cardinal c the natural map

$$\mathrm{Hom}_R(G, G)^{(c)} \longrightarrow \mathrm{Hom}_R(G^c, G)$$

is an isomorphism. Actually this definition requires us to work under the axiomatic assumption $V = L$ that the mathematical world is constructible. See [7] for details on $V = L$. Beyond mentioning it here there is no further reference to this logical assumption in this paper.

At the time of this writing there is essentially only one known example of a module that is both self-small and self-slender.

Theorem 10.2.1 *The reduced torsion-free finite rank abelian groups (rtffr) G are self-small and self-slender.*

Proof Because the *rtffr* group G has finite rank it contains a finite linearly independent subset $\{x_1, \cdots, x_n\}$ such that $G/\langle x_1, \ldots, x_n\rangle$ is a torsion group. Then in any map $f : G \to G^{(\mathcal{I})}$ there is a finite subset $\mathcal{F} \subset \mathcal{I}$ such that $f(x_1, \ldots, x_n) \subset G^{(\mathcal{F})}$. One readily shows that $f(G) \subset G^{(\mathcal{F})}$ which implies that G is self-small. The dual result, that an rtffr group G is self-slender, follows immediately from [14, Proposition 94.2]. $\square$

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By [11, Theorem 2.1.11] or by [12, Theorem 3.2(2)], if G is a self-small right R-module then

$$\mathrm{h}_G(\cdot) : G\text{-}\mathbf{Plex} \longrightarrow \mathbf{Mod}\text{-}E$$

is a (covariant) category equivalence. By [11, Theorem 2.1.12] or by [12, Comment 3.3], the inverse of $\mathrm{h}_G(\cdot)$ is the functor

$$\mathrm{t}_G(\cdot) : \mathbf{Mod}\text{-}E \longrightarrow G\text{-}\mathbf{Plex}$$

given as follows. For a right E-module M choose a fixed projective resolution

$$\mathcal{P}(M) = \qquad \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0$$

and define

$$\mathrm{t}_G(M) = \mathcal{P}(M) \otimes_E G.$$

One proves that $\mathrm{t}_G(\mathcal{P}(M))$ is a G-plex. A map $f : M \longrightarrow N$ in $\mathbf{Mod}\text{-}E$ lifts to a chain map $\bar{f} : \mathcal{P}(M) \longrightarrow \mathcal{P}(N)$ which is unique up to homotopy class. There is a chain map

$$\bar{f} \otimes_E 1_G : \mathcal{P}(M) \otimes_E G \longrightarrow \mathcal{P}(N) \otimes_E N$$

of G-plexes. Then define $\mathrm{t}_G(f)$ to be the homotopy equivalence class of $\bar{f} \otimes_E 1_G$.

$$\mathrm{t}_G(f) = [\bar{f} \otimes 1_G].$$

These identities define the functor $\mathrm{t}_G(\cdot)$. The definition of $\mathrm{t}_G(f)$ is why we have required that the maps in **Complex** be homotopy equivalence classes of chain maps.

We have thus described the diamond in Diagram (2) with vertices (read top down, left to right) G-**Plex**, **Mod-**E, **M-Spaces**, and **SAb**.

Dually, if G is self-slender then [11, Theorem 2.3.3] or [12, Theorem 8.2(2)] states that under the logical assumption $V = L$, $\mathrm{Hom}_R(\cdot, G)$ induces a contravariant category equivalence

$$\mathrm{h}^G(\cdot) : G\text{-}\mathbf{Coplex} \longrightarrow E\text{-}\mathbf{Mod}$$

Its inverse is induced by $\mathrm{Hom}_E(\cdot, G)$ and is denoted by

$$\mathrm{h}^G(\cdot) : E\text{-}\mathbf{Mod} \longrightarrow G\text{-}\mathbf{Coplex}.$$

Because G is self-small, [11, Corollary 2.2.8], (or see [12]), shows us that for each G-plex $\mathcal{Q}$

$$\mathrm{Tor}_E^k(M, G) = H_k^P \circ \mathrm{t}_G(M)$$

for each integer $k > 0$. Hence

$$\mathrm{Tor}_E^*(\cdot, G) = H_*^P \circ \mathrm{t}_G(\cdot)$$

so that the triangle in Diagram (2) defined by G-**Plex**, **Mod-**E, and **SAb** commutes. The inverse relationship between $\mathrm{h}_G(\cdot)$ and $\mathrm{t}_G(\cdot)$ proves that

Corollary 10.2.2 *Let G be a self-small right R-module. The Torsion functor factors as*

$$\mathrm{Tor}_E^*(\mathrm{h}_G(\cdot), G) = H_*^P(\cdot).$$

In a dual manner the triangle defined by E-**Mod**, G-**coPlex**, and **SAb** in Diagram (2) commutes. Specifically, we can apply [11, Corollary 2.3.6]. Because G is self-slender, for each integer $k > 0$, $\mathrm{Ext}_E^k(\cdot, G)$ factors through $H_k^c(\cdot) : G\text{-}\mathbf{coPlex} \longrightarrow \mathbf{SAb}$.

$$\mathrm{Ext}_E^k(\cdot, G) = H_k^c \circ \mathrm{h}^G(\cdot)$$

Hence

Corollary 10.2.3 *Let G be a self-slender right R-module. The Extension functor factors as*

$$\mathrm{Ext}_E^*(\cdot, G) = H_*^c \circ \mathrm{h}^G(\cdot).$$

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10.3 The Construction Function

In this section we investigate the nature of the construction map and the homology map. Homotopy type is an equivalence relation $\sim$ on the set **M-Spaces** so we let

$$\textbf{M-Spaces}/\sim \quad = \text{ the set of homotopy equivalence classes of M-spaces.}$$

If we agree two sequences of abelian groups $(\cdots, A_2, A_1)$ and $(\cdots, B_2, B_1)$ are isomorphic iff $A_k \cong B_k$ for each integer $k > 0$, then

$$\textbf{SAb}/\cong \quad = \text{ the set of isomorphism classes of sequences of abelian groups.}$$

Theorem 10.3.1 *1. If $S \in$ **SAb** then there is a simply connected M-space $X = C_*(S)$ such that $H_*^M(X) \cong S$.*

2. If X and Y are simply connected M-spaces, and if $H_k^M(X) \cong H_k^M(Y)$ for each $k \geq 2$ then $X \sim Y$.

Proof 1. Given $S = (\cdots, A_2, A_1) \in$ **SAb** there are simply connected M-spaces $C_k(A_k) = X_k$, $k = 1, 2, \cdots$ whose homology groups satisfy

$$H_k^M(X_k) \cong A_k$$

and such that $H_p^X(X_k) = 0$ for integers $p \neq k > 0$. Let $X = \bigvee_{k>0} X_k$ and then observe that X is simply connected. Furthermore, for an integer $p > 0$ we have

$$H_p^X(X) \cong H_p^X(\bigvee_{k>0} X_k) \cong \bigoplus_{k>0} H_p^X(X_k) \cong H_p^X(X_p)$$

because $H_p^X(\cdot)$ changes one point unions into direct sums, [15, Corollary 2.25]. Hence

$$\begin{aligned} H_*^M(X) &\cong (\cdots, H_2^X(X), H_1^X(X)) \\ &\cong (\cdots, H_2^X(X_2), H_1^X(X_1)) \\ &\cong (\cdots, A_2, A_1) \\ &= S. \end{aligned}$$

2. Let X and Y be simply connected M-spaces such that $H_k^M(X) \cong H_k^M(Y)$ for each integer $k \geq 2$. Since X and Y are M-spaces there are, for each integer $k > 0$, M-spaces X_k and Y_k concentrated at k such that

$$X \sim \bigvee_{k>0} X_k \text{ and } Y \sim \bigvee_{k>0} Y_k.$$

Since X and Y are simply connected,

$$H_1^X(X) = H_1^X(X_1) = 0 = H_1^X(Y_1).$$

By our definition of M-spaces, $X_1 = C_1(0) = C_1(H_1^X(X_1))$ so that X_1 contracts to a point. Similarly, Y_1 contracts to a point. Thus $X_1 \sim Y_1$. Furthermore, for any integer $p \geq 2$

$$H_p^X(X_p) \cong H_p^X(X) \cong H_p^X(Y) \cong H_p^X(Y_p)$$

so that $X_p \sim Y_p$ by Lemma 10.1.2. Hence $X \sim Y$. $\square$

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10.4 The Greek Maps

In this section we will determine the rules for the Greek maps α, β, γ, δ, and ϵ.

Theorem 10.4.1 *Suppose that G is a self-small and self-slender right R-module.*

1. *Let $\mathcal{Q}, \mathcal{Q}' \in G$-**Plex** be such that $H_1^P(\mathcal{Q}) = H_1^P(\mathcal{Q}') = 0$. Then $\alpha(\mathcal{Q}) \sim \alpha(\mathcal{Q}')$ iff $H_k^P(\mathcal{Q}) \cong H_k^P(\mathcal{Q}')$ for each integer $k > 0$.*
2. *Let $\mathcal{Q} \in G$-**Plex** be such that $H_1^P(\mathcal{Q}) = 0$ and let $X \in$ **M-Space** be simply connected. Then $\alpha(\mathcal{Q}) \sim X$ iff $H_k^P(\mathcal{Q}) \cong H_k^M(X)$ for each integer $k > 0$.*

Proof 1. Let $\mathcal{Q}, \mathcal{Q}' \in G$-**Plex** be such that $\alpha(\mathcal{Q}) \sim \alpha(\mathcal{Q}')$, and let $k > 0$ be an integer. Then $H_*^M(\alpha(\mathcal{Q})) \cong H_*^M(\alpha(\mathcal{Q}'))$. Because G is self-small and self-slender, the commutativity of Diagram (2) implies that $H_*^P(\mathcal{Q}) \cong H_*^M(\alpha(\mathcal{Q}))$ so that $H_*^P(\mathcal{Q}) \cong H_*^P(\mathcal{Q}')$.

Conversely, suppose that $H_*^P(\mathcal{Q}) \cong H_*^P(\mathcal{Q})$. Since G is self-small and self-slender, the commutativity of Diagram (2) implies that $H_*^M(\alpha(\mathcal{Q})) \cong H_*^M(\alpha(\mathcal{Q}'))$ so that

$$C_*(H_*^M(\alpha(\mathcal{Q}))) \cong C_*(H_*^M(\alpha(\mathcal{Q}'))).$$

Since $H_1^P(\mathcal{Q}) = H_1^P(\mathcal{Q}') = 0$, Lemma 10.1.2 implies that $\alpha(\mathcal{Q}) \sim \alpha(\mathcal{Q}')$.

2. If $\alpha(\mathcal{Q}) \sim X$ then $H_*^M(\alpha(\mathcal{Q})) \cong H_*^M(X)$ so that $H_*^P(\mathcal{Q}) \cong H_*^M(X)$ by the commutativity of Diagram (2).

Conversely suppose that $H_*^P(\mathcal{Q}) \cong H_*^M(X)$. By the commutativity of Diagram (2), because X is simply connected, and by Theorem 10.3.1

$$\alpha(\mathcal{Q}) \sim C_*(H_*^P(\mathcal{Q})) \sim C_*(H_*^M(X)) \sim X.$$

This completes the proof. □

Reading the above corollary a different way we see that $\alpha^{-1}(X)$ is the set of G-plexes $\mathcal{Q}$ whose homology groups are the homology groups of X. A similar set of results is true for δ.

Theorem 10.4.2 *Suppose that G is a self-small and self-slender right R-module.*

1. *Let $\mathcal{W}, \mathcal{W}' \in G$-**Coplex** be such that $H_1^c(\mathcal{W}) = H_1^c(\mathcal{W}') = 0$. Then $\delta(\mathcal{W}) \sim \delta(\mathcal{W}')$ iff $H_*^c(\mathcal{W}) \cong H_*^c(\mathcal{W}')$.*
2. *Let $\mathcal{W} \in G$-**coPlex** be such that $H_1^c(\mathcal{W}) = 0$ and let $X \in$ **M-Space** be simply connected. Then $\delta(\mathcal{W}) \sim X$ iff $H_*^c(\mathcal{W}) \cong H_*^M(X)$.*

Let **Complex**$/H_*$ denote the set of equivalence classes $[\mathcal{Q}] = \{\mathcal{Q}' \mid H_*^P(\mathcal{Q}) \cong H_*^P(\mathcal{Q}')\}$ for complexes $\mathcal{Q}$. Similar quotients are defined for categories of complexes or topological spaces.

The homology functor in the next result is $H_*^F(\cdot)$: **Free Complex** $\longrightarrow$ **SAb** and does not appear in Diagram (2). It is used, however, to define $D_* = C_* \circ H_*^F$.

Theorem 10.4.3 *Suppose that G is a self-small and self-slender right R-module.*

1. *Let $\mathcal{F}, \mathcal{F}' \in$ **Free Complex** be such that $H_1^F(\mathcal{F}) = H_1^F(\mathcal{F}') = 0$. Then $D_*(\mathcal{F}) \sim D_*(\mathcal{F}')$ iff $H_*^F(\mathcal{F}) \cong H_*^F(\mathcal{F}')$.*
2. *Let $X \in$ **M-Spaces** be simply connected. There is an $\mathcal{F} \in$ **Free Complex** such that $D_*(\mathcal{F}) \sim X$.*

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Proof 1. Let $\mathcal{F}, \mathcal{F}' \in$ **Free Complex** and let $H_1^F(\mathcal{F}) \sim H_1^F(\mathcal{F}')$. Suppose that $H_*^F(\mathcal{F}) \cong H_*^F(\mathcal{F}')$. Then $C_*(H_*^F(\mathcal{F})) \sim C_*(H_*^F(\mathcal{F}'))$. Since G is self-small and self-slender $D_* = C_* \circ H_*^F$ so that $D_*(\mathcal{F}) \sim D_*(\mathcal{F}')$.

Conversely, reverse the above argument.

2. Let $X \in$ **M-Spaces** be simply connected. Since G is self-small and self-slender the free complex $\beta(X)$ has homology groups $H_*^F(\beta(X)) = H_*^M(X)$ so that

$$D_*(\beta(X)) \sim C_*(H_*^F(\beta(X))) \sim C_*(H_*^M(X)) \sim X$$

by Lemma 10.1.2. □

10.5 Coherent Modules and Complexes

It is interesting to ask how much of the equivalences $\mathrm{h}_G(\cdot)$ and $\mathrm{h}^G(\cdot)$ in Diagram (2) are preserved if we delete the hypotheses self-small and self-slender. We will show in this section that a surprisingly large portion of G-**Plex** is equivalent to a readily definable full subcategory of **Mod-**$\mathrm{End}_R(G)$ if G is simply a right R-module. A dual result for $\mathrm{h}^G(\cdot)$ is also found.

The G-plex $\mathcal{Q}$ is called a *coherent G-plex* if for each integer $k > 0$, Q_k is a direct summand of a finite direct sum of copies of G. For any G, $0 \longrightarrow G$ is a coherent G-plex. Given nonzero $n \in \mathbb{Z}$ and $G = \mathbb{Q}/\mathbb{Z}$, $\cdots \xrightarrow{n} \mathbb{Q}/\mathbb{Z} \xrightarrow{n} \mathbb{Q}/\mathbb{Z}$ is a coherent G-plex. We let

$$G\text{-}\mathbf{CohPlex} = \text{ the category of coherent } G\text{-plexes.}$$

Dually a *coherent G-coplex* is a G-coplex $\mathcal{W}$ in which each term W_k is a direct summand of a finite product of copies of G. We let

$$G\text{-}\mathbf{CohCoplex} = \text{ the category of coherent } G\text{-coplexes.}$$

$$\begin{tikzcd}
 & G\text{-}\mathbf{CohPlex} \arrow[rr, "\gamma"] \arrow[dl, shift right, "\mathrm{h}_G"'] \arrow[dd, "H_*^P"] \arrow[dr, "\alpha"] & & \mathbf{Free\ Complex} \arrow[dl, shift left, "D_*"] \\
\mathbf{Coh}\text{-}E \arrow[ur, shift right, "\mathrm{t}_G"'] \arrow[dr, "\mathrm{Tor}_E^*(\cdot, G)"'] & & \mathbf{M\text{-}Spaces} \arrow[ur, shift left, "\beta"] \arrow[dl, shift right, "H_*^M"'] & \\
 & \mathbf{SAb} \arrow[ur, shift right, "C_*"'] & & \mathrm{G}\text{-}\mathbf{CohCoplex} \arrow[ll, "H_*^c"] \arrow[ul, "\delta"'] \arrow[uu, "\epsilon"'] \arrow[dl, shift right, "\mathrm{h}^G"'] \\
 & & E\text{-}\mathbf{Coh} \arrow[ul, "\mathrm{Ext}_E^*(\cdot, G)"] \arrow[ur, shift right, "\mathrm{h}^G"'] &
\end{tikzcd}$$

Diagram (3)

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A right E-module M is called a *coherent right E-module* if there is a projective resolution $\mathcal{P}(M)$ of M in which each term P_k is a finitely generated projective right E-module, e.g., over a right Noetherian ring E each finitely generated module is a coherent right E-module. We let

$$\textbf{Coh-}E = \text{ the category of coherent right } E\text{-modules}$$

and dually we let

$$E\textbf{-Coh} = \text{ the category of coherent left } E\text{-modules.}$$

The functors $\mathrm{h}_G(\cdot)$ and $\mathrm{t}_G(\cdot)$ in Diagram (3) are inverse equivalences by [11, Theorem 2.1.12], and the functors $\mathrm{h}^G(\cdot)$ are inverse contravariant equivalences by [11, Theorem 2.3.6]. We leave it to the reader to mimic the proof of the commutativity of Diagram (2) to prove that Diagram (3) commutes. We note again that Diagram (3) is constructed with very few hypotheses on G.

10.6 Complete Sets of Invariants

The purpose behind diagrams like Diagram (1), (2), and (3) is to make contributions to one area of mathematics by studying another area. Thus we want to study **M-Spaces** by studying G and we want to study G by studying E-modules or **M-Spaces**. This is the strategy of a problem called the *Langlands Program* in which one seeks concrete connections between two seemingly unrelated areas of mathematics. We feel that Diagrams (1), (2), and (3) are partial solutions to the Langlands Program.

Let $\mathbf{X}$ be a set and let $\sim$ be an equivalence relation on $\mathbf{X}$. A *complete set of invariants for* $\mathbf{X}$ *up to* $\sim$ is a set $\mathbf{Y}$ for which there exists a bijection $\psi : \mathbf{X}/\sim \longrightarrow \mathbf{Y}$ from the equivalence classes in $\mathbf{X}/\sim$ onto the set $\mathbf{Y}$. If $\mathbf{Y}$ is a set of groups then we say that $\mathbf{Y}$ is a complete set of *algebraic invariants for* $\mathbf{X}$ up to $\sim$. If $\mathbf{Y}$ is a set of topological spaces then we say that $\mathbf{Y}$ is a complete set of *topological invariants for* $\mathbf{X}$ up to $\sim$. For example, as a consequence of Jonsson's Theorem, if G is an rtffr group then the strongly indecomposable quasi-summands of G and their multiplicities in G form a complete set of algebraic invariants for G up to *quasi-isomorphism*. See [3]. An important unanswered question in abelian group theory is to find a set of accessible *numeric* invariants for strongly indecomposable $A \in \mathbf{Ab}$ up to quasi-isomorphism. We show that the class of homotopy equivalence classes of M-spaces is a complete set of topological invariants for $\mathbf{Ab}$.

Theorem 10.6.1 *Let A and A' be abelian groups, and let $k > 1$ be an integer.*

1. *$C_k(A) \sim C_k(A')$ iff $A \cong A'$.*
2. *If $C_k(A) \sim \bigvee_{i\in\mathcal{I}} X_i$ for some some M-spaces $\{X_i \mid i \in \mathcal{I}\}$ then $A = \bigoplus_{i\in\mathcal{I}} H_k^M(X_i)$.*
3. *If $A \cong \bigoplus_{i\in\mathcal{I}} A_i$ for some groups $\{A_i \mid i \in \mathcal{I}\}$ then $C_k(A) \sim \bigvee_{i\in\mathcal{I}} C_k(A_i)$.*

Proof 1. Suppose that $C_k(A) \sim C_k(A')$. By Lemma 10.1.1 we have

$$A \cong H_k^M(C_k(A)) \cong H_k^M(C_k(A')) \cong A'.$$

The converse is clear since the function C_k takes isomorphic groups to homotopic M-spaces.

2. If $C_k(A) \sim \bigvee_{i\in\mathcal{I}} X_i$ then by Lemma 10.1.1

$$A \cong H_k^M(C_k(A)) \cong H_k^M\left(\bigvee_{i\in\mathcal{I}} X_i\right) \cong \bigoplus_{i\in\mathcal{I}} H_k^M(X_i)$$

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since homology functors take one point unions to direct sums.

3. Consider the M-space $\bigvee_{i\in\mathcal{I}} C_k(A_i)$. An application of $H_k^M(\cdot)$ and Lemma 10.1.1 yields

$$H_k^M\left(\bigvee_{i\in\mathcal{I}} C_k(A_i)\right) \cong \bigoplus_{i\in\mathcal{I}} H_k^M(C_k(A_i)) \cong \bigoplus_{i\in\mathcal{I}} A_i \cong A \cong H_k^M(C_k(A)).$$

By Lemma 10.1.1, $\bigvee_{i\in\mathcal{I}} C_k(A_i) \sim C_k(A)$, which completes the proof. □

Theorem 10.6.2 *Let A and A' be abelian groups. Then*

$$A \cong A' \textit{ iff } C_2(A) \sim C_2(A').$$

Thus $C_2(A)$ is a complete set of topological invariants for A.

Proof Apply Theorem 10.6.1(1). □

Theorem 10.6.3 *Let X, X' be simply connected M-spaces. Then*

$$X \sim X' \textit{ iff } H_*^M(X) \cong H_*^M(X').$$

Thus the sequence of groups $H_*^M(X)$ is a complete set of algebraic invariants for X.

Proof Apply Theorem 10.3.1(2). □

10.7 Unique Decompositions

Theorem 10.6.3 will lead us to an Azumaya-Krull-Schmidt Theorem for abelian groups and topological spaces. We say that an M-space X is *M-indecomposable* if given M-spaces U and V such that $X = U \vee V$ then either U or V contracts to a point. We consider only those nontrivial M-indecomposable M-spaces, and we consider those M-spaces that are concentrated at an integer $k > 1$.

Theorem 10.7.1 *Suppose that X is a simply connected M-space. Then X is indecomposable iff X is concentrated at k for some integer $k > 0$ and $H_k^M(X)$ is an indecomposable abelian group.*

Proof Apply Theorems 10.3.1 and 10.6.3. □

These indecomposable M-spaces yield a unique decomposition for M-spaces. A topological space X *possesses a unique M-decomposition* if

1. There is a one point union $X \sim \bigvee_{i\in\mathcal{I}} X_i$ for some index set $\mathcal{I}$ and some *M-indecomposable* M-spaces X_i, $i \in \mathcal{I}$;

2. If $X = \bigvee_{j\in\mathcal{J}} Y_j$ for some index set $\mathcal{J}$ and some M-indecomposable M-spaces Y_j, $j \in \mathcal{J}$ then there is a bijection $\pi : \mathcal{I} \longrightarrow \mathcal{J}$ such that $X_i \sim Y_{\pi(j)}$ for each $i \in \mathcal{I}$.

The abelian group A is said to *possess a unique decomposition* if

1. There is a direct sum $A = \bigoplus_{i\in\mathcal{I}} A_i$ for some set $\{A_i \mid i \in \mathcal{I}\}$ of *indecomposable* abelian groups;

2. If $A = \bigoplus_{j\in\mathcal{J}} B_j$ is a direct sum of indecomposable abelian groups then there is a bijection $\pi : \mathcal{I} \longrightarrow \mathcal{J}$ such that $A_i \cong B_{\pi(i)}$ for each $i \in \mathcal{I}$.

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For example, the finitely generated abelian groups possess a unique decomposition, as do the modules that are finite direct sums of modules with local endomorphism rings. See [2] and [14] for yet larger classes of abelian groups that possess unique decomposition.

Our next result characterizes topologically those abelian groups that possess a unique decomposition.

Theorem 10.7.2 *Let A be an abelian group. The following are equivalent.*

1. *A possesses a unique decomposition.*
2. *$C_2(A)$ possesses a unique M-decomposition.*
3. *$C_k(A)$ possesses a unique M-decomposition for each integer $k > 1$.*

Proof $3 \Rightarrow 2$ is clear.

$2 \Rightarrow 1$ Suppose that $C_2(A)$ possesses a unique M-decomposition. There is an index $\mathcal{I}$ and a set $\{X_i \mid i \in \mathcal{I}\}$ of simply connected M-indecomposable M-spaces such that $C_2(A) = \bigvee_{i\in\mathcal{I}} X_i$. By Lemma 10.1.1

$$A \cong H_2^M(C_2(A)) \cong \bigoplus_{i\in\mathcal{I}} H_2^M(X_i).$$

If $H_2^M(X_i) = B \oplus C$ for some abelian groups B and C then by Lemma 10.1.2 and Theorem 10.6.1(3)

$$X_k \sim C_2(H_2^M(X_k)) \sim C_2(B) \vee C_2(C).$$

Since X_i is M-indecomposable $C_2(C)$ contracts to a point, so that by Lemma 10.1.1, $C \cong H_2^M(C_2(C)) = 0$. Thus $H_2^M(X_i)$ is an indecomposable abelian group for each $i \in \mathcal{I}$.

Let $A_i = H_2^M(X_i)$ for each $i \in \mathcal{I}$. To see that the direct sum decomposition $A \cong \bigoplus_{\mathcal{I}} A_i$ is unique suppose that

$$A \cong \bigoplus_{j\in\mathcal{J}} B_j$$

for some indecomposable abelian groups B_j. Then by Theorem 10.6.1(2)

$$C_2(A) \sim \bigvee_{i\in\mathcal{I}} C_2(A_i) \sim \bigvee_{j\in\mathcal{J}} C_2(B_j)$$

and each $C_2(B_j)$ is M-indecomposable. Because part 2 states that X possesses a unique M-decomposition we see that there is a bijection $\pi : \mathcal{I} \longrightarrow \mathcal{J}$ such that $C_2(A_i) \cong C_2(B_{\pi(i)})$ for each $i \in \mathcal{I}$. Thus

$$A_i \cong H_2^M(C_2(A_i)) \cong H_2^M(C_2(B_{\pi(i)})) \cong B_{\pi(i)}$$

by Lemma 10.1.1, whence A possesses a unique decomposition.

$1 \Rightarrow 3$ is proved in exactly the same manner that we proved $2 \Rightarrow 1$. This completes the logical cycle. □

An open question in the Theory of Abelian Groups is to characterize the groups A for which the condition $A \oplus B \cong A \oplus C$ implies $B \cong C$. Groups that satisfy this property, called *the cancellation property*, include those A with local endomorphism ring, and those A that are finitely generated abelian groups. We characterize the cancellation property topologically.

Theorem 10.7.3 *Let A be an abelian group. The following are equivalent.*

1. *Suppose that $A \oplus B \cong A \oplus C$ for some abelian groups B and C. Then $B \cong C$.*
2. *Suppose that $C_2(A) \vee X \sim C_2(A) \vee Y$ for some simply connected M-spaces X and Y. Then $X \sim Y$.*

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Proof $1 \Rightarrow 2$ Assume part 1 and suppose that $C_2(A) \vee X \sim C_2(A) \vee Y$ for some simply connected M-spaces X and Y. Then $C_2(A) \vee X$ and $C_2(A) \vee Y$ are simply connected M-spaces. Apply $H_2(\cdot)$ to see that

$$H_2(C_2(A)) \oplus H_2(X) \cong H_2(C_2(A) \vee X) \cong H_2(C_2(A) \vee Y) \cong H_2(C_2(A)) \oplus H_2(Y).$$

By Lemma 10.1.1, $H_2(C_2(A)) \cong A$, so that

$$A \oplus H_2(X) \cong A \oplus H_2(Y).$$

Then by part 1, $H_2(X) \cong H_2(Y)$, whence $X \sim Y$ by Theorem 10.6.3. This proves that part 2 is true. The converse is proved in a similar manner. □

By reversing our point of view we can characterize the unique M-decompositions of M-spaces in terms of their homology groups.

Theorem 10.7.4 *Let X be a simply connected M-space. The following are equivalent.*

1. *X possesses a unique M-decomposition.*
2. *For each integer $k > 0$, $H_k^M(X)$ possesses a unique decomposition.*

Proof $2 \Rightarrow 1$ Assume part 2. We show that X possesses a decomposition into M-indecomposable M-spaces. By Lemma 10.1.1, for each integer $k > 0$ there is an M-space X_k concentrated at k such that

$$H_k^M(X) \cong H_k^M(X_k).$$

Notice that since X is simply connected $H_1^M(X)$ and X_1 are trivial. By hypothesis $H_k^M(X_k)$ possesses a unique decomposition so that

$$H_k^M(X_k) \cong \bigoplus_{i \in \mathcal{I}_k} A_{ik}$$

for some indexed set $\{A_{ik} \,\big|\, \mathcal{I}_k\}$ of indecomposable abelian groups. Let $X_{i1} = C_1(A_{i1}) = C_1(0)$ be a point and for $k > 1$ let

$$X_{ik} = C_k(A_{ik}).$$

By Lemma 10.1.1

$$H_k^M(X_{ik}) \cong A_{ik}$$

and because the A_{ik} are indecomposable, Theorems 10.3.1(2) and 10.3.1(3) imply that each X_{ik} is M-indecomposable. Thus

$$H_k^M(X_k) = \bigoplus_{i \in \mathcal{I}_k} A_{ik} = H_k^M\left(\bigvee_{i \in \mathcal{I}_k} X_{ik}\right). \tag{10.2}$$

By construction the M-spaces X_{ik} are simply connected spaces, so that

$$X_k \sim \bigvee_{i \in \mathcal{I}_k} X_{ik}$$

by Theorem 10.6.3. Let Y be the simply connected M-space

$$Y = \bigvee_{k>0} X_k = \bigvee_{k>0} (\vee_{i \in \mathcal{I}_k} X_{ik}).$$

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We will show that $X \sim Y$.

Inasmuch as

$$H_*^M(X) = (\cdots, H_2^M(X_2), H_1^M(X_1) = 0) \tag{10.3}$$

and since $H_p^M(X_k) = 0$ for each $p \neq k$ we have

$$H_k^M\left(\bigvee_{\ell>0} X_\ell\right) = H_k^M(X_k).$$

Then

$$H_*^M(X) \cong (H_k^M(X_k)\,\big|\, k > 0) \cong H_*^M\left(\bigvee_{k>0} X_k\right) \cong H_*^M(Y)$$

so by Theorem 10.6.3, $X \sim Y$.

Now suppose that we have another decomposition Y' of Y into simply connected M-indecomposable M-spaces

$$Y' = \bigvee_{k>0} (\vee_{j\in\mathcal{J}_k} X'_{jk})$$

where each X'_{jk} is concentrated at k. Then $Y \sim Y'$ so that

$$\bigoplus_{j\in\mathcal{J}_k} H_k^M(X'_{jk}) = H_k^M\left(\bigvee_{j\in\mathcal{J}_k} X'_{jk}\right) = H_k^M(Y) = H_k^M\left(\bigvee_{i\in\mathcal{I}_k} X_{ik}\right) = \bigoplus_{i\in\mathcal{I}_k} H_k^M(X_{ik})$$

for each integer $k > 0$. Because X'_{ik} and X_{ik} are M-indecomposable, Theorem 10.7.1 shows us that $H_k^M(X'_{ik})$ and $H_k^M(X_{ik})$ are indecomposable abelian groups. Since $H_k^M(Y) \cong H_k^M(X)$ by hypothesis possesses a unique decomposition, we conclude that there is a bijection $\pi : \mathcal{I}_k \longrightarrow \mathcal{J}_k$ such that $H_k^M(X_{ik}) \cong H_k^M(X'_{\pi(i)k})$. Then by Lemma 10.1.2, $X'_{ik} \sim X_{ik}$ for each pair of integers i, k, and therefore X possesses a unique M-decomposition Y. The converse is proved by reversing the above argument. This completes the proof. □

Corollary 10.7.5 *Let X be a simply connected M-space such that $H_k^M(X)$ is a finitely generated abelian group for each integer $k > 0$. Then X possesses a unique M-decomposition.*

Proof Each finitely generated abelian group possesses a unique decomposition. Now apply the above Theorem. □

Corollary 10.7.6 *Let X be a compact simply connected M-space. Then X possesses a unique M-decomposition.*

Proof Evidently, for each integer $k > 0$, $H_k^M(X)$ is finitely generated if X is compact. Now apply the previous corollary. □

10.8 Homological Dimensions

Let fd_E denote the flat dimension of a left $\mathrm{End}_R(G)$-module. Let id_E denote the injective dimension of a left $\mathrm{End}_R(G)$-module. The right R-module G is *E-flat* if G is a flat left $\mathrm{End}_R(G)$-module. For a fixed integer $k > 0$, *projective Euclidean k-space* is a subspace of a quotient space of a union of Euclidean k-spaces $\mathbb{R}^k$ with the quotient topology.

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Theorem 10.8.1 *Let $k > 0$ be an integer and let G be a self-small and self-slender right R-module. The following are equivalent.*

1. $\mathrm{fd}_E(G) \leq k$
2. *Each M-space $X \in$ image α is a subspace of projective Euclidean $k + 1$-space.*

Proof Assume part 1. So that the notation agrees with that of **SAb** let

$$A_k(\cdot) = \mathrm{Tor}_E^k(\cdot, G) \text{ for each integer } k > 0$$

and let $A_*(\cdot) = \mathrm{Tor}_E^*(\cdot, G)$. Observe that $k + j > 1$ for each integer $j > 0$. Follow this argument by tracing through Diagram (2).

Assume that $\mathrm{fd}_E(G) \leq k$, let $X = \alpha(\mathcal{Q}) \in$ image α, and choose $M \in$ **Mod**-$\mathrm{End}_R(G)$ such that $\mathrm{h}_G(\mathcal{Q}) = M$. Then $A_{k+j}(\cdot) = 0$ for each integer $j > 0$. In the construction $C_{k+j}(\cdot)$ we have

$$C_{k+j}(A_{k+j}(\cdot)) = C_{k+j}(0) = \text{ a point}$$

while for $p = 1, \cdots, k$, $C_p(A_p(\cdot))$ is a quotient space of a one point union of $p + 1$-disks. Since a $p+1$-disk embeds in $\mathbb{R}^{k+1}$ for each $p = 1, \cdots, k$, $C_p(A_p(\cdot))$ is a subspace of projective Euclidean $k + 1$-space. Hence

$$C_* \circ A_*(M) = \bigvee_{p>0} C_p(A_p(M)) = \bigvee_{p=1}^{k} C_p(A_p(M))$$

is a subspace of projective Euclidean $k + 1$-space. Finally since Diagram (3) is commutative

$$X = \alpha(\mathcal{Q}) = (C_* \circ A_* \circ \mathrm{h}_G)(\mathcal{Q}) = C_* \circ A_*(M)$$

is a subspace of projective Euclidean $k + 1$-space. This proves part 2.

The converse is proved by reversing the argument. □

Corollary 10.8.2 *Let G be self-small and self-slender. Then $\mathrm{fd}_E(G)$ is finite iff there is an integer k such that each $X \in$ image α embeds in a projective Euclidean k-space.*

Given an abelian group A the M-space $C_1(A)$ is a subspace of projective Euclidean 2-space. If $A = 0$ then $C_1(A)$ is a point. If $A \neq 0$ then in any construction $C_1(A)$, $H_1^M(C_1(A)) \cong A \neq 0$ so that $C_1(A)$ does not contract to a point. This and a couple of lines will prove

Corollary 10.8.3 *Let G be self-small and self-slender. Then G is E-flat iff $C_1(H_1^P(\mathcal{Q}))$ contracts to a point for each G-plex $\mathcal{Q}$.*

Proof The equivalences follow from the commutativity of Diagram (3). G is flat iff $\mathrm{Tor}_E^1(M, G) = 0$ for each right $\mathrm{End}_R(G)$-module M iff $H_1^P(\mathcal{Q}) = 0$ for each G-plex $\mathcal{Q}$ iff $C_1(H_1^P(\mathcal{Q})) = C_1(0)$ contracts to a point for each G-plex $\mathcal{Q}$. □

Theorem 10.8.4 *Let G be self-small and self-slender. For each G-plex $\mathcal{Q}$, $C_1(H_1^P(\mathcal{Q}))$ is compact iff G is E-flat.*

Proof G is not flat iff there is a right $\mathrm{End}_R(G)$-module M such that $\mathrm{Tor}_E^1(M, G) \neq 0$ iff $\mathrm{Tor}_E^1(M^{(\aleph_o)}, G)$ is infinite for some right $\mathrm{End}_R(G)$-module M iff the M-Space X corresponding to $\mathrm{Tor}_E^1(M, G)^{(\aleph_o)}$ is not compact. □

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Dualizing flat dimension we arrive at injective dimension.

Theorem 10.8.5 *Let G be self-small and self-slender. The following are equivalent.*

1. $\mathrm{id}_E(G) \leq k$
2. *Each $X \in$ image δ is a subspace of projective Euclidean $k+1$-space.*

Corollary 10.8.6 *Let G be self-small and self-slender. Then $\mathrm{id}_E(G)$ is finite iff there is an integer k such that each $X \in$ image δ embeds in projective Euclidean k-space.*

Corollary 10.8.7 *Let G be self-small and self-slender. Then $\mathrm{id}_E(G) \leq 1$ iff for each G-coplex $\mathcal{W}$, $\delta(\mathcal{W})$ is a subspace of projective Euclidean 2-space.*

Corollary 10.8.8 *Let G be self-small and self-slender. If α is a surjection then $\mathrm{fd}_E(G) = \infty$.*

Proof We use the fact derived from the commutativity of Diagram (2) that $\mathrm{fd}_E(G)$ is the supremum of the integers k such that $H^P_{k+1}(\mathcal{Q}) = 0$ for each G-plex $\mathcal{Q}$.

If α is a surjection then $C_* \circ \mathrm{Tor}^*_E(\mathrm{h}_G(\cdot), G)$ is a surjection so that there is a G-plex $\mathcal{Q}$ that maps to

$$X = X_1 \vee X_2 \vee \cdots$$

where for each integer $k \neq 0$, X_k is an indecomposable M-space concentrated at k such that $H^M_k(X_k) \neq 0$. Let $\mathcal{Q} \in G$-**Plex** be such that $\alpha(\mathcal{Q}) = X$, and then let $M = \mathrm{h}_G(\mathcal{Q})$. By the commutativity of Diagram (2)

$$\mathrm{Tor}^k_E(M, G) \cong \mathrm{Tor}^k_E(\mathrm{h}_G(\mathcal{Q}), G) \cong H^M_k(\mathcal{Q}) = H^M_k(X) = H^M_k(X_k) \neq 0$$

for each integer $k > 0$. Hence $\mathrm{fd}_E(G) = \infty$. □

Corollary 10.8.9 *Let G be a self-small and self-slender right R-module. If δ is a surjection then $\mathrm{id}_E(G) = \infty$.*

Corollary 10.8.10 *Let G be a self-small and self-slender abelian group. There is at least one noncompact $X \in$ image δ.*

Proof G, being self-slender, is reduced, hence not divisible. There is an $M \neq 0$ such that $\mathrm{Ext}^1_E(M, G) \neq 0$ and so $\mathrm{Ext}^1_E(M^{(\aleph_o)}, G) \cong \mathrm{Ext}^1_E(M, G)^{(\aleph_o)}$ correspondes under C_* to a space that is the union of countably many copies of a nontrivial M-space X. Such a space is not compact. □

For example, if the reduced torsion-free finite rank abelian group G is a flat left $\mathrm{End}_R(G)$-module then image $\alpha = \{0\} \neq$ image δ.

10.9 Miscellaneous

Let us examine the commutative rings R that are self-slender R-modules. This class of rings includes the rings R whose additive structure is a reduced torsion-free finite rank abelian group, and the countable integral domains that are not fields. If R is a commutative ring then

$$\textbf{Mod-}R = R\textbf{-Mod}.$$

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Let

$$R\textbf{-Plex}$$

denote the category of R-plexes whose terms are direct summands of $\oplus_c R$ for some cardinal c. This and the commutativity of Diagram (2) proves

Theorem 10.9.1 *Let R be a commutative ring that is a self-slender R-module, and let $R = G = \mathrm{End}_R(G)$. There is a contravariant category equivalence*

$$\mathrm{h}^R \circ \mathrm{h}_R : R\textbf{-Plex} \longrightarrow R\textbf{-coPlex}.$$

Let us examine Diagram (2) under the hypothesis that $R = G = \mathrm{End}_R(G) = \mathbb{Z}$. Let

2-Plex

denote the category whose objects are exact free complexes

$$0 \longrightarrow \bigoplus_c \mathbb{Z} \longrightarrow \bigoplus_d \mathbb{Z}$$

for cardinals c, d. We observe that $\mathrm{Tor}^k_{\mathbb{Z}}(\cdot, \mathbb{Z}) = 0 = H^C_k(\cdot)$ for each integer $k > 0$. Since each abelian group ($\mathbb{Z}$-module) has a projective resolution with at most two nonzero terms, **2-Plex** is category equivalent to $\mathbb{Z}$**-Plex**.

Now, since the left global dimension of $\mathbb{Z}$ is ≤ 1 each $\mathbb{Z}$-coPlex has the homotopy type of a $\mathbb{Z}$-coplex

$$\prod_c \mathbb{Z} \longrightarrow \prod_d \mathbb{Z} \longrightarrow 0 \tag{10.4}$$

for some cardinals c, d. Thus $\mathbb{Z}$**-coPlex** is equivalent to the full subcategory

2-coPlex

of $\mathbb{Z}$**-coPlex** whose objects are the $\mathbb{Z}$-coplexes of the form (10.4). An application of Theorem 10.9.1 then proves

Theorem 10.9.2 *There is a contravariant category equivalence*

$$\mathrm{h}^{\mathbb{Z}} \circ \mathrm{h}_{\mathbb{Z}} : \textbf{2-Plex} \longrightarrow \textbf{2-coPlex}.$$

Remark 10.9.3 We end this chapter with a comment. Diagram (1) has an empty space in it that has to be filled in. Notice that the upper right corner of Diagram (1) contains **Free Complex** and associated functions. But the lower left corner of the diagram is empty. At the time of this writing there is neither category nor function to complete this corner of Diagram (1).

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Chapter 11

Injective Modules and Prime Ideals of Universal Enveloping Algebras

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Abstract In this paper we study injective modules over universal enveloping algebras of finite-dimensional Lie algebras over fields of arbitrary characteristic. Most of our results are dealing with fields of prime characteristic but we also elaborate on some of their analogues for solvable Lie algebras over fields of characteristic zero. It turns out that analogous results in both cases are often quite similar and resemble those familiar from commutative ring theory.

Subject classifications: 17B35, 17B50, 17B55, 17B56.

Introduction

In this paper we investigate the injective modules and their relation to prime ideals in universal enveloping algebras of finite-dimensional Lie algebras. Especially, in the case that the ground field is of prime characteristic we obtain several results that seem to be new. It should be remarked that most of the results of the first two sections and the last section are already contained in an unpublished manuscript of the author (cf. [13]) but the entire third section and Theorem 11.4.5 are completely new. In the following we will describe the contents of the paper in more detail.

The first section provides the framework for the paper. We begin by recalling the well-known result from noetherian ring theory that every injective module decomposes uniquely (up to isomorphism and order of occurrence) into a direct sum of indecomposable injective modules. Then it is shown that universal enveloping algebras of finite-dimensional Lie algebras over fields of prime characteristic are FBN rings. As a consequence, indecomposable injective modules are in bijection with prime ideals. Moreover, it is proved that the universal enveloping algebra of a finite-dimensional Lie algebra over a field of prime characteristic is a Matlis ring (i.e., every indecomposable injective module is the injective hull of a prime factor ring of the universal enveloping algebra considered as a one-sided module) if and only if the underlying Lie algebra is abelian. A similar result might also hold in characteristic zero but we were neither able to prove this nor to find it in

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the literature.

In the second section we study certain finiteness conditions for injective hulls. It is well known from a result obtained by Donkin [10] and independently by Dahlberg [7] that injective hulls of locally finite modules over universal enveloping algebras of finite-dimensional solvable Lie algebras over fields of characteristic zero are again locally finite. We show that the converse of this result holds, i.e., the locally finiteness of injective hulls of locally finite modules in characteristic zero implies that the underlying Lie algebra is solvable. In fact, the locally finiteness of the injective hull of the one-dimensional trivial module already implies that the underlying Lie algebra is solvable. This generalizes an observation of Donkin in [10]. Moreover, we prove that every essential extension of a locally finite module over a universal enveloping algebra of any finite-dimensional Lie algebra over a field of prime characteristic is locally finite by applying a result of Jategaonkar [18] in conjunction with the result from the first section saying that universal enveloping algebras of finite-dimensional Lie algebras over fields of prime characteristic are FBN rings. In particular, injective hulls of locally finite modules are always locally finite. By generalizing slightly another result of Jategaonkar [19], we also show that injective hulls of artinian modules over universal enveloping algebras of finite-dimensional Lie algebras over a field of prime characteristic are always artinian. Finally, it is established that for the universal enveloping algebra of a non-zero finite-dimensional Lie algebra over a field of prime characteristic non-zero noetherian modules are never injective by proving that the injective dimension of a non-zero noetherian module coincides with the dimension of the underlying Lie algebra. On the other hand, there are artinian and locally finite modules of any possible injective dimension.

In the third section we consider certain locally finite submodules of the linear dual of a universal enveloping algebra. We start off by showing how an argument from [7] can be changed slightly to make it work over arbitrary fields of any characteristic and therefore obtaining a different (and in our opinion more transparent) proof of a result due to Levasseur [25]. Then we give a very short proof of the main result of [21] by using the locally finiteness of injective hulls of locally finite modules over universal enveloping algebras of finite-dimensional solvable Lie algebras in characteristic zero in an essential way. In fact, this argument was motivated by our proof of the injectivity of the continuous dual of the universal enveloping algebra of an arbitrary finite-dimensional Lie algebra over a field of prime characteristic. As an immediate consequence, we obtain that in prime characteristic the cohomology with values in the continuous dual vanishes in every positive degree. In particular, Koszul's cohomological vanishing theorem does remain valid in prime characteristic. These results seem to be new. Moreover, the modular cohomological vanishing theorem is much stronger than its analogue in characteristic zero which follows from a recent result of Schneider (cf. [30]) and says that the cohomology with values in the continuous dual vanishes in degrees one and two.

The last section closes the circle of ideas by coming back to the correspondence between injective modules and prime ideals. It is verified that universal enveloping algebras of finite-dimensional Lie algebras over fields of prime characteristic are injectively homogeneous in the sense of [4]. As a consequence of the general theory of injectively homogeneous rings developed in [4] we obtain a nice description of a minimal injective resolution of the universal enveloping algebra as a module over itself in terms of the injective hulls of its prime factor rings considered as one-sided modules. In particular, this enables us to show that the last term of such a minimal injective resolution is isomorphic to the continuous dual which was proved by Barou and Malliavin [2] for finite-dimensional solvable Lie algebras over algebraically fields of characteristic zero.

Throughout this paper we will assume that all associative rings have a unity element and that all modules over associative rings are unital.

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11.1 Injective Modules and Prime Ideals

Since the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{a}$ is left and right noetherian (cf. [17, Theorem V.6]), finding all injective left and right $U(\mathfrak{a})$-modules reduces to the classification of the indecomposable ones (see [29, Theorem 2.5, Proposition 2.6, and Proposition 2.7]):

Proposition 11.1.1 *Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over an arbitrary field. Then the following statements hold:*

(1) *Every injective left or right $U(\mathfrak{a})$-module is a direct sum of indecomposable injective submodules.*

(2) *If I is an indecomposable injective left or right $U(\mathfrak{a})$-module, then $\mathrm{End}_{\mathfrak{a}}(I)$ is local. In particular, the decomposition in the first part is unique up to isomorphism and order of occurrence of the direct summands.* □

In order to be able to parameterize the indecomposable injective left or right $U(\mathfrak{a})$-modules, one needs the following concept from non-commutative ring theory. A left and right noetherian associative ring R is called a *FBN ring* if every essential left ideal and every essential right ideal of every prime factor ring of R contains a non-zero two-sided ideal (which, in fact, is essential). While classifying the indecomposable injective $U(\mathfrak{a})$-modules by analogy with the commutative case (see [29, Proposition 3.1]), one should be aware that the injective hull of $U(\mathfrak{a})/\mathcal{P}$ (considered as a left or right $U(\mathfrak{a})$-module) is not necessarily indecomposable for every prime ideal $\mathcal{P}$ of $U(\mathfrak{a})$. For example, the injective hull of $U(\mathfrak{a})/\mathrm{Ann}_{U(\mathfrak{a})}(S)$ is isomorphic to the direct sum of d copies of the injective hull of any simple $\mathfrak{a}$-module S of dimension $d > 1$ (cf. the proof of Theorem 11.1.3 and Theorem 11.4.5).

Let M be a non-zero $U(\mathfrak{a})$-module. A two-sided ideal $\mathcal{P}$ is said to be *associated to M* if there exists a submodule N of M such that $\mathcal{P}$ equals the annihilator of every non-zero submodule of N. It is well known that $\mathcal{P}$ is necessarily prime and that for an indecomposable injective module I there exists a unique prime ideal $\mathcal{P}_I$ associated to I (cf. [3]).

If $\mathfrak{a}$ is a *finite-dimensional* Lie algebra over a field of *prime* characteristic, then $U(\mathfrak{a})$ is a *finitely generated* $C(U(\mathfrak{a}))$-module (cf. [35, Theorem 5.1.2]). Hence one has the following well-known facts which are crucial for the results obtained in this paper:

(IC) $U(\mathfrak{a})$ is *integral* over its center $C(U(\mathfrak{a}))$ (cf. [35, Theorem 6.1.4]). More generally, there exists a subalgebra $\mathcal{O}(\mathfrak{a}) \cong \mathbb{F}[t_1, \ldots, t_{\dim_{\mathbb{F}} \mathfrak{a}}]$ of $C(U(\mathfrak{a}))$ such that $U(\mathfrak{a})$ is *integral* over every subring C of $U(\mathfrak{a})$ with $\mathcal{O}(\mathfrak{a}) \subseteq C \subseteq C(U(\mathfrak{a}))$.

(PI) $U(\mathfrak{a})$ is a *PI ring* (cf. [14, p. xi]).

The next result shows that the indecomposable injective modules over universal enveloping algebras in prime characteristic can be classified by their associated prime ideals.

Theorem 11.1.2 *Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over a field of prime characteristic. Then the universal enveloping algebra $U(\mathfrak{a})$ is a FBN ring. In particular, there is a one-to-one correspondence between the indecomposable injective $U(\mathfrak{a})$-modules and the prime ideals of $U(\mathfrak{a})$ given by $I \mapsto \mathcal{P}_I$, where $\mathcal{P}_I$ is the unique prime ideal associated to I.*

Proof The first assertion follows from [14, Proposition 8.1(b)] and the second assertion is a consequence of the first and [23, Theorem 3.5]. □

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Question. Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over a field of characteristic *zero*. It would be interesting to know when $U(\mathfrak{a})$ is a FBN ring? Is $U(\mathfrak{a})$ only an FBN ring if $\mathfrak{a}$ is *abelian*?

An associative ring R is called a *Matlis ring* if every indecomposable injective left or right R-module is isomorphic to the injective hull of $R/\mathcal{P}$ (considered as a left or right R-module) for some prime ideal $\mathcal{P}$ of R. Every left and right noetherian Matlis ring is a FBN ring (see [23, Corollary 3.6]), but the converse is not true as follows from Theorem 11.1.2 and the next result.

Theorem 11.1.3 *Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over a field of prime characteristic. Then the universal enveloping algebra $U(\mathfrak{a})$ is a Matlis ring if and only if $\mathfrak{a}$ is abelian.*

Proof Since both conditions are independent of the ground field $\mathbb{F}$, we can assume that $\mathbb{F}$ is algebraically closed. Suppose that $U(\mathfrak{a})$ is a Matlis ring. According to [22, Corollary 14], every prime ideal of $U(\mathfrak{a})$ is completely prime. Let S be a simple $\mathfrak{a}$-module and set $\mathbb{D} := \mathrm{End}_{\mathfrak{a}}(S)$. Since S is finite-dimensional (cf. [35, Theorem 5.2.4]), $\mathbb{D}$ is a finite-dimensional division algebra over $\mathbb{F}$, and thus $\mathbb{D} = \mathbb{F}$. Then the density theorem (cf. [20, Theorem 16, p. 95]) implies that

$$U(\mathfrak{a})/\mathrm{Ann}_{U(\mathfrak{a})}(S) \cong \mathrm{End}_{\mathbb{F}}(S) \cong \mathrm{Mat}_d(\mathbb{F}),$$

where $d := \dim_{\mathbb{F}} S$. Since S is simple, $\mathrm{Ann}_{U(\mathfrak{a})}(S)$ is primitive (i.e., prime), and thus, $\mathrm{Ann}_{U(\mathfrak{a})}(S)$ is completely prime. It follows that $\mathrm{Mat}_d(\mathbb{F})$ has no zero divisors, i.e., $d = 1$. Hence every simple $\mathfrak{a}$-module is one-dimensional. By virtue of a result due to Jacobson, there exists a (finite-dimensional) faithful semisimple $\mathfrak{a}$-module (see [35, Theorem 5.5.2]). Therefore, we have

$$[\mathfrak{a}, \mathfrak{a}] \subseteq \bigcap_{S \in \mathrm{Irr}(\mathfrak{a})} \mathrm{Ann}_{\mathfrak{a}}(S) = 0,$$

where $\mathrm{Irr}(\mathfrak{a})$ denotes the set of isomorphism classes of simple $\mathfrak{a}$-modules, i.e., $\mathfrak{a}$ is abelian. Finally, the converse is just [29, Proposition 3.1]. □

Remark 11.1.4 The proof of Theorem 11.1.3 applied to a composition factor S of the adjoint module of a finite-dimensional Lie algebra $\mathfrak{a}$ over a field of characteristic zero shows that in this case the universal enveloping algebra $U(\mathfrak{a})$ can only be a Matlis ring if $\mathfrak{a}$ is *solvable* (cf. also [3, p. 49]). This still leaves the question as to whether Theorem 11.1.3 is also true in characteristic zero.

11.2 Injective Hulls

In this section several finiteness properties of injective hulls are considered. Let R be an associative ring and let M be a left or right R-module. An injective module I is called an *injective hull* (or an *injective envelope*) of M if there exists an R-module monomorphism $\iota : M \to I$ such that the image $\mathrm{Im}(\iota)$ of ι is an essential submodule of I. (By abuse of language, the pair (I, ι) is also called an injective hull of M.)

It is well known that every module has an injective hull (cf. [14, Theorem 4.8(a)]). Moreover, injective hulls satisfy the following *universal properties* (cf. [14, Theorem 4.8(b) and (c)] or [32, Theorem 3.30]):

Let M be an R-module and let $(I_R(M), \iota_M)$ be an injective hull of M.

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(I) If I is an injective R-module and ι is an R-module monomorphism from M into I, then every R-module homomorphism η from $I_R(M)$ into I with $\eta \circ \iota_M = \iota$ is a monomorphism. (Since I is injective and ι_M is an R-module monomorphism, there *always* exists an R-module homomorphism from $I_R(M)$ into I with $\eta \circ \iota_M = \iota$!)

(E) If N is an R-module and φ is an R-module monomorphism from M into N such that $\varphi(M)$ is an essential submodule of N, then every R-module homomorphism ν from N into $I_R(M)$ with $\nu \circ \varphi = \iota_M$ is a monomorphism. (Since $I_R(M)$ is injective and φ is an R-module monomorphism, there *always* exists an R-module homomorphism from N into $I_R(M)$ with $\nu \circ \varphi = \iota_M$!)

(I) says that injective hulls are *minimal injective extensions* and (E) says that injective hulls are *maximal essential extensions*. In particular, injective hulls are uniquely determined up to isomorphism (cf. [14, Proposition 4.9]).

Recall that a module is said to be *locally finite* if every finitely generated (or equivalently, every cyclic) submodule is finite-dimensional.

Theorem 11.2.1 *Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over a field of prime characteristic. Then every essential extension of a locally finite $\mathfrak{a}$-module is locally finite.*

Proof Let M be a locally finite $\mathfrak{a}$-module, let E be an essential extension of M, and let e be any non-zero element of E. Then $E' := U(\mathfrak{a})e$ is an essential extension of $M' := E' \cap M$. Since $U(\mathfrak{a})$ is noetherian, $M' \subseteq E'$ is finitely generated. Because M is by assumption locally finite, $M' \subseteq M$ is finite-dimensional. By virtue of Theorem 11.1.2, we can apply [18, Corollary 3.6] or the main result of [33] which both show that E' is also finite-dimensional, i.e., E is locally finite. $\square$

The next result is an immediate consequence of Theorem 11.2.1.

Corollary 11.2.2 *If $\mathfrak{a}$ is a finite-dimensional Lie algebra over a field of prime characteristic, then the injective hull of every locally finite $\mathfrak{a}$-module is locally finite.* $\square$

It is well known that Corollary 11.2.2 is also true for a finite-dimensional *solvable* Lie algebra over an arbitrary field of characteristic *zero* (see [10, Theorem 2.2.3] and [7, Corollary 12]), but it does *not* hold for a finite-dimensional *semisimple* Lie algebra over a field of characteristic *zero* (see [10, Remark after the proof of Proposition 2.2.2] and [8, Remark 1]). More precisely, we have the following result.

Theorem 11.2.3 *Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over a field of characteristic zero. Then the following statements are equivalent:*

(1) *$\mathfrak{a}$ is solvable.*

(2) *The injective hull of the one-dimensional trivial $\mathfrak{a}$-module is locally finite.*

(3) *The injective hull of every locally finite $\mathfrak{a}$-module is locally finite.*

Proof The implication (1)$\Longrightarrow$(3) is just [10, Theorem 2.2.3] or [7, Corollary 12] and the implication (3)$\Longrightarrow$(2) is trivial. Hence it only remains to show the implication (2)$\Longrightarrow$(1).

Suppose that the injective hull $I_\mathfrak{a}(\mathbb{F})$ of the one-dimensional trivial $\mathfrak{a}$-module $\mathbb{F}$ is locally finite. Since the ground field is assumed to have characteristic zero, the Levi decomposition theorem (cf. [17, p. 91]) yields the existence of a semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{a}$ (a so-called *Levi factor* of $\mathfrak{a}$) such that $\mathfrak{a}$ is the semidirect product of $\mathfrak{s}$ and its solvable radical $\mathrm{Solv}(\mathfrak{a})$. According to [7, Proposition 4], the restriction $I := I_\mathfrak{a}(\mathbb{F})_{|\mathfrak{s}}$ is an injective $U(\mathfrak{s})$-module. Since $I_\mathfrak{a}(\mathbb{F})$ is a locally finite $U(\mathfrak{a})$-module, I is a locally finite $U(\mathfrak{s})$-module.

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Since I is injective, it follows from the universal property (I) of injective hulls that $\mathbb{F} \subseteq I_{\mathfrak{s}}(\mathbb{F}) \subseteq I$. If $0 \neq m \in I_{\mathfrak{s}}(\mathbb{F})$, then the cyclic submodule $M := U(\mathfrak{s})m$ of I is finite-dimensional. Since $I_{\mathfrak{s}}(\mathbb{F})$ is an essential extension of $\mathbb{F}$ and M is a non-zero submodule of $I_{\mathfrak{s}}(\mathbb{F})$, $M \cap \mathbb{F} \neq 0$. Then for dimension reasons, $M \cap \mathbb{F} = \mathbb{F}$, i.e., $\mathbb{F} \subseteq M$. By virtue of Weyl's completely reducibility theorem (cf. [17, Theorem III.8, p. 79]), $\mathbb{F}$ has a complement in M, i.e., there exists a submodule C of M such that $M = \mathbb{F} \oplus C$. In particular, $\mathbb{F} \cap C = 0$ which implies that $C = 0$ because C is a submodule of $I_{\mathfrak{s}}(\mathbb{F})$. Consequently, $M = \mathbb{F}$ and therefore $\mathbb{F} = I_{\mathfrak{s}}(\mathbb{F})$. Hence $\mathbb{F}$ is an injective $U(\mathfrak{s})$-module and thus also an injective $U(\mathbb{F}s)$-module for every element $s \in \mathfrak{s}$ (cf. [7, Proposition 4]). Finally $\mathrm{Ext}^1_{U(\mathbb{F}s)}(\mathbb{F}, \mathbb{F}) \cong H^1(\mathbb{F}s, \mathbb{F}) \neq 0$ for every $0 \neq s \in \mathfrak{s}$ yields $\mathfrak{s} = 0$, i.e., $\mathfrak{a} = \mathrm{Solv}(\mathfrak{a})$ is solvable. □

Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over a field of characteristic zero. Donkin [10, Theorem 2.2.3] proved that the largest locally finite submodule $I_{\mathfrak{a}}(M)_{\mathrm{loc}}$ of the injective hull of any *finite-dimensional* $\mathfrak{a}$-module M is *artinian*. In particular, if $\mathfrak{a}$ is solvable, then injective hulls of *finite-dimensional* $\mathfrak{a}$-modules are *artinian*. Furthermore, Dahlberg [8] showed that the injective hull of every *artinian* $\mathfrak{sl}_2(\mathbb{C})$-module is *locally artinian*. In prime characteristic the following stronger result holds.

Theorem 11.2.4 *If $\mathfrak{a}$ is a finite-dimensional Lie algebra over a field of prime characteristic, then the injective hull of every artinian $\mathfrak{a}$-module is artinian.*

Proof Let M be an artinian $\mathfrak{a}$-module. Then the socle $\mathrm{Soc}_{\mathfrak{a}}(M)$ of M is also artinian, i.e., a *finite* direct sum of simple modules. According to $I_{\mathfrak{a}}(M) \cong I_{\mathfrak{a}}(\mathrm{Soc}_{\mathfrak{a}}(M))$ and the additivity of $I_{\mathfrak{a}}(-)$, the assertion is an immediate consequence of (PI) and [19, Theorem 2]. □

Non-zero *noetherian* $\mathfrak{a}$-modules are very often *not injective*. This was proved in [5, Corollary 2.3] for every (not necessarily commutative) *local* noetherian associative ring and motivated the first part of Proposition 11.2.5 below. In particular, injective hulls of noetherian (or even finite-dimensional) $\mathfrak{a}$-modules are not noetherian. Moreover, for artinian and locally finite $\mathfrak{a}$-modules any possible injective dimension can occur.

Proposition 11.2.5 *Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$ of prime characteristic. Then the following statements hold:*

(1) *For every non-zero finitely generated (= noetherian) $\mathfrak{a}$-module M, we have*

$$\mathrm{inj.dim}_{U(\mathfrak{a})} M = \dim_{\mathbb{F}} \mathfrak{a}.$$

(2) *For every integer $0 \leq r \leq \dim_{\mathbb{F}} \mathfrak{a}$ there exists an artinian $\mathfrak{a}$-module M_r such that*

$$\mathrm{inj.dim}_{U(\mathfrak{a})} M_r = r.$$

(3) *For every integer $0 \leq r \leq \dim_{\mathbb{F}} \mathfrak{a}$ there exists a locally finite $\mathfrak{a}$-module N_r such that*

$$\mathrm{inj.dim}_{U(\mathfrak{a})} N_r = r.$$

Proof (1): Since M is noetherian, it has a maximal submodule N. Therefore $S := M/N$ is simple, and thus finite-dimensional (cf. [35, Theorem 5.2.4]). By virtue of [12, Theorem 4.2(3)], there exists an $\mathfrak{a}$-module V such that $\mathrm{Ext}^d_{U(\mathfrak{a})}(V, S) \neq 0$, where $d := \dim_{\mathbb{F}} \mathfrak{a}$. Then the long exact cohomology sequence implies the exactness of

$$\mathrm{Ext}^d_{U(\mathfrak{a})}(V, M) \longrightarrow \mathrm{Ext}^d_{U(\mathfrak{a})}(V, S) \longrightarrow \mathrm{Ext}^{d+1}_{U(\mathfrak{a})}(V, N).$$

Because of $\mathrm{gl.dim}\, U(\mathfrak{a}) = d$ (cf. [6, Theorem 8.2]), the right-hand term vanishes. One concludes that $\mathrm{Ext}^d_{U(\mathfrak{a})}(V, M) \neq 0$, i.e., $\mathrm{inj.dim}_{U(\mathfrak{a})} M \geq d$. The reverse inequality follows from $\mathrm{inj.dim}_{U(\mathfrak{a})} M \leq \mathrm{gl.dim}\, U(\mathfrak{a}) = d$.

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(2): Put $d := \dim_{\mathbb{F}} \mathfrak{a}$ and let M_d be any non-zero finite-dimensional $\mathfrak{a}$-module. By the first part, we have $\operatorname{inj.dim}_{U(\mathfrak{a})} M_d = d$. According to Theorem 11.2.4, the injective hull $I_{\mathfrak{a}}(M_d)$ and therefore $M_{d-1} := I_{\mathfrak{a}}(M_d)/M_d$ are artinian. From the long exact cohomology sequence and the injectivity of $I_{\mathfrak{a}}(M_d)$ one concludes for an arbitrary $\mathfrak{a}$-module X that

$$\operatorname{Ext}^d_{U(\mathfrak{a})}(X, M_{d-1}) \cong \operatorname{Ext}^{d+1}_{U(\mathfrak{a})}(X, M_d) = 0$$

because $\operatorname{inj.dim}_{U(\mathfrak{a})} M_d = d$. Hence $\operatorname{inj.dim}_{U(\mathfrak{a})} M_{d-1} \leq d - 1$ (cf. [32, Theorem 9.8]). By another application of [32, Theorem 9.8], there exists an $\mathfrak{a}$-module X_d such that $\operatorname{Ext}^d_{U(\mathfrak{a})}(X_d, M_d) \neq 0$. Then the long exact cohomology sequence implies

$$\operatorname{Ext}^{d-1}_{U(\mathfrak{a})}(X_d, M_{d-1}) \cong \operatorname{Ext}^d_{U(\mathfrak{a})}(X_d, M_d) \neq 0,$$

i.e., $\operatorname{inj.dim}_{U(\mathfrak{a})} M_{d-1} = d - 1$, and the assertion follows by induction.

(3): The proof is the same as for (2) except that one uses Corollary 11.2.2 instead of Theorem 11.2.4 to conclude that $N_{d-1} := I_{\mathfrak{a}}(N_d)/N_d$ is locally finite. □

Remark 11.2.6 Dually, non-zero *artinian* $\mathfrak{a}$-modules are *never projective* if $\mathfrak{a} \neq 0$ and for noetherian $\mathfrak{a}$-modules any possible projective dimension can occur (see [13]).

Since every *simple* module over a *finite-dimensional* Lie algebra over a field of prime characteristic is *finite-dimensional* (cf. [35, Theorem 5.2.4], the following is an immediate consequence of Proposition 11.2.5(1).

Corollary 11.2.7 *Let* $\mathfrak{a}$ *be a finite-dimensional Lie algebra over a field* $\mathbb{F}$ *of prime characteristic and let* S *be a simple* $\mathfrak{a}$*-module. Then* $\operatorname{inj.dim}_{U(\mathfrak{a})} S = \dim_{\mathbb{F}} \mathfrak{a}$. □

11.3 Locally Finite Submodules of the Coregular Module

Let $\mathfrak{a}$ be a Lie algebra over a field $\mathbb{F}$ of arbitrary characteristic. Then the *linear dual* $U(\mathfrak{a})^* := \operatorname{Hom}_{\mathbb{F}}(U(\mathfrak{a}), \mathbb{F})$ of $U(\mathfrak{a})$ is a left and a right $U(\mathfrak{a})$-module, the so-called *coregular module* of $U(\mathfrak{a})$ (cf. [9, 2.7.7]). It is well known that $U(\mathfrak{a})^*$ is injective as a left and right $U(\mathfrak{a})$-module (cf. [25, Proposition 1]).

Let $U(\mathfrak{a})^\circ$ denote the *continuous dual* of $U(\mathfrak{a})$ which is the largest locally finite submodule of the left and right $U(\mathfrak{a})$-module $U(\mathfrak{a})^*$. It is well known that $U(\mathfrak{a})^\circ$ also consists of all linear forms on $U(\mathfrak{a})$ that vanish on some two-sided ideal of finite codimension in $U(\mathfrak{a})$ (cf. [26, p. 51]).

Finally, let $U(\mathfrak{a})^\natural$ denote the set of all linear forms on $U(\mathfrak{a})$ that vanish on a certain power of the augmentation ideal $U(\mathfrak{a})_+$ of $U(\mathfrak{a})$. Then one has the following inclusions where $\mathbb{F}^*$ is identified with the linear forms on $U(\mathfrak{a})$ that vanish on $U(\mathfrak{a})_+$ (cf. [9, Lemma 2.5.1]):

$$\mathbb{F} \cong \mathbb{F}^* \subseteq U(\mathfrak{a})^\natural \subseteq U(\mathfrak{a})^\circ \subseteq U(\mathfrak{a})^* .$$

If $\mathfrak{a} \neq 0$, then all these inclusions are proper.

The following is also well known (cf. [25, Lemme 2]).

Lemma 11.3.1 *If* $\mathfrak{a}$ *is a finite-dimensional Lie algebra over an arbitrary field, then* $U(\mathfrak{a})^\natural$ *is an essential extension of the one-dimensional trivial left and right* $U(\mathfrak{a})$*-module.* □

For the convenience of the reader we include a proof of the following result.

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Theorem 11.3.2 (cf. [25, Théorème 3] or [7, Theorem 3]) *If $\mathfrak{a}$ is a finite-dimensional nilpotent Lie algebra over an arbitrary field, then $U(\mathfrak{a})^\natural$ is an injective hull of the one-dimensional trivial left and right $U(\mathfrak{a})$-module.*

Proof Since $\mathbb{F} \cong \mathbb{F}^* \subseteq U(\mathfrak{a})^*$ and $U(\mathfrak{a})^*$ is injective, the universal property (I) of injective hulls implies that $I_\mathfrak{a}(\mathbb{F}) \subseteq U(\mathfrak{a})^*$. It follows from [7, Proposition 1] that $I_\mathfrak{a}(\mathbb{F})$ is locally finite. Consider $\varphi \in I_\mathfrak{a}(\mathbb{F})$. Then $E := U(\mathfrak{a})\varphi$ is a finite-dimensional extension of $\mathbb{F}$. An application of Fitting's lemma (cf. [17, Theorem II.4, p. 39]) shows that $\mathfrak{a}$ acts nilpotently on E and it follows from the Engel-Jacobson theorem (cf. [35, Corollary 1.3.2]) that a certain power of the augmentation ideal $U(\mathfrak{a})_+$ annihilates E. Consequently, $\varphi \in U(\mathfrak{a})^\natural$ and therefore $I_\mathfrak{a}(\mathbb{F}) \subseteq U(\mathfrak{a})^\natural$. Finally, the other inclusion follows from Lemma 11.3.1 and the universal property (E) of injective hulls. □

Remark 11.3.3 It is observed in [25, Remarque 2 after Théorème 3] that $U(\mathfrak{a})^\natural$ is *not* injective for the two-dimensional non-nilpotent Lie algebra. It would be interesting to know whether the injectivity of $U(\mathfrak{a})^\natural$ implies that $\mathfrak{a}$ is nilpotent.

The isomorphism $H^n(\mathfrak{a}, U(\mathfrak{a})^\natural) \cong \operatorname{Ext}^n_{U(\mathfrak{a})}(\mathbb{F}, U(\mathfrak{a})^\natural)$ in conjunction with Theorem 11.3.2 and [32, Theorem 7.6] yields the following cohomological vanishing theorem due to Koszul:

Corollary 11.3.4 (cf. [21, Théorème 6]) *If $\mathfrak{a}$ is a finite-dimensional nilpotent Lie algebra over an arbitrary field, then*

$$H^n(\mathfrak{a}, U(\mathfrak{a})^\natural) = 0$$

for every positive integer n. □

Question. Does the vanishing $H^n(\mathfrak{a}, U(\mathfrak{a})^\natural)$ for every positive integer n imply that $\mathfrak{a}$ is nilpotent?

Let us now consider arbitrary finite-dimensional Lie algebras over fields of prime characteristic.

Theorem 11.3.5 *If $\mathfrak{a}$ is a finite-dimensional Lie algebra over a field of prime characteristic, then the continuous dual $U(\mathfrak{a})^\circ$ is injective as a left and right $U(\mathfrak{a})$-module.*

Proof Since $U(\mathfrak{a})^\circ \subseteq U(\mathfrak{a})^*$ and $U(\mathfrak{a})^*$ is injective, the universal property (I) of injective hulls implies that $I_\mathfrak{a}(U(\mathfrak{a})^\circ) \subseteq U(\mathfrak{a})^*$. Because $U(\mathfrak{a})^\circ$ is locally finite, it follows from Corollary 11.2.2 that $I_\mathfrak{a}(U(\mathfrak{a})^\circ)$ is also locally finite. But since by definition $U(\mathfrak{a})^\circ$ is the *largest* locally finite submodule of $U(\mathfrak{a})^*$, $U(\mathfrak{a})^\circ = I_\mathfrak{a}(U(\mathfrak{a})^\circ)$ is injective. □

The isomorphism $H^n(\mathfrak{a}, U(\mathfrak{a})^\circ) \cong \operatorname{Ext}^n_{U(\mathfrak{a})}(\mathbb{F}, U(\mathfrak{a})^\circ)$ in conjunction with Theorem 11.3.5 and [32, Theorem 7.6] yields the following cohomological vanishing theorem:

Corollary 11.3.6 *If $\mathfrak{a}$ is a finite-dimensional Lie algebra over a field of prime characteristic, then*

$$H^n(\mathfrak{a}, U(\mathfrak{a})^\circ) = 0$$

for every positive integer n. □

Remark 11.3.7 The case $n = 1$ of Corollary 11.3.6 was already proved by Masuoka [30, Proposition 5.1]. It follows from Corollary 11.3.6 in conjunction with [21, Théorème 2] that every cohomology class of a finite-dimensional Lie algebra over a field of prime characteristic with coefficients in a finite-dimensional module is annihilable. This result was proved in a completely different way by Dzumadil'daev [11, Theorem 3.1, pp. 467–470].

The equivalence of (1), (3), and (4) in the next result is essentially due to Koszul (see [21, Théorème 7 and p. 536]. Moreover, for an *algebraically closed* ground field the implication (1)$\Longrightarrow$(2) follows from [2, Théorème 3.6 and Théorème 4.10] (see also [26, Proposition 3.4 and Proposition 3.6] for $\mathbb{F} = \mathbb{C}$).

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Theorem 11.3.8 *Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over a field of characteristic zero. Then the following statements are equivalent:*

(1) $\mathfrak{a}$ *is solvable.*

(2) *The continuous dual $U(\mathfrak{a})^\circ$ is injective as a left and right $U(\mathfrak{a})$-module.*

(3) $H^n(\mathfrak{a}, U(\mathfrak{a})^\circ) = 0$ *for every positive integer n.*

(4) $H^3(\mathfrak{a}, U(\mathfrak{a})^\circ) = 0$.

Proof The proof of the implication (1)$\Longrightarrow$(2) is the same as for Theorem 11.3.5 except that one uses Theorem 11.2.3 instead of Corollary 11.2.2 in order to conclude that $I_\mathfrak{a}(U(\mathfrak{a})^\circ)$ is locally finite. Since (2)$\Longrightarrow$(3) is clear and (4) is just a special case of (3), it remains to show the implication (4)$\Longrightarrow$(1).

Suppose that $H^3(\mathfrak{a}, U(\mathfrak{a})^\circ) = 0$ and let M be an arbitrary finite-dimensional $\mathfrak{a}$-module. Then the isomorphism $\mathrm{Hom}_\mathbb{F}(U(\mathfrak{a}), M) \cong U(\mathfrak{a})^* \otimes_\mathbb{F} M$ (where M is considered as a *trivial* $\mathfrak{a}$-module) implies that $H^3(\mathfrak{a}, \mathrm{Hom}_\mathbb{F}(U(\mathfrak{a}), M)_{\mathrm{loc}}) = 0$ where $\mathrm{Hom}_\mathbb{F}(U(\mathfrak{a}), M)_{\mathrm{loc}}$ denotes the largest locally finite submodule of $\mathrm{Hom}_\mathbb{F}(U(\mathfrak{a}), M)$. According to [21, Théorème 2], it follows that every cohomology class in $H^3(\mathfrak{a}, M)$ is annihilable and thus [21, 5), p. 536] yields that $\mathfrak{a}$ is solvable. □

Remark 11.3.9 The above proof of the implication (1)$\Longrightarrow$(2) is not only much more direct than in [2] or [26] but also answers affirmatively a question posed at the end of the third section in [2]. Moreover, it should be noted that the implication (2)$\Longrightarrow$(1) in Theorem 11.3.8 can also be obtained directly from the universal property (I) of injective hulls and Theorem 11.2.3.

Recently, H.-J. Schneider has generalized the implication (1)$\Longrightarrow$(2) in Theorem 11.3.8 even further. Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over a field of characteristic zero and let $\mathrm{Solv}(\mathfrak{a})$ denote the solvable radical of $\mathfrak{a}$. Then Schneider proves that the restriction $[U(\mathfrak{a})^\circ]_{|\mathrm{Solv}(\mathfrak{a})}$ of $U(\mathfrak{a})^\circ$ to $\mathrm{Solv}(\mathfrak{a})$ is injective (cf. [30, Theorem 5.3]). This in conjunction with the Hochschild-Serre spectral sequence (cf. [16, Theorem 6]) and the two Whitehead lemmata (cf. [17, Theorem III.13]) implies that $H^1(\mathfrak{a}, U(\mathfrak{a})^\circ) = 0 = H^2(\mathfrak{a}, U(\mathfrak{a})^\circ)$ (see [30, Proposition 5.1 and Theorem 5.2]). But Theorem 11.3.8 shows that $H^3(\mathfrak{a}, U(\mathfrak{a})^\circ) \neq 0$ if $\mathfrak{a}$ is *not solvable* which generalizes [30, Remark 5.9].

It follows from the universal properties (E) and (I) of injective hulls in conjunction with Lemma 11.3.1 and Theorem 11.3.5 that

$$\mathbb{F} \cong \mathbb{F}^* \subseteq U(\mathfrak{a})^\natural \subseteq I_\mathfrak{a}(\mathbb{F}) \subseteq U(\mathfrak{a})^\circ .$$

Note that the cocommutative Hopf algebra structure on $U(\mathfrak{a})$ induces a commutative algebra structure on $U(\mathfrak{a})^*$ which over a field $\mathbb{F}$ of characteristic zero can be identified with the algebra of power series in $\dim_\mathbb{F} \mathfrak{a}$ variables (cf. [9, Proposition 2.7.5]) and the continuous dual $U(\mathfrak{a})^\circ$ is a subalgebra of $U(\mathfrak{a})^*$.

Let $\mathfrak{a}$ be a finite-dimensional *solvable* Lie algebra over the complex numbers. Then Levasseur [26, Théorème 2.2] has shown that $I_\mathfrak{a}(\mathbb{F})$ is isomorphic to a polynomial algebra in $\dim_\mathbb{F} \mathfrak{a}$ variables on which $\mathfrak{a}$ acts via derivations.

Conjecture. Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$. If $\mathfrak{a}$ is solvable and $\mathrm{char}(\mathbb{F}) = 0$ or if $\mathfrak{a}$ is arbitrary and $\mathrm{char}(\mathbb{F}) > 0$, then $I_\mathfrak{a}(\mathbb{F})$ is isomorphic to a polynomial algebra in $\dim_\mathbb{F} \mathfrak{a}$ variables on which $\mathfrak{a}$ acts via derivations.

If $\mathfrak{a}$ is abelian, then this follows from [31, Theorem 2] and in [7, Section 4] there are examples confirming this for Lie algebras of small dimensions.

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11.4 Minimal Injective Resolutions

Let $\mathcal{I}$ be a two-sided ideal of an associative ring R. Then

$$\text{u.gr}(\mathcal{I}) := \sup\{n \in \mathbb{N}_0 \mid \text{Ext}^n_R(R/\mathcal{I}, R) \neq 0\}$$

and

$$\text{l.gr}(\mathcal{I}) := \inf\{n \in \mathbb{N}_0 \mid \text{Ext}^n_R(R/\mathcal{I}, R) \neq 0\}$$

are called *upper grade* and *lower* (or *homological*) *grade* of $\mathcal{I}$, respectively. A left and right noetherian associative ring R is left (resp. right) *injectively homogeneous* over a central subring C if R is integral over C, $\text{inj.dim}_R R < \infty$ (resp. $\text{inj.dim} R_R < \infty$) and $\text{u.gr}(\mathcal{M}) = \text{u.gr}(\mathcal{N})$ for all maximal ideals $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \cap C = \mathcal{N} \cap C$. In [4] it was demonstrated that for associative rings integral over a central subring the class of injectively homogeneous rings is a natural generalization of the class of commutative *Gorenstein rings*. Moreover, [4, Corollary 3.6] shows that R is injectively homogeneous over its center $C(R)$ if and only if R is injectively homogeneous over *every* subring $C \subseteq C(R)$ over which R is integral, and by virtue of [4, Corollary 4.4], R is left injectively homogeneous if and only if R is right injectively homogeneous.

Lemma 11.4.1 *If $\mathfrak{a}$ is a finite-dimensional Lie algebra over a field of prime characteristic, then $U(\mathfrak{a})$ is injectively homogeneous over every subring C of $U(\mathfrak{a})$ with $\mathcal{O}(\mathfrak{a}) \subseteq C \subseteq C(U(\mathfrak{a}))$.*

Proof Let $\mathcal{M}$ be a maximal ideal of $U(\mathfrak{a})$. Then $\wp := \mathcal{M} \cap C(U(\mathfrak{a}))$ is also maximal [35, Corollary 6.3.4], and thus Hilbert's Nullstellensatz yields that $C(U(\mathfrak{a}))/\wp$ is finite-dimensional. Since $U(\mathfrak{a})$ is finitely generated over $C(U(\mathfrak{a}))$, we conclude that $M := U(\mathfrak{a})/\mathcal{M}$ is also finite-dimensional. Set $d := \dim_{\mathbb{F}} \mathfrak{a}$. According to [12, Theorem 4.2(3)], there exists a simple $\mathfrak{a}$-module S such that $\text{Ext}^d_{U(\mathfrak{a})}(M, S) \neq 0$. If $\mathcal{A}$ denotes the annihilator of a generator of S in $U(\mathfrak{a})$, we obtain a short exact sequence $0 \to \mathcal{A} \to U(\mathfrak{a}) \to S \to 0$ of $U(\mathfrak{a})$-modules. The long exact cohomology sequence implies the exactness of

$$\text{Ext}^d_{U(\mathfrak{a})}(M, U(\mathfrak{a})) \longrightarrow \text{Ext}^d_{U(\mathfrak{a})}(M, S) \longrightarrow \text{Ext}^{d+1}_{U(\mathfrak{a})}(M, \mathcal{A}).$$

Because of $\text{gl.dim}\, U(\mathfrak{a}) = d$ (cf. [6, Theorem 8.2]), the right-hand term vanishes. We conclude that $\text{Ext}^d_{U(\mathfrak{a})}(M, U(\mathfrak{a})) \neq 0$, i.e., $\text{u.gr}(\mathcal{M}) \geq \text{l.gr}(\mathcal{M}) \geq d$. The reverse inequality follows from $\text{u.gr}(\mathcal{M}) \leq \text{gl.dim}\, U(\mathfrak{a}) = d$. Hence $\text{u.gr}(\mathcal{M}) = d$ for every maximal ideal of $U(\mathfrak{a})$. This and (IC) yield the assertion. □

Remark 11.4.2 Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over a field of characteristic zero. According to a theorem of Latyšev [24], $U(\mathfrak{a})$ is a PI algebra if and only if $\mathfrak{a}$ is abelian. Since every algebra which is a finitely generated module over its center is a PI algebra (cf. [14, p. xi]), $U(\mathfrak{a})$ is injectively homogeneous over its center if and only if $\mathfrak{a}$ is abelian.

One consequence of Lemma 11.4.1 is that $\text{inj.dim}\, U(\mathfrak{a})_\wp < \infty$ for every semiprime ideal $\wp$ of every subring C of $U(\mathfrak{a})$ with $\mathcal{O}(\mathfrak{a}) \subseteq C \subseteq C(U(\mathfrak{a}))$ (cf. [1, Fundamental Theorem (e), p. 10] and [4, Theorem 4.1]). More importantly for the purpose of this paper, it is an immediate consequence of Lemma 11.4.1 and [4, Theorem 5.5] that the minimal injective resolution of $U(\mathfrak{a})$ has the same form as for *commutative Gorenstein rings* (cf. [1, Fundamental Theorem (f), p. 10]). Recall that a *minimal injective resolution* of a module M is a long exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \xrightarrow{d_0} I_1 \longrightarrow \cdots \longrightarrow I_n \xrightarrow{d_n} I_{n+1} \longrightarrow \cdots$$

such that I_n is an injective hull of $\text{Ker}(d_n)$ for every non-negative integer n.

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Theorem 11.4.3 *Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$ of prime characteristic. If $0 \longrightarrow U(\mathfrak{a}) \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_d \longrightarrow 0$ is a minimal injective resolution of $U(\mathfrak{a})$ as a left or right $U(\mathfrak{a})$-module, then*

$$I_n \cong \bigoplus_{\mathrm{ht}(\mathcal{P})=n} I_{\mathfrak{a}}(U(\mathfrak{a})/\mathcal{P})$$

for every $0 \leq n \leq d := \dim_{\mathbb{F}} \mathfrak{a}$. $\square$

Remark 11.4.4 If $\mathfrak{a}$ is a finite-dimensional Lie algebra over a field of characteristic zero, then the structure of a minimal injective resolution of $U(\mathfrak{a})$ is even in the *solvable* case more complicated than in Theorem 11.4.3 (cf. [27, 28]).

Let $\mathfrak{a}$ be a finite-dimensional solvable Lie algebra over an algebraically closed field of characteristic zero. Then the last term of a minimal injective resolution of $U(\mathfrak{a})$ is isomorphic to the continuous dual $U(\mathfrak{a})^\circ$ of $U(\mathfrak{a})$ (see [2, Théorème 3.6 and Théorème 4.10] and also [26, Proposition 3.4 and Proposition 3.6] for $\mathbb{F} = \mathbb{C}$). We conclude the paper by applying Theorem 11.4.3 in order to prove the analogue of this result in prime characteristic.

Theorem 11.4.5 *Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over an algebraically closed field of prime characteristic. If $0 \longrightarrow U(\mathfrak{a}) \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_d \longrightarrow 0$ is a minimal injective resolution of $U(\mathfrak{a})$ as a left or right $U(\mathfrak{a})$-module, then $I_d \cong U(\mathfrak{a})^\circ$.*

Proof By virtue of Corollary 11.2.2, injective hulls of locally finite modules are locally finite. Since $\mathbb{F}$ is algebraically closed, this enables one to prove that

$$U(\mathfrak{a})^\circ \cong \bigoplus_{S \in \mathrm{Irr}(\mathfrak{a})} I_{\mathfrak{a}}(S)^{\oplus \dim_{\mathbb{F}} S}$$

as left or right $U(\mathfrak{a})$-module, where $\mathrm{Irr}(\mathfrak{a})$ denotes the set of isomorphism classes of simple $\mathfrak{a}$-modules (cf. [15, 1.5] for the analogous statement in terms of coalgebras and comodules). On the other hand, it follows from (PI) and [34, Theorem 4] that a prime ideal $\mathcal{P}$ of $U(\mathfrak{a})$ has maximal height d if and only if $\mathcal{P}$ is maximal. But every maximal ideal $\mathcal{P}$ of $U(\mathfrak{a})$ is primitive, i.e., there is a simple $\mathfrak{a}$-module S such that $\mathcal{P} = \mathrm{Ann}_{U(\mathfrak{a})}(S)$. Then the density theorem (cf. [20, Theorem 16, p. 95]) yields that

$$U(\mathfrak{a})/\mathcal{P} = U(\mathfrak{a})/\mathrm{Ann}_{U(\mathfrak{a})}(S) \cong \mathrm{End}_{\mathbb{F}}(S) \cong S^{\oplus \dim_F S^*}$$

as a left or right $U(\mathfrak{a})$-module. In particular, simple $\mathfrak{a}$-modules are isomorphic if and only if their annihilators in $U(\mathfrak{a})$ coincide. According to (PI) and Kaplansky's theorem (cf. [20, Theorem 50]), every primitive ideal of $U(\mathfrak{a})$ is maximal and therefore

$$I_d \cong \bigoplus_{\mathrm{ht}(\mathcal{P})=d} I_{\mathfrak{a}}(U(\mathfrak{a})/\mathcal{P}) \cong \bigoplus_{S \in \mathrm{Irr}(\mathfrak{a})} I_{\mathfrak{a}}(S)^{\oplus \dim_{\mathbb{F}} S^*} \cong U(\mathfrak{a})^\circ .$$

$\square$

Question. Does Theorem 11.4.5 remain valid for *arbitrary* ground fields of prime characteristic?

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Chapter 12

Commutative Ideal Theory without Finiteness Conditions: Irreducibility in the Quotient Field

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Abstract Let R be an integral domain and let Q denote the quotient field of R. We investigate the structure of R-submodules of Q that are Q-irreducible, or completely Q-irreducible. One of our goals is to describe the integral domains that admit a completely Q-irreducible ideal, or a nonzero Q-irreducible ideal. If R has a nonzero finitely generated Q-irreducible ideal, then R is quasilocal. If R is integrally closed and admits a nonzero principal Q-irreducible ideal, then R is a valuation domain. If R has an m-canonical ideal and admits a completely Q-irreducible ideal, then R is quasilocal and all the completely Q-irreducible ideals of R are isomorphic. We consider the condition that every nonzero ideal of R is an irredundant intersection of completely Q-irreducible submodules of Q and present eleven conditions that are equivalent to this. We classify the domains for which every nonzero ideal can be represented uniquely as an irredundant intersection of completely Q-irreducible submodules of Q. The domains with this property are the Prüfer domains that are almost semi-artinian, that is, every proper homomorphic image has a nonzero socle. We characterize the Prüfer and Noetherian domains that possess a completely Q-irreducible ideal or a nonzero Q-irreducible ideal.

Subject classifications: Primary 13A15, 13F05.

Keywords: irreducible ideal, completely irreducible ideal, injective module, Prüfer domain, m-canonical ideal.

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12.1 Introduction

This article continues a study of commutative ideal theory in rings without finiteness conditions begun in [15], [16], [17] and [26]. In [15] and [16] we examine irreducible and completely irreducible ideals of commutative rings. In the present article we investigate stronger versions of these two notions of irreducibility for ideals of integral domains. In particular, we consider irreducibility of an ideal of an integral domain when it is viewed as a submodule of the quotient field of the domain.

All rings in this paper are commutative and contain a multiplicative identity. Our notation is as in [18]. Let R be a ring and let C be an R-module. An R-submodule A of C is *C-irreducible* if $A = B_1 \cap B_2$, where B_1 and B_2 are R-submodules of C, implies that either $B_1 = A$ or $B_2 = A$. An R-submodule A of C is *completely C-irreducible* (or *completely irreducible* when the module C is clear from context) if $A = \bigcap_{i \in I} B_i$, where $\{B_i\}_{i \in I}$ is a family of R-submodules of C, implies $A = B_i$ for some $i \in I$.

In the case where the module C is the ring R, an ideal A of R is R-irreducible as a submodule of R precisely if A is *irreducible* as an ideal in the conventional sense that A is not the intersection of two strictly larger ideals. It is established by Fuchs in [14, Theorem 1] that a proper irreducible ideal A of the ring R is a *primal ideal* in the sense that the set of elements of R that are non-prime to A form an ideal P that is necessarily a prime ideal and is called the *adjoint prime ideal* of A. One then says that A is P-primal. For such an ideal A, it is the case that $A = A_{(P)}$, where $A_{(P)} = \bigcup_{x \in R \setminus P} (A :_R x)$.

In Remark 12.1.1 we record several general facts about completely C-irreducible submodules. The straightforward proofs are omitted.

Remark 12.1.1 For a proper submodule A of C the following are equivalent:

1. A is completely C-irreducible.
2. There exists an element $x \in C \setminus A$ such that $x \in B$ for every submodule B of C that properly contains A.
3. C/A has a simple essential socle, that is, C/A is a cocyclic R-module.
4. C/A is subdirectly irreducible in the sense that in any representation of C/A as a subdirect product of R-modules, one of the projections to a component is an isomorphism.

It is also straightforward to see that every submodule of a module C is an intersection of completely C-irreducible submodules of C. Thus a nonzero module C contains proper completely C-irreducible submodules.

The main focus of our present study is the case where R is an integral domain and $C = Q$ is the quotient field of R. (Throughout this paper Q is understood to be the quotient field of the integral domain R.) We are thus interested in Q-irreducible and completely Q-irreducible submodules of Q. We are particularly interested in determining conditions on an integral domain R in order that R admit a completely Q-irreducible ideal, or a nonzero Q-irreducible ideal. The zero ideal of R is always Q-irreducible, but if $R \neq Q$, the zero ideal of R is not completely Q-irreducible. In the case where R admits completely Q-irreducible ideals, or nonzero Q-irreducible ideals, we are interested in describing the structure of such ideals. Ideals with either of these properties are necessarily primal ideals.

It is frequently the case that an integral domain R may fail to have any fractional ideals that are completely Q-irreducible, or any nonzero ideals that are Q-irreducible. If $R = \mathbb{Z}$ is the ring of integers, then every nonzero proper Q-irreducible R-submodule of Q is completely Q-irreducible and has the form $p^n\mathbb{Z}_{p\mathbb{Z}}$, where p is a prime integer and n is an integer. Thus for $R = \mathbb{Z}$ every nonzero

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proper Q-irreducible R-submodule of Q is a fractional ideal of a valuation overring of R. Moreover, every nonzero fractional R-ideal has a unique representation as an irredundant intersection of infinitely many completely Q-irreducible R-submodules of Q. It follows that R has no nonzero fractional ideal that is Q-irreducible.

In Section 12.2 we establish basic properties of irreducible submodules of an R-module C with special emphasis on the case where $C = Q$. We prove in Theorem 12.2.5 that if R admits a nonzero principal Q-irreducible fractional ideal, then R is quasilocal, and R is integrally closed if and only if R is a valuation domain. In Theorem 12.2.11 we give several necessary conditions for an integral domain to possess a nonzero Q-irreducible ideal. If A is a nonzero Q-irreducible ideal, we prove that $\mathrm{End}(A)$ is quasilocal, and that A is a primal ideal of $\mathrm{End}(A)$ with adjoint prime the maximal ideal of $\mathrm{End}(A)$. If the integral domain R admits a nonzero finitely generated Q-irreducible ideal, we prove that R is quasilocal. Moreover, every nonzero Q-irreducible ideal of a Noetherian domain is completely Q-irreducible.

In Section 12.3 we review some relevant results and examples regarding completely Q-irreducible fractional ideals. Over a quasilocal domain, an m-canonical ideal (if it exists) is an example of a completely Q-irreducible ideal. If R has an m-canonical ideal and admits a completely Q-irreducible ideal, we prove that R is quasilocal and all completely Q-irreducible ideals of R are isomorphic. We classify the Noetherian domains that admit a nonzero Q-irreducible ideal.

In Proposition 12.4.3 of Section 12.4 we show that a proper submodule A of the quotient field Q of a domain is an irredundant intersection of Q-irreducible submodules if and only if the injective hull of Q/A is an interdirect sum of indecomposable injectives.

In Section 12.5 we continue to examine irredundant intersections of Q-irreducible submodules in Q. We draw on the literature to give in Theorem 12.5.2 eleven different module- and ideal-theoretic conditions that are equivalent to the assertion that every nonzero ideal of a domain is an irredundant intersection of completely irreducible submodules of Q. We show in particular that such a domain is locally almost perfect, and from this observation we answer in the negative a question of Bazzoni and Salce of whether every locally almost perfect domain R has the property that Q/R is semi-artinian (Example 12.5.5). In Theorem 12.5.9 we classify the domains for which every nonzero ideal can be represented *uniquely* as an irredundant intersection of completely Q-irreducible submodules of Q. The domains having this property have Krull dimension at most one and are necessarily Prüfer, that is, every nonzero finitely generated ideal is invertible. They may be described precisely as the Prüfer domains R that are almost semi-artinian, that is, every proper homomorphic image of R has a nonzero socle.

In light of Theorem 12.5.9 it is useful to describe the completely irreducible submodules of the quotient field of a Prüfer domain. This is done in Theorem 12.6.2. Also in Section 12.6 we characterize the Prüfer domains that possess a completely Q-irreducible ideal, or a nonzero Q-irreducible ideal. We prove that a Prüfer domain R that admits a nonzero Q-irreducible ideal also admits a completely Q-irreducible ideal, and this holds if and only if every proper R-submodule of Q is a fractional R-ideal.

In Section 12.7 we discuss several open questions, and in an appendix we correct some errors in the article [17] that were pointed out to us by Jung-Chen Liu and her student Zhi-Wei Ying. We are grateful to them for showing us these mistakes.

12.2 The Structure of Q-irreducible Ideals

We begin with several general results.

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Proposition 12.2.1 *Let R be a ring and C an R-module. The following statements are equivalent for a proper R-submodule A of C.*

(i) A is a completely C-irreducible R-submodule of C.

(ii) There exists $x \in C \setminus A$ such that for all $y \in C \setminus A$ we have $x \in A + Ry$.

(iii) A is C-irreducible and there exists a maximal ideal M of R such that $A \subset (A :_C M)$, where $(A :_C M) = \{y \in C : yM \subseteq A\}$.

Furthermore, if R is a domain, A is torsionfree and C is the divisible hull of A, then statements (i)-(iii) are equivalent to:

(iv) There is a maximal ideal M of R such that $A = AR_M$ and A is completely C-irreducible as an R_M-submodule of C.

Proof (i) $\Rightarrow$ (ii) Let A^* be the intersection of all R-submodules of C properly containing A. Then $A \subset A^*$, and A^*/A is a simple R-module. Hence $A^* = Rx + A$ for some $x \in Q \setminus A$, and (ii) follows.

(ii) $\Rightarrow$ (iii) By (ii) there exists $x \in C \setminus A$ such that $A^* := A + Rx$ is contained in every R-submodule of C properly containing A. Hence A^*/A is a simple R-module and $A^*/A \cong R/M$ for some maximal ideal M of R. Thus $A^* \subseteq (A :_C M)$ so that $(A :_C M) \neq A$.

(iii) $\Rightarrow$ (i) Since A is irreducible, $(A :_C M)/A \cong R/M$ and every proper submodule containing A contains $(A :_C M)$, proving (i).

(i) $\Rightarrow$ (iv) Since R is a domain and A is torsion-free, $A = \bigcap_{M \in \mathrm{Max}(R)} A_M$, where each A_M is identified with its image in $C = QA$. Because A is completely C-irreducible, $A = A_M$ for some maximal ideal M of R. The assumption that A is completely C-irreducible as an R-module clearly implies A is completely C-irreducible as an R_M-submodule of C.

(iv) $\Rightarrow$ (iii) Since we have established the equivalence of (i)-(iii), and since by assumption A is a completely irreducible R_M-submodule of C, we have by (iii) (applied to the R_M-module A) that there exists $x \in (A :_C MR_M) \setminus A$. Now since $A = A_M$, we have $A \neq (A :_C MR_M) = (A :_C M)$. Thus it remains to observe that A is C-irreducible. Suppose $A = B \cap D$ for some R-submodules B and D of C. Then $A = A_M = B_M \cap D_M$, so since by assumption A is irreducible as an R_M-submodule of C, it must be that $A = B_M$ or $A = D_M$. Thus $B \subseteq A$ or $D \subseteq A$, proving that A is irreducible. $\square$

Remark 12.2.2 Let R be an integral domain that is properly contained in its quotient field Q.

(i) By Remark 12.1.1, every R-submodule of Q is an intersection of completely irreducible submodules of Q. In particular, every ideal of R is an intersection of completely irreducible submodules of Q.

(ii) A fractional ideal A of R is completely Q-irreducible if and only if A is not the intersection of fractional R-ideals that properly contain A. If A is a fractional R-ideal and $A \neq Q$, then A is completely Q-irreducible if and only if there exists $x \in Q \setminus A$ such that x is in every fractional ideal that properly contains A.

(iii) A maximal ideal P of R is completely Q-irreducible if it is Q-irreducible. This is immediate from Proposition 12.2.1, since $P \subsetneq R \subseteq (P :_Q P)$.

In Lemma 12.2.3, we establish several general facts about Q-irreducible and completely Q-irreducible ideals.

Lemma 12.2.3 *Let A be a proper ideal of the integral domain R. Then*

(i) *A is Q-irreducible if and only if for each nonzero $r \in R$ the ideal rA is irreducible.*

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(ii) *For a nonzero $q \in Q$, the fractional ideal qA is Q-irreducible if and only if A is Q-irreducible. Therefore the property of being Q-irreducible is an invariant of isomorphism classes of fractional R-ideals.*

(iii) *A is Q-irreducible if and only if there is a prime ideal P of R such that $A = AR_P$ and A is a Q-irreducible ideal of R_P. It then follows that P is uniquely determined by A and A is P-primal.*

(iv) *For a nonzero $q \in Q$, the fractional ideal qA is completely Q-irreducible if and only if A is completely Q-irreducible. Therefore the property of being completely Q-irreducible is an invariant of isomorphism classes of fractional R-ideals.*

(v) *If A is completely R-irreducible and if for each nonzero $r \in R$ the ideal rA is irreducible, then A is completely Q-irreducible.*

Proof (i) Assume A is Q-irreducible and r is a nonzero element of R. If $rA = B \cap C$ for ideals B and C of R, then $A = r^{-1}B \cap r^{-1}C$. Since A is Q-irreducible, either $A = r^{-1}B$ or $A = r^{-1}C$. Hence either $rA = B$ or $rA = C$ and rA is irreducible. Conversely, assume A is not Q-irreducible. Then there exist R-submodules B and C of Q that properly contain A such that $A = B \cap C$. We may assume that B and C are fractional ideals of R. Then there exists a nonzero $r \in R$ such that rB and rC are integral ideals of R. Moreover, $A = B \cap C$ implies $rA = rB \cap rC$ and $A \subset B$ implies $rA \subset rB$ and similarly $A \subset C$ implies $rA \subset rC$. Therefore rA is reducible. This completes the proof of (i).

Statements (ii) and (iv) are clear since $A = \bigcap_{i \in I} B_i$ if and only if $qA = \bigcap_{i \in I} qB_i$ and multiplication by q (or by q^{-1}) preserves strict inclusion.

(iii) Assume A is Q-irreducible. Then A is P-primal for some prime ideal P of R, so that $A = A_{(P)} = AR_P \cap R$. Since A is Q-irreducible, this forces $A = AR_P$. Clearly then A is Q-irreducible as an R_P-module since it is Q-irreducible as an R-module. Conversely, suppose that $A = AR_P$ and A is Q-irreducible as an ideal of R_P. If $A = B \cap C$ for some R-submodules B and C of Q, then $A = AR_P = BR_P \cap CR_P$, and since A is a Q-irreducible R_P-submodule of Q, $A = BR_P$ or $A = CR_P$. Thus $B \subseteq A$ or $C \subseteq A$, which completes the proof.

(v) Since A is completely R-irreducible, there exists an element $x \in R \setminus A$ such that x is in every ideal of R that properly contains A. Let $A^* = A + xR$. If A is not completely Q-irreducible, then there exists an R-submodule B of Q that properly contains A but does not contain x. Since there are no ideals properly between A and A^*, $A = A^* \cap B$ and this intersection is irredundant. We may assume that B is a fractional ideal of A. Then there exists a nonzero $r \in R$ such that rB is an integral ideal of R. Therefore $rA = rA^* \cap rB$ is an irredundant intersection. It follows that rA is not irreducible. □

Remark 12.2.4 With regard to Lemma 12.2.3 we have:

1. If A is a nonzero Q-irreducible ideal of R and P is as in Lemma 12.2.3(iii), then $R_P \subseteq \mathrm{End}(A)$ and $rA = A$ for each $r \in R \setminus P$. It follows that A is contained in every ideal of R not contained in P. Thus if P is a maximal ideal of R and A is P-primary with $A = AR_P$, then R is quasilocal.

2. It is also true that if A and B are isomorphic R-submodules of Q, then A is (completely) Q-irreducible if and only if B is (completely) Q-irreducible. For A and B are R-isomorphic if and only if there exists $q \in Q$ such that $A = qB$.

Theorem 12.2.5 *If the integral domain R has a nonzero principal fractional ideal that is Q-irreducible, then R is quasilocal and every principal ideal of R is Q-irreducible. If R is integrally closed, then*

(i) R is Q-irreducible if and only if R is a valuation domain, and

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(ii) R is completely Q-irreducible if and only if R is a valuation domain with principal maximal ideal.

Proof (i) Lemma 12.2.3 implies that R has a nonzero principal fractional ideal that is (completely) Q-irreducible if and only if every nonzero principal fractional ideal of R is (completely) Q-irreducible. Suppose R has distinct maximal ideals M and N. Then there exist $x \in M$ and $y \in N$ such that $x + y = 1$. It follows that $xyR = xR \cap yR$ is an irredundant intersection. By Lemma 12.2.3(i), R is not Q-irreducible.

(ii) Suppose that R is integrally closed and Q-irreducible but is not a valuation domain. Then there exists $x \in Q$ such that neither x nor $1/x$ is in R. Let $\mathcal{F}$ be the set of valuation overrings of R that contain x and let $\mathcal{G}$ be the set of valuation overrings of R that contain $1/x$. Let $A = \bigcap_{V \in \mathcal{F}} V$ and $B = \bigcap_{W \in \mathcal{G}} W$. Then $x \in A$ implies $R \subsetneq A$ and $1/x \in B$ implies $R \subsetneq B$. Observe that every valuation overring of R is a member of at least one of the sets $\mathcal{F}$ or $\mathcal{G}$. Since R is integrally closed, we have $R = A \cap B$, a contradiction to the assumption that R is Q-irreducible. Conversely, it is clear that if R is a valuation domain, then R is integrally closed and Q-irreducible.

(iii) By (ii) we need only observe the well-known fact that a valuation domain R is completely Q-irreducible if and only if the maximal ideal of R is principal. (See for example [3].) □

Remark 12.2.6 There exist integral domains R that are completely Q-irreducible and are not integrally closed. If R is a one-dimensional Gorenstein local domain, then R, and every nonzero principal fractional ideal of R, is completely Q-irreducible. Thus, for example, if k is a field and a and b are relatively prime positive integers, then the subring $R := k[[t^a, t^b]]$ of the formal power series ring $k[[t]]$ is completely Q-irreducible.

Theorem 12.2.5(ii) characterizes among integrally closed domains R the ones that are valuation domains as precisely those R that are Q-irreducible. As a corollary to Proposition 12.2.1, we have the following additional characterizations of the valuation property in terms of Q-irreducibility.

Corollary 12.2.7 *The following are equivalent for a domain R with quotient field Q.*

(i) R is a valuation domain.

(ii) Every irreducible ideal is Q-irreducible.

(iii) Every completely irreducible ideal is completely Q-irreducible.

(iv) There exists a maximal ideal of R that is Q-irreducible.

(v) There exists a maximal ideal of R that is completely Q-irreducible.

Proof (i) $\Rightarrow$ (ii) If R is a valuation domain, then it is easy to see that irreducible ideals are Q-irreducible since the R-submodules of Q are linearly ordered.

(ii) $\Rightarrow$ (iii) If A is a completely irreducible ideal of R, then there is a maximal ideal M of R such that $(A :_R M) \neq A$. Thus $(A :_Q M) \neq A$, and since A is by (ii) Q-irreducible, we have from Proposition 12.2.1 (iii) that A is Q-irreducible.

(iii) $\Rightarrow$ (iv) This is clear from the fact that maximal ideals are completely irreducible.

(iv) $\Rightarrow$ (v) This follows from Remark 12.2.2(iii).

(v) $\Rightarrow$ (i) Let M be a completely Q-irreducible maximal ideal of R. For every nonzero $r \in R$, rM is completely irreducible by Proposition 12.2.3. It is shown in Lemma 5.1 of [16] that this property characterizes valuation domains, so the proof is complete. □

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Corollary 12.2.8 *Let P be a prime ideal of a domain R. Then P is Q-irreducible if and only if $P = PR_P$ and R_P is a valuation domain. Thus if P is Q-irreducible, then R_P/P is the quotient field of R/P, and R is a pullback of R/P and the valuation domain R_P. Moreover P is completely Q-irreducible as an ideal of R_P.*

Proof Suppose that P is Q-irreducible. By Lemma 12.2.3, $P = PR_P$ and PR_P is a Q-irreducible ideal of R_P. Hence, by Corollary 12.2.7, R_P is a valuation domain.

Conversely, assume $P = PR_P$ and R_P is a valuation domain. By Corollary 12.2.7, P is a Q-irreducible ideal of R_P. Hence, by Lemma 12.2.3, P is Q-irreducible. It follows from Remark 12.2.2(iii) that $P = PR_P$ is a completely Q-irreducible ideal of R_P. □

Remark 12.2.9 With $P = PR_P$ as in Corollary 12.2.8, if $R \neq R_P$, then P as an ideal of R is not completely Q-irreducible. For Proposition 12.2.1 (iii) implies that a completely Q-irreducible prime ideal is a maximal ideal, and by Remark 12.2.4(i), if P is maximal and Q-irreducible, then $R = R_P$. It can happen however that P is Q-irreducible and nonmaximal. This is the case, for example, if P is a nonmaximal prime of a valuation domain R.

Remark 12.2.10 Pullbacks arising as in Corollary 12.2.8 have been well studied; for a recent survey see [20]. For example, a consequence of our Corollary 12.2.8 and Theorem 4.8 in [19] is that if a domain R has a Q-irreducible prime ideal P, then R is coherent if and only if R/P is coherent.

Theorem 12.2.11 *Assume that A is a nonzero Q-irreducible ideal of the integral domain R. Then*

(i) If A is not principal, then AA^{-1} is contained in the Jacobson radical of R.

(ii) $\mathrm{End}(A)$ *is a quasilocal integral domain.*

Let M denote the maximal ideal of $\mathrm{End}(A)$.

(iii) A is an M-primal ideal of $\mathrm{End}(A)$.

(iv) If M is finitely generated as an ideal of $\mathrm{End}(A)$*, then A is completely Q-irreducible as an ideal both of R and of* $\mathrm{End}(A)$.

(v) If A is a finitely generated ideal of R, then R is quasilocal and the maximal ideal of R is the adjoint prime of A.

(vi) If both A and its adjoint prime are finitely generated ideals, then A is completely Q-irreducible.

Proof (i) Let $x \in A^{-1}$ and suppose that there is a maximal ideal N of R not containing xA. Then there exists $y \in N$ such that $xA + yR = R$. It follows that $xyA = xA \cap yR$. By Lemma 12.2.3(ii), xyA is irreducible. Therefore either $xyA = xA$ or $xyA = yR$. If $xA = xyA$, then $xA \subseteq yR \subseteq N$, a contradiction, while if $xyA = yR$, then $xA = R$ and A is principal. We conclude that every maximal ideal of R contains xA. Therefore AA^{-1} is contained in the Jacobson radical of R.

(ii) and (iii) Since A is Q-irreducible as an ideal of R, it is also Q-irreducible as an ideal of $\mathrm{End}(A)$. By Lemma 12.2.3(iii), there is a prime ideal M of $\mathrm{End}(A)$ such that $A = A\,\mathrm{End}(A)_M$. Thus $\mathrm{End}(A)_M \subseteq \mathrm{End}(A)$, which implies that M is the unique maximal ideal of $\mathrm{End}(A)$. Also by Lemma 12.2.3(iii), A is M-primal.

(iv) Let $x_1, \ldots, x_n$ generate M. By Lemma 12.2.1(iii), to show that A is completely Q-irreducible it suffices to prove that $(A :_Q M) \neq A$. Now $(A :_Q M) = x_1^{-1}A \cap \cdots \cap x_n^{-1}A$, so if $(A :_Q M) = A$, then the Q-irreduciblity of A implies $x_i^{-1}A = A$ for some i. In this case, $x_i^{-1} \in \mathrm{End}(A)$, which is impossible since $x_i \in M$, the maximal ideal of $\mathrm{End}(A)$.

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(v) By Lemma 12.2.3(ii), $A = AR_P$ for some prime ideal P of R. Thus $R_P \subseteq \mathrm{End}(A)$. But A is a finitely generated ideal of R implies that $\mathrm{End}(A)$ is an integral extension of R. This forces $R = R_P$, so that P is the unique maximal ideal of R.

(vi) By (v), R is quasilocal with maximal ideal M, and M is the adjoint prime of A. As in the proof of (iv), we have $A \subset (A :_Q M)$. Therefore Lemma 12.2.1(ii) implies that A is completely Q-irreducible. □

Corollary 12.2.12 *Every nonzero Q-irreducible ideal over a Noetherian domain is completely Q-irreducible. If the Noetherian domain R admits a completely Q-irreducible ideal, then R is local and* $\dim R \leq 1$.

Proof Suppose that A is a nonzero Q-irreducible ideal of R. By Theorem 12.2.11(vi) A is a completely Q-irreducible ideal of R, and hence also of $\mathrm{End}(A)$. By Theorem 12.2.11(ii), $\mathrm{End}(A)$ is quasilocal. Since R is Noetherian, $\mathrm{End}(A)$ is a finitely generated integral extension of R. Therefore R is local.

If $\dim R > 1$, then there exists a nonzero nonmaximal prime ideal P of R. Let $x \in P$ with $x \neq 0$. Then xM is completely irreducible by Lemma 12.2.3(iv). However, by Corollary 1.4 in [16] a completely irreducible ideal of a Noetherian local domain is primary for the maximal ideal, contradicting $xM \subseteq P$. Therefore $\dim R \leq 1$. □

Corollary 12.2.13 *If the integral domain R admits an invertible Q-irreducible ideal, then every invertible ideal of R is principal and completely Q-irreducible.*

Proof Suppose that A is an invertible Q-irreducible ideal of R. By Theorem 12.2.11(i) A is principal. Let B be an invertible ideal of R. Since A is invertible, $A = (B :_Q: (B :_Q A))$. Moreover, $(B :_Q A)$ is an invertible, hence finitely generated, fractional ideal of R. Hence there are elements $q_1, \ldots, q_k \in Q$ such that $A = (B :_Q (q_1, \ldots, q_k)R) = q_1^{-1}B \cap \cdots \cap q_k^{-1}B$. Since A is Q-irreducible, there exists $i \in \{1, \ldots, k\}$ such that $B = q_i A$. Hence B is principal and R-isomorphic to A. By Lemma 12.2.3, B is Q-irreducible. □

Remark 12.2.14 Statement (ii) of Theorem 12.2.11 is true also when A is a completely irreducible submodule of Q. For by Lemma 12.2.1(iv) (with A viewed as a completely irreducible $\mathrm{End}(A)$-submodule of Q) there is a maximal ideal M of $\mathrm{End}(A)$ such that $A = A_M$. This forces $\mathrm{End}(A)_M \subseteq \mathrm{End}(A)$, so $\mathrm{End}(A)$ is quasilocal.

12.3 Completely Q-Irreducible and m-Canonical Ideals

As noted in Remark 12.2.2 every ideal of a domain is the intersection of completely irreducible submodules of the quotient field. Thus for a given domain there exists an abundance of completely irreducible submodules of Q. However, as we observe in Section 12.1, a domain need not possess a completely Q-irreducible ideal (see also Example 12.3.7).

In this section we examine the existence and structure of completely Q-irreducible ideals. We also consider the class of "m-canonical" ideals. A nonzero fractional ideal A of a domain R is an *m-canonical* fractional ideal if for all nonzero ideals B of R, $B = (A :_Q (A :_Q B))$. This terminology is from [1] and [25]. Different terminology is used in [3] and [18] to express the same concept. An ideal A is, in our terminology, m-canonical if and only if, in the terminology of [3] and [18], R is an "A-divisorial" domain and $\mathrm{End}(A) = R$. Notice that the property of being an m-canonical ideal is invariant with respect to R-isomorphism for fractional ideals of R.

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It follows from [25, Lemma 4.1] that an m-canonical ideal of a quasilocal domain is completely Q-irreducible. A deeper result is due to S. Bazzoni [3]: *A fractional ideal A of a quasilocal domain R is m-canonical if and only if A is completely Q-irreducible,* $\mathrm{End}(A) = R$ *and for all nonzero $r \in R$, A/rA satisfies the dual AB-5* of Grothendieck's AB-5.* (An R-module B satisfies AB-5* if for any submodule C of B and inverse system of submodules $\{B_i\}_{i \in I}$ of B, it is the case that $\bigcap_{i \in I}(C + B_i) = C + \bigcap_{i \in I}(B_i)$.)

As examples later in this section show, a domain need not possess an m-canonical ideal. However if R admits an m-canonical ideal, then all completely Q-irreducible ideals of R are isomorphic:

Proposition 12.3.1 *Let R be a domain that is not a field. If R has an m-canonical ideal A, then every completely Q-irreducible ideal of R is isomorphic to A. Consider the following statements.*

(i) R has an m-canonical ideal.

(ii) Any two completely Q-irreducible ideals of R are isomorphic.

Then $(i) \Rightarrow (ii)$. If every completely irreducible proper submodule of Q is a fractional ideal of R, then $(ii) \Rightarrow (i)$.

Proof Suppose that R has an m-canonical ideal A. If B is a nonzero ideal of R, then $B = \bigcap_q q^{-1}A$, where q ranges over all nonzero elements of $(A :_Q B)$. Thus if B is completely Q-irreducible, then $B = q^{-1}A$ for some $0 \neq q \in (A :_Q B)$. Thus every proper completely Q-irreducible ideal is isomorphic to A, and (i) $\Rightarrow$ (ii).

Assume that any two completely Q-irreducible ideals are isomorphic and every completely Q-irreducible proper submodule of Q is a fractional ideal of R. Let A be a completely irreducible R-ideal. By Remark 12.1.1 every ideal of R is an intersection of completely Q-irreducible submodules of Q and therefore of completely Q-irreducible fractional ideals of R. Thus every ideal of R is an intersection of ideals isomorphic to A; that is, for any ideal B, there exists a set $X \subseteq Q$ such that $B = \bigcap_{q \in X} qA$. It follows that $B = (A :_Q (A :_Q B))$. Hence A is an m-canonical ideal. □

Remark 12.3.2 An integral domain may have an m-canonical ideal, but not admit a completely Q-irreducible fractional ideal. For example, if R is a Dedekind domain having more than one maximal ideal, then R admits an m-canonical ideal, but does not have any completely Q-irreducible fractional ideals. Indeed, as we observe in Proposition 12.3.3, if R has an m-canonical ideal and admits a completely Q-irreducible ideal, then R is quasilocal.

Proposition 12.3.3 *If R has an m-canonical ideal and a completely Q-irreducible ideal, then R is quasilocal.*

Proof Let A be a completely Q-irreducible ideal of R. By Proposition 12.3.1, A is an m-canonical ideal. Therefore $R = \mathrm{End}(A)$. By Theorem 12.2.11, $\mathrm{End}(A)$ is quasilocal. Therefore R is quasilocal. □

Remark 12.3.4 If A is a proper R-submodule of Q, then A is contained in a completely irreducible proper submodule of R. Thus if every completely irreducible proper submodule of Q is a fractional ideal of R, then every proper submodule of Q is a fractional ideal of R. The latter property holds for R if and only if there exists a valuation overring of R which is a fractional ideal of R [31, Theorem 79].

Routine arguments show that a nonzero fractional ideal of a valuation domain is m-canonical if and only if it is completely Q-irreducible. Also in the Noetherian case, the condition AB-5* is redundant, as we note next. The following proposition is essentially due in the case of Krull dimension 1 to Matlis [32] and in the general case with the assumption that $\mathrm{End}(A) = R$ to Bazzoni

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[3]. Bazzoni's proof shows that you can omit in our context the assumption that $\mathrm{End}(A) = R$. We outline how to do this in the proof. We also include a different proof of the step (iv) $\Rightarrow$ (iii).

Proposition 12.3.5 (Bazzoni [3, Theorem 3.2], Matlis [32, Theorem 15.5]) *The following statements are equivalent for a nonzero fractional ideal A of a Noetherian local domain (R, M) that is not a field.*

(i) Q/A is an injective R-module.

(ii) R has Krull dimension 1 and $(A : M)/A$ is a simple R-module.

(iii) A is an m-canonical ideal.

(iv) A is Q-irreducible.

Proof (i) $\Rightarrow$ (ii) By Proposition 4.4 in [33] a Noetherian domain that admits an ideal of injective dimension 1 necessarily has Krull dimension 1. Thus $\dim(R) = 1$, so we may apply Theorem 15.5 in [32] to obtain (ii).

(ii) $\Rightarrow$ (iii) This is contained in Theorem 15.5 of [32].

(iii) $\Rightarrow$ (i) If A is an m-canonical ideal, then necessarily $\mathrm{End}(A) = R$, so Theorem 3.2 of [3] applies.

(iii) $\Rightarrow$ (iv) An m-canonical ideal of a quasilocal domain is completely Q-irreducible [25, Lemma 4.1].

(iv) $\Rightarrow$ (iii) Suppose that A is Q-irreducible. By Corollary 12.2.12 $\dim R = 1$ and A is completely Q-irreducible. By Theorem 12.2.11 $\mathrm{End}(A)$ is a quasilocal domain. Since R is Noetherian, $\mathrm{End}(A)$ is Noetherian. Thus by Theorem 3.2 in [3] A is an m-canonical ideal of $\mathrm{End}(A)$.

By [32, Theorem 15.7] a Noetherian local domain of Krull dimension 1 has an m-canonical ideal if and only if the total quotient ring of the completion of the domain is Gorenstein. Therefore the total quotient ring of the completion of $\mathrm{End}(A)$ is Gorenstein. Now $\mathrm{End}(A)$ is an overring of R that is finitely generated as a module over R. Hence there exists a nonzero $x \in R$ such that $x\,\mathrm{End}(A) \subseteq R$. It follows that the total quotient ring T of the completion of R coincides with the completion of $\mathrm{End}(A)$. Thus T is a Gorenstein ring, and by the result cited above, R has an m-canonical ideal, say B. By Proposition 12.3.1 B is isomorphic to A, so A is an m-canonical ideal of R. □

Remark 12.3.6 Let R be a Noetherian domain of positive dimension. If R admits a nonzero Q-irreducible ideal, then R is local and $\dim R = 1$. Every proper R-submodule of Q is a fractional R-ideal if and only if the integral closure $\overline{R}$ of R is local (so a DVR) and is a finitely generated R-module. In this case every proper R-submodule of Q that is completely Q-irreducible is a fractional R-ideal. There exist, however, other one-dimensional Noetherian local domains R that admit completely Q-irreducible ideals. By Proposition 12.3.5, R admits a completely Q-irreducible ideal if and only if the total quotient ring of the completion of R is Gorenstein. In particular, this is true if R is Gorenstein. There exist examples where R is Gorenstein and $\overline{R}$ is not local, or not a finitely generated R-module, or both. For such an R, nonzero principal fractional ideals of R are completely Q-irreducible, and there also exist completely Q-irreducible proper R-submodules of Q that are not fractional R-ideals.

Example 12.3.7 *A one-dimensional Noetherian local domain need not possess a nonzero Q-irreducible ideal.* As noted in the proof of Proposition 12.3.5 it suffices to exhibit a Noetherian local domain R of Krull dimension 1 such that the total quotient ring of R is not Gorenstein. Such examples can be found in Proposition 3.1 of [12] and Theorem 1.26 and Corollary 1.27 of [27]. A specific example, based on [27], is obtained as follows. Let x, y, z be algebraically independent over the field k and let $R = k[x, y, z]_{(x,y,z)}$. Let $f, g \in xk[[x]]$ be such that x, f, g are algebraically

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independent over k. Let $u = y - f$ and $v = z - g$. Then $P := (u, v)k[[x, y, z]]$ is a prime ideal of height 2 of the completion $\widehat{R} = k[[x, y, z]]$ of R having the property that $P \cap R = (0)$. If $\mathbf{q}$ is a P-primary ideal of $\widehat{R}$, it follows from [27, Theorem 1.26] that $(\widehat{R}/\mathbf{q}) \cap k(x, y, z)$ is a one-dimensional Noetherian local domain having $\widehat{R}/\mathbf{q}$ as its completion. If we take $\mathbf{q} = P^2 = (u^2, uv, v^2)\widehat{R}$, then the total quotient ring of $\widehat{R}/\mathbf{q}$ is not Gorenstein.

Remark 12.3.8 (i) It is an open question whether a completely Q-irreducible ideal of a quasilocal integrally closed domain R is an m-canonical ideal if $\mathrm{End}(A) = R$ [3, Question 5.5]. The answer is affirmative when $A = R$: this is Theorem 2.3 of [3].

(ii) In [3] Bazzoni relates the question in (i) to a 1968 question of Heinzer [24]: If R is a domain for which every nonzero ideal is divisorial, is the integral closure of R a Prüfer domain? To obtain that R has a Prüfer integral closure the weaker requirement that R be completely Q-irreducible is not sufficient, as we note below in Example 12.3.10.

(iii) Bazzoni constructs in Example 2.11 of [3] an example of a quasilocal domain R such that R is completely Q-irreducible but not m-canonical. By Lemma 12.3.5 and (i) such a domain is neither Noetherian nor integrally closed.

The $D + M$ construction provides a source of interesting examples of completely Q-irreducible ideals. The following example is from [25, Remark 5.3], as strengthened in [1]. We recall it here, since it is relevant to Example 12.3.10.

Example 12.3.9 Let $k \subset F$ be a proper extension of fields and V be a valuation domain (that is not a field) of the form $V = F + M$, where M is the maximal ideal of V. Define $R = k + M$. Then R is a quasilocal domain with maximal ideal M. If U is any k-subspace of F of codimension 1, then the fractional ideal $A = U + M$ is a completely Q-irreducible fractional ideal of R since every R-submodule of the quotient field Q of R that properly contains A contains also V.

It is proved in Theorem 3.2 of [1] that if F is an algebraic extension of k with $[F : k]$ infinite, then there exist codimension 1 subspaces U and W of F such that $U + M$ and $W + M$ are non-isomorphic completely Q-irreducible fractional ideals of R. Thus by Proposition 12.3.1 R does not possess an m-canonical ideal. Indeed, it is shown in Theorem 3.1 of [1] that R has an m-canonical ideal if and only if $[F : k]$ is finite.

We shall see in Theorem 12.6.3 that it is possible for a domain R to possess a completely Q-irreducible ideal A and not be quasilocal. It follows from this result that $\mathrm{End}(A)$ need not equal R. However, in this situation, R is not quasilocal. The next example shows that even when R is quasilocal, it is possible for a completely Q-irreducible ideal to have an endomorphism ring not equal to R.

Gilmer and Hoffmann in [21] establish the existence of an integral domain R that admits a unique minimal overring, but has the property that the integral closure of R is not Prüfer. In Example 12.3.10 we modify this example to establish the existence of an integral domain R that has infinitely many distinct fractional overrings R_t, $t \in \mathbb{N}$, such that each R_t is completely Q-irreducible as a fractional ideal of R. Since R_t is a fractional overring of R, $\mathrm{End}(R_t) = R_t$. We remark that Bazzoni in [3, Section 4] has abstracted and greatly generalized the example of [21].

Example 12.3.10 Let K be a field and let $L = K((X))$ be the quotient field of the formal power series ring $K[[X]]$. Every nonzero element of L has a unique expression as a Laurent series $\sum_{n \geq k} a_n X^n$, where k is an integer, the $a_n \in K$ and $a_k \neq 0$. Let Y be an indeterminate over L and let $V = L[[Y]]$ denote the formal power series ring in Y over the field L. Thus V is a rank-one discrete valuation domain (DVR) of the form $L + M$, where $M = YL[[Y]]$ is the maximal ideal of V. Let $R = K + M^2$. It is well known and readily established that R is a one-dimensional quasilocal domain with maximal ideal M^2. For t a positive integer, let W_t be the set of all elements

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$f \in K((X))$ such that $f = 0$ or the coefficient of X^{-t} in the Laurent expansion of f is 0. Notice that W_t is a K-subspace of L and $L = W_t \oplus KX^{-t}$ as K vector spaces. Let $R_t = K + W_tY + M^2$. Then R_t is an overring of R and $Y^2R_t \subseteq M^2$, so R_t is a fractional ideal of R.

We show that R_t is completely Q irreducible as a fractional R-ideal by proving that $X^{-t}Y$ is in every fractional ideal of R that properly contains R_t. Let $f \in Q \setminus R_t$. Since $Q = L((Y))$, there exists an integer j such that $f = \sum_{n \geq j} b_n Y^n$, where each $b_n \in L$ and $b_j \neq 0$. Notice that $f \notin R_t$ implies $j \leq 1$. Since $L = K((X))$, there exists an integer k such that $b_j = \sum_{n \geq k} a_n X^n$, where each $a_n \in K$ and $a_k \neq 0$. Since a_k is a unit of R, the fractional ideal $R_t + Rf = R_t + a_k^{-1} f$, so we may assume that $a_k = 1$. If $j < 0$, then $X^{-k-t}Y^{1-j} \in M^2 \subset R$ and $X^{-k-t}Y^{1-j}f = X^{-t}Y + \alpha Y + \beta Y^2$, where $\alpha \in K[[X]]$ and $\beta \in V = L[[Y]]$. Since $\alpha \in W_t$, $\alpha Y + \beta Y^2 \in R_t$. Hence $X^{-t}Y \in R_t + Rf$ if $j < 0$. If $j = 0$ and $k \neq 0$, then $X^{-k-t}Y^{1-j} \in W_tY \subset R_t$ and $X^{-k-t}Yf = X^{-t}Y + \alpha Y + \beta Y^2$, where $\alpha Y + \beta Y^2 \in R_t$, so $X^{-t}Y \in R_t + Rf$ in this case. If $j = 0$ and $k = 0$, replace f by $f - 1$ to obtain a situation where $k > 0$ and $j \geq 0$. If $j = 1$, then $f \notin R_t$ implies $b_1 \notin W_t$. Hence $b_1 = c + dX^{-t}$, where $c \in W_t$ and $0 \neq d \in K$. Hence $f - cY = dX^{-t}Y + \alpha Y^2$, where $\alpha \in L[[Y]]$. Therefore also in this case $X^{-t}Y \in R_t + Rf$. We conclude that R_t is completely Q-irreducible.

In Example 12.3.10 the completely Q-irreducible fractional ideals that are constructed have endomorphism rings integral over the base ring. In Example 12.3.13 we exhibit a Noetherian local domain R and a completely Q-irreducible R-submodule A of Q such that $\mathrm{End}(A)$ is not integral over R. We first give a partial characterization of when valuation overrings are (completely) Q-irreducible.

Theorem 12.3.11 *Let V be a valuation overring of the domain R. Then the following two statements hold for V.*

(i) If V/R is a divisible R-module, then V is a Q-irreducible R-submodule of Q. Moreover, V has a principal maximal ideal if and only if V is a completely Q-irreducible R-submodule of Q.

(ii) Suppose that V is a DVR. Then V is a completely Q-irreducible R-submodule of Q if and only if V/R is a divisible R-module.

Proof (i) The assumption that V/R is divisible implies that every R-submodule of Q containing V is also a V-submodule of Q. For if $x \notin V$, then $1/x \in V$. Since V/R is divisible, $V = (1/x)V + R$. Thus $V + xR = xV$. Hence $V + xR$ is a V-submodule of Q. This implies that any R-submodule of Q containing V is a V-module. Since V is Q-irreducible as a V-submodule of Q, it follows that V is Q-irreducible as an R-submodule of Q.

If the valuation domain V has principal maximal ideal, then, by Theorem 12.2.5, V is a completely Q-irreducible V-submodule of Q. Therefore V is a completely Q-irreducible R-submodule of Q.

Conversely, if V is a completely Q-irreducible R-submodule of Q, then necessarily V is a completely Q-irreducible V-submodule of Q. By Corollary 12.2.7 every principal ideal of V is Q-irreducible. Hence by Theorem 12.2.5 V has a principal maximal ideal.

(ii) Suppose that V is a completely Q-irreducible R-submodule of Q. Let $0 \neq x \in R$. We claim that $V = R + xV$. Consider the ideal $C = (R + xV :_Q V)$ of V. Since V is a DVR, C is isomorphic to V. Also, $C = \cap_{y \in V} y^{-1}(R + xV)$, so since C is completely Q-irreducible, C is isomorphic to $R + xV$. Thus V and $R + xV$ are isomorphic as R-modules, and since these two modules are rings, this forces $R + xV = V$, proving that V/R is divisible. The converse follows from (i). □

Remark 12.3.12 Let V be a DVR overring of the integral domain R and let P be the center of V on R. Necessary and sufficient conditions in order that V/R be a divisible R-module are that (i) PV is the maximal ideal of V, and (ii) the canonical inclusion map of $R/P \hookrightarrow V/PV$ is an

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isomorphism. By Theorem 12.3.11(ii), these conditions are also necessary and sufficient in order that V be completely Q-irreducible as an R-submodule of Q.

Example 12.3.13 Let K be a field, and let X and Y be indeterminates for K. Define R to be the ring $K[X, Y]_{(X,Y)}$. We construct a valuation overring V of R such that V is a completely Q-irreducible R-submodule of Q. Let $g(X) \in XK[[X]]$ be such that X and $g(X)$ are algebraically independent over K. Define a mapping v on $K[X, Y]\backslash\{0\}$ by $v(f(X, Y))$ = smallest exponent of X appearing in the power series $f(X, g(X))$. Then v extends to a rank-one discrete valuation on $K(X, Y)$ centered on $(X, Y)R$ and having residue field K. (More details regarding this construction can be found in Chapter VI, Section 15, of [37].) Since the valuation ring V of v has maximal ideal $(X, Y)V$ and residue field $V/(X, Y)V = K$, it follows that $V = R + (X, Y)^k V$ for all $k > 0$. Since V is a DVR, $V = R + fV$ for every nonzero $f \in R$. Hence V/R is a divisible R-module. By Theorem 12.3.11, V is a completely Q-irreducible R-submodule of Q.

12.4 Q-irreducibility and Injective Modules

Let N be a submodule of the torsion-free R-module M. N is said to be an *RD-submodule (relatively divisible)* if $rN = N \cap rM$ for all $r \in R$. An R-module X is called *RD-injective* if every homomorphism from an RD-submodule N of any R-module M can be extended to a homomorphism $M \to X$. Every R-module M can be embedded as an RD-submodule in an RD-injective module, and among such RD-injectives there is a minimal one, unique up to isomorphisms over M, called the *RD-injective hull* $\widehat{M}$ of M. If M is torsion-free, then so are both $\widehat{M}$ and $\widehat{M}/M$.

The R-topology of an R-module M is defined by declaring the submodules rM for all $0 \neq r \in R$ as a subbase of open neighborhoods of 0. If M is torsion-free, then it is Hausdorff in the R-topology if and only if it is reduced (i.e., it has no divisible submodules $\neq 0$). M is R-complete if it is complete (Hausdorff) in the R-topology. If M is reduced torsion-free, then it is an RD-submodule of its R-completion $\widetilde{M}$. Observe that for a prime ideal P the R-completion and R_P-completion of R_P are identical. The R-completion $\widetilde{M}$ of a torsion-free R-module M is an RD-submodule of the RD-injective hull $\widehat{M}$ such that $\widehat{M}/\widetilde{M}$ is reduced torsion-free.

Lemma 12.4.1 *For a proper R-submodule A of Q the following conditions are equivalent:*

(i) *A is Q-irreducible;*

(ii) *the injective hull $E(Q/A)$ of the R-module Q/A is indecomposable;*

(iii) *the RD-injective hull $\widehat{A}$ of A is indecomposable.*

Proof (i) $\Leftrightarrow$ (ii) An injective module is indecomposable exactly if it is uniform.

(ii) $\Leftrightarrow$ (iii) This equivalence is a consequence of Matlis' category equivalence between the category of h-divisible torsion R-modules T and the category of reduced R-complete torsion-free R-modules M, given by the correspondences

$$T \mapsto \mathrm{Hom}_R(Q/R, T) \quad \text{and} \quad M \mapsto Q/R \otimes_R M$$

which are inverse to each other. Under the category equivalence, Q/A and the R-completion $\widetilde{A}$ of A correspond to each other, and so do the injective hull of Q/A and the RD-injective hull $\widehat{A}$ of A. As equivalence preserves direct decompositions, the claim is evident. $\square$

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Let I be an ideal of the ring R. It is well known that if $E(R/I)$ is indecomposable, then I is irreducible. Note that $E(R/I)$ can also be written as $E(Q/A)$ for a Q-irreducible R-submodule A of Q. In fact, $E(R/I)$ is a summand of $E(Q/I)$, so we can write: $E(Q/I) = E(R/I) \oplus E$ for an injective R-module E. The kernel of the projection of Q/I into the first summand is of the form A/I for a Q-irreducible submodule A of Q, and then $E(R/I) = E(Q/A)$.

Conversely, if A is a Q-irreducible proper submodule of Q, and $x \in Q \setminus A$, then the set $I = \{r \in R \mid rx \in A\}$ is a primal ideal of R such that $E(Q/A) = E(R/I)$. The adjoint prime P of the primal ideal I may be called the *prime associated to* A: this is uniquely determined by A, though I depends on the choice of x.

Lemma 12.4.2 *Every indecomposable injective R-module can be written as $E(Q/A)$ for a Q-irreducible R-submodule of A of Q. Moreover, there is a unique prime ideal P of R such that $E(Q/A) \cong E(R/I)$ for a P-primal ideal I of R, and P is a maximal ideal whenever A is completely Q-irreducible.*

We can add that I can be replaced by P if and only if P is a strong Bourbaki associated prime for I. Indeed, $E(R/I) = E(R/P)$ if and only if there are elements $r \in R \setminus I$ and $s \in R \setminus P$ such that $(I :_R r) = (P :_R s)$. Since $(P :_R s) = P$, this is equivalent to $P = (I :_R r)$, that is, P is a strong Bourbaki associated prime of I.

It is clear that every proper submodule of Q is the intersection of Q-irreducible submodules. This intersection is in general redundant. A criterion for irredundancy is as follows.

Proposition 12.4.3 *A proper submodule A of Q admits an irredundant representation as an intersection of Q-irreducible submodules if and only if $E(Q/A)$ is an interdirect sum of indecomposable injectives.*

Proof Suppose $A = \bigcap_{i\in I} A_i$ is an irredundant intersection with Q-irreducible submodules A_i of Q. Setting $B_i = \bigcap_{j\in I, j\neq i} A_j$, it is clear that the submodule generated by B_i/A $(i \in I)$ in Q/A is their direct sum. Hence $E(Q/A)$ contains the direct sum of the injective hulls $E(Q/A_i) \cong E(B_i/A)$. As Q/A embeds in the direct product of the Q/A_i, $E(Q/A)$ embeds in the direct product of the $E(Q/A_i)$. Thus $E(Q/A)$ is an interdirect sum of the $E(Q/A_i)$ (these are evidently indecomposable).

Conversely, suppose $E(Q/A)$ is an interdirect sum of indecomposable injectives E_i $(i \in I)$. Since E_i is a uniform module, we have $(Q/A) \cap E_i \neq 0$ for each $i \in I$. Clearly, A_i (defined by $A_i/A = (Q/A) \cap \prod_{j\in I, j\neq i} E_j$) is a submodule of Q, which is maximal disjoint from E_i, so Q-irreducible. The intersection $A = \bigcap_{i\in I} A_i$ is evidently irredundant. □

12.5 Irredundant Decompositions and Semi-Artinian Modules

In this section we examine domains for which every nonzero submodule of Q is an irredundant intersection of completely irreducible submodules of Q. Such domains are closely related to the class of almost perfect rings.

A ring R is *perfect* if every R-module has a projective cover; equivalently (since our rings are assumed to be commutative), R satisfies the descending chain condition on principal ideals [2]. In their study [6] of strongly flat covers of modules, Bazzoni and Salce introduced the class of *almost perfect domains*, consisting of those domains R for which every proper homomorphic image of R is perfect. Every noetherian domain of Krull dimension 1 is almost perfect, but the class of almost perfect domains includes also non-noetherian non integrally closed domains– see for example Section 3 of [5].

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There are a number of applications of perfect and almost perfect domains in the literature, most of which are motivated by the rich module theory for these classes of rings [5, 6, 10]. In this section we emphasize different features of the module and ideal theory of almost perfect domains, namely, the close connection with irredundant decompositions into completely irreducible submodules.

If R is a ring, then an R-module A is (almost) *semi-artinian* if every (proper) homomorphic image of A has a nonzero socle. In a semi-artinian module every irreducible submodule is completely irreducible (see for example [9, Lemma 2.4]), but this property does not characterize semi-artinian modules [16, Example 1.7].

As indicated by Lemma 12.5.1 below, the semi-artinian property is both necessary and sufficient for irredundant decompositions into completely irreducible submodules. Bazzoni and Salce note in [5] that:

$$R \text{ almost perfect} \Rightarrow Q/R \text{ semi-artinian} \Rightarrow R \text{ locally almost perfect.}$$

They show also that R is almost perfect if and only if R is h-local and every localization of R at a maximal ideal is almost perfect. In general, the first implication cannot be reversed [5, Example 2.1]. Smith asserts in [36] that the converse of the second implication is always true, but as noted in [5, p. 288] the proof is incorrect. Thus Bazzoni and Salce raise the question in [5, p. 288] of whether the converse is always true; namely, if R is locally almost perfect, is Q/R necessarily semi-artinian?

We give an example in this section to show that the answer is negative, and we characterize in Theorem 12.5.2(vi) and (vii) precisely when a locally almost perfect domain R has Q/R semi-artinian. We collect also in this theorem a number of different characterizations of domains R for which Q/R is semi-artinian.

The following lemma is a special case of a lattice theoretic result [9, Theorem 4.1]. A number of other properties of irredundant intersections of completely irreducible submodules of semi-artinian modules can be deduced from this same article.

Lemma 12.5.1 (Dilworth-Crawley [9]) *Let R be a ring and A be an R-module. Then A is (almost) semi-artinian if and only if every (nonzero) submodule of A is an irredundant intersection of completely irreducible submodules of A.*

In order to formulate (vii) of the next theorem, we recall that a topological space X is *scattered* if every nonempty subspace of X contains an isolated point.

Theorem 12.5.2 *The following statements are equivalent for a domain R with quotient field Q.*

(i) Q/R is semi-artinian.

(ii) Every nonzero torsion module is semi-artinian.

(iii) R is almost semi-artinian.

(iv) Q is almost semi-artinian.

(v) For each nonzero proper ideal A of R, there is a maximal ideal that is a strong Bourbaki associated prime of A.

(vi) R is locally almost perfect and for each nonzero radical ideal J of R, there is a maximal ideal of R/J that is principal.

(vii) R is locally almost perfect and for each nonzero radical ideal J of R, $\mathrm{Spec}(R/J)$ *is scattered.*

(viii) For each torsion R-module T, every submodule of T is an irredundant intersection of completely irreducible submodules of T.

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(ix) *For each torsion-free module A, every nonzero submodule of A is an irredundant intersection of completely irreducible submodules of QA.*

(x) *Each nonzero submodule of Q is an irredundant intersection of completely irreducible submodules of Q.*

(xi) *Each nonzero ideal of R is an irredundant intersection of completely irreducible submodules of Q.*

(xii) *Each nonzero ideal of R is an irredundant intersection of completely irreducible ideals.*

Proof The equivalence of (i)-(iv) can be found in [10, Theorem 4.4.1]. It follows then from Lemma 12.5.1 that (i) - (iv) are equivalent to (viii), (ix), (x) and (xii). The equivalence of (vi) and (vii) is a consequence of Corollary 2.10 in [26]. To complete the proof it is enough to show that (v) and (vi) are equivalent to (i) and that (xi) is equivalent to (iii).

(i) $\Rightarrow$ (vi) Since Q/R is semi-artinian, R is locally almost perfect. We have already established that (i) is equivalent to (xii). That (xii) implies (vi) is a consequence of Corollary 2.10 of [26].

(vi) $\Rightarrow$ (v) Suppose that A is a proper nonzero ideal of R. Since for every nonzero radical ideal J of R, R/J has a maximal ideal of R that is principal, every nonzero ideal of R has a Zariski-Samuel associated prime M [26, Theorem 2.8]; that is, $M = \sqrt{A :_R x}$ for some $x \in R \backslash A$. Since R has Krull dimension 1, M is a maximal ideal of R. By (vi) $R_M/(A_M :_{R_M} x)$ contains a simple R_M-module. Thus there exists $y \in R \setminus (A_M :_{R_M} x)$ such that $MR_M = (A_M :_{R_M} x) :_{R_M} y = A_M :_{R_M} xy$. Since $A :_R x \subseteq A :_R xy \subseteq M$ and $\sqrt{A :_R x} = M$, it follows that M is the only maximal ideal of R containing $A :_R xy$. Thus since $A_M :_{R_M} xy = MR_M$, it is the case that $A :_R xy = M$.

(v) $\Rightarrow$ (iii) If A is a proper nonzero ideal of R and M is a strong Bourbaki associated prime of A, then $A :_R M \neq A$, so R/A contains a simple R-module.

(iii) $\Rightarrow$ (xi) Since (iii) is equivalent to (x), it is sufficient to note that (x) implies (xi).

(xi) $\Rightarrow$ (iii) Let A be a proper nonzero ideal of R. Then there exists a completely irreducible submodule C of Q such that $A = C \cap D$ is an irredundant intersection for some submodule D of Q. Let $x \in D \setminus C$. Now $(C :_Q M)/C$ is the essential socle of Q/C, so if $y \in (C :_Q M) \setminus C$, then $y \in xR + C$. Thus $rx \in yR + C$ for some $r \in R$ such that $rx \notin C$. Consequently, $rxM \subseteq C$, and since $x \in D$, it is the case that $rxM \subseteq A$ with $rx \notin A$. Thus $rx + A$ is a nonzero member of the socle of R/A. Statement (iii) now follows. □

An integral domain R is *almost Dedekind* if for each maximal ideal M of R, R_M is a DVR. In [35, Theorem 3.2] it is shown that if X is a Boolean (i.e., compact Hausdorff totally disconnected) topological space, then there exists an almost Dedekind domain R with nonzero Jacobson radical such that $\mathrm{Max}(R)$ is homeomorphic to X. Thus we obtain the following corollary to Theorem 12.5.2(vii).

Corollary 12.5.3 *The following statements are equivalent for a Boolean topological space X.*

(i) *X is a scattered space.*

(ii) *There exists a domain R with nonzero Jacobson radical such that Q/R is semi-artinian and* $\mathrm{Max}(R)$ *is homeomorphic to X.*

Remark 12.5.4 In Example 2.1 of [5] an example is given of a domain R for which Q/R is semi-artinian but R is not almost perfect. Using the corollary, we may obtain many such examples. Indeed, let X be an infinite Boolean scattered space. Then there exists an almost Dedekind domain R such that $\mathrm{Max}(R)$ is homeomorphic to X and R is not a Dedekind domain. In particular, R is not h-local, since an h-local almost Dedekind domain is Dedekind. Thus Q/R is semi-artinian but R is not almost perfect.

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It is not difficult to exhibit infinite Boolean scattered spaces. For example, let X be a well-ordered set such that not every element has an immediate successor. Then X is a scattered space with respect to the order topology on X, and the isolated points of X are precisely the smallest element of X and the immediate successors of elements in X (see [28, Example 17.3, p. 272]).

In [5] Bazzoni and Salce raise the question of whether every locally almost perfect domain R has the property that Q/R is semi-artinian. Using Theorem 12.5.2 we give an example to show that this is not the case.

Example 12.5.5 Let X be a Boolean space that is not scattered (e.g., let X be the Stone-Čech compactification of the set of natural numbers with the discrete topology). As noted above, there exists an almost Dedekind domain R such that $\mathrm{Max}(R)$ is homeomorphic to X and R has nonzero Jacobson radical. Then R is locally almost perfect but by Theorem 12.5.2(vii) Q/R is not semi-artinian.

In [15] it is shown that every irreducible ideal of an almost perfect domain is primary. A similar argument yields:

Lemma 12.5.6 *If R is a locally almost perfect domain, then every proper irreducible ideal is primary.*

Proof Let A be a nonzero irreducible ideal. Then A is primary if and only if any strictly ascending chain of the form $A \subset A :_R b_1 \subset A :_R b_1b_2 \subset \cdots \subset A :_R b_1b_2\cdots b_n \subseteq \cdots$ for $b_1, b_2, \ldots, b_n, \ldots \in R$ terminates [14]. Suppose there is an infinite such strictly ascending chain, and let M be a maximal ideal containing every residual $A :_R b_1b_2\cdots b_n$. Since R_M is an almost perfect domain, R_M/A_M has the descending chain condition for principal ideals. Thus there exists $n > 0$ such that $A_M :_R b_1b_2\cdots b_n = A_M :_R b_1b_2\cdots b_{n+1}$. If $r \in A :_R b_1b_2\cdots b_{n+1}$, then there exists $x \in R \setminus M$ such that $xr \in A :_R b_1b_2 \ldots b_n$. An irreducible ideal of a domain of Krull dimension 1 is contained in a unique maximal ideal (see for example [26, Lemma 2.7]), so necessarily A is M-primal. Thus x is prime to A and it follows that $r \in A :_R b_1b_2\cdots b_n$. However, this forces $A :_R b_1b_2\cdots b_n = A :_R b_1b_2\cdots b_{n+1}$, contrary to assumption. Thus A is primary. □

Theorem 12.5.7 *If R is an almost semi-artinian domain, then every ideal of R is an irredundant intersection of primary completely irreducible ideals.*

Proof The theorem follows from Lemma 12.5.6 and Theorem 12.5.2(xii). □

We characterize next the domains R for which every nonzero submodule of Q can be represented uniquely as an irredundant intersection of completely Q-irreducible R-submodules.

An R-module B is *distributive* if for all submodules A_1, A_2 and A_3 of B, $(A_1 \cap A_2) + A_3 = (A_1+A_3)\cap(A_2\cap A_3)$. The module B is *uniserial* if its submodules are linearly ordered by inclusion. An R-module is distributive if and only if for all maximal ideals M of R, B_M is a uniserial R_M-module [29].

Lemma 12.5.8 *Let R be a ring and B be an R-module. Let $\mathcal{A}$ be the set of all R-submodules of B that are finite intersections of completely irreducible submodules of B. Then the module B is distributive if and only if for each $A \in \mathcal{A}$, the representation of A as an irredundant intersection of completely irreducible submodules of B is unique.*

Furthermore, if a submodule B of a distributive R-module can be represented as a (possibly infinite) irredundant intersection of irreducible submodules, then this representation is unique.

Proof Suppose that each representation of $A \in \mathcal{A}$ as an irredundant intersection of completely Q-irreducible submodules of B is unique. Then this property holds also for the R_M-submodules of B_M for each maximal ideal M of R. Thus by the remark preceding the theorem, to prove that B is distributive it suffices to show that B_M is a uniserial R_M-module. Thus we may reduce to the case

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where R is a quasilocal domain with maximal ideal M and show that B is a uniserial R-module. If B is not uniserial, there exist incomparable completely B-irreducible submodules C_1 and C_2 of B. Define $A = C_1 \cap C_2$, $C_1^* = C_1 :_B M$ and $C_2^* = C_2 :_B M$. By Lemma 12.2.1, $C_1 \subset C_1^*$ and $C_2 \subset C_2^*$. Now there exist $x \in (C_1^* \cap C_2) \setminus A$ and $y \in (C_1 \cap C_2^*) \setminus A$. (This follows from the irreduciblity of the C_i and the modularity of the lattice of submodules of Q; see for example Noether [34, Hilfssatz II].) We have Soc $B/A = (A + xR + yR)/A$ is a 2-dimensional vector space over R/M and $x + y \notin C_1 \cup C_3$. Let C_3 be an R-submodule of B containing $A + (x + y)R$ that is maximal with respect to $x \notin C_3$. Then C_3 is completely B-irreducible, distinct from C_1 and C_2 and $A = C_1 \cap C_3$. Yet $A \in \mathcal{A}$, so this contradiction means that the submodules of B are comparable. The converse and the last assertion follow from the fact that in a complete distributive lattice, an irredundant meet decomposition into meet-irreducible elements is unique [8, pp. 5-6] . □

Theorem 12.5.9 *The following are equivalent for a domain R with quotient field Q.*

(i) Every nonzero submodule of Q can be represented uniquely as an irredundant intersection of completely irreducible submodules of Q.

(ii) Every nonzero ideal of R can be represented uniquely as an irredundant intersection of completely irreducible submodules of Q.

(iii) Every nonzero proper ideal of R can be represented uniquely as an irredundant intersection of completely irreducible ideals of R.

(iv) R is an almost Dedekind domain such that for each radical ideal J of R, R/J has a finitely generated maximal ideal.

(v) R is an almost semi-artinian Prüfer domain.

Proof (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (iii) This follows from Theorem 12.5.2 and Lemma 12.5.8.

(iii) $\Leftrightarrow$ (iv) This is proved in [26, Corollaries 2.10 and 3.9].

(iv) $\Rightarrow$ (v) This follows from Theorem 12.5.2.

(v) $\Rightarrow$ (i) Since R is a Prüfer domain, Q is a distributive R-module. Thus (i) is a consequence of Theorem 12.5.2 and Lemma 12.5.8. □

12.6 Prüfer Domains

In light of Theorem 12.5.9 it is of interest to describe the completely irreducible submodules of the quotient field of a Prüfer domain. We do this in Theorem 12.6.2. We need for the proof of this theorem a description of the completely irreducible ideals of a Prüfer domain. This is a special case of Theorem 5.3 in [16]: *A proper ideal A of a Prüfer domain is completely irreducible if and only if $A = MB_{(M)}$ for some maximal ideal M and nonzero principal ideal B of R.*

Lemma 12.6.1 *Let R be an integral domain and let A be a flat R-submodule of Q. If A is Q-irreducible, then* End(A) *is quasilocal and is Q-irreducible as an R-submodule of Q.*

Proof Since A is a flat R-submodule of Q, it is the case that $A(B \cap C) = AB \cap AC$ for all R-submodules B and C of Q [7, I.2, Proposition 6]. Suppose now that End$(A) = B \cap C$ for R-submodules B and C of Q. Then $A = A\,\mathrm{End}(A) = A(B \cap C) = AB \cap AC$, and since A is Q-irreducible, $A = AB$ or $A = AC$. Thus $B \subseteq \mathrm{End}(A)$ or $C \subseteq \mathrm{End}(A)$, so that End$(A)$ is

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Q-irreducible. Finally, if $\mathrm{End}(A)$ is not quasilocal, then there exist two nonzero non-units $x, y \in \mathrm{End}(A)$ such that $x\,\mathrm{End}(A) + y\,\mathrm{End}(A) = \mathrm{End}(A)$. Thus $xy\,\mathrm{End}(A) = x\,\mathrm{End}(A) \cap y\,\mathrm{End}(A)$, so $\mathrm{End}(A) = y^{-1}\,\mathrm{End}(A) \cap x^{-1}\,\mathrm{End}(A)$. Since $\mathrm{End}(A)$ is Q-irreducible, this forces x or y to be a unit, a contradiction. □

Theorem 12.6.2 *Let R be a Prüfer domain. Then*

(i) the Q-irreducible R-submodules of Q are precisely the R-submodules of Q that are also R_P-submodules for some prime ideal P, and

(ii) the completely Q-irreducible proper R-submodules of Q are precisely the R-submodules of Q that are isomorphic to MR_M for some maximal ideal M of R.

Conversely, either of statements (i) and (ii) characterizes among the class of domains those that are Prüfer.

Proof (i) If A is Q-irreducible submodule of Q, then by Lemma 12.6.1 $\mathrm{End}(A)$ is quasilocal. Since R is a Prüfer domain, there is a prime ideal P of R such that $R_P = \mathrm{End}(A)$ and A is an R_P-submodule of Q. Conversely, if P is a prime ideal of R, A is an R_P-submodule of Q and $A = B \cap C$ for some R-submodules B and C of Q, then $A = BR_P \cap CR_P$. Since R_P is a valuation domain $A = BR_P$ or $A = CR_P$. Thus $A = B$ or $A = C$ and A is Q-irreducible.

(ii) Suppose that R is a Prüfer domain and let A be a completely Q-irreducible proper R-submodule of Q. Then by Proposition 12.2.1, $A = AR_M$ for some maximal ideal M of R and A is a completely Q-irreducible submodule of R_M. Since R_M is a valuation domain, there exists $q \in Q$ such that $qA \subseteq R_M$. Moreover, qA is a completely irreducible ideal of R_M, so by Lemma 5.1 of [16], $qA = xMR_M$ for some $x \in R_M$. Hence A is isomorphic to MR_M.

On the other hand, if A is an R-submodule of the form xMR_M for some $x \in Q$, then A is a completely irreducible fractional ideal of the valuation domain R_M [16, Lemma 5.1]. Since R_M is a valuation domain, A is a completely Q-irreducible of R_M. Thus by Proposition 12.2.1, A is a completely Q-irreducible R-submodule of Q.

It is easy to see that statement (i) characterizes Prüfer domains. For let M be a maximal ideal of R, and observe that since by (i) the ideals of R_M are irreducible, they are linearly ordered.

Finally, suppose that each completely Q-irreducible proper R-submodule of Q is isomorphic for some maximal ideal M to the maximal ideal of R_M. Let M be a maximal ideal of R. Then by assumption rMR_M is an irreducible ideal of R_M for all $r \in R$. By Lemma 5.1 of [16], R_M must be a valuation domain. Thus R is a Prüfer domain since every localization of R at a maximal ideal is a valuation domain. □

In Theorem 12.6.3, we describe the Prüfer domains that have a completely Q-irreducible ideal.

Theorem 12.6.3 *The following statements are equivalent for a Prüfer domain R.*

(i) There exists a completely Q-irreducible ideal of R.

(ii) There exists a nonzero Q-irreducible ideal of R.

(iii) There is a nonzero prime ideal contained in the Jacobson radical of R.

(iv) Every proper R-submodule of Q is a fractional ideal of R.

Proof (i) ⇒ (ii) This is clear.

(ii) ⇒ (iii) Suppose that A is a Q-irreducible ideal of R. By Lemma 12.2.3, $A = AR_P$ for some prime ideal of R. If A is an invertible ideal of R, then by Theorem 12.2.11 P is the unique maximal ideal of R, so that statement (iii) is clearly true. It remains to consider the case where A is not invertible. By Theorem 12.2.11, if x is a nonzero element in A^{-1}, then xA is contained in

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the Jacobson radical of R. Since by Lemma 12.2.3(ii), xA is Q-irreducible we may assume without loss of generality that A itself is contained in the Jacobson radical of R.

Now let $\{N_i\}$ be the set of maximal ideals of R. Since AR_P is an ideal of R and A is contained in each N_i, it follows that for each i, $AR_P R_{N_i} = AR_{N_i} \subset R_{N_i}$. Thus there is prime ideal P_i contained in P and N_i that contains A (the ideal P_i can be chosen to be the contraction of the maximal ideal of the ring $R_P R_{N_i}$ that contains A). Because R is a Prüfer domain, the prime ideals contained in P are linearly ordered by inclusion. Thus if $Q = \bigcap_i P_i$, then Q is a nonzero prime ideal of R (for it contains A) and Q is contained in every maximal ideal of R.

(iii) $\Rightarrow$ (i) Let P be a nonzero prime ideal of R contained in the Jacobson radical of R. Since R is a Prüfer domain, $P = PR_P$, so if $0 \neq x$ is in P it follows that xR_M is contained in P. Thus xMR_M is contained in $P = PR_M$. Moreover by Proposition 12.6.2 xMR_M is a completely Q-irreducible R-submodule of Q.

(iii) $\Rightarrow$ (iv) Statement (iv) is equivalent to the assertion that there exists a valuation overring $V \subset Q$ of R such that $(R :_Q V) \neq 0$ [31, Theorem 79]. If R satisfies (iii), then a nonzero prime ideal P contained in the Jacobson radical of R has the property that $PR_P = P$. Thus V can be chosen to be R_P.

(iv) $\Rightarrow$ (ii) By the theorem of Matlis cited in (iii) $\Rightarrow$ (iv), there exists a valuation ring V with $(R :_Q V) \neq 0$. Thus since R is a Prüfer domain there is a prime ideal P with $V = R_P$ and $rR_P \subseteq R$ for some nonzero $r \in R$. By Proposition 12.6.2, rR_P is a Q-irreducible ideal of R. $\square$

Remark 12.6.4 If R is a Prüfer domain with nonzero Jacobson radical ideal J, then there exists a unique largest prime ideal P contained in J. If M is a maximal ideal of R, then $PR_M = PR_P$ since R_M is a valuation domain. Thus $P = \bigcap_{M \in \mathrm{Max}(R)} PR_M = PR_P$. It follows that R_P/P is the quotient field of R/P. Using this observation it is not hard to see that a Prüfer domain R satisfies the equivalent conditions of Theorem 12.6.3 if and only if R occurs in a pullback diagram of the form

$$\begin{tikzcd} R \arrow[r] \arrow[d] & D \arrow[d, "\alpha"'] \\ V \arrow[r, "\beta"] & K \end{tikzcd}$$

where

- α is injective and D is a Prüfer domain such that the Jacobson radical of D does not contain a nonzero prime ideal,
- K is isomorphic to the quotient field of D, and
- β is surjective with V a valuation domain.

Thus if D is any Prüfer domain with quotient field Q and X is an indeterminate for Q, then $D + XQ[[X]]$ is a Prüfer domain satisfying the equivalent conditions of Theorem 12.6.3.

12.7 Questions

We conclude with several questions that we have not been able to resolve. Other questions touching on similar issues can be found in [1], [3] and [25].

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Question 12.7.1 *What conditions on a domain R guarantee that any two completely Q-irreducible fractional ideals are necessarily isomorphic?*

Proposition 12.3.1 gives an answer to this question in the case where every proper submodule of Q is a fractional R-ideal. By Theorem 12.6.2 if R is a valuation domain, then all completely Q-irreducible ideals of R are isomorphic. If R is a Noetherian local domain, then by Propositions 12.3.1 and 12.3.5 any two Q-irreducible ideals are isomorphic.

Question 12.7.2 *What integral domains R admit a completely Q-irreducible ideal? a nonzero Q-irreducible ideal?*

The Noetherian and Prüfer cases of Question 12.7.2 are settled in Proposition 12.3.5 and Theorem 12.6.3, respectively.

Question 12.7.3 *If R admits a nonzero Q-irreducible ideal, does R also admit a completely Q-irreducible ideal?*

The answer to Question 12.7.3 is yes if R is Prüfer or Noetherian.

Question 12.7.4 *If A is a (completely) irreducible submodule of the quotient field of a quasilocal domain R, what can be said about* $\mathrm{End}(A)$*? For a completely Q-irreducible ideal A of a quasilocal domain R does it follow that* $\mathrm{End}(A)$ *is integral over R?*

Theorem 12.6.3 along with the fact that if A is completely irreducible, then $\mathrm{End}(A)$ is quasilocal, shows that if R is not quasilocal, then $\mathrm{End}(A)$ need not be integral over R even if R is a Prüfer domain.

Theorem 12.2.11, Example 12.3.10 and Example 12.3.13 are relevant to Question 12.7.4.

Question 12.7.5 *If R is a (Noetherian) domain, what are the completely irreducible submodules of Q?*

Theorem 12.6.2 answers Question 12.7.5 in the case where R is Prüfer.

Question 12.7.6 *If A is a completely Q-irreducible R-submodule of Q, when is A a fractional ideal of R? of* $\mathrm{End}(A)$*?*

If R is a valuation domain, then every proper submodule of Q is a fractional ideal of R. The case where R is a one-dimensional Noetherian domain is deeper, but has been resolved independently by Bazzoni and Goeters. A consequence of Theorem 3.4 of [3] is that if A is a completely Q-irreducible submodule of Q such that $\mathrm{End}(A)$ is Noetherian and has Krull dimension 1, then (by Theorem 12.2.11) $\mathrm{End}(A)$ is local and (by the cited result of Bazzoni) A is a fractional ideal of $\mathrm{End}(A)$. Indeed, a more general result due to H. P. Goeters is true: If A is a submodule of the quotient field of a local Noetherian domain of Krull dimension 1, then A is a fractional ideal of $\mathrm{End}(A)$ [22, Lemma 1]. Recently, Goeters has extended this to all quasilocal Matlis domains [23].

12.8 Appendix: Corrections to [17]

In this appendix we correct several mistakes from our earlier paper [17]. We include also a stronger version of Lemma 3.2 of this paper. The main corrections concern Lemmas 2.1(iv) and 3.2 of [17]. The notation and terminology of this appendix is that of [17].

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The proof of statement (iv) of Lemma 2.1 of [17] is incorrect. Statement (iv) should be modified in the following way:

(iv) *For each nonzero nonmaximal prime ideal P of R, if $\{M_i\}$ is the collection of maximal ideals of R not containing P, then $R_P \subseteq (\bigcap_i R_{M_i})R_M$ for each maximal ideal M of R containing P.*

Having changed statement (iv), we modify now the original proofs of (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) in the following way. For (iii) $\Rightarrow$ (iv) we note that by Theorem 3.2.6 of [11] $\mathrm{End}(P) = R_P \cap (\bigcap_i R_{M_i})$ and $\mathrm{End}(P_M) = R_P$. Thus by (iii) $R_P = \mathrm{End}(P_M) = \mathrm{End}(P)_M = R_P \cap (\bigcap_i R_{M_i})R_M$, and (iv) follows.

For the proof of (iv) $\Rightarrow$ (v), we have as in the original proof that

$$R_P = \mathrm{End}(A)_M = (\bigcap_{Q \in \mathcal{X}_A} R_Q)R_M \cap (\bigcap_N R_N)R_M.$$

We claim that $\bigcap_{Q \in \mathcal{X}_A} R_Q \subseteq R_P$. If this is not the case then since R_M is a valuation domain it must be that $R_P \subset (\bigcap_{Q \in \mathcal{X}_A} R_Q)R_M$ (proper containment). Hence from the above representation of $\mathrm{End}(A)_M$ we deduce that since R_P is a valuation domain, $R_P = (\bigcap_N R_N)R_M$. Thus $(\bigcap_N R_N)R_M \subset (\bigcap_{Q \in \mathcal{X}_A} R_Q)R_M$. By (iv), $R_{Q'} \subseteq (\bigcap_N R_N)R_M$ since no N contains Q'. However $Q' \in \mathcal{X}_A$, so this implies $R_{Q'} \subset R_{Q'}R_M$, but since M contains Q', $R_{Q'}R_M = R_{Q'}$. This contradiction implies that $\bigcap_{Q \in \mathcal{X}_A} R_Q \subseteq R_P$, so every element $r \in P$ is contained in some $Q \in \mathcal{X}_A$. Consequently, no element of P is prime to A.

Reference is made in the first paragraph of the proof of Lemma 3.3 of [16] to the original version of statement (iv). In particular it is claimed that since R has the separation property, $P_i S$ is a maximal ideal of S. This can be justified now using the following more general fact, which does not appear explicitly in [17]:

Lemma 12.8.1 *A Prüfer domain R has the separation property if and only if for each collection $\{P_i : i \in I\}$ of incomparable prime ideals, the ideals P_i extend to maximal ideals of $S := \bigcap_{i \in I} R_{P_i}$.*

Proof If R has the separation property, then for each $j \in I$, $\mathrm{End}(P_j) = R_{P_j} \cap (\bigcap_N R_N)$ by Theorem 3.2.6 of [11], where N ranges over the maximal ideals of R that do not contain P_j. Thus $\mathrm{End}(P_j) \subseteq S$ since the P_i's are comaximal. By Lemma 2.1(ii) of [17] P_j is a maximal ideal of $\mathrm{End}(P_j)$, and since R is a Prüfer domain, either P_j extends to a maximal ideal SP_j of S or $SP_j = S$. The latter case is impossible since $S \subseteq R_{P_j}$. Thus SP_j is a maximal ideal of S. The converse follows from Theorem 3.2.6 of [15] and Lemma 2.1(ii) of [17]. □

A second reference to the original version of Lemma 2.1(iv) is made in the first paragraph of the proof of (i) $\Rightarrow$ (ii) of Theorem 3.7. In this paragraph it is claimed that since $\mathrm{End}(A)_M = R_P$, the elements of P are not prime to A. Since (by Theorem 2.3 of [17]) R has the separation property, this claim is immediate from Lemma 2.1(v) and the original argument that appealed to Lemma 2.1(iv) is unnecessary.

The argument in the third paragraph of the proof of Lemma 3.2 of [17] is incorrect, but rather than patch this argument we give below a stronger version of this lemma. It requires a slight strengthening of Lemma 3.1 of [17].

Lemma 12.8.2 *(cf. Lemma 3.1 of [17]) Let A be an ideal of a Prüfer domain R. Suppose Q is a prime ideal of R that contains A, and P is a prime ideal such that* $\mathrm{End}(A)_Q = R_P$. *If $P \in \mathrm{Ass}(A)$, then* $\mathrm{End}(A)_Q = \mathrm{End}(A_Q)$.

Proof Since $P \in \mathrm{Ass}(A)$, $A_{(P)}$ is a primal ideal with adjoint prime P, and it follows that A_P is a P_P-primal ideal. By [17, Lemma 1.4], $\mathrm{End}(A_P) = R_P$. Thus $\mathrm{End}(A_P) = \mathrm{End}(A)_Q$, so $A\,\mathrm{End}(A_P) = A\,\mathrm{End}(A)_Q$ implies $A_P = A_Q$. Consequently, $\mathrm{End}(A_Q) = \mathrm{End}(A_P) = R_P = \mathrm{End}(A)_Q$. □

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Lemma 12.8.3 *(cf. Lemma 3.2 of [17]) Let R be a Prüfer domain with field of fractions F, let X be an R-submodule of F, and let M be a maximal ideal of R. Then $\mathrm{End}(X)_M = R_P$ for some $P \in \mathrm{Spec}\, R$ with $P \subseteq M$. Assume that P is the union of prime ideals P_i, where each P_i is the radical of a finitely generated ideal. Then $\mathrm{End}(X)_Q = \mathrm{End}(X_Q)$ for all prime ideals Q such that $P \subseteq Q \subseteq M$.*

Proof Since $R_M \subseteq \mathrm{End}(X)_M$ and R_M is a valuation domain, $\mathrm{End}(X)_M = R_P$ for some prime ideal $P \subseteq M$. If $\mathrm{End}(X)_M = F$, then clearly $\mathrm{End}(X)_M = \mathrm{End}(X_M)$, so we assume $\mathrm{End}(X)_M \neq F$ and thus $P \neq (0)$. Let Q be a prime ideal of R such that $P \subseteq Q \subseteq M$. Since $\mathrm{End}(X)_M = R_P$, we have $\mathrm{End}(X)_Q = R_P$. Now $R_P = \mathrm{End}(X)_Q \subseteq \mathrm{End}(X_Q) \subseteq \mathrm{End}(X_P)$, so to prove Lemma 12.8.3, it suffices to show that $\mathrm{End}(X_P) \subseteq R_P$.

Let $S = \mathrm{End}(X)$. Now $PS \subseteq PR_P$, so $PS \neq S$. Since S is an overring of the Prüfer domain R, S is a flat extension of R, so PS is a prime ideal of S and $S_{PS} = R_P$. Also, PS is the union of the prime ideals P_iS, and each P_iS is the radical of a finitely generated ideal of S.

Let L be a prime ideal of S such that $L \subseteq PS$ and such that $L = \sqrt{I}$, where I is a finitely generated ideal of S. We prove there exists a nonzero $q \in F$ such that qX_L is an ideal of S_L that is primary for L_L. The invertible ideal I^2 of S is an intersection of principal fractional ideals of S. Since $\mathrm{End}(X) = S$, each principal fractional ideal of S is an intersection of S-submodules of F of the form qX, $q \in F$. Since $I^2 \subseteq L$, I^2 is an intersection of ideals of S of the form $L \cap qX$, where $q \in F$. Since $I^2 \subset I \subseteq L$ (where $\subset$ denotes proper containment), there exists $q \in F$ such that $I^2 \subseteq L \cap qX \subset L$. Hence there exists a maximal ideal N of S with $L \subseteq N$ such that $I_N^2 \subseteq L_N \cap qX_N \subset L_N$. Since S_N is a valuation domain, the S_N-modules qX_N and L_N are comparable and $I_N^2 \subseteq L_N \cap qX_N \subset L_N$ implies $I_N^2 \subseteq qX_N \subset L_N$. Now $\sqrt{I^2} = \sqrt{I} = L$ and $I^2 \subseteq N$ implies $L \subseteq N$. Thus $I_L^2 \subseteq qX_L \subseteq L_L$, and we conclude that $\sqrt{qX_L} = L_L$.

We observe next that $X_P \neq F$. Since $P \neq 0$, there exists i such that $P_i \neq 0$ and $L := P_iS \subseteq PS$, where $L = \sqrt{I}$ for some finitely generated ideal I of S. As we have established in the paragraph above, there exists a nonzero $q \in F$ such that qX_L is an ideal of S_L. Thus $qX_P \subseteq qX_L \subseteq S_L$, so it is not possible that $X_P = F$.

Fix some member L of the chain $\{P_iS\}$. Since $X_P \neq F$, $L \subseteq PS$ and R_P is a valuation domain, there exists a nonzero element s of S such that $sX \subseteq L_L$. Since $\mathrm{End}(X_P) = \mathrm{End}(sX_P)$ and we wish to show that $\mathrm{End}(X_P) \subseteq R_P$ we may assume without loss of generality that $s = 1$; that is, we assume for the rest of the proof that $X \subseteq L_L$. Define $A = X \cap S$. Then A is an ideal of S. Moreover A is contained in L since $A_L \subseteq X_L \subseteq L_L$.

With the aim of applying Lemma 12.8.2, we show that $PS \in \mathrm{Ass}(A)$. For each i define $L_i = P_iS$. It suffices to show each L_i with $L \subseteq L_i \subseteq PS$ is in $\mathrm{Ass}(A)$, since this implies that $PS = \bigcup_{L_i \supseteq L} L_i$ is a union of members of $\mathrm{Ass}(A)$. Let i be such that $L \subseteq L_i$. Since L_i is the radical of a finitely generated ideal of S, there exists (as we have established above) a nonzero $q \in F$ such that qX_{L_i} is an ideal of S_{L_i} that is primary for $(L_i)_{L_i}$. Now $A_{L_i} = X_{L_i} \cap S_{L_i}$. Since S_{L_i} is a valuation domain, $A_{L_i} = X_{L_i}$ or $S_{L_i} \subseteq X_{L_i}$. By assumption, $X \subseteq L_L$. Since $L \subseteq L_i$, it follows that $X_{L_i} \subseteq L_L$, so it is impossible that $S_{L_i} \subseteq X_{L_i}$. Thus $A_{L_i} = X_{L_i}$. Consequently, $qX_{L_i} = qA_{L_i}$ and qA_{L_i} is an ideal of S_{L_i} that is primary for $(L_i)_{L_i}$. Since S_{L_i} is a valuation domain, it follows that $qA_{L_i} = A_{L_i} : s$ for some $s \in S$. Thus $(L_i)_{L_i} \in \mathrm{Ass}(A_{L_i})$, so $L_i \in \mathrm{Ass}(A)$. This proves $PS \in \mathrm{Ass}(A)$.

Since $A = X \cap S$ is an ideal of S, $S \subseteq \mathrm{End}(A)$. For each maximal ideal N of S, either $A_N = X_N$ or $A_N = S_N$. It follows that $\mathrm{End}(A) \subseteq \mathrm{End}(X) = S$, so $\mathrm{End}(A) = S$. Thus $\mathrm{End}(A)_P = S_P = R_P$, and by Lemma 12.8.2, $\mathrm{End}(A_P) = R_P$. (We have used here that $S_{SP} = R_P$.) Now $A_P = X_P \cap S_P = X_P \cap R_P$. Since R_P is a valuation domain, $A_P = X_P$ or $R_P \subseteq X_P$. The latter case is impossible since $X_P \subseteq X_L \subseteq L_L$. Thus $A_P = X_P$. We conclude that $\mathrm{End}(X_P) = \mathrm{End}(A_P) = R_P$. □

Finally we make two corrections to the proof of Lemma 3.3. The third paragraph should read: *Define $A = JR_Q \cap R$. Then $AS = JR_Q \cap S$ is QS-primary. In particular, QS is the unique minimal prime of AS and $A \not\subseteq P_iS \cap R = P_i$ for each $i \geq 1$.*

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Also, in the fifth paragraph an exponent is incorrect: x_i needs to be chosen in $A_i \setminus (P_1 \cup \cdots \cup P_i \cup A^{i+1})$. Then in the eighth paragraph, we have $x_{i+1}S_N \subset x_i S_N$ since $x_i \in A^i \setminus A^{i+1}$ and $A^{i+1}S_N \cap R = A^{i+1}R_Q \cap R = A^{i+1}$.

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Chapter 13

Covers and Relative Purity over Commutative Noetherian Local Rings

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Abstract We relate the concepts of $\mathcal{C}$-pure sequence and $\mathcal{C}$-cover in the category of modules over a commutative noetherian local ring, and study τ_η-closed modules and the existence of τ_η-closed covers of modules over regular local rings.

13.1 Preliminaries

Throughout this paper R denotes a commutative ring with identity and τ a hereditary torsion theory in the category of R-modules, R-Mod. By $\mathcal{L}(\tau)$ we denote the Gabriel topology associated to the hereditary torsion theory τ. $Q_\tau(-)$ is the localization functor associated to τ and $\tau(-)$ the τ-torsion functor. By $E_\tau(M)$ we shall denote the τ-injective envelope of an R-module M.

Definition 13.1.1 **([1])** Let $\mathcal{C}$ be a class of R-modules closed under isomorphisms. We say that E in $\mathcal{C}$ is a $\mathcal{C}$*-precover* of an R-module X if there exists an homomorphism $\phi\colon E \longrightarrow X$ such that $Hom_R(E', X) \to Hom_R(E', E)$ is surjective for every $E' \in \mathcal{C}$. If furthermore every $f : E \to E$ such that $\phi f = \phi$ is an isomorphism, then $\phi\colon E \longrightarrow X$ is said to be a $\mathcal{C}$*-cover*.

Remark (a) A $\mathcal{C}$-cover of an R-module, if it exists, is unique up to isomorphisms.
(b) The concept of $\mathcal{C}$-envelope can be defined in a dual manner (cf. [1]).

This work is mainly concerned with τ-injective covers and τ-torsionfree τ-injective covers in R-Mod.

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13.2 τ_I-Closed Modules

Let I be a non-zero ideal of R. Let τ_I be the hereditary torsion theory in R-Mod with Gabriel filter $\mathcal{L}(\tau_I) = \{J \le R | I^n \subseteq J \ \ for \ some \ n \in \mathbb{N}\}$. An R-module M is said to be τ_I*-closed* if it is τ_I-torsionfree and τ_I-injective.

Proposition 13.2.1 *Let M be a τ_I-torsionfree R-module. If*

$$Ext_R^1(R/I^n, M) = 0 \ \forall n \in \mathbb{N}$$

then $Ext_R^1(R/J, M) = 0 \ \forall J \in \mathcal{L}(\tau_I)$, so M is τ_I-injective.

Proof Let $J \in \mathcal{L}(\tau_I)$. Then there exists $n \in \mathbb{N}$ such that $I^n \subseteq J$. If we consider the exact sequence $0 \to J/I^n \to R/I^n \to R/J \to 0$ and apply $Hom_R(-, M)$ we get

$$0 \to Hom_R(R/J, M) \to Hom_R(R/I^n, M) \to Hom_R(J/I^n, M) \to$$

$$\to Ext_R^1(R/J, M) \to Ext_R^1(R/I^n, M) = 0.$$

Since J/I^n is τ_I-torsion and M is τ_I-torsionfree it follows that

$$Hom_R(J/I^n, M) = 0,$$

so $Ext_R^1(R/J, M) = 0$ □

Proposition 13.2.2 *Let M be a τ_I-torsionfree R-module. Suppose that I is generated by a regular sequence. The following statements are equivalent.*

1) M is τ_I-injective.

2) $Ext_R^1(R/I, M) = 0$.

Proof We only need to check 2) $\Rightarrow$ 1).

By Proposition 13.2.1 this will follow if we prove that $Ext_R^1(R/I^n, M) = 0 \ \forall n \in \mathbb{N}$.

From the exact sequence $0 \to I^{n-1}/I^n \to R/I^n \to R/I^{n-1} \to 0$ we get the exact

$$Ext_R^1(R/I^{n-1}, M) \to Ext_R^1(R/I^n, M) \to Ext_R^1(I^{n-1}/I^n, M). \tag{13.1}$$

Since I is generated by a regular sequence we have that I^{i-1}/I^i is a free R/I-module $\forall i > 0$ (see [5, Theorem 19.9]). Hence

$$Ext_R^1(I^{n-1}/I^n, M) \cong \prod Ext_R^1(R/I, M) = 0.$$

Then, from the sequence (13.1) it follows inductively that $Ext_R^1(R/I^n, M) = 0$. □

Proposition 13.2.3 *If I is generated by a regular sequence then an R-module M is τ_I-torsionfree if and only if $Hom_R(R/I, M) = 0$.*

Proof It is enough to check that $Hom_R(R/I^n, M) = 0$ for all $n > 0$ when $Hom_R(R/I, M) = 0$. From the exact sequence

$$0 \to I/I^2 \to R/I^2 \to R/I \to 0$$

we get

$$0 = Hom_R(R/I, M) \to Hom_R(R/I^2, M) \to Hom_R(I/I^2, M).$$

By hypothesis I^i/I^{i+1} is a free R/I-module for all $i > 0$, hence $Hom_R(I/I^2, M) = 0$ and so $Hom_R(R/I^2, M) = 0$. Again the proof follows by an inductive argument. □

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The following Corollary is now easy to prove.

Corollary 13.2.4 *Let I be an ideal of R generated by a regular sequence of length greater than or equal to 2 and let M be an R-module. The following statements are equivalent.*

(a) M is τ_I-closed.

(b) $Hom_R(R/I, M) = Ext^1_R(R/I, M) = 0$.

If R is noetherian and M is finitely generated then the above statements are equivalent to:

(c) I contains an M-regular sequence of length greater than or equal to 2.

13.3 Relative Purity over Local Rings

Let (R, η) be a commutative noetherian local ring and F an R-module. By F^ν we shall denote the Matlis dual module of F, $Hom_R(F,\ E(R/\eta))$.

If $\mathcal{C}$ is a subcategory of R-Mod, by ${}^\nu\mathcal{C}$ we mean the subcategory of R-mod whose objects are N^ν.

Definition 13.3.1 An R-module M is said to be *Matlis pure-injective* (respectively Matlis reflexive) if the evaluation map $M \to M^{\nu\nu}$ splits (respectively if the evaluation map is an isomorphism).

Definition 13.3.2 If $\mathcal{C}$ is a subcategory of R-Mod, an exact sequence $0 \to X' \to X \to X'' \to 0$ of R-modules is said to be $\mathcal{C}$-pure if $E \otimes_R X' \to E \otimes_R X$ is a monomorphism for all $E \in \mathcal{C}$.

The following six results are easy modifications of the corresponding results of [3, Section 3].

Proposition 13.3.3 ([3, Proposition 5]) *Let $\mathcal{C}$ be a subcategory of R-Mod and $0 \to X \xrightarrow{\phi} F \to K \to 0$ an exact sequence of R-modules with F in ${}^\nu\mathcal{C}$. The following assertions are equivalent.*

(1) $0 \to X \to F \to K \to 0$ is $\mathcal{C}$-pure.

(2) $F^\nu \xrightarrow{\phi^\nu} X^\nu \to 0$ is a $\mathcal{C}$-precover.

Proposition 13.3.4 ([3, Proposition 6]) *Let $\mathcal{C}$ be a subcategory of R-Mod such that $\mathcal{C} \subseteq {}^{\nu\nu}\mathcal{C}$ and $0 \to K \to E \xrightarrow{\phi} M \to 0$ an exact sequence of R-modules with $E \in \mathcal{C}$. If K is Matlis reflexive and $0 \to M^\nu \to E^\nu \to K^\nu \to 0$ is $\mathcal{C}$-pure then $E \xrightarrow{\phi} M \to 0$ is a $\mathcal{C}$-precover.*

Proposition 13.3.5 ([3, Proposition 7]) *Let $\mathcal{C}$ be a subcategory of R-Mod closed under extensions and such that $\mathcal{C} \subseteq {}^{\nu\nu}\mathcal{C}$. Let $0 \to K \to E \xrightarrow{\phi} M \to 0$ be an exact sequence of R-modules with $E \in \mathcal{C}$ and K Matlis pure-injective. If $E^{\nu\nu} \xrightarrow{\phi^{\nu\nu}} M^{\nu\nu} \to 0$ is a $\mathcal{C}$-cover then $E \xrightarrow{\phi} M \to 0$ is a $\mathcal{C}$-cover.*

Proposition 13.3.6 ([3, Lemma 3]) *Let τ be a hereditary torsion theory in R-Mod. An R-module M is τ-closed if and only if $M^{\nu\nu}$ is τ-closed.*

Corollary 13.3.7 ([3, Corollary 8]) *Let τ be a hereditary torsion theory in R-Mod and $0 \to K \to E \to M \to 0$ an exact sequence of R-modules with E τ-torsionfree τ-injective and K Matlis reflexive. If $0 \to M^\nu \to E^\nu \to K^\nu \to 0$ is T-pure where T is the class of all τ-closed R-modules, then $E \to M \to 0$ is a τ-closed precover.*

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Corollary 13.3.8 ([3, Corollary 9]) *Let τ be a hereditary torsion theory in R-Mod and $0 \to K \to E \to M \to 0$ an exact sequence of R-modules with E τ-torsionfree τ-injective and K Matlis pure-injective. If $E^{\nu\nu} \xrightarrow{\phi^{\nu\nu}} M^{\nu\nu} \to 0$ is a τ-closed cover then $E \xrightarrow{\phi} M \to 0$ is a τ-closed cover.*

Recall that a module M is τ-flat if $id \otimes f : M \otimes_R A \to M \otimes_R B$ is a monomorphism for every monomorphism $f : A \to B$ with τ-torsion cokernel. It is a well-known fact that τ-flat modules may be characterized as those modules M whose Matlis dual M^ν is τ-injective.

Lemma 13.3.9 *If M is τ-injective and E is injective then $Hom_R(M, E)$ is τ-flat.*

Proof Let $0 \to I \to R \to R/I \to 0$ be exact with R/I τ-torsion. Since M is τ-injective, we have the exact sequence

$$0 \to Hom_R(R/I, M) \to Hom_R(R, M) \to Hom_R(I, M) \to 0.$$

Applying $Hom_R(-, E)$ we get

$$0 \to Hom_R(Hom_R(I, M), E) \to Hom_R(Hom_R(R, M), E) \to$$
$$Hom_R(Hom_R(R/I, M), E) \to 0.$$

Since R is noetherian, we have that I, R, and R/I are finitely presented. So using the natural isomorphism

$$Hom_R(Hom_R(N, M), E) \cong N \otimes_R Hom_R(M, E)$$

with N finitely presented and E an injective R-module, we get the desired condition. □

The next result is an immediate generalization of [2, Proposition 1.1].

Proposition 13.3.10 *If M is any R-module and E is an injective R-module, then $Hom_R(M, E)$ has a τ-flat precover.*

Corollary 13.3.11 *The Matlis dual of every module (M^ν) has a τ-flat precover for any torsion theory τ in R-Mod.*

Corollary 13.3.12 *Every Matlis pure-injective R-module has a τ-flat precover for any torsion theory τ in R-Mod.*

13.4 Relative Purity over Regular Local Rings

Since a noetherian local ring (R, η) is regular if and only if η is generated by a regular sequence, applying Corollary 13.2.4 we have the following result.

Corollary 13.4.1 *Let (R, η) be a regular local ring. An R-module M is τ_η-closed if and only if*

$$Hom_R(R/\eta, M) = Ext^1(R/\eta, M) = 0.$$

Proposition 13.4.2 ([4, Section 3]) *Let (R, η) be a d-dimensional regular local ring. For each R-module N there exist isomorphisms*

$$Tor^R_{d-i}(R/\eta, N) \cong Ext^i_R(R/\eta, N), \quad \forall\, 0 \le i < d.$$

If M is finitely generated then M is τ_η-closed if and only if the projective dimension of M is less than or equal to $d - 2$.

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Now, by Corollary 13.2.4 we have that if (R, η) is a regular local ring of Krull dimension greater than or equal to 2, then R is τ_η-closed. On the other hand, if (R, η) has Krull dimension exactly 2 then if R is τ_η-closed it is indeed regular. In the last case, by the above proposition a finitely generated module M is τ_η-closed if and only if M projective.

Theorem 13.4.3 *Let (R, η) be a complete regular local ring of Krull dimension 2.*

a) For each finitely generated R-module M, every exact sequence of the form $0 \to K \to R^{(n)} \xrightarrow{p} M \to 0$ has the property that $p : R^{(n)} \to M$ is a τ_η-closed precover and a τ_η-injective precover.

b) For each finitely generated R-module M (respectively finitely generated and τ_η-torsionfree R-module M), the projection $E(M) \to E(M)/M$ is a τ_η-injective precover (respectively a τ_η-closed and τ_η-injective precover).

Proof a) We use Proposition 13.3.4. Since R is complete and K is finitely generated, it follows that K is Matlis reflexive. By Proposition 13.3.6 $\mathcal{C} \subseteq {}^{\nu\nu}\mathcal{C}$ where $\mathcal{C}$ is the class of τ_η-closed R-modules or the class of τ_η-injective R-modules. Finally we see that the sequence $0 \to M^\nu \to R^{(n)\nu} \to K^\nu \to 0$ is pure relative to the class of all τ_η-injective R-modules (and so pure relative to the class of all τ_η-closed R-modules): since $K^\nu = Hom_R(K, E(R/\eta))$ is τ_η-torsion, it follows that, for every τ-flat R-module E, $Tor_1^R(K^\nu, E) = 0$. But, by Proposition 13.4.2, τ_η-flat is equivalent to τ_η-injective, so the above exact sequence has the desired condition. By Proposition 13.3.4, the result follows.

b) This can be proved using the same arguments of (a). □

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Chapter 14

Torsionless Linearly Compact Modules

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Abstract The aim of this paper is to answer a problem raised in a recent monograph by Robert Colby and Kent Fuller [3, pp. 129, 130] concerning R-torsionless linearly compact R-modules; see the introduction for a precise definition of this class of modules. Over a ring R these modules are particular submodules of products R^κ. Are $\mathbb{Z}^{(\omega)}$ and $P = \mathbb{Z}^\omega$ $\mathbb{Z}$-torsionless linearly compact (for $R = \mathbb{Z}$)? Is this class closed under direct sums? Both questions can be answered to the negative. In fact we show much more and characterize $\mathbb{Z}$-torsionless linearly compact groups: They are the free groups of finite rank. The same result holds for all principal ideal domains which are neither fields nor complete discrete valuation rings.

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14.1 Introduction

Linearly compact modules are crucial objects for the structure theory of modules based on (extensions of) Morita duality; see Colby and Fuller [3, Section 4] for example. Linear compactness can easily be defined by inverse limits: A module M is linearly compact if with any related system $\sigma_\alpha : M \longrightarrow M_\alpha$ $(\alpha \in I)$ of epimorphisms as in Proposition 14.1.1 also the unique homomorphism $\sigma : M \longrightarrow \overline{M}$ is surjective; see [3, p.75]. It turned out that proofs using linearly compact modules often only require a weaker condition to obtain similarly strong results. This can be seen in recent publications [4, 5] by Colpi and Fuller. Thus Colby and Fuller suggested in their nice monograph [3, Section 5.7] to replace linear compactness by the weaker hypothesis torsionless linear compactness. Here the trivial cokernels C (of the surjective maps above) are replaced by cokernels C which may not be 0 but have trivial dual $C^* = 0$. This notion was inspired by the version that appeared in [8].

Colby and Fuller [3, Chapter 5.7, 5.8] succeeded to lay the ground for an extended theory and naturally posed related questions which we want to deal with. Thus we recall the central notions of a torsionless linearly compact R-module in detail from the new monograph [3]. If M is an R-

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module, then traditionally and also in this paper $M^* = \mathrm{Hom}_R(M, R)$ denotes the dual module of M. Following Bass [1] an R-module M is torsionless if $M \subseteq R^\kappa$ for some cardinal κ. This is half of our central definition. The other half depends on the notion of inverse systems. Let us fix our notations.

Let $(I, \leq)$ be an *inverse directed set*, i.e., a partially ordered set so that for all $\beta, \gamma \in I$, there is $\alpha \in I$ with $\alpha \leq \beta, \gamma$. A set of R-modules and maps $(M_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \in I)$ is an *inverse system of modules* if $\pi_\alpha^\beta : M_\beta \longrightarrow M_\alpha$ is an R-homomorphism, and whenever $\alpha < \beta < \gamma$, then $\pi_\alpha^\gamma = \pi_\beta^\gamma \pi_\alpha^\beta$ (maps are acting on the right). An R-module and R-homomorphisms $(\overline{M}, \pi_\alpha : \alpha \in I)$ is the *inverse limit* of this inverse system, if $\pi_\alpha : \overline{M} \longrightarrow M_\alpha$ is an R-homomorphism ($\alpha \in I$), and whenever $\alpha < \beta$, then $\pi_\alpha = \pi_\beta \pi_\alpha^\beta$. Recall the well-known proposition, which we apply several times just below.

Proposition 14.1.1 *Let $(M_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \in I)$ be an inverse system of modules with inverse limit $(\overline{M}, \pi_\alpha : \alpha \in I)$. For any* related inverse system *$\sigma_\alpha : M \longrightarrow M_\alpha$ $(\alpha \in I)$ with $\sigma_\alpha = \sigma_\beta \pi_\alpha^\beta$ for all $\alpha < \beta$ there is a unique homomorphism $\sigma : M \longrightarrow \overline{M}$ with $\sigma_a = \sigma\pi_\alpha$ $(\alpha \in I)$.*

Thus the system has a unique inverse limit $\overline{M} = \varprojlim_I M_\alpha$ with homomorphisms π_α. We can write

$$\overline{M} = \{m = \sum_{\alpha \in I} m_\alpha \in \prod_{\alpha \in I} M_\alpha \text{ such that } m_\beta \pi_\alpha^\beta = m_\alpha \ \forall\, \alpha < \beta \in I\} \subseteq \prod_{\alpha \in I} M_\alpha$$

as a submodule of the product and

$$\pi_\beta : \overline{M} \longrightarrow M_\beta \ (\sum_{\alpha \in I} m_\alpha \longrightarrow m_\beta).$$

It follows from the definition of an inverse limit that we may assume that the maps $\pi_\alpha^\beta : M_\beta \longrightarrow M_\alpha$ are epimorphisms (replacing M_α by $\mathrm{Im}\, \pi_\alpha^\beta$). Now we are ready to complete our central definition with the above notations.

Definition 14.1.2 An R-module M is *R-torsionless linearly compact* (we will say that M is an *R-TLC*–module and a *TLC-group* if $R = \mathbb{Z}$) if the following two conditions hold:

(i) M is a submodule of a cartesian product R^κ for a suitable cardinal κ.

(ii) If $(M_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \in I)$ is an inverse system and if there is a related inverse system $\sigma_\alpha : M \longrightarrow M_\alpha$ $(\alpha \in I)$ of homomorphisms with cokernel having trivial dual $[(M_\alpha/M\sigma_\alpha)^* = 0]$, then also $\sigma : M \longrightarrow \overline{M}$ has cokernel with trivial dual $[(\overline{M}/M\sigma)^* = 0]$.

We want to prove the following theorem for abelian groups. By $P = \mathbb{Z}^\omega$ we denote the Baer-Specker group and $S = \mathbb{Z}^{(\omega)}$ is the free group of countable rank, hence $S \subseteq P$ canonically.

Theorem 14.1.3 *If $M \subseteq P$, then M is a* TLC*-group if and only if M is free of finite rank.*

The result has an immediate consequence.

Corollary 14.1.4 *A group is a* TLC*-group if and only if it is free of finite rank.*

Thus TLC-groups are well known and as a consequence the natural questions raised by Colby, Fuller [3, p. 129] are answered for $R = \mathbb{Z}$: For example, the groups $M = S$ or $M = P$ are not TLC-groups and the class is not closed under infinite direct sums; see [3, pp. 129, 130, questions (a),…,(d)]. But in this case the class obviously is closed under taking finite direct sums and extensions. The ring $\mathbb{Z}$ can be replaced by any principal ideal domain which is neither a field nor a complete discrete valuation ring. We would like to thank Kent Fuller for drawing our attention to these problems.

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14.2 Proof of the Theorem

We also state the following two easy and well-known propositions used in this section; their proof can be found in [6, p. 330, 331, Proposition 1.2, 1.3], for instance. (The notion of a direct system is dual to the inverse system above. Also dually we can replace homomorphisms of the direct system by injective maps. When passing from one system to the other we will keep the same indexing set $(I, \leq)$, but the relevant maps act in the opposite direction.) Recall from the introduction that $M^* = \mathrm{Hom}(M, R)$. If $\rho : M \longrightarrow N$, then $\rho^* : N^* \longrightarrow M^*$ denotes the canonical map induced by ρ.

Proposition 14.2.1 *Suppose $(M_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \in I)$ is an inverse system of modules. Then $(M_\alpha^*, (\pi_\alpha^\beta)^* : \alpha \leq \beta \in I)$ is a direct system of modules.*

Proposition 14.2.2 *Suppose $(M_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \in I)$ is a direct system of modules and let $(\overline{M}, \pi_\alpha : \alpha \in I)$ be its direct limit. Then $(M_\alpha^*, (\pi_\alpha^\beta)^* : \alpha \leq \beta \in I)$ is an inverse system of modules and $(\overline{M}^*, \pi_\alpha^* : \alpha \in I)$ is its inverse limit.*

We first consider the part of Theorem 14.1.3 showing that finitely generated free groups are TLC-groups. For this direction we must check the condition of our test lemma for TLC-groups, which is [3, Lemma 5.7.6] restricted to abelian groups.

Lemma 14.2.3 (Test Lemma) *Suppose that the abelian group M satisfies the following three conditions.*

(i) *M is reflexive.*

(ii) *If $X \subseteq \mathbb{Z}^\kappa$ and $M \longrightarrow X \longrightarrow C \longrightarrow 0$ is an exact sequence with $C^* = 0$, then X is reflexive as well.*

(iii) *If $\eta : L \longrightarrow M^*$ is a monomorphism, then $(L^*/M^{**}\eta^*)^* = 0$.*

Then M is a TLC-*group.*

For convenience we include the short proof which is more direct for abelian groups.

Proof We assume the notation from Proposition 14.1.1 and let $\overline{M} = \varprojlim_I M_\alpha$ be the inverse limit with homomorphisms $\pi_\alpha : M \longrightarrow M_\alpha$ $(\alpha \in I)$. Showing that M is a TLC-group we also assume $M_\alpha \subseteq \mathbb{Z}^\kappa$ for all $\alpha \in I$ and

$$M \xrightarrow{\sigma_\alpha} M_\alpha \longrightarrow C_\alpha \longrightarrow 0$$

with $C_\alpha^* = 0$ is the related system of maps. Thus $\sigma_\alpha^* : M_\alpha^* \longrightarrow M^*$ is injective and there is a unique monomorphism $\tau : \varinjlim_I M_\alpha \longrightarrow M^*$ by the dual result of Proposition 14.1.1. If $D = (\varinjlim_I M_\alpha^*)^*/M^{**}\tau^*$, then $D^* = 0$ by hypothesis (iii). By hypothesis (ii) for $X = M_\alpha$ follows that M_α is reflexive. Thus there is an isomorphism $\nu : \overline{M} \longrightarrow \varprojlim_I M_\alpha^{**} \longrightarrow (\varinjlim_I M_\alpha^*)^*$. Let $\delta : M \longrightarrow M^{**}$ be the evaluation map which is also an isomorphism by (i). We obtain the following diagram

$$\begin{tikzcd}
M \arrow[r, "\sigma"] \arrow[d, "\delta"'] & \overline{M} \arrow[r] \arrow[d, "\nu"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
M^{**} \arrow[r, "(\varinjlim \pi_\alpha^*)^*"] & (\varinjlim_I M_\alpha^*)^* \arrow[r] & D \arrow[r] & 0
\end{tikzcd}$$

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with the induced isomorphism γ. Now we apply * to the last diagram and pass to its dual diagram. From $D^* = 0$ and γ^* follows $C^* = 0$. Hence M is a TLC-group. $\square$

Finally we check the three conditions (i), (ii), (iii) of the Test Lemma 14.2.3 for finitely generated free abelian groups M. Clearly M is reflexive.

To show (ii) consider $X \subseteq \mathbb{Z}^\kappa$ and the sequence $M \xrightarrow{\varphi} X \longrightarrow C \longrightarrow 0$ and note that $M\varphi \subseteq X \subseteq \mathbb{Z}^\kappa$ is also finitely generated. If $M' = (M\varphi)_*$ denotes the pure subgroup of $\mathbb{Z}^\kappa$ purely generated by $M\varphi$, then M' has finite rank. It follows that M' is free of finite rank because $\mathbb{Z}^\kappa$ is $\aleph_1$-free (see Fuchs [7, Vol. 1, p. 94, Theorem 19.2]), hence M' is finitely generated and must split because $\mathbb{Z}^\kappa$ is also separable; see [7, Vol. 2, Section 87]. Let $Z^\kappa = M' \oplus D$. If $x + M\varphi \in X/M\varphi \setminus M'/M\varphi$, then there are $y \in M'$ and $0 \neq z \in D$ with $x = y + z$ and there is a homomorphism $\psi : \mathbb{Z}^\kappa \longrightarrow \mathbb{Z}$ with $M\varphi \subseteq M' \subseteq \operatorname{Ker}\psi$ and $z\psi \neq 0$. Hence ψ induces a non-trivial homomorphism $X/M\varphi \longrightarrow \mathbb{Z}$. This is a contradiction because $X/M\varphi \cong C$ and $C^* = 0$ by the above short exact sequence. Thus $M\varphi \subseteq X \subseteq M' \subseteq \mathbb{Z}^\kappa$ and X is also finitely generated and free, hence reflexive; (ii) follows.

If $\eta : L \longrightarrow M^*$ is a monomorphism, then $0 \longrightarrow L \longrightarrow M^* \longrightarrow D \longrightarrow 0$ is a short exact sequence, and D is a direct sum of a finite group E and a free group. It follows $0 \longrightarrow M \longrightarrow L^* \longrightarrow E \longrightarrow 0$ from $\operatorname{Ext}(D, \mathbb{Z}) \cong E$, $\operatorname{Ext}(M, \mathbb{Z}) = 0$ and $M \cong M^{**}$. In particular $E^* = 0$ and (iii) also holds. We derived the

Corollary 14.2.4 *All free groups of finite rank are* TLC-*groups.*

For the converse direction we recall that intersections of decreasing chains are inverse limits; see [7, Vol. 1, p. 62, Example 3]. This follows immediately from the preliminary remarks and Proposition 14.1.1.

Proposition 14.2.5 *Let $\{G_\alpha : \alpha \in \delta\}$ be a decreasing chain of subgroups of some group G with $G_\delta = \bigcap_{\alpha\in\delta} G_\alpha$. If $\alpha < \beta \in \delta$, then let $\pi_\alpha^\beta : G_\beta \longrightarrow G_\alpha$ be the injection map. Then*

$$\varprojlim_{\delta} G_\alpha \subseteq \prod_{\alpha\in\delta} G_\alpha$$

is the collection of all vectors with constant entry, thus with constant entry in $G_\delta = \bigcap_{\alpha\in\delta} G_\alpha$. Thus $\varprojlim_{\delta} G_\alpha = G_\delta$.

Next we will deal with subgroups M of the Baer-Specker group $P = \mathbb{Z}^\omega = \prod_{i\in\omega} \mathbb{Z}e_i$; recall that $S = \bigoplus_{i\in\omega} \mathbb{Z}e_i$ is its canonical free subgroup. The subgroups $P_n = \prod_{i\geq n} \mathbb{Z}e_i$ $(n \in \omega)$ of P generate the Hausdorff product topology on P. If $M \subseteq P$, then $\overline{M}$ denotes the closure of M in the product topology.

Lemma 14.2.6 *If $M \subseteq P$ is a subgroup and not finitely generated, then $\overline{M}$ is isomorphic to P and there is an isomorphism α of $\overline{M}$ onto P with $S \subseteq M\alpha \subseteq P$.*

Proof Subgroups of P of finite rank are finitely generated (and free), because P is $\aleph_1$-free, see [7]. If M is not finitely generated, then it must have infinite rank. An important observation by Nunke [10, p. 68, Lemma 2 (b)] applies; see also Chase [2]. There is an isomorphism of $\overline{M}$ with P, which carries M onto a subgroup of P containing S. $\square$

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Lemma 14.2.7 *If $M \subseteq P$ is not finitely generated then we can find $P' \cong P$ such that $P \subseteq P'$, and there is a descending chain $\{G_i : i \in \omega\}$ and $\bigcap_{i\in\omega} G_i = G_\omega$ of subgroups of P' such that*

(i) $M \subseteq G_\omega$ and $G_\omega/M \cong \mathbb{Z}$, hence $G_\omega \cong \mathbb{Z} \oplus M$ and $G_\omega^* \neq 0$.

(ii) G_i/M is divisible of rank 1, hence $(G_i/M)^* = 0$.

Proof We apply the previous lemma to M, which is not finitely generated, and get $S \subseteq M \subseteq \overline{M} \cong P$. If $P' = \prod_{i\in\omega} \mathbb{Z}e'_i$ is a copy of P and J_p is the ring of p-adic integers, then we consider the map

$$P' \longrightarrow J_p \ (e'_n \longrightarrow p^n).$$

This map extends linearly to $S' = \bigoplus_{i\in\omega} \mathbb{Z}e'_i$ and is continuous in the product topology on P' and the p-adic topology on J_p. Since S' is dense in P' it extends uniquely to an epimorphism from P' to J_p. Its kernel is a product $\prod_{i\in\omega} \mathbb{Z}(pe'_i - e'_{i+1})$. We put $e_i = pe'_i - e'_{i+1}$ and thus identify their product with $\overline{M}$. Hence

$$S \subseteq M \subseteq \overline{M} \subseteq P' \text{ and } P'/\overline{M} = J_p.$$

Moreover $0 \longrightarrow \overline{M}/S \longrightarrow P'/S \longrightarrow P'/\overline{M} \longrightarrow 0$ are canonical maps and $\overline{M}/S$ (by Hulanicki, see [7]) and $P'/\overline{M}$ are cotorsion. Thus also P'/S is cotorsion and in particular P'/M is cotorsion.

Now consider $1 \in J_p = P'/\overline{M}$ and its preimage $x \in P'$. Thus $0 \neq x + M \in P'/M$ is a torsion-free element which is not divisible (because its image 1 is torsion-free and p-reduced). By Harrison's characterization of cotorsion groups (see Fuchs [7, Vol. 1, p. 238]) we can write

$$P'/M = A \oplus C \oplus D$$

where D is divisible, A is torsion-free, algebraically compact and C is the adjusted part. Now let M_* be the pure closure of M in P'. As noted above, the element $x + \overline{M} = 1 \in P'/\overline{M}$ is not p-divisible, so $x + M$ does not belong to the maximal divisible subgroup D. The adjusted part C is the $\mathbb{Z}$-adic closure of the torsion subgroup $T = M_*/M$, hence C is divisible modulo T. Thus $x + M$ must have a non-trivial component in A and we may assume that $x + M \in A$ which is the completion of a product of J_p's for various primes p; so $x + M \in J_p$ (w.l.o.g.) which here is a direct summand of A. Now we are ready to use some simple structure theory.

Let $\mathbb{Q}_p \subseteq J_p$ be the p-localization of $\mathbb{Z}$, hence $\mathbb{Q}_p/\mathbb{Z} = \bigoplus_{j\in\omega} Z_{q_j^\infty}$, where $\{q_j : j \in \omega\}$ is the list of all primes different from p. Choose preimages $\mathbb{Z} \subseteq Q^i \subseteq \mathbb{Q}_p$ such that $Q^i/\mathbb{Z} = \bigoplus_{j\geq i} Z_{q_j^\infty}$. Moreover choose preimages $G_i \subseteq P'$ such that

$$G_i/M = Q^i \subseteq \mathbb{Q}_p \subseteq J_p \subseteq A \subseteq P'/M.$$

The family $\{G_i \subseteq P' : i \in \omega\}$ constitutes a descending chain of subgroups of P' satisfying the conditions of the lemma with $G_\omega = x\mathbb{Z} \oplus M$. □

Combining Lemma 14.2.7 and Proposition 14.2.5 we have the

Corollary 14.2.8 *Any* TLC*-subgroup of the Baer-Specker group is free of finite rank.*

Proof We rewrite the conditions for the infinitely generated group M in the last lemma using the notation of Definition 14.1.2: $\sigma_i = \mathrm{id} : M \longrightarrow G_i$ has cokernel $G_i/M = Q^i$ with trivial dual, $\sigma : M \longrightarrow G_\omega$ has cokernel $G_\omega/M = \mathbb{Z}$ with nontrivial dual. Thus M is not a TLC-subgroup. □

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If $M \subseteq \mathbb{Z}^\kappa = \prod_{i\in\kappa} \mathbb{Z}e_i$ is a subgroup of a product for some infinite cardinal κ which is not finitely generated, then there is a countable infinite set of independent elements $x_k = \sum_{i\in\kappa} x_{ik}e_i \in M$ ($k \in \omega$). Inductively we can find a countable set $I \subset \kappa$ such that the elements $x_k \restriction I = \sum_{i\in I} x_{ik}e_i \in \mathbb{Z}^I$ ($k \in \omega$) are independent. Thus there is an epimorphism $\pi : \mathbb{Z}^\kappa \longrightarrow P$ such that $M\pi$ is not finitely generated. In the last proof we replace σ_i by $\pi\sigma_i$ and σ by $\pi\sigma$; hence M is not a TLC-group. This proves Corollary 14.1.4.

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Chapter 15

Big Indecomposable Mixed Modules over Hypersurface Singularities

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15.1 Introduction

This research began as an effort to determine exactly which one-dimensional local rings have indecomposable finitely generated modules of arbitrarily large constant rank. The approach, which uses a new construction of indecomposable modules via the bimodule structure on certain Ext groups, turned out to be effective mainly for hypersurface singularities. The argument was eventually replaced by a direct, computational approach [6], which applies to all one-dimensional Cohen-Macaulay local rings.

In this paper we resurrect the Ext argument to build indecomposable modules of large rank over hypersurface singularities of any dimension $d \geq 1$. The main point of the construction is that, modulo an indecomposable finite-length part, the modules constructed are maximal Cohen-Macaulay modules. Thus, even when there are no indecomposable maximal Cohen-Macaulay modules of large rank, we can build short exact sequences

$$0 \to T \to X \to F \to 0,$$

in which T and X are indecomposable, T has finite length, and F is maximal Cohen-Macaulay of arbitrarily large constant rank. The main result (Theorem 15.3.3) on building indecomposables is quite general, and it is likely that there are other contexts where it will prove useful.

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In order to state our main application, we establish some terminology. Let k be a field. By a *hypersurface singularity* we mean a commutative Noetherian local ring $(R, \mathfrak{m}, k)$ whose $\mathfrak{m}$-adic completion $\widehat{R}$ is isomorphic to $S/(f)$, where $(S, \mathfrak{n}, k)$ is a complete regular local ring and f is a non-zero element of $\mathfrak{n}^2$. A Noetherian local ring $(R, \mathfrak{m}, k)$ is *Dedekind-like* [10, Definition 2.5] provided R is one-dimensional and reduced, the integral closure $\overline{R}$ is generated by at most 2 elements as an R-module, and $\mathfrak{m}$ is the Jacobson radical of $\overline{R}$. (Examples include discrete valuation rings and rings such as $k[[x, y]]/(xy)$ and $\mathbb{R}[[x, y]]/(x^2 + y^2)$.) If $(R, \mathfrak{m}, k)$ is a complete hypersurface singularity containing a field, we will call R an *(A_1)-singularity* provided R is isomorphic to a ring of the form

$$k[[x_0, \dots, x_d]]/(g + v_1x_2^2 + v_1v_2x_3^2 + \dots + v_1v_2v_3 \cdot \dots \cdot v_{d-1}x_d^2), \qquad (\dagger)$$

where each v_i is a unit of $k[[x_0, \dots, x_i]]$, $g \in k[[x_0, x_1]]$ and $k[[x_0, x_1]]/(g)$ is Dedekind-like (but not a discrete valuation ring). By adjusting g and multiplying the defining equation by v_1^{-1}, we could eliminate the unit v_1. However, the form (†) is more convenient notationally and in fact will be essential in Corollary 15.6.5. If k is algebraically closed and of characteristic different from 2, we can make the change of variables $\sqrt{v_1 \cdot \dots \cdot v_{i-1}}x_i \mapsto x_i$ $(i = 2, \dots, d)$ and put g in the form $x_0^2 + x_1^2$, so that R acquires the more palatable form $k[[x_0, \dots, x_d]]/(x_0^2 + x_1^2 + \dots + x_d^2)$.

We consider the following property of a commutative Noetherian local ring $(R, \mathfrak{m}, k)$:

For every positive integer m, there exist an integer $n \geq m$ and an indecomposable maximal Cohen-Macaulay R-module F such that $F_\mathfrak{p} \cong R_\mathfrak{p}^{(n)}$ for every prime ideal $\mathfrak{p} \neq \mathfrak{m}$. (‡)

At the opposite extreme, we say that a Gorenstein local ring $(R, \mathfrak{m}, k)$ has *bounded Cohen-Macaulay type* provided there is a bound on the number of generators required for indecomposable maximal Cohen-Macaulay R-modules. (We restrict to Gorenstein rings to avoid any possible conflict with the terminology of [17]. Cf. [17, Lemma 1.4].) In our context, at least in the complete case, there is a dichotomy, the proof of which will be deferred to §4:

Theorem 15.1.1 *Let $(R, \mathfrak{m}, k)$ be a hypersurface singularity of positive dimension, containing a field of characteristic different from* 2. *If $\widehat{R}$ does not have bounded Cohen-Macaulay type, then both R and $\widehat{R}$ satisfy.* (‡)

The rings of bounded Cohen-Macaulay type of course include those of *finite Cohen-Macaulay type* (those having only finitely many indecomposable maximal Cohen-Macaulay modules up to isomorphism). Among excellent Gorenstein rings containing a field, the rings of finite Cohen-Macaulay type have been classified completely (cf. [16, §0]). It turns out ([17] and Corollary 15.6.5 below) that if $(R, \mathfrak{m}, k)$ is a complete hypersurface singularity containing a field of characteristic different from 2, then R has bounded but infinite Cohen-Macaulay type if and only if R is either an (A_∞)- or (D_∞)-singularity, that is, R is isomorphic to a ring as in (†) but with $g =$ either x_1^2 or $x_0x_1^2$.

We now state our main application of Theorem 15.3.3. The proof will be given in §6.

Theorem 15.1.2 *Let $(R, \mathfrak{m}, k)$ be a hypersurface singularity of positive dimension, containing a field of characteristic different from* 2. *Assume that the $\mathfrak{m}$-adic completion $\widehat{R}$ has bounded Cohen-Macaulay type but is not an (A_1)-singularity. Given any positive integer m, there exist an integer $n \geq m$ and a short exact sequence of finitely generated R-modules*

$$0 \to T \to X \to F \to 0, \qquad (15.1)$$

in which

(a) T is an indecomposable finite-length module,

(b) X is indecomposable,

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(c) *F is maximal Cohen-Macaulay, and*

(d) $F_{\mathfrak{p}} \cong X_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{(n)}$ *for every prime ideal* $\mathfrak{p} \neq \mathfrak{m}$.

Putting Theroems 15.1.1 and 15.1.2 together, we have the following:

Corollary 15.1.3 *Let $(R, \mathfrak{m}, k)$ be a hypersurface singularity of positive dimension, containing a field of characteristic different from 2. Assume $\widehat{R}$ is not an (A_1)-singularity. Given any integer m, there exist an integer $n \geq m$, an indecomposable finitely generated R-module X, and a finite-length submodule $T \subsetneq X$ (possibly $T = 0$) such that X/T is maximal Cohen-Macaulay and $X_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{(n)}$ for every prime ideal $\mathfrak{p} \neq \mathfrak{m}$.*

We have been unable to determine whether or not the conclusion of Corollary 15.1.3 holds if R is an (A_1)-singularity, but we expect that it always fails. More precisely, we conjecture that if R is an (A_1)-singularity then there is a bound b, depending only on $\dim(R)$, such that for every short exact sequence 15.1 satisfying (a) – (c) and every non-maximal prime ideal, $X_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module of rank at most b. This is true in dimension one [11], where one can take $b = 2$.

Here is a brief outline of the paper: In §2 and §3 we establish our main result, Theorem 15.3.3, on building indecomposable modules. In §4 we review some known results on syzygies and double branched covers, and we prove Theorem 15.1.1. In §5 we work through some details of a construction of large indecomposable finite-length modules, and in §6 we assemble the results of §3 – §5 to prove Theorem 15.1.2.

15.2 Bimodules

In this section let R be a commutative Noetherian ring, and let A and B be module-finite R-algebras (not necessarily commutative). Let ${}_A E_B$ be an $A-B$-bimodule. We assume E is R-*symmetric*, that is, $re = er$ for $r \in R$ and $e \in E$. Furthermore we assume that E is module-finite over R. The Jacobson radical of a (not necessarily commutative) ring C is denoted by $\mathrm{J}(C)$, and the ring C is said to be *local* provided $C/\mathrm{J}(C)$ is a division ring, equivalently [4, Proposition 1.10], the set of non-units of C is closed under addition. (The emergence of local rings in this non-commutative sense has forced the annoying repetition of "commutative Noetherian local ring" where most commutative people would say simply "local ring".) The following lemma assembles some useful trivialities that allow us to transfer ring properties across the bimodule E.

Lemma 15.2.1 *Let $\alpha : {}_A A \to {}_A E$ and $\beta : B_B \to E_B$ be module homomorphisms, and assume that $\alpha(1_A) = \beta(1_B)$. Put $C := \beta^{-1}(\alpha(A))$.*

(1) *If $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $\alpha(a_i) = \beta(b_i)$, $i = 1, 2$, then $\alpha(a_1 a_2) = \beta(b_1 b_2)$.*

(2) *C is an R-subalgebra of B.*

(3) $\mathrm{Ker}(\beta) \cap C$ *is an ideal of C; thus $D := \beta(C)$ has a unique ring structure making $\beta' : C \twoheadrightarrow D$ (the map induced by β) a ring homomorphism.*

(4) *Assume $\alpha(A) \subseteq \beta(B)$. Then the map $\alpha' : A \twoheadrightarrow D$ induced by α is a ring homomorphism (where D has the ring structure of* (3)).

Proof (1) We have $\alpha(a_1 a_2) = a_1 \alpha(a_2) = a_1 \beta(b_2) = a_1 \beta(1_B b_2) = a_1 \beta(1_B) b_2 = a_1 \alpha(1_A) b_2 = \alpha(a_1 1_A) b_2 = \alpha(a_1) b_2 = \beta(b_1) b_2 = \beta(b_1 b_2)$. This proves (1), and it follows that C is a subring of

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B. A similar argument, using the fact that E is R-symmetric, shows that $1_B r \in C$ for each $r \in R$. Thus C is an R-subalgebra of B.

For (3), let $b_1, b_2 \in C$, with $b_2 \in \mathrm{Ker}(\beta)$. Choosing $a_1, a_2 \in A$ as in (1), we have $\beta(b_1 b_2) = \alpha(a_1 a_2) = a_1\alpha(a_2) = a_1\beta(b_2) = 0$. Since $\mathrm{Ker}(\beta) \cap C$ is clearly a right ideal of C, it is an ideal. To prove (4), let $a_1, a_2 \in A$, and choose $b_1, b_2 \in B$ as in (1). Then $\alpha(a_1 a_2) = \beta(b_1 b_2) = \beta(b_1)\beta(b_2) = \alpha(a_1)\alpha(a_2)$. □

Theorem 15.2.2 *With notation of Lemma 15.2.1, assume $\alpha(1_A) = \beta(1_B)$ and $\mathrm{Ker}(\beta) \subseteq \mathrm{J}(B)$. If A is local and $\alpha(1_A) \neq 0$, then C is local.*

Proof Suppose first that $\alpha(A) \subseteq \beta(B)$. With D as in the lemma, we have surjective ring homomorphisms

$$A \overset{\alpha'}{\twoheadrightarrow} B \overset{\beta'}{\twoheadleftarrow} C.$$

Therefore D is a (non-trivial) local ring, and to show that C is local, it will suffice to show that $\mathrm{Ker}(\beta') \subseteq \mathrm{J}(C)$. Since $\mathrm{Ker}(\beta) \subseteq \mathrm{J}(B)$, it is enough to show that $\mathrm{J}(B) \cap C \subseteq \mathrm{J}(C)$. As B is a module-finite R-algebra, left invertibility and right-invertibility are the same in B (thus we simply use the word "invertible"). Suppose now that $x \in \mathrm{J}(B) \cap C$. To show that $x \in \mathrm{J}(C)$ we must show that $z := 1 + yx$ is invertible in C for each $y \in C$. Since z *is* invertible in B, write $bz = 1$, with $b \in B$. Since B is module-finite over R, b is integral over R, say, $b^n + r_1 b^{n-1} + \cdots + r_{n-1} b + r_n = 0$, with $r_i \in R$. Multiplying this equation by z^{n-1}, we see that $b \in C$, as desired.

For the general case, put $G = \alpha^{-1}(\beta(B))$. By (2) of Lemma 15.2.1 (with the roles of A and B interchanged), G is an R-subalgebra of A. To see that C is local, it will suffice to show that every non-unit of G is a non-unit of A. Since A is integral over R, the argument in the preceding paragraph does the job. □

15.3 Extensions

Here we establish a context for Theorem 15.2.2. Let R be a commutative Noetherian ring, and let T and F be finitely generated R-modules. Put $A := \mathrm{End}_R(T)$ and $B := \mathrm{End}_R(F)$. Note that each of the R-modules $\mathrm{Ext}^n_R(F, T)$ has a natural $A - B$-bimodule structure. Indeed, any $f \in B$ induces an R-module homomorphism $f^* : \mathrm{Ext}^n_R(F, T) \to \mathrm{Ext}^n_R(F, T)$. For $x \in \mathrm{Ext}^n_R(F, T)$ put $x \cdot f = f^*(x)$. The left A-module structure is defined similarly, and the fact that $\mathrm{Ext}^n_R(F, T)$ is a bimodule follows from the fact that $\mathrm{Ext}^n(\_\,,\_)$ is an additive bifunctor. Note that $\mathrm{Ext}^n_R(F, T)$ is R-symmetric, since, for $r \in R$, multiplications by r on F and on T induce the same endomorphism of $\mathrm{Ext}^n_R(F, T)$.

Put $E = \mathrm{Ext}^1_R(F, T)$, regarded as equivalence classes of short exact sequences $0 \to T \to X \to F \to 0$. Let $\alpha : {}_AA \to {}_AE$ and $\beta : B_B \to E_B$ be module homomorphisms satisfying $\alpha(1_A) = \beta(1_B) =: [\xi]$. Then α and β are, up to signs, the connecting homomorphisms in the long exact sequences of Ext obtained by applying $\mathrm{Hom}_R(\_\,, T)$ and $\mathrm{Hom}_R(F, \_\,)$, respectively, to the equivalence class $[\xi]$ of the short exact sequence ξ. (When one computes Ext via resolutions one must adorn maps with appropriate $\pm$ signs, in order to ensure naturally the connecting homomorphisms. In what follows, the choice of sign will not be important.)

Recall that T is a *torsion* module provided it is killed by some non-zerodivisor of R, and that F is *torsion-free* provided every non-zerodivisor of R is a non-zerodivisor on F.

Lemma 15.3.1 *Let R be a commutative Noetherian ring, T a finitely generated torsion module, and F a finitely generated torsion-free module. Let A, B, E be as above, and let $\alpha : {}_AA \to {}_AE$ and*

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$\beta : B_B \to E_B$ be module homomorphisms with $\alpha(1_A) = [\xi] = \beta(1_B)$, where ξ is the short exact sequence

$$0 \to T \xrightarrow{i} X \xrightarrow{\pi} F \to 0. \qquad (\xi)$$

Let $\rho : \mathrm{End}_R(X) \to \mathrm{End}_R(F) =: B$ be the canonical homomorphism (reduction modulo torsion). Then the image of ρ is exactly the ring $C := \beta^{-1}\alpha(A) \subseteq B$.

Proof By applying various Hom functors to ξ, we obtain the following diagram of exact sequences:

$$\begin{tikzcd}
 & \mathrm{Hom}_R(F, X) \arrow[r] \arrow[d] & \mathrm{Hom}_R(X, X) \arrow[d, "\pi_*"] & \\
0 \arrow[r] & B \arrow[r, "\chi", "\cong"'] \arrow[d, "\beta"'] & \mathrm{Hom}_R(X, F) \arrow[r] \arrow[d, "i_*"'] & 0 \\
A \arrow[r, "\alpha"] & E \arrow[r, "\pi^*"] & \mathrm{Ext}^1_R(X, T) &
\end{tikzcd}$$

The top square commutes, and the bottom square commutes up to sign. Clearly $\rho = \chi^{-1}\pi_*$, and an easy diagram chase shows that the image of $\chi^{-1}\pi_*$ is C. □

Lemma 15.3.2 *Keep the notation and hypotheses of Lemma 15.3.1. Suppose C has no idempotents other than 0 and 1. If $X = U \oplus V$ (a decomposition as R-modules), then either U or V is a torsion module.*

Proof Suppose $X = U \oplus V$, with both U and V non-zero, and let $f \in \mathrm{End}_R(X)$ be the projection on U (relative to the decomposition $X = U \oplus V$). Then π induces an isomorphism $\overline{\pi} : U/U_{\mathrm{tors}} \oplus V/V_{\mathrm{tors}} \to F$, and $\rho(f) \in \mathrm{End}_R(F)$ is the projection on $\overline{\pi}(U/U_{\mathrm{tors}})$. If U/U_{tors} and V/V_{tors} were both non-zero, $\rho(f)$ would be a non-trivial idempotent of C, contradiction. □

The next theorem is our main result on construction of indecomposable modules.

Theorem 15.3.3 *Let T be a finitely generated torsion module and F a finitely generated torsion-free module over a commutative Noetherian ring R. Assume $A := \mathrm{End}_R(T)$ is local, put $B = \mathrm{End}_R(F)$, and assume that there is a right B-module homomorphism $\beta : B \to \mathrm{Ext}^1_R(F, T)$ with $\mathrm{Ker}(\beta) \subseteq \mathrm{J}(B)$. In the resulting short exact sequence*

$$0 \to T \to X \to F \to 0, \qquad (\xi)$$

where $\beta(1_B) = [\xi] \in \mathrm{Ext}^1_R(F, T)$, the module X is indecomposable.

Proof Let $\alpha : A \to \mathrm{Ext}^1_R(F, T)$ be the left A-module homomorphism taking 1_A to $[\xi]$. Since T is indecomposable (as its endomorphism ring is local) we may assume that $F \neq 0$. Then $\alpha(1_A) = \beta(1_B) \neq 0$. Now Theorem 15.2.2 implies that C is local. Suppose now that $X = U \oplus V$ with U and V non-zero. By Lemma 15.3.2 either U or V is torsion, say, $U \subseteq T$. Then U is a direct summand of T, whence $U = T$. But then the short exact sequence ξ splits, contradicting $\alpha(1_A) \neq 0$. □

The modules T and F in the theorem could be replaced by a torsion and torsion-free module with respect to any torsion theory for finitely generated R-modules. For example, one could take T to be any non-zero finite-length module and F a module of positive depth. The key property we need is that $\mathrm{Hom}_R(T, F) = 0$, to ensure, in Lemma 15.3.1, that T is a fully invariant submodule of X and that the map χ in the proof is surjective.

For lack of a convenient reference, we record the following result:

Lemma 15.3.4 *Let M be a finitely generated module over a commutative Noetherian local ring $(R, \mathfrak{m})$, let Γ be an R-subalgebra of $\mathrm{End}_R(M)$, and let $g \in \Gamma$. If $g(M) \subseteq \mathfrak{m}M$, then $g \in \mathrm{J}(\Gamma)$.*

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Proof It will suffice to show that $1 + hg$ is a unit of Γ for every $h \in \Gamma$. For each $x \in M$ we have $x = (1+hg)(x) - h(g(x)) \in (1+hg)(M) + \mathfrak{m}M$. By Nakayama's lemma, $1+hg$ is surjective and therefore (as M is Noetherian) an automorphism. The inverse (in $\operatorname{End}_R(M)$) of $1 + hg$ is integral over R and therefore is in $R[1 + hg] \subseteq \Gamma$. □

15.4 Syzygies and Double Branched Covers

We begin by assembling some known results from the literature. In this section "local ring" always means "commutative Noetherian local ring".

Let $(R, \mathfrak{m}, k)$ be local ring. Given a finitely generated R-module M, we denote by $\operatorname{syz}^n_R(M)$ the n^{th} syzygy of M with respect to a minimal free resolution of M. If we write $\operatorname{syz}^n_R(M) = F \oplus R^{(a)}$, where F has no non-zero free summand, then the module F is called the n^{th} *reduced syzygy* of M and is denoted by $\operatorname{redsyz}^n_R(M)$. Both $\operatorname{syz}^n_R(M)$ and $\operatorname{redsyz}^n_R(M)$ are well defined up to isomorphism. Moreover, if $0 \to G \oplus R^{(b)} \to R^{(b_{n-1})} \to \cdots \to R^{(b_0)} \to M \to 0$ is exact (not necessarily minimal) and G has no non-zero free summand, then $G \cong \operatorname{redsyz}^n_R(M)$. These observations follow easily from Schanuel's lemma [19, §19, Lemma 3], and direct-sum cancellation over local rings [3]. We denote by $\mu_R(M)$ the number of generators required for the R-module M.

Lemma 15.4.1 *Let $(S, \mathfrak{n}, k)$ be a local ring, let z be a non-zerodivisor in $\mathfrak{n}$, and put $R = S/(z)$. Let M be a finitely generated R-module. Given positive integers p, q, we have* $\operatorname{redsyz}^p_S(\operatorname{redsyz}^q_R(M)) \cong \operatorname{redsyz}^p_S(\operatorname{syz}^q_R(M)) \cong \operatorname{redsyz}^{p+q}_S(M)$.

Proof Since $\operatorname{syz}^1_S(R) \cong S$, the first isomorphism is clear; therefore we focus on the second. By induction it suffices to treat the case $p = q = 1$. Letting $M_1 = \operatorname{syz}^1_R(M)$ and $m = \mu_R(M)$, we have a short exact sequence of R-modules $0 \to M_1 \to R^{(m)} \xrightarrow{\alpha} M \to 0$. We fit this sequence into a commutative exact diagram:

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r, "j"] \arrow[d, "\psi"'] & S^{(n)} \arrow[r, "\beta"] \arrow[d, "\phi"'] & M_1 \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & S^{(m)} \arrow[r, "z"] & S^{(m)} \arrow[r, "\pi"] & R^{(m)} \arrow[r] \arrow[d, "\alpha"] & 0 \\
 & & & M \arrow[d] & \\
 & & & 0 &
\end{tikzcd}
$$

Here the top short exact sequence is obtained by mapping some free S-module onto M_1. Thus $\operatorname{redsyz}^1_S(\operatorname{syz}^1_R(M))$ is obtained from N by tossing out all free summands. The map ϕ is a lifting of $i\beta$, and ψ is the induced map on kernels. A routine diagram chase shows that the sequence

$$0 \to N \xrightarrow{\begin{bmatrix} \psi \\ -j \end{bmatrix}} S^{(m)} \oplus S^{(n)} \xrightarrow{[z\ \phi]} S^{(m)} \xrightarrow{\alpha\pi} M \to 0$$

is exact. Thus $\operatorname{redsyz}^2_S(M)$ too is obtained from N by removing free summands. □

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Proposition 15.4.2 (Herzog, [7]) *Let R be an indecomposable maximal Cohen-Macaulay module over a Gorenstein local ring $(R, \mathfrak{m}, k)$. Then $\operatorname{syz}^n_R(M)$ is indecomposable for all n.*

Proof For $n = 1$ this is [7, Lemma 1.3]; for $n \geq 2$ we use induction. □

Recall [19, p. 107] that the *multiplicity* of a finitely generated module M over a local ring $(R, \mathfrak{m})$ is defined by $\mathrm{e}(\mathfrak{m}, M) = \lim_{n\to\infty} \frac{d!}{n^d} \ell_R(M/\mathfrak{m}^n M)$, where ℓ_R denotes length. The multiplicity of R is defined by $\mathrm{e}(R) = \mathrm{e}(\mathfrak{m}, R)$. For a hypersurface singularity $R = S/(f)$, where $(S, \mathfrak{n})$ is a regular local ring and $0 \neq f \in \mathfrak{n}$, $\mathrm{e}(R)$ is the largest integer n for which $f \in \mathfrak{n}^n$ (cf. [20, (40.2)]); in particular, $\mathrm{e}(R) = 1$ if and only if R is a regular local ring.

Proposition 15.4.3 (Kawasaki [9, Theorem 4.1]) *Let $(R, \mathfrak{m})$ be a hypersurface singularity of dimension d and with multiplicity $\mathrm{e}(R) \geq 3$. Then, for every integer $t > \mathrm{e}(R)$, the maximal Cohen-Macaulay R-module $\operatorname{syz}^{d+1}_R(R/\mathfrak{m}^t)$ is indecomposable and requires at least $\binom{d+t-1}{d-1}$ generators.*

Next we review the basic properties of double branched covers. These results could be extracted from Knörrer's paper [13], but we will use the exposition in Yoshino's book [23]. The reader should be aware that Yoshino uses the notation syz^n for the n^{th} *reduced* syzygy. It will be important to us to know that certain syzygies are *automatically* devoid of free direct summands, and thus we need to appeal to Yoshino's proofs rather than merely the statements of his results.

Let $(R, \mathfrak{m}, k)$ be a complete hypersurface singularity, that is, a ring of the form $S/(f)$, where $(S, \mathfrak{n}, k)$ is a complete regular local ring and f is a non-zero element of $\mathfrak{n}$. A *double branched cover* of R is a hypersurface singularity $R^\# := S[[z]]/(f + z^2)$, where z is an indeterminate. Warning: Despite the persuasive notation, $R^\#$ is not always uniquely defined up to isomorphism. For example, $\mathbb{R}[[x, y]]/(x^2) = \mathbb{R}[[x, y]]/(-x^2)$, yet $\mathbb{R}[[x, y, z]]/(z^2 + x^2) \not\cong \mathbb{R}[[x, y, z]](z^2 - x^2)$. Thus, for example, when we write $A \cong R^\#$, we mean that A is isomorphic to the double branched cover of R with respect to *some* presentation $R \cong S/(f)$. This ambiguity is the reason for the occurrence of the units v_i in the definition of (A_1)-singularity.

The element z is a non-zerodivisor on $R^\#$, and by killing z we get a surjective ring homomorphism $R^\# \twoheadrightarrow R$. Thus every R-module can be viewed as an $R^\#$-module. Given a maximal Cohen-Macaulay R-module M, we let $M^\# = \operatorname{syz}^1_{R^\#}(M)$. Since the depth of M is $\dim(R^\#) - 1$, $M^\#$ is a maximal Cohen-Macaulay $R^\#$-module. Also, given a maximal Cohen-Macaulay $R^\#$-module N, we get a maximal Cohen-Macaulay R-module N/zN.

Proposition 15.4.4 *Let $R = \widehat{R} = S/(f)$ and $R^\# = S[[z]]/(f + z^2)$ as above, let M be a maximal Cohen-Macaulay R-module with no summand isomorphic to R, and let N be a maximal Cohen-Macaulay $R^\#$-module. Then:*

(a) $\operatorname{syz}^1_R(M)$ *has no summand isomorphic to R.*

(b) $M^\#$ *has no summand isomorphic to $R^\#$.*

(c) $M^\#/zM^\# \cong M \oplus \operatorname{syz}^1_R(M)$.

(d) If $\operatorname{char}(R) \neq 2$, *then* $(N/zN)^\# \cong N \oplus \operatorname{syz}^1_{R^\#}(N)$.

Proof For (a), we refer to [23, Chapter 7]: Since M has no free summand, it is the cokernel of a reduced matrix factorization (φ, ψ). Then $\operatorname{syz}^1_R(M)$ is the cokernel of (ψ, φ) and, by [23, (7.5.1)], $\operatorname{syz}^1_R(M)$ has no non-zero free summand.

For (c) and (d), we refer to the proofs of (12.4.1) and (12.4.2) in [23]. The blanket assumption of [23, Chapter 12] that S is a ring of power series over an algebraically closed field of characteristic 0 is not needed; however the proof of (12.4.2) *does* require that $\frac{1}{2} \in R$.

If (b) were false, we could kill z and get a surjection $M^\#/zM^\# \twoheadrightarrow R$. Since R is local, either M or $\operatorname{syz}^1_R(M)$ would have a non-zero free summand by (c), and this would contradict either the hypotheses or (a). □

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The following result from [17] (respectively [13], [23, Theorem 12.5]) is an easy consequence:

Corollary 15.4.5 ([17, Proposition 1.5]) *) Let $R = \widehat{R} = S/(f)$ and $R^\# = S[[z]]/(f + z^2)$ as above, and assume* $\mathrm{char}(k) \neq 2$. *Then $R^\#$ has bounded (respectively finite) Cohen-Macaulay type if and only if R has bounded (respectively finite) Cohen-Macaulay type.*

Both here and in §6, we will need the following lemma, whose proof is embedded in the proof of [17, Proposition 1.8],:

Lemma 15.4.6 *Let $(R, \mathfrak{m}, k)$ be a complete hypersurface singularity containing a field, with* $\mathrm{char}(k)$ $\neq 2$. *Assume* $\mathrm{e}(R) = 2$ *and $d := \dim(R) \geq 2$. Then there is a complete hypersurface singularity A of dimension $d - 1$ such that $R \cong A^\#$.*

Proof Write $R = S/(f)$, where $S = k[[x_0, \ldots, x_d]]$. Write $f = \sum_{i=0}^{\infty} f_i$, where each f_i is a homogeneous polynomial in $x_0, \ldots, x_d$ of degree i. We have $f_0 = f_1 = 0$ and $f_2 \neq 0$. We may assume, after a linear change of variables, that f_2 contains a term of the form cx_d^2, where c is a non-zero element of k. Now consider f as a power series in one variable, x_d, over $S' := k[[x_0, \ldots, x_{d-1}]]$. As such, the constant term and the coefficient of x_d are in the maximal ideal of S'. The coefficient of x_d^2 is of the form $c + g$, where g is in the maximal ideal of S'. Therefore, by [15, Chapter IV, Theorem 9.2], f can be written uniquely in the form

$$f(x_d) = u(x_d^2 + b_1 x_d + b_2),$$

where the b_i are elements of the maximal ideal of S' and u is a unit of S.

We may ignore the presence of u, as it does not change R. Then, since $\mathrm{char}(k) \neq 2$, we can complete the square and, after a linear change of variables, write $f = x_d^2 + h(x_0, \ldots, x_{d-1})$ for some power series $h \in S'$. Putting $A := S'/(h)$, we have $R \cong A^\#$. □

Our final task in this section is to prove Theorem 15.1.1. We will proceed by induction on the dimension, but in order to make the induction proceed more smoothly we will prove a formally strong assertion, which we formulate in Theorem 15.4.8 below. Let us say that a finitely generated module M over a local ring $(R, \mathfrak{m}, k)$ is *free of constant rank (or constant rank n) on the punctured spectrum* provided there is an integer n such that $M_\mathfrak{p} \cong R_\mathfrak{p}^{(n)}$ for every prime ideal $\mathfrak{p} \neq \mathfrak{m}$. We will need the following "connectedness" result.

Lemma 15.4.7 *Let $(R, \mathfrak{m}, k)$ be a local ring, T an R-module of finite length, and $F =$* $\mathrm{redsyz}_R^t(T)$ *for some $t \geq 0$. Then F is free of constant rank on the punctured spectrum. If, in addition, R is a complete hypersurface singularity with* $\mathrm{e}(R) = 2$ *and* $\dim(R) \geq 2$, *then any direct summand of F is free of constant rank on the punctured spectrum.*

Proof The first assertion is trivial. For the second, write $R = S/(f)$, where $(S, \mathfrak{n}, k)$ is a regular local ring and $f \in \mathfrak{n}^2 - \mathfrak{n}^3$. Let G be a direct summand of F. Of course G_P is free for every $P \neq \mathfrak{m}$, and the only issue is whether the rank function is constant. If f is irreducible or if $f = ug^2$ for some unit u and some $g \in \mathfrak{n} - \mathfrak{n}^2$, then R has a unique minimal prime ideal Q. Since every (non-maximal) prime P contains Q, we have $\mathrm{rank}(G_P) = \mathrm{rank}(G_Q)$ for all P. The only other possibility is that $f = f_1 f_2$ where f_1 and f_2 are prime elements, neither dividing the other. Now R has two minimal primes $Q_1 = (\overline{f_1})$ and $Q_2 = (\overline{f_2})$. Let $\mathcal{P}$ be any prime ideal of S minimal over (f_1, f_2). Since $\mathcal{P}$ has height 2 and $\dim(S) \geq 3$, $\mathcal{P} \neq \mathfrak{n}$. Then $P := \mathcal{P}/(f)$ is a non-maximal prime ideal of R, and it contains both Q_1 and Q_2. It follows that G has the same rank at Q_1 and at Q_2 and therefore has constant rank on the punctured spectrum. □

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Since over a Cohen-Macaulay ring every t^{th} syzygy, for $t \geq \dim(R)$, is maximal Cohen-Macaulay, 15.1.1 is an immediate consequence of the following Theorem:

Theorem 15.4.8 *Let $(R, \mathfrak{m}, k)$ be a hypersurface singularity of dimension $d \geq 1$, containing a field of characteristic different from 2. Suppose $\widehat{R}$ does not have bounded Cohen-Macaulay type. For every integer m, there exist a finite-length R-module T and an integer $t \geq \dim(R)$ such that some direct summand of $\operatorname{redsyz}^t_R(T)$ is indecomposable and is free of constant rank at least m on the punctured spectrum.*

Proof We may harmlessly assume that $m \geq 2$. Suppose first that R is complete. If $d = 1$, choose any integer $n \geq m$. By [18, Proposition 1.1], there is an indecomposable maximal Cohen-Macaulay (= torsion-free) R-module F such that $K \otimes_R F \cong K^{(n)}$, where K is the total quotient ring of R. Thus we get an injection $j : F \to K^{(n)}$ such that j_P is an isomorphism for each non-maximal prime ideal P. Now choose a non-zero divisor c such that $c \cdot j(F) \subseteq R^{(n)}$. This gives an injection $F \hookrightarrow R^{(n)}$ whose cokernel T has finite length. Since F is indecomposable and $n \geq 2$, we see that $F \cong \operatorname{redsyz}^1_R(T)$ as desired.

Still assuming R is complete, suppose $d \geq 2$. If $\mathrm{e}(R) \geq 3$, we can use Proposition 15.4.3 to get the required module F. Obviously R is not a regular local ring, so we may assume that $\mathrm{e}(R) = 2$. By Lemma 15.4.6, $R \cong A^{\#}$ for a suitable complete hypersurface singularity A of dimension $d - 1$. Recall that $A \cong R/(z)$ for some non-zerodivisor z.

By Corollary 15.4.5, A does not have bounded Cohen-Macaulay type. The inductive hypothesis provides a finite-length A-module T, an integer $t - 1 \geq d - 1$, and an indecomposable direct summand G of $\operatorname{redsyz}^{t-1}_A(T)$ having constant rank at least $2m$ on the punctured spectrum. Then $G^{\#} := \operatorname{syz}^1_R(G) = \operatorname{redsyz}^1_R(G)$, by Proposition 15.4.4. It follows from Lemma 15.4.1 that $G^{\#}$ is a direct summand of $\operatorname{redsyz}^t_R(T)$ and therefore, by Lemma 15.4.7, is free of constant rank on the punctured spectrum. Letting $b = \mu_R(G) = \mu_A(G)$, we have a short exact sequence $0 \to G^{\#} \to R^{(b)} \to G \to 0$. Localizing at a prime P not containing z, we see that $G^{\#}_P \cong R^{(b)}_P$. Note that $b \geq 2m$.

By Proposition 15.4.4, $G^{\#}/zG^{\#} \cong G \oplus \operatorname{syz}^1_A(G)$. Since, by Proposition 15.4.2, $\operatorname{syz}^1_A(G)$ is indecomposable, it follows that $G^{\#}$ must be a direct sum of at most two indecomposable modules. By Lemma 15.4.7, $G^{\#}$ has a direct summand of constant rank at least m on the punctured spectrum. This finishes the proof in the case that R is complete.

In the general case, choose a finite-length $\widehat{R}$-module T, an integer $t \geq \dim(R)$, and an indecomposable direct summand F of $\operatorname{redsyz}^t_{\widehat{R}}(T)$ with constant rank at least m on the punctured spectrum. Then T has finite length as an R-module, and we put $H := \operatorname{syz}^t_R(T)$. Write $H = H_1 \oplus \cdots \oplus H_s$ with each H_i indecomposable. Since $\widehat{H} \cong \operatorname{syz}^t_{\widehat{R}}(T)$, the Krull-Schmidt theorem implies that F is a direct summand of some $\widehat{H_i}$. Moreover, Lemma 15.4.7 implies that $\widehat{H_i}$ is free of constant rank, say c, on the punctured spectrum of $\widehat{R}$, and of course $c \geq \operatorname{rank}(F) \geq m$. Let p be any non-maximal prime ideal of R, and choose a prime P of $\widehat{R}$ lying over p (cf. [19, Theorem 7.3]). The R_p-module $(H_i)_p$ then becomes free of rank c after the flat local base change $R_p \to \widehat{R}_P$. By faithfully flat descent [5], (2.5.8), $(H_i)_p \cong R^{(c)}_p$. □

15.5 Finding a Suitable Finite-Length Module

The main technical step in the proof of Theorem 15.1.2 is to find, in dimension one, an indecomposable finite-length module T such that $\operatorname{redsyz}^1(T)$ has large rank. The idea of the construction goes back to the 70's, in papers by Drozd [2] and Ringel [21]. Our development depends

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on an explicit description, by Klingler and Levy [10] of the endomorphism rings of these modules. A *Drozd ring*, [10, Definition 2.4], is a commutative Artinian local ring $(\Lambda, \mathcal{M})$ such that $\mu_\Lambda(\mathcal{M}) = \mu_\Lambda(\mathcal{M}^2) = 2$, $\mathcal{M}^3 = 0$, and there is an element $x \in \mathcal{M} - \mathcal{M}^2$ with $x^2 = 0$. The prototype is the ring $k[[x, y]]/(x^2, xy^2, y^3)$ where k is a field.

Lemma 15.5.1 *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with $\dim(R) = 1$ and $\mu_R(\mathfrak{m}) = 2$. If R is not Dedekind-like, then R has a Drozd ring as a homomorphic image.*

Proof Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. All hypotheses on R transfer to $\widehat{R}$ (cf. [12, Lemma 11.8]). Moreover, if we can produce a surjection φ from $\widehat{R}$ onto a Drozd ring Λ, then the composition $R \hookrightarrow \widehat{R} \to \varphi \to \Lambda$ is surjective. Therefore we may assume that R is complete.

It will suffice to show that R is not a homomorphic image of a complete local Dedekind-like ring. To see this, we note that R is not a Klein ring (cf. [10, Definition 2.8]) since Klein rings are Artinian. Also, since $\mu_R(\mathfrak{m}) = 2$, R does not have an Artinian triad (cf. [10, Definition 2.4]) as a homomorphic image. By Klingler and Levy's "dichotomy theorem" [10, Theorem 3.1], R maps onto a Drozd ring.

We now assume, by way of contradiction, that D is a complete local Dedekind-like ring and $\sigma : D \twoheadrightarrow R$ is a surjective ring homomorphism.

Suppose first that R is reduced. Of course $\mathrm{Ker}(\sigma) \neq 0$; since both D and R are one-dimensional, D is not a domain. Since the integral closure $\overline{D}$ of D is generated by 2 elements as a D-module, D has exactly two minimal primes P, Q, and both D/P and D/Q are discrete valuation rings. Since R is reduced, either P or Q must be the kernel of σ. But then R is a discrete valuation ring and hence is Dedekind-like, contradiction.

Now assume that R is not reduced. Since $\mathrm{e}(D) \leq 2$ and R and D have the same dimension, it follows that $\mathrm{e}(R) \leq 2$. Since R is Cohen-Macaulay but not a discrete valuation ring, $\mathrm{e}(R)$ must be 2. Write $R = S/I$, where S is a complete regular local ring. Since $\mu_R(\mathfrak{m}) = 2$, we can choose S to be two-dimensional. Since R has depth 1, the Auslander-Buchsbaum formula [19, Theorem 19.1] says that R has projective dimension one as an S-module. Thus I is principal, say, $I = Sf$, where $f \in \mathfrak{n}^2 - \mathfrak{n}^3$. Since R is not reduced, we have, up to a unit, $f = x^2$, where $x \in \mathfrak{n} - \mathfrak{n}^2$. Choosing an element $y \in S$ such that $\mathfrak{n} = (x, y)$, we see that R maps onto the Drozd ring $S/(x^2, xy^2, y^3)$. $\square$

Lemma 15.5.2 *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay local ring with $\mu_R(\mathfrak{m}) = 2$. Assume R is not Dedekind-like. Given any integer n, there is an indecomposable finite-length module T such that $F := \mathrm{redsyz}^1_R(T)$ is free of constant rank greater than n on the punctured spectrum.*

Proof Choose, using Lemma 15.5.1, an ideal I such that $\Lambda := R/I$ is a Drozd ring. Fix elements $x, y \in \mathfrak{m}$ such that $\mathfrak{m} = Rx + Ry$ and $x^2 \in I$. When there is no danger of confusion we denote the images of these elements in Λ simply by x and y.

Fix a positive integer n, and let ϕ be the $n \times n$ invertible matrix (over R or Λ) with 1's on the diagonal and superdiagonal and 0's elsewhere. We will follow the development in [10] closely, with the exception that our matrices act on the left and we write vectors in $\Lambda^{(n)}$ as columns. Put

$$Q := \frac{\Lambda^{(n)}}{y\Lambda^{(n)}} \oplus \frac{\Lambda^{(n)}}{xy\Lambda^{(n)}} \oplus \Lambda^{(n)} \tag{15.2}$$

and let $\mathcal{R}$ denote the R-submodule of Q consisting of elements of the form

$$(\mathbf{b}x + y\Lambda^{(n)}, -\mathbf{b}y^2 - \mathbf{d}y + \mathbf{c}x + xy\Lambda^{(n)}, \mathbf{d}x - (\phi\mathbf{c})y^2) \tag{15.3}$$

where $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ range over $\Lambda^{(n)}$. Finally, put $T := Q/\mathcal{R}$. Of course T is a torsion R-module, since it is killed by $\mathfrak{m}^3$.

To show that T is indecomposable, suppose f is an idempotent endomorphism of T. We will show that f is either 0 or 1. Let $\Gamma = \{g \in \mathrm{End}_\Lambda(\Lambda^{(n)}) \mid g(\mathcal{R}) \subseteq \mathcal{R}\}$. Since the obvious surjection

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$\sigma : \Lambda^{(3n)} \to T$ is a projective cover, the induced map $\Gamma \to \mathrm{End}_\Lambda(T)$ is surjective, and by Lemma 15.3.4 its kernel is contained in $\mathrm{J}(\Gamma)$. Since Γ is left Artinian, idempotents lift modulo the Jacobson radical (cf. [14, (4.12), (21.28)]). Thus let $F \in \Gamma$ be an idempotent lifting f. It will suffice to show that F is either 0 or 1. Now we invoke [10, Lemma 4.8], which implies that F has the following block form:

$$F = \begin{bmatrix} F_{11} & * & * \\ \alpha & F_{22} & * \\ \beta & \gamma & F_{33} \end{bmatrix},$$

where

(1) each block is an $n \times n$ matrix,

(2) $F_{11} \equiv F_{22} \equiv F_{33} \pmod{\mathcal{M}}$,

(3) $\phi F_{11} \equiv F_{11}\phi \pmod{\mathcal{M}}$, and

(4) the entries of α, β, and γ are in $\mathcal{M}$.

(Our matrix is the transpose of the matrix displayed in [10, 4.8.1], since ours operates on the left.)

Letting bars denote reduction modulo $\mathcal{M}$, we have

$$\overline{F} = \begin{bmatrix} \overline{F_{11}} & * & * \\ 0 & \overline{F_{11}} & * \\ 0 & 0 & \overline{F_{11}} \end{bmatrix}.$$

Since $\overline{F_{11}}$ commutes with the non-derogatory matrix $\overline{\phi}$, $\overline{F_{11}}$ belongs to $k[\overline{\phi}]$, which is a local ring. Moreover, since $\overline{F}^2 = \overline{F}$, it follows that $\overline{F_{11}}^2 = \overline{F_{11}}$. Therefore $\overline{F_{11}}^2 = 0$ or 1. An easy computation then shows that $\overline{F} = 0$ or 1. By Lemma 15.3.4 the kernel of the map $\mathrm{End}_\Lambda(\Lambda^{(3n)}) \to \mathrm{End}_k(k^{(3n)})$ is contained in the Jacobson radical of $\mathrm{End}_\Lambda(\Lambda^{(3n)})$. It follows that $F = 0$ or 1, as desired.

Let $L := \mathrm{syz}^1_R(T)$, and write $L = R^{(r)} \oplus F$, where F has no non-zero free direct summand. To complete the proof, it will suffice to show that $\mathrm{rank}(F) \geq \frac{n}{e-1}$, where $e = \mathrm{e}_R(R)$. Put $s := \mathrm{rank}(F)$ and $m := \mu_R(F)$. It follows, e.g., from [8, (1.6)], that $m \leq es$. The statement of [8, (1.6)] assumes that k is infinite. This is not a problem, since none of m, e, s is changed by the flat local base change $R \to R(X) := R[X]_{\mathfrak{m}[X]}$.) Now $\mu_R(L) = r + m = 3n - s + m$, whence $\mu_R(L) - 3n \leq (e-1)s$. Therefore it will suffice to show that $\mu_R(L) \geq 4n$. Since $\mu_R(\mathfrak{m}) = 2$, the following lemma completes the proof: □

Lemma 15.5.3 *There is a surjective R-homomorphism from L onto $\mathfrak{m}^{(2n)}$.*

Proof Let Q be as in 15.2, and let $\rho : R^{(n)} \oplus R^{(n)} \oplus R^{(n)} \to Q$ be the natural homomorphism. Then $L = \rho^{-1}(\mathcal{R})$. Let $\pi : R^{(n)} \oplus R^{(n)} \oplus R^{(n)} \to R^{(n)} \oplus R^{(n)}$ be the projection on the first two coordinates. We will show that $\pi(L) = \mathfrak{m}^{(n)} \oplus \mathfrak{m}^{(n)}$. Since $\Lambda^{(3n)} \to T$ is a projective cover [10, (4.6.4)], $\mu_R(T) = 3n$. Therefore $L \subseteq \mathfrak{m}(R^{(n)} \oplus R^{(n)} \oplus R^{(n)})$, and it follows that $\pi(L) \subseteq \mathfrak{m}^{(n)} \oplus \mathfrak{m}^{(n)}$.

For the reverse inclusion, fix i, $1 \leq i \leq n$, and let $\mathbf{e}_i \in R^{(n)}$ be the i^{th} unit vector. It will suffice to show that the four elements $(\mathbf{e}_i x, 0)$, $(\mathbf{e}_i y, 0)$, $(0, \mathbf{e}_i x)$, and $(0, \mathbf{e}_i y)$ are all in $\pi(L)$.

We have $(\mathbf{e}_i x, 0) = \pi(\mathbf{e}_i x, 0, -\mathbf{e}_i x)$, and clearly $(\mathbf{e}_i x, 0, -\mathbf{e}_i x) \in L$. (Take the elements $\mathbf{b}, \mathbf{c}, \mathbf{d}$ in 15.3 to be the images, in $\Lambda^{(n)}$, of $\mathbf{e}_i, 0, -\mathbf{e}_i y$, respectively.) Since $\rho(y\mathbf{e}_i, 0, 0) = 0 \in \mathcal{R}$, $(\mathbf{e}_i y, 0) \in \pi(L)$. Next, we have $(0, \mathbf{e}_i x) = \pi(0, \mathbf{e}_i x, -(\phi\mathbf{e}_i)y^2) \in \pi(L)$. (Take $\mathbf{c}$ to be the image of $\mathbf{e}_i$, and take $\mathbf{b} = \mathbf{d} = 0$.) Finally, $(0, \mathbf{e}_i y) = \pi(0, \mathbf{e}_i y, -\mathbf{e}_i x) \in \pi(L)$. (Take $\mathbf{b} = \mathbf{c} = 0$, and let $\mathbf{d}$ be the image of $-\mathbf{e}_i$.) This completes the proof of Lemma 15.5.3, and therefore of Lemma 15.5.2 as well. □

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15.6 The Main Application

We begin with three preparatory lemmas, the first of which is an iterated version of Lemma 15.4.6.

Lemma 15.6.1 *Let $(R, \mathfrak{m}, k)$ be a complete hypersurface singularity containing a field of characteristic different from 2. Assume $d := \dim(R) \geq 2$ and that R has bounded Cohen-Macaulay type. Then R is isomorphic to a ring of the form*

$$k[[x_0, \dots, x_d]]/(g + v_1x_2^2 + v_1v_2x_3^2 + \dots + v_1v_2 \cdot \dots \cdot v_{d-1}x_d^2),$$

where each v_i is a unit of $k[[x_0, \dots, x_i]]$ and $g \in k[[x_0, x_1]]$. Moreover, if we put $R_1 := k[[x_0, x_1]]/(g)$ and $R_i := k[[x_0, x_1, \dots, x_i]]/(g + v_1x_2^2 + v_1v_2x_3^2 + \dots + v_1v_2 \cdot \dots \cdot v_{i-1}x_i^2)$ for $2 \leq i \leq d$, we have $R_i \cong R_{i-1}^{\#}$ for $2 \leq i \leq d$.

Proof By Proposition 15.4.3, $\mathrm{e}(R) \leq 2$. Therefore $\mathrm{e}(R) = 2$ since R is not a regular local ring. Write $R_d = k[[x_0, \dots, x_d]]/(f)$. As in the proof of Lemma 15.4.6, we can do a linear change of variables to get $f = u_d(x_d^2 + g_{d-1})$, where u_d is a unit and $g_{d-1} \in k[[x_0, \dots, x_{d-1}]]$. With $A = k[[x_0, \dots, x_{d-1}]]/(g_{d-1})$, we see that $R_d \cong A^{\#}$. By Corollary 15.4.5, A has bounded Cohen-Macaulay type. Also, $g_{d-1} \in (x_0, \dots, x_{d-1})^2$ (else R would be regular), so A is not regular. Continuing (if $d \geq 3$), we note that the next change of variables, in $k[[x_0, \dots, x_{d-1}]]$, does not affect x_d. Eventually, we get units $u_i \in k[[x_0, \dots, x_i]]$ and $g_1 \in k[[x_0, x_1]]$ such that

$$R \cong k[[x_0, \dots, x_d]]/u_d(x_d^2 + u_{d-1}(x_{d-1}^2 + u_{d-2}(\dots(x_3^2 + u_2(x_2^2 + g_1))\dots))).$$

Let $v_i = u_i^{-1}$ for each i, and put $v_1 = 1$. Multiplying the defining equation by $v_1v_2 \cdot \dots \cdot v_d$ and putting $g = g_1$, we obtain the desired form. The "Moreover" assertion is clear, once we multiply the defining equation for R_i by $(v_1 \cdot \dots \cdot v_{i-1})^{-1}$. □

Lemma 15.6.2 *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring, M a finitely generated R-module, and F a maximal Cohen-Macaulay R-module. Put $B = \mathrm{End}_R(F)$. Then, for all integers $i \geq 0$ and $j \geq 1$, we have*

$$\mathrm{Ext}_R^{i+j}(F, \mathrm{redsyz}_R^i(M)) \cong \mathrm{Ext}_R^{i+j}(F, \mathrm{syz}_R^i(M)) \cong \mathrm{Ext}_R^j(F, M) \text{ as right } B\text{-modules.} \tag{15.4}$$

Proof Since R is Gorenstein and F is maximal Cohen-Macaulay, we have $\mathrm{Ext}_R^j(F, R) = 0$ for $j \geq 1$. Thus we may as well use actual syzygies instead of reduced syzygies. The desired isomorphism is obtained inductively, by applying $\mathrm{Hom}_R(F, \_)$ to the short exact sequences $0 \to \mathrm{syz}_R^{j+1}(M) \to R^{(n_j)} \to \mathrm{syz}_R^j(M) \to 0$. The resulting isomorphisms are B-linear, by naturality of the connecting homomorphisms in the long exact sequence of Ext. □

Lemma 15.6.3 ([23, (7.2)]) *Let $(R, \mathfrak{m}, k)$ be a complete hypersurface singularity, and let M be a maximal Cohen-Macaulay R-module having no non-zero free summand. Then M has a periodic minimal free resolution, with period at most 2.*

Finally, we state and prove Theorem 15.1.2 in the following slightly stronger form:

Theorem 15.6.4 *Let $(R, \mathfrak{m}, k)$ be a hypersurface singularity of dimension $d \geq 1$, containing a field of characteristic different from 2. Assume that $\widehat{R}$ has bounded Cohen-Macaulay type but is not an (A_1)-singularity. Put $t = d$ if d is odd and $t = d + 1$ if d is even. Given any positive integer m there is a short exact sequence of finitely generated R-modules*

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$$0 \to T \to X \to F \to 0, \tag{15.5}$$

in which

(a) *T is an indecomposable finite-length module,*

(b) *X is indecomposable,*

(c) *$F \cong \operatorname{redsyz}^t_R(T)$, and*

(d) *F and X are free of (the same) constant rank at least m on the punctured spectrum.*

Proof We may assume that $m \geq 2$. Suppose for the moment that we have proved the theorem in the complete case, and let T, X, F be $\widehat{R}$-modules fitting into the exact sequence 15.5 and satisfying (a) – (d) (for $\widehat{R}$). Write $F \oplus \widehat{R}^{(b)} \cong \operatorname{syz}^t_{\widehat{R}}(T)$. In the general case, let $H = \operatorname{redsyz}^t_R(T)$, and write $H \oplus R^{(a)} \cong \operatorname{syz}^t_R(T)$. Then $\widehat{H} \oplus \widehat{R}^{(b)} \cong \operatorname{syz}^t_{\widehat{R}}(T)$. Since R is not isomorphic to a direct summand of H, it follows, e.g., from [22, Proposition 2], that $\widehat{R}$ is not isomorphic to a direct summand of $\widehat{H}$. Therefore $\widehat{H} \cong \operatorname{redsyz}^t_{\widehat{R}}(T) \cong F$. Since $\operatorname{Ext}^1_R(T, H)$ has finite length as an R-module, we have $\operatorname{Ext}^1_R(T, H) = (\operatorname{Ext}^1_R(T, H))\widehat{\ } = \operatorname{Ext}^1_{\widehat{R}}(\widehat{T}, \widehat{H}) = \operatorname{Ext}^1_{\widehat{R}}(T, F)$. This means that the extension 15.5 over $\widehat{R}$ is actually the completion of an extension $0 \to T \to Y \to H \to 0$ of R-modules. It follows that $\widehat{Y} \cong X$ and hence that Y is indecomposable. Finally, the argument in the last three sentences of §4 shows that Y and H are free on the punctured spectrum, of the same rank as X and F.

Thus we may assume from now on that R is complete. We write $R = R_d$ in the form (†), using Lemma 15.6.1. With the R_i as in Lemma 15.6.1, we make the identifications $R_i = R^{\#}_{i-1}$ and $R_{i-1} = R_i/(z_i)$. None of the rings R_i is an (A_1)-singularity; in particular, R_1 is not Dedekind-like. By Lemma 15.5.2, there is a finite-length R_1-module T whose first reduced syzygy F_1 is free of constant rank at least m on the punctured spectrum.

We now define R_i-modules F_i inductively. For $i = 2, \ldots, d$, let $F_i = F^{\#}_{i-1} (= \operatorname{syz}^1_{R_i}(F_{i-1}))$. Applying Proposition 15.4.4 and Lemma 15.4.1 inductively, we see that F_i has no non-zero free direct summand and that

$$F_i \cong \operatorname{redsyz}^i_{R_i}(T) \text{ for } i = 1, \ldots, d. \tag{15.6}$$

Therefore F_i is free of constant rank on the punctured spectrum of R_i. To estimate the size of this rank, we look at the short exact sequence $0 \to F_i \to R^{(b_{i-1})} \to F_{i-1} \to 0$, where $b_{i-1} = \mu_{R_{i-1}}(F_{i-1})$. By localizing at a prime ideal P not containing z_i, we learn that the rank of F_i is exactly b_{i-1}. Since a module that is free of rank r on the punctured spectrum obviously needs at least r generators, we have inequalities $b_{d-1} \geq \cdots \geq b_1 \geq m$.

Next, we let $G_1 = \operatorname{syz}^1_{R_1}(F_1)$. By Proposition 15.4.4 G_1 has no non-zero free direct summand, and $\mu_{R_1}(G_1) = \mu_{R_1}(F_1) = b_1$ by Lemma 15.6.3. For $i = 2, \ldots, d$ we define $G_i = G^{\#}_{i-1}$ $(= \operatorname{syz}^1_{R_i}(G_{i-1}))$. By Proposition 3.4, G_i has no non-zero free summands, that is, $G_i = \operatorname{redsyz}^1_{R_i}(G_{i-1})$. Using Lemma 15.4.1, we see that

$$G_i = \operatorname{redsyz}^1_{R_i}(F_i), \text{ for } i = 1, \ldots, d. \tag{15.7}$$

The argument in the last paragraph shows that the rank of G_i (on the punctured spectrum of R_i) is at least m, for $i = 2, \ldots, d$. (Fortunately, we don't care about the rank of G_1.)

Recall that $R = R_d$. Suppose first that d is odd (possibly $d = 1$). We put $F := F_d$ and $B := \operatorname{End}_R(F)$. Since d is odd, we have, by periodicity (Lemma 15.6.3),

$$F \cong \operatorname{syz}^d_R(G_d) \cong \operatorname{redsyz}^d_R(G_d). \tag{15.8}$$

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Applying Lemma 15.6.2 to 15.6 and 15.8, we obtain isomorphisms of right B-modules

$$\mathrm{Ext}^1_R(F, T) \cong \mathrm{Ext}^{d+1}_R(F, F) \cong \mathrm{Ext}^1_R(F, G_d).$$

By 15.7, there is a short exact sequence

$$0 \to G_d \to R^{(b)} \xrightarrow{\varphi} F \to 0,$$

where $b = b_d = \mu_R(F)$. Applying $\mathrm{Hom}_R(F, \_\_)$, we get an exact sequence of B-modules

$$\mathrm{Hom}_R(F, R^{(b)}) \xrightarrow{\varphi_*} B \xrightarrow{\delta} \mathrm{Ext}^1_R(F, G_d).$$

Combining this with (5.4.5), we obtain an exact sequence of right B-modules

$$\mathrm{Hom}_R(F, R^{(b)}) \xrightarrow{\varphi_*} B \xrightarrow{\beta} \mathrm{Ext}^1_R(F, T).$$

If $f : F \to F$ is in the image of φ_*, then $f(F) \subseteq \mathfrak{m}F$, as F has no non-zero free summands. By Lemma 15.3.4, $\mathrm{Ker}(\beta) \subseteq \mathrm{J}(B)$, and now Theorem 15.3.3 provides the desired exact sequence 15.5.

If d is even, then $\mathrm{syz}^{d+1}_R(F_d) \cong G_d$ by periodicity (Lemma 15.6.3). But $G_d \cong \mathrm{redsyz}^{d+1}_{R_d}(T)$ by Lemma 15.4.1. Two applications of Lemma 15.6.2 now show that $\mathrm{Ext}^1_R(G_d, F_d) \cong \mathrm{Ext}^1_R(G_d, T)$ as right $\mathrm{End}_R(G_d)$-modules. Therefore, when we apply $\mathrm{Hom}_R(G_d, \_\_)$ to the short exact sequence

$$0 \to F_d \to R^{(t)} \to \psi \to G_d \to 0,$$

we obtain an exact sequence

$$\mathrm{Hom}_R(G_d, R^{(t)}) \to \psi_* \to \mathrm{End}_R(G_d) \to \beta \to \mathrm{Ext}^1_R(G_d, T)$$

of right $\mathrm{End}_R(G_d)$-modules. We put $F = G_d$ and proceed exactly as in the case where d is odd. $\square$

We conclude with the following result, a reformulation of the main results of [17]:

Corollary 15.6.5 *Let $(R, \mathfrak{m}, k)$ be a complete hypersurface singularity containing a field of characteristic different from 2. Then R has bounded but infinite Cohen-Macaulay type if and only if R is isomorphic to a ring of the form (†), where g is either x_1^2 or $x_0x_1^2$.*

Proof By [1, Proposition 4.2] (cf. also [17]), $k[[x_0, x_1]]/(x_1^2)$ and $k[[x_0, x_1]]/(x_0x_1^2)$ have bounded but infinite Cohen-Macaulay type. The "if" direction now follows from Lemma 15.6.1 and Corollary 15.4.5.

For the converse, suppose R has bounded but infinite Cohen-Macaulay type. Using Lemma 15.6.1, we can put R into the form (†). By Corollary 15.4.5, the ring $A := k[[x_0, x_1]]/(g)$ has bounded but infinite Cohen-Macaulay type. The arguments in [17] show that after a change of variables in $k[[x_0, x_1]]$ we have either $g = ux_1^2$ or $g = ux_0x_1^2$ for some unit $u \in k[[x_0, x_1]]$. Now multiply the defining equation for R by u^{-1}, and replace v_1 by $u^{-1}v_1$, to get the desired form. $\square$

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Chapter 16

Every Endomorphism of a Local Warfield Module is the Sum of Two Automorphisms

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Abstract In this paper, we prove the title statement, except of course for $p = 2$ (where it is false); all modules here are over the ring of integers localized at a prime p. The same result was proved by R. Göbel and A. Opdenhövel for modules having finite torsion-free rank.

Subject classifications: 20K21, 20K30.

Keywords: local Warfield module, endomorphism, Axiom 3, knice submodule, primitive element, $*$-valuated coproduct.

16.1 Introduction

Our notation and terminology are in agreement with [1] and [3]. As in [1], all modules considered are over $\mathbb{Z}_{(p)}$, the ring of integers localized at a prime $p \neq 2$. We shall rely heavily on [3] for the basic properties of knice submodules, primitive elements and $*$-valuated coproducts; facts established in [3] about these concepts are used freely without further reference.

In the interest of brevity, we refer to the well-written and informative paper [1] for the historical development of the problem considered here pertaining to endomorphisms being the sum of two automorphisms and, in addition, for a discussion of related problems. It should be mentioned that the main result achieved by R. Göbel and A. Opdenhövel in [1] is remarkable inasmuch as, even though Warfield modules were studied intensely over the intervening time period, it took approximately thirty years to extend the result (stated in the title) from torsion modules in [2] to mixed modules of finite torsion-free rank. The purpose of this note is to remove the restriction on rank. Therefore, we

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intend to prove the following.

Main Theorem Every endomorphism of a p-local Warfield module, with $p \neq 2$, is the sum of two automorphisms.

Our proof of the Main Theorem requires the Axiom 3 characterization of a local Warfield module established in [3]. Recall that a collection of submodules $\mathcal{C}$ of a $\mathbb{Z}_{(p)}$-module G is an Axiom 3 system if the following three conditions are satisfied.

(H1) $\mathcal{C}$ contains the trivial submodule 0.

(H2) If $\{A_i\}_{i\in I} \subseteq \mathcal{C}$, then $\sum_{i\in I} A_i \in \mathcal{C}$.

(H3) If $A \in \mathcal{C}$ and if B is any countable submodule of G, there is an $A' \in \mathcal{C}$ such that $A + B \subseteq A'$ and A'/A is countable.

Theorem 16.1.1 ([3]) *A module over $\mathbb{Z}_{(p)}$ is a Warfield module if and only if it has an Axiom 3 system of knice submodules.*

16.2 The Key Lemma

Using Theorem 16.1.1 and the familiar infinite combinatorics associated with Axiom 3 (see, for example, the proof of Theorem 3.4 in [3]), we shall quickly conclude the Main Theorem from the following lemma. Indeed, the mere *statement* of the lemma can be viewed as a substantial part of the proof of the Main Theorem since this statement essentially contains the detailed strategy for its proof. The overall strategy, of course, is to build bridges from one member A of an Axiom 3 system $\mathcal{C}$ of knice submodules of G to a larger member A' of $\mathcal{C}$ until we reach G, itself. In order to do this, we choose the A's in $\mathcal{C}$ judiciously so that they are φ-invariant submodules on which the endomorphism φ is the sum of two automorphisms.

Lemma 16.2.1 *Let G be a $\mathbb{Z}_{(p)}$-module with $p \neq 2$ and let A be a knice submodule of G. Suppose that φ is an endomorphism of G that maps A into itself and that π and χ are height-preserving automorphisms of A for which $\varphi = \pi + \chi$ on A. Further suppose that π and χ have been extended to height-preserving isomorphisms $B \to B'$ and $B \to B''$, respectively, so that their sum agrees with the mapping φ from B into G. Assume that B, B' and B'', respectively, are finite extensions in G of $*$-valuated coproducts of the form*

$$B_0 = A \oplus \langle x_1\rangle \oplus \langle x_2\rangle \oplus \cdots \oplus \langle x_k\rangle,$$

$$B_0' = A \oplus \langle \pi(x_1)\rangle \oplus \langle \pi(x_2)\rangle \oplus \cdots \oplus \langle \pi(x_k)\rangle,$$

$$B_0'' = A \oplus \langle \chi(x_1)\rangle \oplus \langle \chi(x_2)\rangle \oplus \cdots \oplus \langle \chi(x_k)\rangle,$$

where the x_i's, $\pi(x_i)$'s and $\chi(x_i)$'s are all primitive elements of G; naturally, we allow $k = 0$, that is, we allow for the set of x_i's to be vacuous. Then, if F is any finitely generated submodule of G, there exists a set (possibly vacuous) of primitive elements x_i, y_i and z_i, where $k+1 \leq i \leq k+m$, for which

$$C_0 = B_0 \oplus \langle x_{k+1}\rangle \oplus \langle x_{k+2}\rangle \oplus \cdots \oplus \langle x_{k+m}\rangle,$$

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$$C_0' = B_0' \oplus \langle y_{k+1} \rangle \oplus \langle y_{k+2} \rangle \oplus \cdots \oplus \langle y_{k+m} \rangle,$$

$$C_0'' = B_0'' \oplus \langle z_{k+1} \rangle \oplus \langle z_{k+2} \rangle \oplus \cdots \oplus \langle z_{k+m} \rangle$$

are $*$-valuated coproducts. Moreover, there exist such submodules C_0, C_0' and C_0'' that have finite extensions C, C' and C'', respectively, all of which contain F and such that π and χ can be extended to height-preserving isomorphisms $C \to C'$ and $C \to C''$, where the extended π and χ satisfy: (1) $\pi(x_i) = y_i$ and $\chi(x_i) = z_i$ for $k+1 \le i \le k+m$, and (2) $\pi + \chi = \varphi$ (as a mapping of C into G).

Proof At the outset, we note that from Lemma 3.5 in [3] we can conclude that the relative Ulm and Warfield invariants of G with respect to B are the same of those with respect to B' and B''.

Clearly, it suffices to prove the lemma for the case where $F = \langle x \rangle$ is cyclic, for the general result then follows by induction on the number of generators for F.

First, we prove that π and χ, respectively, can be extended in the desired way to height-preserving isomorphisms $C \to C'$ and $C \to C''$, where $x \in C$. After this is accomplished, we then show that we can also capture x in C' and C''. For clarity, we refer to these respective projects as the *Domain Extension* and the *Image Extension*. Note that the Domain Extension and the Image Extension are not completely symmetrical because φ is only an endomorphism, not an automorphism, and hence not reversible.

Domain Extension. There is no loss of generality in assuming that the coset $x + B$ has infinite order or finite order p. We distinguish the two cases.

Case 1: $x + B$ has order p. The proof for this case is similar to an argument given in [2], where Main Theorem was proved when G is torsion. (As is well known, a torsion local Warfield module is a totally projective p-group.) The proof below is a modified and somewhat abbreviated version of the argument found in [2] that basically applies to the mixed case, as long as the coset has finite order. First, however, some general observations are required. Since A is knice in G by hypothesis and since the x_i's are primitive, the $*$-valuated coproduct B_0 must be a knice submodule of G. Hence, B itself is knice in G since it is a finite extension of B_0. Likewise, the submodules B' and B'' are knice in G. Since B is knice, there is no loss of generality in assuming that x is proper with respect to B. Set $|x| = \alpha$ where, as usual, $|x|$ denotes the height of x in G.

We need to find an element $y \in G$ that satisfies conditions (1)–(8) below. The existence of a y satisfying these conditions will immediately enable us to extend π and χ, respectively, to height-preserving isomorphisms

$$\pi : \langle B, x \rangle \to \langle B', y \rangle$$
$$\chi : \langle B, x \rangle \to \langle B'', \varphi(x) - y \rangle$$

with $\pi(x) = y$ and $\chi(x) = \varphi(x) - y$, so the property $\pi + \chi = \varphi$ on $\langle B, x \rangle$ is retained.

(1) $|y| = \alpha$.

(2) $y + B'$ has order p.

(3) $py = \pi(px)$.

(4) y is proper with respect to B'.

(5) $|\varphi(x) - y| = \alpha$.

(6) $\varphi(x) - y + B''$ has order p.

(7) $\varphi(px) - py = \chi(px)$.

(8) $\varphi(x) - y$ is proper with respect to B''.

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To show the existence of such a y, we consider two subcases.

Case 1.1: $|px| \neq \alpha + 1$ (which, by convention, excludes $\alpha = \infty$). In this case, there exists an element in $G[p]$ of height α which is proper with respect to B. Since the relative Ulm invariants of G with respect to B are the same as those with respect to B', there exists an element $s \in G[p]$ of height α which is proper with respect to B'. Likewise, there exists $t \in G[p]$ of height α which is proper with respect to B''. At least one of s, t and $s + t$ must be proper with respect to *both* B' and B'', so we may assume that s already enjoys this property. Now choose $w \in p^{\alpha+1}G$ so that $\pi(px) = pw$ and set $y = w + s$. It is easily verified that y satisfies conditions (1)–(4). Since the elements x, $\varphi(x)$ and y all have height greater than or equal to α, conditions (5)–(8) will be satisfied if we can show that condition

(9) $|\varphi(x) - y + b''| \leq \alpha$

holds for each $b'' \in B''$. In this connection, we note that (7) follows from (3) and the fact that $\varphi = \pi + \chi$ on B. If (9) should fail for the original choice of y given above, we need only change the definition of y to $y = w + 2s$ in which case (9) is satisfied because s is proper with respect to B''. In other words, condition (9) cannot fail for both choices of y, either one of which is satisfactory.

Since conditions (1)–(8) are assumed to hold now for y, there exist height-preserving isomorphic extensions $\langle B, x\rangle \to \langle B', y\rangle$ and $\langle B, x\rangle \to \langle B'', \varphi(x) - y\rangle$ of π and χ, respectively, for which $\pi(x) = y$ and $\chi(x) = \varphi(x) - y$.

Case 1.2: $|px| = \alpha + 1$ (which includes $\alpha = \infty$). If $|p(x + b)| \neq \alpha + 1$ for some $b \in p^\alpha G \cap B$, we can replace x by $x + b$ and return to Case 1.1. Thus, we may assume that $|p(x + b)| = \alpha + 1$ for each $b \in p^\alpha G \cap B$. In this situation, we need only choose $y \in p^\alpha G$ so that $\pi(px) = py$. If we extend π and χ by letting $\pi(x) = y$ and $\chi(x) = \varphi(x) - y$, it is routine to verify that π and χ remain height-preserving isomorphisms.

We have shown in Case 1 (comprised of Case 1.1 and Case 1.2) that we can extend π and χ so that the designated element x is contained in their domains.

Case 2: $x + B$ has infinite order. In view of the previously established Case 1, it suffices to prove that we can extend π and χ, as in Case 1, so that some nonzero multiple of x is contained in C, as opposed to x itself. We can then reach x by an application of Case 1.

Since B is a knice submodule of G, we know that there exist primitive elements $g_1, g_2, \ldots, g_r$ and a $*$-valuated coproduct

$$B \oplus \langle g_1\rangle \oplus \langle g_2\rangle \oplus \cdots \oplus \langle g_r\rangle$$

that contains $p^n x$ for some integer $n \geq 0$. Indeed, it is enough to have some nonzero multiple of each g_i contained in C. (At this point we warn the reader that our replacing elements with suitable multiples may become somewhat monotonous in the argument to follow.) Since the induction hypothesis survives, we need only show that we can capture g_1 in C. Hence, there is no loss of generality in replacing x by g_1, thereby assuming from the outset that x is primitive with $B \oplus \langle x\rangle$ a $*$-valuated coproduct.

Let the height sequence of x be denoted by $\|x\| = \bar{\alpha}$. Since the relative Warfield invariants of G with respect to B are the same as those with respect to B' and B'', there exist primitive elements y and z in G such that $B' \oplus \langle y\rangle$ and $B'' \oplus \langle z\rangle$ are $*$-valuated coproducts and such that, for some $e \geq 0$, $p^e x$, $p^e y$ and $p^e z$ all have the same height sequence $p^e\bar{\alpha}$. Thus, by replacing the original x, y and z by appropriate multiples, we may assume that x, y and z all have the same height sequence, namely $\bar{\alpha}$.

It is important to be able to choose $y = z$ in the preceding discussion. But since $y = (y + z) - z$ where $y + z \in G(\bar{\alpha})$, Proposition 2.8 in [3] implies that there is a multiple w of $y + z$ or a multiple w of z such that $B' \oplus \langle w\rangle$ is a $*$-valuated coproduct with w primitive – otherwise $B' \oplus \langle y\rangle$ being a $*$-valuated coproduct would be contradicted. Likewise, since $z = (y + z) - y$, there is a primitive multiple w of $y+z$ or a multiple w of y such that $B'' \oplus \langle w\rangle$ is a $*$-valuated coproduct. Consequently,

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after replacement by some appropriate multiple of y, z, or $y+z$, we can choose y and z so that $y = z$. Thus, replacing x by the corresponding multiple, we may assume that $B \oplus \langle x\rangle$, $B' \oplus \langle y\rangle$ and $B'' \oplus \langle y\rangle$ are each $*$-valuated coproducts where x and y are primitive with the same height sequence $\bar{\alpha}$. What we actually ultimately desire, however, is that $\varphi(x) - y$ is primitive and that both $B' \oplus \langle y\rangle$ and $B'' \oplus \langle \varphi(x) - y\rangle$ are $*$-valuated coproducts. If this condition cannot be achieved by replacing y and $\varphi(x) - y$ by appropriate multiples, then as above the equation $y = (\varphi(x) - y) - (\varphi(x) - 2y)$ would imply that $\varphi(x) - 2y$ is primitive, and, after replacement by appropriate multiples, that $B'' \oplus \langle \varphi(x) - 2y\rangle$ is a $*$-valuated coproduct. Now, without changing x, replace y by $2y$ and observe that the defect has been removed; that is, both $B' \oplus \langle y\rangle$ and $B'' \oplus \langle \varphi(x) - y\rangle$ are $*$-valuated coproducts, and both y and $\varphi(x) - y$ are primitive with height sequence $\bar{\alpha}$. We can now extend π and χ in the desired way by letting $\pi(x) = y$ and $\chi(x) = \varphi(x) - y$. This completes the proof of the Domain Extension.

Image Extension. We want to show that if $y \in F$, then π and χ can be extended to height-preserving isomorphisms $\pi : C \to C'$ and $\chi : C \to C''$ so that y is contained in both C' and C'', and so that $\pi + \chi = \varphi$ as a mapping from C into G. Since π and χ are symmetrical in the hypotheses, it suffices to demonstrate that we can capture y in C'. As before, the argument reduces to the case where the coset $y + B'$ has order p and to the case where it has infinite order.

Case 1: $y + B'$ has order p. Since B' is knice, we may assume that y is proper with respect to B'. To accomplish this, we may go outside of F, but we remain inside of C' so no harm is done. Let $|y| = \alpha$ and consider the two usual subcases.

Case 1.1: $|py| \neq \alpha + 1$. Choose $w \in p^{\alpha+1}G$ so that $pw \in B$ and $\pi(pw) = py$. Since the relative Ulm invariants of G with respect to B are the same as those with respect to B', there exists an element $s \in G[p]$ having height α and is proper with respect to B. Define $x = w + s$. An alternate choice for x is $x = w + 2s$. In either case, the extension of π defined by setting $\pi(x) = y$ is a height-preserving isomorphism. Moreover, if y is also proper with respect to B'' (as well as B'), it is straightforward to show that $\varphi(x) - y$ is proper with respect to B'' for at least one of the preceding choices for x. In this case we obtain the desired extensions by letting $\pi(x) = y$ and $\chi(x) = \varphi(x) - y$. Therefore, the argument for Case 1.1 now rests on the assertion that there is no loss of generality in assuming that y is proper with respect to both B' and B''. To prove this assertion, assume that y is not proper with respect to B''; we continue to assume that y is proper with respect to B'. Thus, $|y - b''| \geq \alpha + 1$ for some $b'' \in B''$. Obviously, $|b''| = \alpha$ and $|pb''| \geq \alpha + 2$. Since y is proper with respect to B', so is b''.

As in [3], if H is any submodule of G and α is an ordinal, we define

$$H(\alpha) = p^{\alpha}G[p] \cap (H + p^{\alpha+1}G).$$

Since $\chi\pi^{-1} : B' \to B''$ is a height-preserving isomorphism from B' onto B'' that maps A to itself, there is an induced isomorphism from $B'(\alpha)/A(\alpha)$ onto $B''(\alpha)/A(\alpha)$. Moreover, $B'(\alpha)/A(\alpha)$ and $B''(\alpha)/A(\alpha)$ are finite since B'/A and B''/A are finitely generated (see the proof of Lemma 3.5 in [3]). Therefore, $B'(\alpha) \subseteq B''(\alpha)$ implies equality. If g is any element in $p^{\alpha+1}G$ with $pg = pb''$, then clearly $b'' - g \in B''(\alpha)$. But $b'' - g \notin B'(\alpha)$ because b'' is proper with respect to B'. Hence, $B'(\alpha) \subseteq B''(\alpha)$ cannot hold. This leads to the existence of an element $b' \in B'$ that has height α with $|pb'| \geq \alpha + 2$ that is proper with respect to B''. Now, if we replace y by $y + b'$ we have the desired element y that is proper with respect to both B' and B'', which completes the proof of Case 1.1.

Case 1.2: $|py| = \alpha + 1$. If $|p(y + b')| \neq \alpha + 1$ for some $b' \in p^{\alpha}G \cap B'$, we replace y by $y + b'$ and return to Case 1.1. Thus, assume that $|p(y + b')| = \alpha + 1$ for all $b' \in p^{\alpha}G \cap B'$. In this case, π and χ continue to be height-preserving isomorphisms whose sum is φ if we define extensions

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by $\pi(x) = y$ and $\chi(x) = \varphi(x) - y$, where x is any element in $p^\alpha G$ that satisfies $\pi(px) = py$. Obviously, y is in the image of the extended π, so this case is proven.

Case 2: $y + B'$ has infinite order. From the discussion of Case 2 in the Domain Extension, it is enough to consider the case where y is primitive and $B' \oplus \langle y \rangle$ is a $*$-valuated coproduct. Let $\bar{\alpha}$ denote the height sequence of y. If $B'' \oplus \langle y \rangle$ is also a $*$-valuated coproduct, we have shown in Case 2 of the Domain Extension that the desired extensions of π and χ exist with $\pi(x) = y$. We complete the proof of this case by showing that there exists $b' \in B'$ with the property that $y + b'$ is primitive with height sequence $\bar{\alpha}$ and both $B' \oplus \langle y + b' \rangle$ and $B'' \oplus \langle y + b' \rangle$ are $*$-valuated coproducts. If we can show this, all we have to do is to replace y by $y + b'$, and the proof is finished by the remarks just made.

If replacing y by nonzero multiples of itself does not make $B'' \oplus \langle y \rangle$ a $*$-valuated coproduct, then we may assume that there exists $b'' \in B''$ for which $b'' - y \in G(\bar{\alpha}^*)$. As in [3], for any submodule H of G and any height sequence $\bar{\alpha}$, we define

$$H_{\bar{\alpha}} = (H + G(\bar{\alpha}^*)) \cap G(\bar{\alpha}) = (H \cap G(\bar{\alpha})) + G(\bar{\alpha}^*).$$

Since B'/A and B''/A are finitely generated, $B'_{\bar{\alpha}}/A_{\bar{\alpha}}$ and $B''_{\bar{\alpha}}/A_{\bar{\alpha}}$ are finite. Moreover, the height-preserving isomorphism $\chi\pi^{-1} : B' \to B''$ maps A to itself, so there is an induced isomorphism from $B'_{\bar{\alpha}}/A_{\bar{\alpha}}$ onto $B''_{\bar{\alpha}}/A_{\bar{\alpha}}$. Therefore $B'_{\bar{\alpha}} \subseteq B''_{\bar{\alpha}}$ implies equality. Obviously b'' is in $B''_{\bar{\alpha}}$, but it follows from $B' \oplus \langle y \rangle$ being a $*$-valuated coproduct that b'' is not contained in $B'_{\bar{\alpha}}$. Hence $B'_{\bar{\alpha}} \subseteq B''_{\bar{\alpha}}$ cannot hold. This leads to the existence of an element $b' \in B' \cap G(\bar{\alpha})$ that is not contained in $B''_{\bar{\alpha}}$. By passing to multiples if necessary, we conclude that $y + b'$ is primitive with height sequence $\bar{\alpha}$ and $B' \oplus \langle y + b' \rangle$ and $B'' \oplus \langle y + b' \rangle$ are both $*$-valuated coproducts. $\square$

16.3 Proof of the Main Theorem

Assume that G is a p-local Warfield module (with $p \neq 2$) and that φ is an endomorphism of G. We need to show that φ is the sum of two automorphisms of G.

Using Theorem 16.1.1, select an Axiom 3 system $\mathcal{C}$ of knice submodules of G that satisfies conditions (H1), (H2) and (H3). Let $\mathcal{E}$ be the set of all pairs (π, A) such that $A \in \mathcal{C}$ and π and $\varphi - \pi$ are automorphisms of A that preserve heights in G. Note that $\mathcal{E}$ is not empty since $0 \in \mathcal{C}$ by condition (H1). The set $\mathcal{E}$ can be partially ordered as expected: if (π_1, A_1) and (π_2, A_2) are elements of $\mathcal{E}$, then the first is less than or equal to the second if $A_1 \subseteq A_2$ and π_2 extends π_1. Zorn's Lemma is applicable because condition (H2) implies that $\mathcal{C}$ is closed under unions of ascending chains. Thus, there is a maximal element $(\pi, A) \in \mathcal{E}$ and we set $\chi = \varphi - \pi$.

Suppose that $A \neq G$ and select $x \in G \setminus A$. By Lemma 16.2.1, there exist height-preserving isomorphisms $C_1 \to C'_1$ and $C_1 \to C''_1$ that are finite extensions of π and χ, respectively, where the sum is φ on C_1 and $x \in C_1$. Repeated applications of the Lemma yield ascending sequences of such finite extensions $C_n \to C'_n$ and $C_n \to C''_n$ with the property that

$$C_n \cup C'_n \cup C''_n \subseteq C_{n+1} \cap C'_{n+1} \cap C''_{n+1}$$

for all $n < \omega$. In particular, $\bigcup_{n<\omega} C_n = \bigcup_{n<\omega} C'_n = \bigcup_{n<\omega} C''_n$. Moreover, using condition (H3) on the collection $\mathcal{C}$, we can force $\bigcup_{n<\omega} C_n \in \mathcal{C}$. But this gives us an element of $\mathcal{E}$ larger than our maximal element, so we conclude that $A = G$. This completes the proof of the Main Theorem.

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Chapter 17

Wakamatsu Tilting Modules, U-Dominant Dimension, and k-Gorenstein Modules

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Abstract Let Λ and Γ be left and right noetherian rings and ${}_\Lambda U$ a Wakamatsu tilting module with $\Gamma = \mathrm{End}({}_\Lambda T)$. We introduce a new definition of U-dominant dimensions and show that the U-dominant dimensions of ${}_\Lambda U$ and U_Γ are identical. We characterize k-Gorenstein modules in terms of homological dimensions and the property of double homological functors preserving monomorphisms. We also study a generalization of k-Gorenstein modules, and characterize it in terms of some similar properties of k-Gorenstein modules.

Subject classifications: 16E10, 16E30, 16D90.

Keywords: U-dominant dimension, k-Gorenstein modules, Wakamatsu tilting modules, flat dimension.

17.1 Introduction and Main Results

Let Λ be a ring. We use Mod Λ (resp. Mod Λ^{op}) to denote the category of left (resp. right) Λ-modules, and use mod Λ (resp. mod Λ^{op}) to denote the category of finitely generated left Λ-modules (resp. right Λ-modules).

Definition 17.1.1 [7] For a module M in mod Λ (resp. mod Λ^{op}) and a positive integer k, M is said to have *dominant dimension* at least k, written as $\mathrm{dom.dim}({}_\Lambda M)$ (resp. $\mathrm{dom.dim}(M_\Lambda)) \geq k$, if each of the first k terms in a minimal injective resolution of M is Λ-flat (resp. Λ^{op}-flat).

For a module T in Mod Λ (resp. Mod Λ^{op}), we use add-$\lim_\Lambda T$ (resp. add-$\lim T_\Lambda$) to denote the subcategory of Mod Λ (resp. Mod Λ^{op}) consisting of all modules isomorphic to direct summands of a direct limit of a family of modules in which each is a finite direct sum of copies of ${}_\Lambda T$ (resp. T_Λ). We now introduce a definition of *U-dominant dimension* as follows.

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Definition 17.1.2 Let U be in Mod Λ (resp. Mod Λ^{op}) and k a positive integer. For a module M in Mod Λ (resp. Mod Λ^{op}), M is said to have U-dominant dimension at least k, written as U-dom.dim$({}_{\Lambda}M)$ (resp. U-dom.dim(M_{Λ})) $\geq k$, if each of the first k terms in a minimal injective resolution of M can be embedded into a direct limit of a family of modules in which each is a finite direct sum of copies of ${}_{\Lambda}U$ (resp. U_{Λ}), that is, each of these terms is in add-lim${}_{\Lambda}U$ (resp. add-limU_{Λ}).

Remark 17.1.3 Notice that a module (not necessarily finitely generated) is flat if and only if it is a direct limit of a family of finitely generated free modules (see [15]). So, if putting ${}_{\Lambda}U = {}_{\Lambda}\Lambda$ (resp. $U_{\Lambda} = \Lambda_{\Lambda}$), then the above definition of U-dominant dimension coincides with that of the usual dominant dimension for any ring Λ.

Tachikawa in [19] showed that if Λ is a left and right artinian ring then the dominant dimensions of ${}_{\Lambda}\Lambda$ and Λ_{Λ} are identical. Hoshino in [7] further showed that this result also holds for left and right noetherian rings. Colby and Fuller in [5] gave some equivalent conditions of dom.dim$({}_{\Lambda}\Lambda) \geq$ 1 (or 2) in terms of the properties of double dual functors (with respect to ${}_{\Lambda}\Lambda_{\Lambda}$). These results motivate our interests in establishing the identity of U-dominant dimensions of ${}_{\Lambda}U$ and U_{Γ} (where $\Gamma = \mathrm{End}({}_{\Lambda}U)$) and characterizing the properties of modules with a given U-dominant dimension.

Let T be a module in mod Λ. For a module $A \in$ mod Λ and a non-negative integer n, we say that the grade of A with respect to ${}_{\Lambda}T$, written as grade$_T A$, is at least n if $\mathrm{Ext}^i_{\Lambda}(A, T) = 0$ for any $0 \leq i < n$. We say that the strong grade of A with respect to ${}_{\Lambda}T$, written as s.grade$_T A$, is at least n if grade$_T B \geq n$ for all submodules B of A. The notion of the (strong) grade of modules with respect to a given module in mod Λ^{op} is defined dually.

The following is one of the main results in this paper.

Theorem 17.1.4 *Let Λ and Γ be left and right noetherian rings and ${}_{\Lambda}U$ a Wakamatsu tilting module with $\Gamma = \mathrm{End}({}_{\Lambda}U)$. For a positive integer k, the following statements are equivalent.*

(1) U-dom.dim$({}_{\Lambda}U) \geq k$.

(2) s.grade$_U$Ext$^1_{\Lambda}(M, U) \geq k$ for any $M \in$mod Λ.

(3) Hom$_{\Lambda}(U, E_i)$ is Γ-flat, where E_i is the $(i+1)$-st term in a minimal injective resolution of ${}_{\Lambda}U$, for any $0 \leq i \leq k-1$.

(1)^{op} U-dom.dim$(U_{\Gamma}) \geq k$.

(2)^{op} s.grade$_U$Ext$^1_{\Gamma}(N, U) \geq k$ for any $N \in$mod Γ^{op}.

(3)^{op} Hom$_{\Gamma}(U, E'_i)$ is Λ^{op}-flat, where E'_i is the $(i+1)$-st term in a minimal injective resolution of U_{Γ}, for any $0 \leq i \leq k-1$.

Kato in [14] gave a definition of U-dominant dimension as follows, which is different from that of 17.1.2. For a module M in mod Λ (resp. mod Λ^{op}), M is said to have U-dominant dimension at least k, written as U-dom.dim$({}_{\Lambda}M)$ (resp. U-dom.dim(M_{Λ})) $\geq k$, if each of the first k terms in a minimal injective resolution of M is cogenerated by ${}_{\Lambda}U$ (resp. U_{Λ}), that is, each of these terms can be embedded into a direct product of copies of ${}_{\Lambda}U$ (resp. U_{Λ}). If we adopt the definition of U-dominant dimension given by Kato, then in Theorem 17.1.4 the equivalence of (2), (3), (2)^{op} and (3)^{op} and that (1) implies (3) also hold. However, that (3) does not imply (1) in general. For example, consider Wakamatsu tilting module ${}_{\mathbb{Z}}\mathbb{Z}$ and its injective envelope ${}_{\mathbb{Z}}\mathbb{Q}$, where $\mathbb{Z}$ is the ring of integers and $\mathbb{Q}$ is the field of rational numbers. Then the module ${}_{\mathbb{Z}}\mathbb{Q}$ is flat, but it cannot be embedded into any direct product of copies of ${}_{\mathbb{Z}}\mathbb{Z}$ since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$.

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Corollary 17.1.5 *Let Λ and Γ be left and right noetherian rings and ${}_\Lambda U$ a Wakamatsu tilting module with $\Gamma = \mathrm{End}({}_\Lambda T)$. Then U-dom.dim$({}_\Lambda U) = U$-dom.dim(U_Γ).*

Remark 17.1.6 We do not know whether the conclusion in Corollary 16.1.4 holds for Kato's U-dominant dimension. The answer is positive when Λ and Γ are artinian algebras (see [11, Theorem 1.3]).

Putting ${}_\Lambda U_\Gamma = {}_\Lambda \Lambda_\Lambda$, we immediately get the following result, which is due to Hoshino (see [7, Theorem]).

Corollary 17.1.7 *For a left and right noetherian ring Λ, dom.dim$({}_\Lambda\Lambda)$ =dom.dim(Λ_Λ).*

Definition 17.1.8 [12] For a non-negative integer k, a module $U \in \mathrm{mod}\ \Lambda$ with $\Gamma = \mathrm{End}({}_\Lambda U)$ is called *k-Gorenstein* if $\mathrm{s.grade}_U \mathrm{Ext}^i_\Gamma(N, U) \geq i$ for any $N \in \mathrm{mod}\ \Gamma^{op}$ and $1 \leq i \leq k$. Dually, we may define the notion of k-Gorenstein modules in mod Γ^{op}.

We introduce a new homological dimension of modules as follows.

Definition 17.1.9 Let Λ be a ring and T in Mod Λ. For a module A in Mod Λ, if there exists an exact sequence $\cdots \to T_n \to \cdots \to T_1 \to T_0 \to A \to 0$ in Mod Λ with each $T_i \in$add-$\lim_\Lambda T$ for any $i \geq 0$, then we define T-lim.dim$_\Lambda(A)$ =inf$\{n|$ and there exists an exact sequence $0 \to T_n \to \cdots \to T_1 \to T_0 \to A \to 0$ in Mod Λ with each $T_i \in$add-$\lim_\Lambda T$ for any $0 \leq i \leq n\}$. We set T-lim.dim$_\Lambda(A)$ infinity if no such an integer exists. For Λ^{op}-modules, we may define such a dimension dually.

Remark 17.1.10 Putting ${}_\Lambda T = {}_\Lambda\Lambda$ (resp. $T_\Lambda = \Lambda_\Lambda$), the dimension defined as above is just the flat dimension of modules.

In [21], Wakamatsu showed that the notion of k-Gorenstein modules is left-right symmetric. We give here some other characterizations of k-Gorenstein modules. The following is another main result in this paper.

Theorem 17.1.11 *Let Λ and Γ be left and right noetherian rings and ${}_\Lambda U$ a Wakamatsu tilting module with $\Gamma = \mathrm{End}({}_\Lambda T)$. Then, for a positive integer k, the following statements are equivalent.*

(1) *${}_\Lambda U$ is k-Gorenstein.*

(2) *$\mathrm{s.grade}_U \mathrm{Ext}^i_\Lambda(M, U) \geq i$ for any $M \in \mathrm{mod}\ \Lambda$ and $1 \leq i \leq k$.*

(3) *U-lim.dim$_\Lambda(E_i) \leq i$, where E_i is the $(i+1)$-st term in a minimal injective resolution of ${}_\Lambda U$, for any $0 \leq i \leq k-1$.*

(4) *l.fd$_\Gamma(\mathrm{Hom}_\Lambda(U, E_i)) \leq i$ for any $0 \leq i \leq k-1$, where l.fd denotes the left flat dimension.*

(5) *$\mathrm{Ext}^i_\Gamma(\mathrm{Ext}^i_\Lambda(\ , U), U)$ preserves monomorphisms in mod Λ for any $0 \leq i \leq k-1$.*

(1)^{op} *U_Γ is k-Gorenstein.*

(2)^{op} *$\mathrm{s.grade}_U \mathrm{Ext}^i_\Gamma(N, U) \geq i$ for any $N \in \mathrm{mod}\ \Gamma^{op}$ and $1 \leq i \leq k$.*

(3)^{op} *U-lim.dim$_\Gamma(E'_i) \leq i$, where E'_i is the $(i+1)$-st term in a minimal injective resolution of U_Γ, for any $0 \leq i \leq k-1$.*

(4)^{op} *r.fd$_\Lambda(\mathrm{Hom}_\Gamma(U, E'_i)) \leq i$ for any $0 \leq i \leq k-1$, where r.fd denotes the right flat dimension.*

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(5)^{op} $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_\Gamma(\ ,U),U)$ *preserves monomorphisms in mod* Γ^{op} *for any* $0 \leq i \leq k-1$.

Let Λ and Γ be left and right noetherian rings and ${}_\Lambda U$ a Wakamatsu tilting module with $\Gamma = \mathrm{End}({}_\Lambda U)$. By Theorems 17.1.4 and 17.1.11, if U has U-dominant dimension at least k, then it is k-Gorenstein.

Recall that a left and right noetherian ring Λ is called k-Gorenstein if the flat dimension of the i-th term in a minimal injective resolution of ${}_\Lambda\Lambda$ is at most $i-1$ for any $1 \leq i \leq k$. Auslander showed in [6, Theorem 3.7] that the notion of k-Gorenstein rings is left-right symmetric. Following Definition 17.1.8 and [6, Theorem 3.7], a left and right noetherian ring Λ is k-Gorenstein if it is k-Gorenstein as a Λ-module. So, by Theorem 17.1.11, we have the following corollary, which develops this Auslander's result.

Corollary 17.1.12 *Let Λ and Γ be left and right noetherian rings. Then, for a positive integer k, the following statements are equivalent.*

(1) *Λ is k-Gorenstein.*

(2) *$s.grade_\Lambda Ext^i_\Lambda(M,\Lambda) \geq i$ for any $M \in$mod Λ and $1 \leq i \leq k$.*

(3) *The flat dimension of the i-th term in a minimal injective resolution of ${}_\Lambda\Lambda$ is at most $i-1$ for any $1 \leq i \leq k$.*

(4) $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_\Lambda(\ ,\Lambda),\Lambda)$ *preserves monomorphisms in mod Λ for any* $0 \leq i \leq k-1$.

(2)^{op} *$s.grade_\Lambda Ext^i_\Lambda(N,\Lambda) \geq i$ for any $N \in$ mod Λ^{op} and $1 \leq i \leq k$.*

(3)^{op} *The flat dimension of the i-th term in a minimal injective resolution of Λ_Λ is at most $i-1$ for any $1 \leq i \leq k$.*

(4)^{op} $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_\Lambda(\ ,\Lambda),\Lambda)$ *preserves monomorphisms in mod Λ^{op} for any* $0 \leq i \leq k-1$.

The paper is organized as follows. In Section 17.2, we give some properties of Wakamatsu tilting modules. For example, let Λ and Γ be left and right noetherian rings and ${}_\Lambda U$ a Wakamatsu tilting module with $\Gamma = \mathrm{End}({}_\Lambda U)$. If ${}_\Lambda U$ is k-Gorenstein for all k, then the left and right injective dimensions of ${}_\Lambda U_\Gamma$ are identical provided that both of them are finite. We shall prove our main results in Section 17.3. As applications of the results obtained in Section 17.3, we characterize in Section 17.4 U-dominant dimension of U at least one and two in terms of the properties of $\mathrm{Hom}(\mathrm{Hom}(\ ,U),U)$ preserving monomorphisms and being left exact, respectively. Motivated by the work of Auslander and Reiten in [3], we study in Section 17.5 a generalization of k-Gorenstein modules, which is however not left-right symmetric. We characterize this generalization in terms of some properties similar to that of k-Gorenstein modules. At the end of this section, we generalize the result of Wakamatsu on the symmetry of k-Gorenstein modules.

17.2 Wakamatsu Tilting Modules

In this section, we give some properties of Wakamatsu tilting modules with finite homological dimensions.

Definition 17.2.1 Let Λ be a ring. A module ${}_\Lambda U$ in mod Λ is called a *Wakamatsu tilting module* if ${}_\Lambda U$ is self-orthogonal (that is, $\mathrm{Ext}^i_\Lambda({}_\Lambda U, {}_\Lambda U) = 0$ for any $i \geq 1$), and possessing an exact sequence:

$$0 \to {}_\Lambda\Lambda \to U_0 \to U_1 \to \cdots \to U_i \to \cdots$$

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such that: (1) all term U_i are direct summands of finite direct sums of copies of ${}_\Lambda U$, that is, $U_i \in \mathrm{add}_\Lambda U$, and (2) after applying the functor $\mathrm{Hom}_\Lambda(\ , U)$ the sequence is still exact. The definition of Wakamatsu tilting modules in mod Λ^{op} is given dually (see [20] or [21]).

Let Λ and Γ be rings. Recall that a bimodule ${}_\Lambda U_\Gamma$ is called a faithfully balanced self-orthogonal bimodule if it satisfies the following conditions:

(1) ${}_\Lambda U \in \mathrm{mod}\ \Lambda$ and $U_\Gamma \in \mathrm{mod}\ \Gamma^{op}$.

(2) The natural maps $\Lambda \to \mathrm{End}(U_\Gamma)$ and $\Gamma \to \mathrm{End}({}_\Lambda U)^{op}$ are isomorphisms.

(3) $\mathrm{Ext}^i_\Lambda({}_\Lambda U, {}_\Lambda U) = 0$ and $\mathrm{Ext}^i_\Gamma(U_\Gamma, U_\Gamma) = 0$ for any $i \geq 1$.

The following result is [21, Corollary 3.2].

Proposition 17.2.2 *Let Λ be a left noetherian ring and Γ a right noetherian ring. For a bimodule ${}_\Lambda U_\Gamma$, the following statements are equivalent.*

(1) ${}_\Lambda U$ is a Wakamatsu tilting module with $\Gamma = \mathrm{End}({}_\Lambda U)$.

(2) U_Γ is a Wakamatsu tilting module with $\Lambda = \mathrm{End}(U_\Gamma)$.

(3) ${}_\Lambda U_\Gamma$ is a faithfully balanced self-orthogonal bimodule.

In the rest of this paper, we shall freely use the properties of Wakamatsu tilting modules in Proposition 17.2.2 without pointing it out explicitly.

Recall from [16] that a module U in mod Λ is called a tilting module of projective dimension $\leq r$ if it satisfies the following conditions:

(1) The projective dimension of ${}_\Lambda U$ is at most r.

(2) ${}_\Lambda U$ is self-orthogonal.

(3) There exists an exact sequence in mod Λ:

$$0 \to \Lambda \to U_0 \to U_1 \to \cdots \to U_r \to 0$$

such that each $U_i \in \mathrm{add}_\Lambda U$ for any $0 \leq i \leq r$.

By Proposition 17.2.2 and [16, Theorem 1.5], we have the following result.

Corollary 17.2.3 *Let Λ be a left noetherian ring, Γ a right noetherian ring and ${}_\Lambda U$ a Wakamatsu tilting module with $\Gamma = \mathrm{End}({}_\Lambda U)$. If the projective dimensions of ${}_\Lambda U$ and U_Γ are finite, then ${}_\Lambda U_\Gamma$ is a tilting bimodule (that is, both ${}_\Lambda U$ and U_Γ are tilting) with the left and right projective dimensions identical.*

For a module A in Mod Λ (resp. Mod Λ^{op}), we use $l.\mathrm{id}_\Lambda(A)$ (resp. $r.\mathrm{id}_\Lambda(A)$) to denote the left (resp. right) injective dimension of A.

Lemma 17.2.4 *Let Λ and Γ be rings and ${}_\Lambda U_\Gamma$ a bimodule.*

(1) If Γ is a right noetherian ring, then $r.id_\Gamma(U) = sup\{l.fd_\Gamma(\mathrm{Hom}_\Lambda(U, E)) \mid {}_\Lambda E$ is injective$\}$. Moreover, $r.id_\Gamma(U) = l.fd_\Gamma(\mathrm{Hom}_\Lambda(U, Q))$ for any injective cogenerator ${}_\Lambda Q$ for Mod Λ.

(2) If Λ is a left noetherian ring, then $l.id_\Lambda(U) = sup\{r.fd_\Lambda(\mathrm{Hom}_\Gamma(U, E')) \mid E'_\Gamma$ is injective$\}$. Moreover, $l.id_\Lambda(U) = r.fd_\Lambda(\mathrm{Hom}_\Gamma(U, Q'))$ for any injective cogenerator Q'_Γ for Mod Γ^{op}.

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Proof (1) By [4, Chapter VI, Proposition 5.3], for any $i \geq 1$, we have the following isomorphism:

$$\mathrm{Tor}_i^{\Gamma}(B, \mathrm{Hom}_{\Lambda}(U, E)) \cong \mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Gamma}^i(B, U), E) \tag{1}$$

for any $B \in \mathrm{mod}\ \Gamma^{op}$ and ${}_{\Lambda}E$ injective.

If $l.\mathrm{fd}_{\Gamma}(\mathrm{Hom}_{\Lambda}(U, E)) \leq n(< \infty)$ for any injective module ${}_{\Lambda}E$, then the isomorphism (1) induces $\mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Gamma}^{n+1}(B, U), E) \cong \mathrm{Tor}_{n+1}^{\Gamma}(B, \mathrm{Hom}_{\Lambda}(U, E)) = 0$. Now taking ${}_{\Lambda}E$ as an injective cogenerator for mod Λ, we see that $\mathrm{Ext}_{\Gamma}^{n+1}(B, U) = 0$ and $r.\mathrm{id}_{\Gamma}(U) \leq n$.

Conversely, if $r.\mathrm{id}_{\Gamma}(U) = n(< \infty)$, then $\mathrm{Ext}_{\Gamma}^{n+1}(B, U) = 0$ for any $B \in \mathrm{mod}\ \Gamma^{op}$ and $\mathrm{Tor}_{n+1}^{\Gamma}(B, \mathrm{Hom}_{\Lambda}(U, E)) = 0$ for any injective module ${}_{\Lambda}E$ by the isomorphism (1).

Let Y be any module in Mod Γ^{op}. Then $Y = \varinjlim Y_{\alpha}$ (where Y_{α} ranges over all finitely generated submodules of Y). It is well known that the functor Tor_i commutes with $\varinjlim$ for any $i \geq 0$, so $\mathrm{Tor}_{n+1}^{\Gamma}(Y, \mathrm{Hom}_{\Lambda}(U, E)) \cong \varinjlim \mathrm{Tor}_{n+1}^{\Gamma}(Y_{\alpha}, \mathrm{Hom}_{\Lambda}(U, E)) = 0$ by the above argument. This implies that $l.\mathrm{fd}_{\Gamma}(\mathrm{Hom}_{\Lambda}(U, E)) \leq n$. Consequently, we conclude that the first equality holds.

The above argument in fact proves the second equality.

(2) It is similar to the proof of (1). □

Let ${}_{\Lambda}U_{\Gamma}$ be a bimodule. For a module A in Mod Λ (resp. Mod Γ^{op}), we call $\mathrm{Hom}_{\Lambda}({}_{\Lambda}A, {}_{\Lambda}U_{\Gamma})$ (resp. $\mathrm{Hom}_{\Gamma}(A_{\Gamma}, {}_{\Lambda}U_{\Gamma})$) the dual module of A with respect to ${}_{\Lambda}U_{\Gamma}$, and denote either of these modules by A^*. For a homomorphism f between Λ-modules (resp. Γ^{op}-modules), we put $f^* = \mathrm{Hom}(f, {}_{\Lambda}U_{\Gamma})$. We use $\sigma_A : A \to A^{**}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$ to denote the canonical evaluation homomorphism. A is called U-torsionless (resp. U-reflexive) if σ_A is a monomorphism (resp. an isomorphism).

Lemma 17.2.5 *Let Λ be a left noetherian ring, Γ any ring and ${}_{\Lambda}U_{\Gamma}$ a bimodule. If $\Lambda = \mathrm{End}(U_{\Gamma})$, U_{Γ} is self-orthogonal and $r.\mathrm{id}_{\Gamma}(U) \leq n$, then $\bigoplus_{i=0}^{n} V_i$ is an injective cogenerator for Mod Λ, where V_i is the $(i+1)$-st term in an injective resolution of ${}_{\Lambda}U$ for any $0 \leq i \leq n$.*

Proof Let A be any module in mod Λ. Since $r.\mathrm{id}_{\Gamma}(U) \leq n$, $\mathrm{Ext}_{\Gamma}^i(X, U) = 0$ for any $X \in \mathrm{mod}\ \Gamma^{op}$ and $i \geq n+1$. Then, by the assumption and [13, Theorem 2.2], it is easy to see that A is U-reflexive provided that $\mathrm{Ext}_{\Lambda}^i(A, U) = 0$ for any $1 \leq i \leq n$.

Let S be any simple Λ-module. Then $\mathrm{Ext}_{\Lambda}^t(S, U) \neq 0$ for some t with $0 \leq t \leq n$ (Otherwise, S is U-reflexive by the above argument and hence $S \cong S^{**} = 0$, a contradiction.)

Let

$$0 \to {}_{\Lambda}U \to V_0 \to V_1 \to \cdots \to V_i \to \cdots$$

be an injective resolution of ${}_{\Lambda}U$. Set $W_t = \mathrm{Im}(V_{t-1} \to V_t)$. We then get the following exact sequences:

$$\mathrm{Hom}_{\Lambda}(S, W_t) \to \mathrm{Ext}_{\Lambda}^t(S, U) \to 0,$$

$$0 \to \mathrm{Hom}_{\Lambda}(S, W_t) \to \mathrm{Hom}_{\Lambda}(S, V_t).$$

Because $\mathrm{Ext}_{\Lambda}^t(S, U) \neq 0$, $\mathrm{Hom}_{\Lambda}(S, W_t) \neq 0$ and $\mathrm{Hom}_{\Lambda}(S, V_t) \neq 0$. So $\mathrm{Hom}_{\Lambda}(S, \bigoplus_{i=0}^{n} V_i) \neq 0$ and hence $\bigoplus_{i=0}^{n} V_i$ is an injective cogenerator for Mod Λ by [1, Proposition 18.15]. □

As an application to Theorem 17.1.11, we have the following result.

Proposition 17.2.6 *Let Λ and Γ be left and right noetherian rings and ${}_{\Lambda}U$ a Wakamatsu tilting module with $\Gamma = \mathrm{End}({}_{\Lambda}U)$. If ${}_{\Lambda}U$ is k-Gorenstein for all k and both $l.id_{\Lambda}(U)$ and $r.id_{\Gamma}(U)$ are finite, then $l.id_{\Lambda}(U) = r.id_{\Gamma}(U)$.*

Proof Assume that $l.\mathrm{id}_{\Lambda}(U) = m < \infty$ and $r.\mathrm{id}_{\Gamma}(U) = n < \infty$. Since ${}_{\Lambda}U$ is k-Gorenstein for all k, by Theorem 17.1.11, we have that $l.\mathrm{fd}_{\Gamma}(\mathrm{Hom}_{\Lambda}(U, \bigoplus_{i=0}^{m} E_i)) \leq m$, where E_i is the $(i+1)$-st term in a minimal injective resolution of ${}_{\Lambda}U$ for any $i \geq 0$.

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By Proposition 17.2.2, ${}_\Lambda U_\Gamma$ is a faithfully balanced self-orthogonal bimodule. If $m < n$, then, by Lemmas 17.2.5 and 17.2.4, we have that $\bigoplus_{i=0}^{n} E_i$ ($\cong \bigoplus_{i=0}^{m} E_i$) is an injective cogenerator for Mod Λ and $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda(U, \bigoplus_{i=0}^{m} E_i)) = n$, which is a contradiction. So we have that $m \geq n$. According to the symmetry of k-Gorenstein modules, we can prove $n \geq m$ similarly. □

Proposition 17.2.7 *Let Λ be a left and right artinian ring and ${}_\Lambda U$ a Wakamatsu tilting module with $\Lambda = \mathrm{End}({}_\Lambda U)$. If ${}_\Lambda U$ is k-Gorenstein for all k, then $l.id_\Lambda(U) = r.id_\Lambda(U)$.*

Proof By Theorem 17.1.11, for any $i \geq 1$ and $M \in$mod Λ or mod Λ^{op}, we have that s.grade$_U$ $\mathrm{Ext}^i_\Lambda(M, U) \geq i$. By Proposition 17.2.2, ${}_\Lambda U_\Lambda$ is a faithfully balanced self-orthogonal bimodule. It then follows from [9, Theorem] and its dual statement that $l.\mathrm{id}_\Lambda(U)$ is finite if and only if $r.\mathrm{id}_\Lambda(U)$ is finite. Now our conclusion follows from Proposition 17.2.6. □

Putting ${}_\Lambda U = {}_\Lambda\Lambda$, we immediately have the following result, which generalizes [2, Corollary 5.5(b)].

Corollary 17.2.8 *Let Λ be a left and right artinian ring. If Λ is k-Gorenstein for all k, then $l.id_\Lambda(\Lambda) = r.id_\Lambda(\Lambda)$.*

17.3 The Proof of Main Results

In this section, we prove Theorems 17.1.4 and 17.1.11.

From now on, Λ and Γ are left and right noetherian rings and ${}_\Lambda U$ is a Wakamatsu tilting module with $\Gamma = \mathrm{End}({}_\Lambda U)$. We always assume that

$$0 \to {}_\Lambda U \to E_0 \to E_1 \to \cdots \to E_i \to \cdots$$

is a minimal injective resolution of ${}_\Lambda U$, and

$$0 \to U_\Gamma \to E'_0 \to E'_1 \to \cdots \to E'_i \to \cdots$$

is a minimal injective resolution of U_Γ and k is a positive integer.

Lemma 17.3.1 *Let ${}_\Lambda E$ be injective. Then $l.fd_\Gamma(Hom_\Lambda(U, E)) = U\text{-}lim.dim_\Lambda(E)$.*

Proof We first prove that U-lim.dim$_\Lambda(E) \leq l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda(U, E))$. Without loss of generality, assume that $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda(U, E)) = n < \infty$. Then there exists an exact sequence:

$$0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to \mathrm{Hom}_\Lambda(U, E) \to 0$$

in Mod Γ with each Q_i Γ-flat for any $0 \leq i \leq n$. By [4, Chapter VI, Proposition 5.3], we have that

$$\mathrm{Tor}^\Gamma_j(U, \mathrm{Hom}_\Lambda(U, E)) \cong \mathrm{Hom}_\Lambda(\mathrm{Ext}^j_\Gamma(U, U), E) = 0$$

for any $j \geq 1$. Then we easily get an exact sequence:

$$0 \to U \otimes_\Gamma Q_n \to \cdots \to U \otimes_\Gamma Q_1 \to U \otimes_\Gamma Q_0 \to U \otimes_\Gamma \mathrm{Hom}_\Lambda(U, E) \to 0.$$

Because each Q_i is a direct limit of finitely generated free Γ-modules, $U \otimes_\Gamma Q_i \in$add-lim$_\Lambda U$ for any $0 \leq i \leq n$. On the other hand, $U \otimes_\Gamma \mathrm{Hom}_\Lambda(U, E) \cong \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Gamma(U, U), E) \cong E$ by [18, p.47]. So we conclude that U-lim.dim$_\Lambda(E) \leq n$.

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We next prove that $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda(U, E)) \leq U\text{-lim.dim}_\Lambda(E)$. Assume that $U\text{-lim.dim}_\Lambda(E) = n < \infty$. Then there exists an exact sequence:

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to E \to 0 \tag{2}$$

in Mod Λ with each X_i in add-$\lim_\Lambda U$ for any $0 \leq i \leq n$. Since ${}_\Lambda U$ is finitely generated, by [17, Theorem 3.2], for any direct system $\{M_\alpha\}_{\alpha \in I}$ and $j \geq 0$, we have that $\mathrm{Ext}^j_\Lambda(U, \varinjlim M_\alpha) \cong \varinjlim \mathrm{Ext}^j_\Lambda(U, M_\alpha)$. From this fact we know that $\mathrm{Ext}^j_\Lambda(U, X_i) = 0$ and $\mathrm{Hom}_\Lambda(U, X_i)$ is in add-$\lim_\Gamma \Gamma$ for any $j \geq 1$ and $0 \leq i \leq n$. So each $\mathrm{Hom}_\Lambda(U, X_i)$ is Γ-flat for any $0 \leq i \leq n$ and by applying the functor $\mathrm{Hom}_\Lambda(U, \ )$ to the exact sequence (2) we obtain the following exact sequence:

$$0 \to \mathrm{Hom}_\Lambda(U, X_n) \to \cdots \to \mathrm{Hom}_\Lambda(U, X_1) \to \mathrm{Hom}_\Lambda(U, X_0) \to \mathrm{Hom}_\Lambda(U, E) \to 0.$$

Hence $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda(U, E)) \leq n$. The proof is finished. □

Lemma 17.3.2 *Let m be an integer with $m \geq -k$. Then the following statements are equivalent.*

(1) U-lim.dim$_\Lambda(\bigoplus_{i=0}^{k-1} E_i) \leq k + m$.

(2) s.grade$_U \mathrm{Ext}^{k+m+1}_\Gamma(N, U) \geq k$ for any $N \in$mod Γ^{op}.

(3) $l.fd_\Gamma(Hom_\Lambda(U, E_i)) \leq k + m$ for any $0 \leq i \leq k - 1$.

Proof By Lemma 17.3.1, we have (1) $\Leftrightarrow$ (3).

(2) $\Rightarrow$ (3) We proceed by using induction on i. Suppose that s.grade$_U \mathrm{Ext}^{k+m+1}_\Gamma(N, U) \geq k$ for any $N \in$mod Γ^{op}. We first prove $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda(U, E_0)) \leq k + m$. By assumption, we have $\mathrm{Hom}_\Lambda(\mathrm{Ext}^{k+m+1}_\Gamma(N, U), U) = 0$. We now claim that $\mathrm{Hom}_\Lambda(\mathrm{Ext}^{k+m+1}_\Gamma(N, U), E_0) = 0$. For if otherwise, then there exists $0 \neq f : \mathrm{Ext}^{k+m+1}_\Gamma(N, U) \to E_0$ and $\mathrm{Im} f \bigcap U \neq 0$ (since U is essential in E_0). Hence, there exists a submodule $X (= f^{-1}(\mathrm{Im} f \bigcap U))$ of $\mathrm{Ext}^{k+m+1}_\Gamma(N, U)$ such that $\mathrm{Hom}_\Lambda(X, U) \neq 0$, which contradicts s.grade$_U \mathrm{Ext}^{k+m+1}_\Gamma(N, U) \geq k$. It follows easily from [4, Chapter VI, Proposition 5.3] that $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda({}_\Lambda U_\Gamma, E_0)) \leq k + m$.

Now suppose $i \geq 1$. Consider the exact sequence:

$$0 \to K_{i-1} \to E_{i-1} \to K_i \to 0$$

where $K_{i-1} = \mathrm{Ker}(E_{i-1} \to E_i)$ and $K_i = \mathrm{Im}(E_{i-1} \to E_i)$. Then for any $X \subset \mathrm{Ext}^{k+m+1}_\Gamma(N, U)$, we have an exact sequence:

$$\mathrm{Hom}_\Lambda(X, E_{i-1}) \to \mathrm{Hom}_\Lambda(X, K_i) \to \mathrm{Ext}^1_\Lambda(X, K_{i-1}) \to 0 \tag{3}$$

Since s.grade$_U \mathrm{Ext}^{k+m+1}_\Gamma(N, U) \geq k$ and $1 \leq i \leq k - 1$, $\mathrm{Ext}^1_\Lambda(X, K_{i-1}) \cong \mathrm{Ext}^i_\Lambda(X, U) = 0$. By induction assumption, $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda(U, E_{i-1})) \leq k + m$. It follows from [4, Chapter VI, Proposition 5.3] that $\mathrm{Hom}_\Lambda(\mathrm{Ext}^{k+m+1}_\Gamma(N, U), E_{i-1}) \cong \mathrm{Tor}^\Gamma_{k+m+1}(N, \mathrm{Hom}_\Lambda(U, E_{i-1})) = 0$. Since E_{i-1} is injective, $\mathrm{Hom}_\Lambda(X, E_{i-1}) = 0$. It follows from the exactness of the sequence (3) that $\mathrm{Hom}_\Lambda(X, K_i) = 0$. Observe that E_i is the injective envelope of K_i; by using a similar argument to the case $i = 0$, we can show that $\mathrm{Hom}_\Lambda(\mathrm{Ext}^{k+m+1}_\Gamma(M, U), E_i) = 0$. Hence, we have that $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda({}_\Lambda U_\Gamma, E_i)) \leq k + m$.

(3) $\Rightarrow$ (2) Suppose that $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda(U, \bigoplus_{i=0}^{k-1} E_i)) \leq k + m$. Then, by [4, Chapter VI, Proposition 5.3], we have that $\mathrm{Hom}_\Lambda(\mathrm{Ext}^{k+m+1}_\Gamma(N, U), \bigoplus_{i=0}^{k-1} E_i) = 0$ for any $N \in$mod Γ^{op}. Let X be any submodule of $\mathrm{Ext}^{k+m+1}_\Gamma(N, U)$. Then $\mathrm{Hom}_\Lambda(X, \bigoplus_{i=0}^{k-1} E_i) = 0$. Put $K_0 = U$ and $K_i = \mathrm{Im}(E_{i-1} \to E_i)$ for any $1 \leq i \leq k - 1$. Then $\mathrm{Hom}_\Lambda(X, K_i) = 0$ for any $0 \leq i \leq k - 1$. It is not difficult to prove that $\mathrm{Ext}^{i+1}_\Lambda(X, K_0) \cong \mathrm{Ext}^1_\Lambda(X, K_i)$ and $\mathrm{Ext}^1_\Lambda(X, K_i) \cong \mathrm{Hom}_\Lambda(X, K_{i+1})$ for any $0 \leq i \leq k-2$. Hence we conclude that $\mathrm{Hom}_\Lambda(X, U) = 0 = \mathrm{Ext}^i_\Lambda(X, U)$ for any $1 \leq i \leq k-1$. This completes the proof. □

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Putting $m = -1$, then by Lemma 17.3.2, we have the following

Corollary 17.3.3 *(1) U-lim.dim$_\Lambda(\bigoplus_{i=0}^{k-1} E_i) \leq k-1$ if and only if s.grade$_U$ $\mathrm{Ext}_\Gamma^k(N, U) \geq k$ for any $N \in$mod Γ^{op} if and only if l.fd$_\Gamma$(Hom$_\Lambda(U, \bigoplus_{i=0}^{k-1} E_i)) \leq k-1$.*

(2) U-lim.dim$_\Lambda(E_i) \leq i$ for any $0 \leq i \leq k-1$ if and only if s.grade$_U$ $\mathrm{Ext}_\Gamma^i(N, U) \geq i$ for any $N \in$mod Γ^{op} and $1 \leq i \leq k$ if and only if l.fd$_\Gamma$(Hom$_\Lambda(U, E_i)) \leq i$ for any $0 \leq i \leq k-1$.

Let M be in mod Λ (resp. mod Γ^{op}) and $P_1 \xrightarrow{f} P_0 \to M \to 0$ be a projective presentation of M in mod Λ (resp. mod Γ^{op}). Then we have an exact sequence:

$$0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to \mathrm{Coker} f^* \to 0.$$

We call $\mathrm{Coker} f^*$ the transpose (with respect to ${}_\Lambda U_\Gamma$) of M, and denote it by $\mathrm{Tr}_U M$.

For a positive integer k, recall from [10] that M is called U-k-torsionfree if $\mathrm{Ext}_\Gamma^i(\mathrm{Tr}_U M, U)$ (resp. $\mathrm{Ext}_\Lambda^i(\mathrm{Tr}_U M, U)) = 0$ for any $1 \leq i \leq k$. We call M U-k-syzygy if there exists an exact sequence $0 \to M \to X_0 \to X_1 \to \cdots \xrightarrow{f} X_{k-1}$ with all X_i in add$_\Lambda U$ (resp. addU_Γ), and denote M by $\Omega_U^k(\mathrm{Coker} f)$. Putting ${}_\Lambda U_\Gamma = {}_\Lambda \Lambda_\Lambda$, then, in this case, the notions of U-k-torsionfree modules and U-k-syzygy modules are just that of k-torsionfree modules and k-syzygy modules respectively (see [3] for the definitions of k-torsionfree modules and k-syzygy modules). We use $\mathcal{T}_U^k(\mathrm{mod}\ \Lambda)$ (resp. $\mathcal{T}_U^k(\mathrm{mod}\ \Gamma^{op})$) and $\Omega_U^k(\mathrm{mod}\ \Lambda)$ (resp. $\Omega_U^k(\mathrm{mod}\ \Gamma^{op})$) to denote the full subcategory of mod Λ (resp. mod Γ^{op}) consisting of U-k-torsionfree modules and U-k-syzygy modules, respcetively. It is not difficult to verify that $\mathcal{T}_U^k(\mathrm{mod}\ \Lambda) \subseteq \Omega_U^k(\mathrm{mod}\ \Lambda)$ and $\mathcal{T}_U^k(\mathrm{mod}\ \Gamma^{op}) \subseteq \Omega_U^k(\mathrm{mod}\ \Gamma^{op})$.

The following result generalizes [3, Proposition 1.6(a)].

Lemma 17.3.4 *The following statements are equivalent.*

(1) grade$_U$ Ext$_\Lambda^{i+1}(M, U) \geq i$ for any $M \in$ mod Λ and $1 \leq i \leq k-1$.

(1)^{op} grade$_U$ Ext$_\Gamma^{i+1}(N, U) \geq i$ for any $N \in$ mod Γ^{op} and $1 \leq i \leq k-1$.

If one of the above equivalent conditions holds, then $\mathcal{T}_U^i(\mathrm{mod}\ \Lambda) = \Omega_U^i(\mathrm{mod}\ \Lambda)$ and $\mathcal{T}_U^i(\mathrm{mod}\ \Gamma^{op}) = \Omega_U^i(\mathrm{mod}\ \Gamma^{op})$ for any $1 \leq i \leq k$.

Proof The equivalence of (1) and (1)^{op} was proved in [12, Lemma 3.3]. The latter assertion follows from [10, Theorem 3.1]. □

Putting $m = 0$, then by Lemma 17.3.2, we have the following result, in which the second assertion is just [3, Proposition 2.2] when ${}_\Lambda U_\Gamma = {}_\Lambda \Lambda_\Lambda$.

Corollary 17.3.5 *(1) U-lim.dim$_\Lambda(\bigoplus_{i=0}^{k-1} E_i) \leq k$ if and only if s.grade$_U$ $\mathrm{Ext}_\Gamma^{k+1}(N, U) \geq k$ for any $N \in$mod Γ^{op} if and only if l.fd$_\Gamma$(Hom$_\Lambda(U, \bigoplus_{i=0}^{k-1} E_i)) \leq k$.*

(2) U-lim.dim$_\Lambda(E_i) \leq i+1$ for any $0 \leq i \leq k-1$ if and only if s.grade$_U$ $\mathrm{Ext}_\Gamma^{i+1}(N, U) \geq i$ for any $N \in$mod Γ^{op} and $1 \leq i \leq k$ if and only if l.fd$_\Gamma$(Hom$_\Lambda(U, E_i)) \leq i+1$ for any $0 \leq i \leq k-1$. In this case, $\mathcal{T}_U^i(\mathrm{mod}\ \Gamma^{op}) = \Omega_U^i(\mathrm{mod}\ \Gamma^{op})$ for any $1 \leq i \leq k$.

Proof Our assertions follow from Lemmas 16.3.2 and 16.3.4. □

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Putting $m = -k$, then by Lemma 17.3.2, we have the following

Corollary 17.3.6 *The following statements are equivalent.*

(1) U*-dom.dim*$({}_\Lambda U) \geq k$.

(2) *s.grade*$_U$ *Ext*$^1_\Gamma(N, U) \geq k$ *for any* $N \in$*mod* Γ^{op}.

(3) *Hom*$_\Lambda(U, E_i)$ *is* Γ*-flat for any* $0 \leq i \leq k-1$.

Dually, we have the following

Corollary 17.3.7 *The following statements are equivalent.*

(1) U*-dom.dim*$(U_\Gamma) \geq k$.

(2) *s.grade*$_U$ *Ext*$^1_\Lambda(M, U) \geq k$ *for any* $M \in$*mod* Λ.

(3) *Hom*$_\Gamma(U, E'_i)$ *is* Λ^{op}*-flat for any* $0 \leq i \leq k-1$.

The following two results are cited from [11].

Lemma 17.3.8 *([11, Corollary 2.5]) Hom*$_\Lambda(U, E_0)$ *is* Γ*-flat if and only if Hom*$_\Gamma(U, E'_0)$ *is* Λ^{op}*-flat.*

Lemma 17.3.9 *([11, Lemma 2.6]) Let X be in mod* Λ *(resp. mod* Γ^{op}*) and n a non-negative integer. If grade*$_U X \geq n$ *and grade*$_U$ *Ext*$^n_\Lambda(X, U)$ *(resp. grade*$_U$ *Ext*$^n_\Gamma(X, U)$*)* $\geq n+1$*, then grade*$_U X \geq n+1$.

Lemma 17.3.10 *If* U*-dom.dim*$(U_\Gamma) \geq k$*, then* U*-dom.dim*$({}_\Lambda U) \geq k$.

Proof When $k = 1$, by Corollary 16.3.7, Hom$_\Lambda(U, E'_0)$ is Λ^{op}-flat. Then, by Lemma 17.3.8, Hom$_\Lambda(U, E_i)$ is Γ-flat. So U-dom.dim$({}_\Lambda U) \geq 1$ by Corollary 16.3.6.

Now suppose $k \geq 2$. By induction assumption, U-dom.dim$({}_\Lambda U) \geq k-1$. So, by Corollary 16.3.6, we have that s.grade$_U$ Ext$^1_\Gamma(N, U) \geq k-1$ for any $N \in$mod Γ^{op}.

Let X be any submodule of Ext$^1_\Gamma(N, U)$. Then grade$_U X \geq k-1$. By assumption and Corollary 16.3.7, grade$_U$ Ext$^i_\Gamma(X, U) \geq k$ for any $i \geq 1$. It follows from Lemma 17.3.9 that grade$_U X \geq k$. So s.grade$_U$ Ext$^1_\Gamma(N, U) \geq k$ and hence U-dom.dim$({}_\Lambda U) \geq k$ by Corollary 16.3.6. □

Proof of Theorem 17.1.4. By Corollary 16.3.6 we have that (1) $\Leftrightarrow$ $(2)^{op}$ $\Leftrightarrow$ (3), and by Lemma 17.3.10 we have that (1) $\Rightarrow$ $(1)^{op}$. The other implications follow from the symmetry. □

We now begin to prove Theorem 17.1.11.

Lemma 17.3.11 *([12, Lemma 3.2]) If s.grade*$_U$ Ext$^{i+1}_\Gamma(X, U) \geq i$ *for any* $X \in$*mod* Λ *(resp. mod* Γ^{op}*) and* $1 \leq i \leq k-1$*, then each k-syzygy module in mod* Λ *(resp. mod* Γ^{op}*) is in* $\Omega^k_U(\text{mod } \Lambda)$ *(resp.* $\Omega^k_U(\text{mod } \Gamma^{op})$*).*

Theorem 17.3.12 *The following statements are equivalent.*

(1) *s.grade*$_U$ Ext$^i_\Lambda(M, U) \geq i$ *for any* $M \in$*mod* Λ *and* $1 \leq i \leq k$.

(2) Ext$^i_\Gamma($Ext$^i_\Lambda(\ , U), U)$ *preserves monomorphisms in mod* Λ *for any* $0 \leq i \leq k-1$.

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Proof We proceed by using induction on k.

(1) $\Rightarrow$ (2) Let

$$0 \to X \xrightarrow{f} Y \to Z \to 0 \tag{4}$$

be an exact sequence in mod Λ.

Suppose $k = 1$. By assumption, s.grade$_U \operatorname{Ext}^1_\Lambda(Z, U) \geq 1$. Since Coker$f^*$ is a submodule of $\operatorname{Ext}^1_\Lambda(Z, U)$, $(\operatorname{Coker} f^*)^* = 0$ and $0 \to X^{**} \xrightarrow{f^{**}} Y^{**}$ is exact.

Now suppose $k \geq 2$. From the exact sequence (4), we get an exact sequence:

$$\operatorname{Ext}^{k-1}_\Lambda(Z, U) \xrightarrow{\alpha} \operatorname{Ext}^{k-1}_\Lambda(Y, U) \xrightarrow{\beta} \operatorname{Ext}^{k-1}_\Lambda(X, U) \xrightarrow{\gamma} \operatorname{Ext}^k_\Lambda(Z, U).$$

Set $A = \operatorname{Im}\alpha$, $B = \operatorname{Im}\beta$ and $C = \operatorname{Im}\gamma$. By (1), we have that grade$_U A \geq k-1$, grade$_U B \geq k-1$ and grade$_U C \geq k$. Then we get the following exact sequences:

$$0 \to \operatorname{Ext}^{k-1}_\Gamma(B, U) \to \operatorname{Ext}^{k-1}_\Gamma(\operatorname{Ext}^{k-1}_\Lambda(Y, U), U),$$

$$0 \to \operatorname{Ext}^{k-1}_\Gamma(\operatorname{Ext}^{k-1}_\Lambda(X, U), U) \to \operatorname{Ext}^{k-1}_\Gamma(B, U).$$

Thus we get a composition of monomorphisms:

$$\operatorname{Ext}^{k-1}_\Gamma(\operatorname{Ext}^{k-1}_\Lambda(X, U), U) \hookrightarrow \operatorname{Ext}^{k-1}_\Gamma(B, U) \hookrightarrow \operatorname{Ext}^{k-1}_\Gamma(\operatorname{Ext}^{k-1}_\Lambda(Y, U), U),$$

which is also a monomorphism.

(2) $\Rightarrow$ (1) Suppose $k = 1$. Let M be in mod Λ and X a submodule of $\operatorname{Ext}^1_\Lambda(M, U)$. Because $\operatorname{Ext}^1_\Lambda(M, U)$ is in mod Γ^{op}, X is also in mod Γ^{op}. So there exist a positive integer t and an exact sequence:

$$0 \to U^t \xrightarrow{f} L \to M \to 0$$

such that the induced exact sequence:

$$L^* \xrightarrow{f^*} (U^t)^* \to \operatorname{Ext}^1_\Lambda(M, U)$$

has the property that $X \cong \operatorname{Coker} f^*$. By assumption, f^{**} is monic, so $X^* \cong \operatorname{Ker} f^{**} = 0$. Hence we conclude that s.grade$_U \operatorname{Ext}^1_\Lambda(M, U) \geq 1$.

Now suppose $k \geq 2$. By induction assumption, for any $M \in$mod Λ, we have that s.grade$_U \operatorname{Ext}^i_\Lambda(M, U) \geq i$ for any $1 \leq i \leq k-1$ and s.grade$_U \operatorname{Ext}^k_\Lambda(M, U) \geq k-1$. By [10, Theorem 3.1], $\Omega^i_U(\text{mod } \Lambda) = \mathcal{T}^i_U(\text{mod } \Lambda)$ for any $1 \leq i \leq k$.

Let

$$\cdots \xrightarrow{g_{i+1}} P_i \xrightarrow{g_i} \cdots \xrightarrow{g_2} P_1 \xrightarrow{g_1} P_0 \to M \to 0$$

be a projective resolution of M in mod Λ. Notice that Cokerg_k is a $(k-1)$-syzygy module in mod Λ, so it is in $\Omega^{k-1}_U(\text{mod } \Lambda)$ by Lemma 17.3.11 and hence in $\mathcal{T}^{k-1}_U(\text{mod } \Lambda)$. Thus $\operatorname{Ext}^i_\Gamma(\operatorname{Coker} g^*_k, U) = 0$ for any $1 \leq i \leq k-1$.

Let X be a submodule of $\operatorname{Ext}^k_\Lambda(M, U)$. Then grade$_U X \geq k-1$. By [9, Lemma 2], there exists an embedding $0 \to X \to \operatorname{Coker} g^*_k$. By assumption, we then have an exact sequence:

$$0 \to \operatorname{Ext}^{k-1}_\Gamma(\operatorname{Ext}^{k-1}_\Lambda(X, U), U) \to \operatorname{Ext}^{k-1}_\Gamma(\operatorname{Ext}^{k-1}_\Lambda(\operatorname{Coker} g^*_k, U), U) = 0,$$

which implies that $\operatorname{Ext}^{k-1}_\Gamma(\operatorname{Ext}^{k-1}_\Lambda(X, U), U) = 0$.

On the other hand, s.grade$_U \operatorname{Ext}^{k-1}_\Gamma(X, U) \geq k-1$ by [21, Theorem 7.5]. So s.grade$_U$ $\operatorname{Ext}^{k-1}_\Gamma(X, U) \geq k$. It follows from Lemma 17.3.9 that grade$_U X \geq k$ and s.grade$_U \operatorname{Ext}^k_\Gamma(M, U) \geq k$. We are done. $\square$

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Proof of Theorem 17.1.11. By definition, we have (1) $\Leftrightarrow$ (2). By Corollary 17.3.5(2), we have that (3) $\Leftrightarrow$ $(2)^{op}$ $\Leftrightarrow$ (4). By Theorem 16.3.12 and [21, Theorem 7.5], we have that (5) $\Leftrightarrow$ (2) $\Leftrightarrow$ $(2)^{op}$. The other implications follow from the symmetry. □

17.4 Exactness of the Double Dual

As applications to the results in Section 17.3, we give in this section some characterizations of $(\ )^{**}$ preserving monomorphisms and being left exact, respectively.

As an immediate consequence of Theorem 17.1.11, we have the following result, which generalizes [5, Theorem 1] and [7, Proposition 3.1].

Proposition 17.4.1 *The following statements are equivalent.*

(1) U-*dom.dim*$({}_\Lambda U) \geq 1$.

(2) *s.grade*$_U$ *Ext*$^1_\Lambda(M, U) \geq 1$ *for any* $M \in$*mod* Λ.

(3) $E_0 \in$*add-lim*$_\Lambda U$.

(4) $(\ )^{**}$ *preserves monomorphisms in mod* Λ.

(1)^{op} U-*dom.dim*$(U_\Gamma) \geq 1$.

(2)^{op} *s.grade*$_U$ *Ext*$^1_\Gamma(N, U) \geq 1$ *for any* $N \in$*mod* Γ^{op}.

(3)^{op} $E'_0 \in$*add-lim*U_Γ.

(4)^{op} $(\ )^{**}$ *preserves monomorphisms in mod* Γ^{op}.

Lemma 17.4.2 *Assume that* U-*dom.dim*$({}_\Lambda U) \geq k$. *Then, for a module* M *in mod* Λ, *grade*$_U$ $M \geq k$ *if* $M^* = 0$.

Proof For any $M \in$mod Λ and $i \geq 1$, we have an exact sequence

$$\mathrm{Hom}_\Lambda(M, E_{i-1}) \to \mathrm{Hom}_\Lambda(M, K_i) \to \mathrm{Ext}^i_\Lambda(M, U) \to 0 \tag{5}$$

where $K_i = \mathrm{Im}(E_{i-1} \to E_i)$.

Suppose U-dom.dim$({}_\Lambda U) \geq k$. Then each E_i is in add-lim$_\Lambda U$ for any $0 \leq i \leq k-1$. So, for a given $M \in$mod Λ with $M^* = 0$, we have that $\mathrm{Hom}_\Lambda(M, E_i) = 0$ by [17, Theorem 3.2] and $\mathrm{Hom}_\Lambda(M, K_i) = 0$ for any $0 \leq i \leq k-1$. Then by the exactness of the sequence (5), $\mathrm{Ext}^i_\Lambda(M, U) = 0$ for any $1 \leq i \leq k-1$, and so grade$_U$ $M \geq k$. □

Lemma 17.4.3 *If* $[\mathrm{Ext}^1_\Lambda(M, U)]^* = 0$ *for any* $M \in$*mod* Λ, *then* N^* *is* U-*reflexive for any* $N \in$*mod* Γ^{op}.

Proof By the dual statements of [10, Proposition 4.2 and Corollary 4.2]. □

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We now characterize U-dominant dimension of U at least two. The following result generalizes [5, Theorem 2] and [8, Proposition E].

Proposition 17.4.4 *The following statements are equivalent.*

(1) U*-dom.dim*$({}_{\Lambda}U) \geq 2$.

(2) $(\ )^{**}$: *mod* $\Lambda \to$ *mod* Λ *is left exact.*

(3) $(\ )^{**}$: *mod* $\Lambda \to$ *mod* Λ *preserves monomorphisms and* $Ext^1_{\Gamma}(\mathrm{Ext}^1_{\Lambda}(X, U), U) = 0$ *for any* $X \in$*mod* Λ.

(1)^{op} U*-dom.dim*$(U_{\Gamma}) \geq 2$.

(2)^{op} $(\ )^{**}$: *mod* $\Gamma^{op} \to$ *mod* Γ^{op} *is left exact.*

(3)^{op} $(\ )^{**}$: *mod* $\Gamma^{op} \to$ *mod* Γ^{op} *preserves monomorphisms and* $Ext^1_{\Lambda}(\mathrm{Ext}^1_{\Gamma}(Y, U), U) = 0$ *for any* $Y \in$*mod* Γ^{op}.

Proof By Theorem 17.1.4, we have (1) $\Leftrightarrow$ (1)^{op}. By symmetry, we only need to prove that (1) $\Rightarrow$ (2) and (2)^{op} $\Rightarrow$ (3) $\Rightarrow$ (1)^{op}.

(1) $\Rightarrow$ (2) Assume that U-dom.dim$({}_{\Lambda}U) \geq 2$ and $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is an exact sequence in mod Λ. By Proposition 17.4.1, α^{**} is monic. By Theorem 17.1.4 and [4, Chapter VI, Proposition 5.3], we have that $\mathrm{Hom}_{\Gamma}(U, E'_0)$ is Λ^{op}-flat and $\mathrm{Hom}_{\Gamma}(\mathrm{Ext}^1_{\Lambda}(C, U), E'_0) = 0$. Since $\mathrm{Coker}\alpha^*$ is isomorphic to a submodule of $\mathrm{Ext}^1_{\Lambda}(C, U)$, $\mathrm{Hom}_{\Gamma}(\mathrm{Coker}\alpha^*, E'_0) = 0$ and $(\mathrm{Coker}\alpha^*)^* = 0$. Then by Lemma 17.4.2, we have that $\mathrm{grade}_U\mathrm{Coker}\alpha^* \geq 2$ and $\mathrm{Ext}^1_{\Gamma}(\mathrm{Coker}\alpha^*, U) = 0$. It follows easily that $0 \to A^{**} \xrightarrow{\alpha^{**}} B^{**} \xrightarrow{\beta^{**}} C^{**}$ is exact.

(2)^{op} $\Rightarrow$ (3) By Proposition 17.4.1, $(\ )^{**}$: mod $\Gamma^{op} \to$ mod Γ^{op} preserves monomorphisms and U-dom.dim$({}_{\Lambda}U) = U$-dom.dim$(U_{\Gamma}) \geq 1$. By Theorem 17.1.4, for any $X \in$ mod Λ, we have that s.grade$_U\mathrm{Ext}^1_{\Lambda}(X, U) \geq 1$ and $[\mathrm{Ext}^1_{\Lambda}(X, U)]^* = 0$.

Let

$$0 \to K \xrightarrow{f} Q \xrightarrow{g} \mathrm{Ext}^1_{\Lambda}(X, U) \to 0$$

be an exact sequence in mod Γ^{op} with Q projective. Then, by (2)^{op}, f^{**} is a monomorphism and hence an isomorphism. So f^{***} is also an isomorphism. On the other hand, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & Q^* \arrow[r, "f^*"] \arrow[d, "\sigma_{Q^*}"] & K^* \arrow[r] \arrow[d, "\sigma_{K^*}"] & \mathrm{Ext}^1_{\Gamma}(\mathrm{Ext}^1_{\Lambda}(X, U), U) \arrow[r] & 0 \\
 & Q^{***} \arrow[r, "f^{***}"] & K^{***} & &
\end{tikzcd}$$

It follows from Lemma 16.4.3 that Q^* and K^* are U-reflexive. So σ_{Q^*} and σ_{K^*} are isomorphisms and hence f^* is an isomorphism. Consequently we have that $\mathrm{Ext}^1_{\Gamma}(\mathrm{Ext}^1_{\Lambda}(X, U), U) = 0$.

(3) $\Rightarrow$ (1)^{op} Suppose that (3) holds. Then U-dom.dim$(U_{\Gamma}) \geq 1$ by Proposition 17.4.1.

Let A be in mod Λ and B any submodule of $\mathrm{Ext}^1_{\Lambda}(A, U)$ in mod Γ^{op}. Since U-dom.dim(U_{Γ}) ≥ 1, by Theorem 17.1.4 and [4, Chapter VI, Proposition 5.3], we have that $\mathrm{Hom}_{\Gamma}(U, E'_0)$ is Λ^{op}-flat and $\mathrm{Hom}_{\Gamma}(\mathrm{Ext}^1_{\Lambda}(A, U), E'_0) = 0$. So $\mathrm{Hom}_{\Gamma}(B, E'_0) = 0$ and hence $\mathrm{Hom}_{\Gamma}(B, E'_0/U_{\Gamma}) \cong \mathrm{Ext}^1_{\Gamma}(B, U_{\Gamma})$. On the other hand, $\mathrm{Hom}_{\Gamma}(B, E'_0) = 0$ implies $B^* = 0$. Then by [13, Lemma 2.1], we have that $B \cong \mathrm{Ext}^1_{\Lambda}(\mathrm{Tr}_U B, U)$ with $\mathrm{Tr}_U B$ in mod Λ. By (3), $\mathrm{Hom}_{\Gamma}(B, E'_0/U) \cong \mathrm{Ext}^1_{\Gamma}(B, U) \cong \mathrm{Ext}^1_{\Gamma}(\mathrm{Ext}^1_{\Lambda}(\mathrm{Tr}_U B, U), U) = 0$. Then by using a similar argument to the proof of (2) $\Rightarrow$ (3) in Lemma 17.3.2, we have that $\mathrm{Hom}_{\Gamma}(\mathrm{Ext}^1_{\Lambda}(A, U), E'_1) = 0$ (note: E'_1 is the injective envelope of E'_0/U). It follows from [4, Chapter VI, Proposition 5.3] that $\mathrm{Hom}_{\Gamma}(U, E'_1)$ is Λ^{op}-flat and thus U-dom.dim$(U_{\Gamma}) \geq 2$ by Theorem 17.1.4. $\square$

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17.5 A Generalization of k-Gorenstein Modules

In this section, we study a generalization of k-Gorenstein modules, which is however not left-right symmetric. We characterize this generalization in terms of some properties similar to that of k-Gorenstein modules. The results obtained here develop the main result of Auslander and Reiten in [3].

We begin with the following equivalent characterizations of U-lim.dim$_\Lambda(E_0) \leq 1$ as follows, which generalizes [8, Proposition D].

Proposition 17.5.1 *The following statements are equivalent.*

(1) U-lim.dim$_\Lambda(E_0) \leq 1$.

(2) σ_X is an essential monomorphism for any U-torsionless module X in mod Λ.

(3) f^{**} is a monomorphism for any monomorphism $f : X \to Y$ in mod Λ with Y U-torsionless.

(4) f^{**} is a monomorphism for any monomorphism $f : X \to Y$ in mod Λ with X and Y U-torsionless.

(5) grade$_U$ $\mathrm{Ext}^1_\Lambda(X, U) \geq 1$ for any $X \in$mod Λ.

(6) s.grade$_U$ $\mathrm{Ext}^2_\Gamma(N, U) \geq 1$ for any $N \in$mod Γ^{op}.

Proof (1) $\Leftrightarrow$ (6) follows from Corollary 16.3.5(2) and (3) $\Rightarrow$ (4) is trivial.

(1) $\Rightarrow$ (2) Suppose U-lim.dim$_\Lambda(E_0) \leq 1$. Then by Lemma 17.3.1, we have that $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda(U, E_0)) \leq 1$.

Assume that X is U-torsionless in mod Λ. Then Coker$\sigma_X \cong \mathrm{Ext}^2_\Gamma(\mathrm{Tr}_U X, U)$ by [13, Lemma 2.1]. By [4, Chapter VI, Proposition 5.3], we have that $\mathrm{Hom}_\Lambda(\mathrm{Coker}\sigma_X, E_0) \cong \mathrm{Hom}_\Lambda(\mathrm{Ext}^2_\Gamma(\mathrm{Tr}_U X, U), E_0) \cong \mathrm{Tor}^\Gamma_2(\mathrm{Tr}_U X, \mathrm{Hom}_\Lambda(U, E_0)) = 0$. Then $A^* = 0$ for any submodule A of Cokerσ_X, which implies that any non-zero submodule of Cokerσ_X is not U-torsionless.

Let B be a submodule of X^{**} with $X \bigcap B = 0$. Then $B \cong B/(X \bigcap B) \cong (X + B)/X$ is isomorphic to a submodule of Cokerσ_X. On the other hand, B is clearly U-torsionless. So $B = 0$ and hence σ_X is essential.

(2) $\Rightarrow$ (3) Let $f : X \to Y$ be monic in mod Λ with Y U-torsionless. Then $f^{**}\sigma_X = \sigma_Y f$ is monic. By (2), σ_X is an essential monomorphism, so f^{**} is monic.

(4) $\Rightarrow$ (5) Let X be in mod Λ and $0 \to Y \xrightarrow{g} P \to X \to 0$ an exact sequence in mod Λ with P projective. It is easy to see that $[\mathrm{Ext}^1_\Lambda(X, U)]^* \cong \mathrm{Ker}g^{**}$. On the other hand, since ${}_\Lambda U_\Gamma$ is a faithfully balanced bimodule, P is U-reflexive and Y is U-torsionless. So g^{**} is monic by (4) and hence $\mathrm{Ker}g^{**} = 0$ and $[\mathrm{Ext}^1_\Lambda(X, U)]^* = 0$.

(5) $\Rightarrow$ (1) Let M be in mod Γ^{op} and $\cdots \to P_1 \to P_0 \to M \to 0$ a projective resolution of M in mod Γ^{op}. Put $N =$Coker$(P_2 \to P_1)$. By [13, Lemma 2.1], $\mathrm{Ext}^2_\Gamma(M, U) \cong \mathrm{Ext}^1_\Gamma(N, U) \cong \mathrm{Ker}\sigma_{\mathrm{Tr}_U N}$. On the other hand, since N is U-torsionless, $\mathrm{Ext}^1_\Lambda(\mathrm{Tr}_U N, U) \cong \mathrm{Ker}\sigma_N = 0$.

Let X be any finitely generated submodule of $\mathrm{Ext}^2_\Gamma(M, U)$ and $f_1 : X \to \mathrm{Ext}^2_\Gamma(M, U)(\cong \mathrm{Ker}\sigma_{\mathrm{Tr}_U N})$ the inclusion, and let f be the composition: $X \xrightarrow{f_1} \mathrm{Ext}^2_\Gamma(M, U) \xrightarrow{g} \mathrm{Tr}_U N$, where g is a monomorphism. Then $\sigma_{\mathrm{Tr}_U N} f = 0$ and $f^*\sigma^*_{\mathrm{Tr}_U N} = (\sigma_{\mathrm{Tr}_U N} f)^* = 0$. But $\sigma^*_{\mathrm{Tr}_U N}$ is epic by [1, Proposition 20.14], so $f^* = 0$. Hence, by applying the functor $\mathrm{Hom}_\Lambda(\ , U)$ to the exact sequence $0 \to X \xrightarrow{f} \mathrm{Tr}_U N \to \mathrm{Coker}f \to 0$, we have that $X^* \cong \mathrm{Ext}^1_\Lambda(\mathrm{Coker}f, U)$ and then $X^{**} \cong [\mathrm{Ext}^1_\Lambda(\mathrm{Coker}f, U)]^* = 0$ by (5), which implies that $X^* = 0$ since X^* is a direct summand of $X^{***}(= 0)$. By using a similar argument to the proof of (2) $\Rightarrow$ (3) in Lemma 17.3.2, we can prove that $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda(U, E_0)) \leq 1$. Therefore U-lim.dim$_\Lambda(E_0) \leq 1$ by Lemma 17.3.1. □

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By Proposition 17.4.1, we have that $E_0 \in$add-lim$_\Lambda U$ if and only if $E_0' \in$add-limU_Γ, that is, U-lim.dim$_\Lambda(E_0) = 0$ if and only if U-lim.dim$_\Gamma(E_0') = 0$. However, in general, we don't have the fact that U-lim.dim$_\Lambda(E_0) \leq 1$ if and only if U-lim.dim$_\Gamma(E_0') \leq 1$ even when ${}_\Lambda U_\Gamma = {}_\Lambda\Lambda_\Lambda$.

Example 17.5.2 We use I_0 and I_0' to denote the injective envelope of ${}_\Lambda\Lambda$ and Λ_Λ, respectively. Consider the following example. Let K be a field and Δ the quiver:

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \xrightarrow{\gamma} 3$$

(1) If $\Lambda = K\Delta/(\alpha\beta\alpha)$. Then $l.\mathrm{fd}_\Lambda(I_0) = 1$ and $r.\mathrm{fd}_\Lambda(I_0') \geq 2$. (2) If $\Lambda = K\Delta/(\gamma\alpha, \beta\alpha)$. Then $l.\mathrm{fd}_\Lambda(I_0) = 2$ and $r.\mathrm{fd}_\Lambda(I_0') = 1$.

Compare the following result with Theorem 17.3.12.

Theorem 17.5.3 *The following statements are equivalent.*

(1) grade$_U$ $\mathrm{Ext}^i_\Lambda(M, U) \geq i$ *for any* M *∈mod* Λ *and* $1 \leq i \leq k$.

(2) $\mathrm{Ext}^i_\Gamma(\mathrm{Ext}^i_\Lambda(\ , U), U)$ *preserves monomorphisms* $X \to Y$ *with both* X *and* Y *torsionless in mod* Λ *for any* $0 \leq i \leq k-1$.

Proof We proceed by using induction on k. The case $k = 1$ follows from Proposition 17.5.1. Now suppose $k \geq 2$.

(1) $\Rightarrow$ (2) Let A be a torsionless module in mod Λ. Then there exists an exact sequence in mod Λ with P projective:

$$0 \to A \to P \to B \to 0.$$

By (1), for any $1 \leq i \leq k-1$, we have that grade$_U\mathrm{Ext}^i_\Lambda(A, U)$ =grade$_U\mathrm{Ext}^{i+1}_\Lambda(B, U) \geq i+1$, which implies that $\mathrm{Ext}^i_\Gamma(\mathrm{Ext}^i_\Lambda(A, U), U) = 0$. The desired conclusion follows trivially.

(2) $\Rightarrow$ (1) By induction assumption, for any M ∈mod Λ, we have that grade$_U\mathrm{Ext}^i_\Lambda(M, U) \geq i$ for any $1 \leq i \leq k-1$ and grade$_U\mathrm{Ext}^k_\Lambda(M, U) \geq k-1$. So it suffices to prove that $\mathrm{Ext}^{k-1}_\Gamma(\mathrm{Ext}^k_\Lambda(M, U), U) = 0$.

Let

$$0 \to K \to P \to M \to 0$$

be an exact sequence in mod Λ with P projective. Then by (2), we have the following exact sequence:

$$0 \to \mathrm{Ext}^{k-1}_\Gamma(\mathrm{Ext}^{k-1}_\Lambda(K, U), U) \to \mathrm{Ext}^{k-1}_\Gamma(\mathrm{Ext}^{k-1}_\Lambda(P, U), U).$$

But the last term in this sequence is always zero, so $\mathrm{Ext}^{k-1}_\Gamma(\mathrm{Ext}^k_\Lambda(M, U), U) \cong \mathrm{Ext}^{k-1}_\Gamma(\mathrm{Ext}^{k-1}_\Lambda(K, U), U) = 0$. $\square$

Compare the following result with [21, Theorem 7.5].

Theorem 17.5.4 *The following statements are equivalent.*

(1) s.grade$_U$ $\mathrm{Ext}^{i+1}_\Gamma(N, U) \geq i$ *for any* N *∈mod* Γ^{op} *and* $1 \leq i \leq k$.

(2) grade$_U$ $\mathrm{Ext}^i_\Lambda(M, U) \geq i$ *for any* M *∈mod* Λ *and* $1 \leq i \leq k$.

Proof We proceed by using induction on k. The case $k = 1$ follows from Proposition 17.5.1. Now suppose $k \geq 2$.

(1) $\Rightarrow$ (2) By induction assumption, for any M ∈mod Λ, we have that grade$_U\mathrm{Ext}^i_\Lambda(M, U) \geq i$ for any $1 \leq i \leq k-1$ and grade$_U\mathrm{Ext}^k_\Lambda(M, U) \geq k-1$. Then $\mathcal{T}^i_U(\mathrm{mod}\ \Lambda) = \Omega^i_U(\mathrm{mod}\ \Lambda)$ for any $1 \leq i \leq k$ by Lemma 17.3.4.

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Let
$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$
be an exact sequence in mod Λ with each P_i projective for any $i \geq 0$. By [9, Lemma 2], we have the following exact sequence:
$$0 \to \mathrm{Ext}^k_\Lambda(M,U) \to \mathrm{Tr}_U\Omega^{k-1}_\Lambda(M) \to P^*_{k+1} \to \mathrm{Tr}_U\Omega^k_\Lambda(M) \to 0. \tag{6}$$
Notice that $\Omega^{k-1}_\Lambda(M)$ is $(k-1)$-syzygy and $\Omega^k_\Lambda(M)$ is k-syzygy, so, by Lemma 16.3.11, $\Omega^{k-1}_\Lambda(M)$ (resp. $\Omega^k_\Lambda(M)$) is in $\Omega^{k-1}_U(\text{mod }\Lambda)$ (resp. $\Omega^k_U(\text{mod }\Lambda)$) and hence is in $\mathcal{T}^{k-1}_U(\text{mod }\Lambda)$ (resp. $\mathcal{T}^k_U(\text{mod }\Lambda)$). It follows that $\mathrm{Ext}^i_\Gamma(\mathrm{Tr}_U\Omega^{k-1}_\Lambda(M),U)=0$ for any $1\leq i\leq k-1$ and $\mathrm{Ext}^i_\Gamma(\mathrm{Tr}_U\Omega^k_\Lambda(M),U)=0$ for any $1\leq i\leq k$. In addition, $P^*_{k+1}\in\mathrm{add}U_\Gamma$, so $\mathrm{Ext}^i_\Gamma(P^*_{k+1},U)=0$ for any $i\geq 1$. Thus from the exact sequence (6) we get an embedding:
$$0 \to \mathrm{Ext}^{k-1}_\Gamma(\mathrm{Ext}^k_\Lambda(M,U),U) \to \mathrm{Ext}^{k+1}_\Gamma(\mathrm{Tr}_U\Omega^k_\Lambda(M),U).$$
Then, by (1), we have that $\mathrm{grade}_U\mathrm{Ext}^{k-1}_\Gamma(\mathrm{Ext}^k_\Lambda(M,U),U)\geq k$. Consequently, $\mathrm{grade}_U\mathrm{Ext}^k_\Lambda(M,U)\geq k$ by Lemma 17.3.9.

(2) $\Rightarrow$ (1) By induction assumption, for any $N\in\text{mod }\Gamma^{op}$, we have that $\text{s.grade}_U\mathrm{Ext}^{i+1}_\Gamma(N,U)\geq i$ for any $1\leq i\leq k-1$ and $\text{s.grade}_U\mathrm{Ext}^{k+1}_\Gamma(N,U)\geq k-1$. Then $\mathcal{T}^i_U(\text{mod }\Gamma^{op})=\Omega^i_U(\text{mod }\Gamma^{op})$ for any $1\leq i\leq k$ by Lemma 17.3.4.

Let X be a submodule of $\mathrm{Ext}^{k+1}_\Gamma(N,U)$. Then $\mathrm{grade}_U X\geq k-1$. By [9, Lemma 2], there exists an exact sequence:
$$0 \to X \xrightarrow{f} \mathrm{Tr}_U\Omega^k_\Gamma(N) \to \mathrm{Coker} f \to 0. \tag{7}$$
Notice that $\Omega^k_\Gamma(N)$ is k-syzygy, so, by Lemma 17.3.11, it is in $\Omega^k_U(\text{mod }\Gamma^{op})$ and hence is in $\mathcal{T}^k_U(\text{mod }\Gamma^{op})$. It follows that $\mathrm{Ext}^i_\Lambda(\mathrm{Tr}_U\Omega^k_\Gamma(N),U)=0$ for any $1\leq i\leq k$. So from the exact sequence (7) we get that $\mathrm{Ext}^{k-1}_\Lambda(X,U)\cong\mathrm{Ext}^k_\Lambda(\mathrm{Coker} f,U)$. By (2), $\mathrm{grade}_U\mathrm{Ext}^{k-1}_\Lambda(X,U)=\mathrm{grade}_U\mathrm{Ext}^k_\Lambda(\mathrm{Coker} f,U)\geq k$. It follows from Lemma 17.3.9 that $\mathrm{grade}_U X\geq k$ and $\text{s.grade}_U\mathrm{Ext}^{k+1}_\Gamma(N,U)\geq k$. □

Recall that a full subcategory $\mathcal{X}$ of mod Λ (resp. mod Γ^{op}) is said to be closed under extensions if the middle term B of any short sequence $0\to A\to B\to C\to 0$ is in $\mathcal{X}$ provided that the end terms A and C are in $\mathcal{X}$.

The following is the main result in this section.

Theorem 17.5.5 *The following statements are equivalent.*

(1) $\text{s.grade}_U\mathrm{Ext}^{i+1}_\Gamma(N,U)\geq i$ *for any* $N\in\text{mod }\Gamma^{op}$ *and* $1\leq i\leq k$.

(2) $U\text{-lim.dim}_\Lambda(E_i)\leq i+1$ *for any* $0\leq i\leq k-1$.

(3) $\text{l.fd}_\Gamma(\mathrm{Hom}_\Lambda(U,E_i))\leq i+1$ *for any* $0\leq i\leq k-1$.

(4) $\mathrm{grade}_U\mathrm{Ext}^i_\Lambda(M,U)\geq i$ *for any* $M\in\text{mod }\Lambda$ *and* $1\leq i\leq k$.

(5) $\mathrm{Ext}^i_\Gamma(\mathrm{Ext}^i_\Lambda(\ ,U),U)$ *preserves monomorphisms* $X\to Y$ *with both* X *and* Y *torsionless in mod* Λ *for any* $0\leq i\leq k-1$.

If one of the above equivalent conditions holds, then $\Omega^i_U(\text{mod }\Gamma^{op})(=\mathcal{T}^i_U(\text{mod }\Gamma^{op}))$ *is closed under extensions for any* $1\leq i\leq k$.

Proof By Corollary 17.3.5(2), we have that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). It follows from Theorems 17.5.4 and 17.5.3 that (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). The last assertion follows from [10, Theorem 3.3]. □

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We use I_i (resp. I_i') to denote the $(i+1)$-st term in a minimal injective resolution of ${}_\Lambda\Lambda$ (resp. Λ_Λ) for any $i \geq 0$. The following corollary generalizes [3]. In [3, Theorem 4.7], the assumption of Λ being a noetherian algebra is necessary for proving (5) $\Rightarrow$ (3). But here the assumption of Λ being a left and right noetherian ring is enough for all of the implications.

Corollary 17.5.6 *The following statements are equivalent.*

(1) s.grade$_\Lambda$Ext$_\Lambda^{i+1}(N, \Lambda) \geq i$ for any $N \in$mod Λ^{op} and $1 \leq i \leq k$.

(2) l.fd$_\Lambda(I_i) \leq i+1$ for any $0 \leq i \leq k-1$.

(3) grade$_\Lambda$Ext$_\Lambda^i(M, \Lambda) \geq i$ for any $M \in$mod Λ and $1 \leq i \leq k$.

(4) $\mathrm{Ext}_\Lambda^i(\mathrm{Ext}_\Lambda^i(\ ,\Lambda), \Lambda)$ *preserves monomorphisms $X \to Y$ with both X and Y torsionless in mod Λ for any $0 \leq i \leq k-1$.*

(5) $\Omega_\Lambda^i(\mathrm{mod}\ \Lambda^{op})$ *is closed under extensions for any $1 \leq i \leq k$.*

(6) add$\Omega_\Lambda^i(\mathrm{mod}\ \Lambda^{op})$ *(the subcategory of mod Λ^{op} whose objects are those modules which are direct summands of i-th syzygies) is closed under extensions for any $1 \leq i \leq k$.*

Proof By Theorem 17.5.5, we have that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). The equivalence of (1), (5) and (6) follows from the dual statements of [3, Theorem 1.7]. $\square$

At the end of this section, we generalize the result of Wakamatsu on the symmetry of k-Gorenstein modules.

Proposition 17.5.7 *Assume that m is a non-negative integer and U-lim.dim$_\Lambda(E_i) \leq i+1$ for any $0 \leq i \leq m-1$.*

(1) If U-lim.dim$_\Gamma(\bigoplus_{i=0}^m E_i') \leq m$, then U-lim.dim$_\Lambda(E_m) \leq m$; Especially, if l.id$_\Lambda(U) \leq m$, then U-lim.dim$_\Lambda(E_m) \leq m$.

(2) For a positive integer k, if U-lim.dim$_\Gamma(\bigoplus_{i=0}^m E_i') \leq m$ and U-lim.dim$_\Gamma(E_{m+j}') \leq m+j$ for any $1 \leq j \leq k-1$, then U-lim.dim$_\Lambda(E_{m+j}) \leq m+j$ for any $0 \leq j \leq k-1$.

Proof The case $m = 0$ follows from Theorem 16.1.8. Now suppose $m \geq 1$.

(1) By Corollaries 17.3.5 and 17.3.3, it suffices to prove that if s.grade$_U$Ext$_\Gamma^{i+1}(N, U) \geq i$ for any $N \in$mod Γ^{op} and $1 \leq i \leq m$ and s.grade$_U$Ext$_\Lambda^{m+1}(M, U) \geq m+1$ for any $M \in$mod Λ, then s.grade$_U$Ext$_\Gamma^{m+1}(N, U) \geq m+1$ for any $N \in$mod Γ^{op}.

Suppose that

$$\cdots \to Q_i \to \cdots \to Q_1 \to Q_0 \to N \to 0 \tag{8}$$

is a projective resolution of N in mod Γ^{op}.

By Lemma 17.3.4, we have that $\mathcal{T}_U^i(\mathrm{mod}\ \Gamma^{op}) = \Omega_U^i(\mathrm{mod}\ \Gamma^{op})$ for any $1 \leq i \leq m+1$. Notice that Coker$(Q_{m+1} \to Q_m)$ is m-syzygy, so, by Lemma 17.3.11, it is in $\Omega_U^m(\mathrm{mod}\ \Gamma^{op})$ and hence is in $\mathcal{T}_U^m(\mathrm{mod}\ \Gamma^{op})$, which implies that $\mathrm{Ext}_\Lambda^i(\mathrm{Tr}_U\Omega_\Gamma^m(N), U) = 0$ for any $1 \leq i \leq m$.

Let X be a submodule of $\mathrm{Ext}_\Gamma^{m+1}(N, U)$. Then grade$_U X \geq m$. By [9, Lemma 2], we have an exact sequence:

$$0 \to X \xrightarrow{f} \mathrm{Tr}_U\Omega_\Gamma^m(N) \to \mathrm{Coker} f \to 0.$$

We then get an embedding $0 \to \mathrm{Ext}_\Lambda^m(X, U) \to \mathrm{Ext}_\Lambda^{m+1}(\mathrm{Coker} f, U)$. By assumption, s.grade$_U$ $\mathrm{Ext}_\Lambda^{m+1}(\mathrm{Coker} f, U) \geq m+1$. So grade$_U\mathrm{Ext}_\Lambda^m(X, U) \geq m+1$. It follows from Lemma 17.3.9 that grade$_U X \geq m+1$ and s.grade$_U\mathrm{Ext}_\Gamma^{m+1}(N, U) \geq m+1$.

By Lemma 17.2.4(2) and the dual statement of Lemma 17.3.1, we have that U-lim.dim$_\Gamma$ $(\bigoplus_{i=0}^k E_i') \leq l$.id$_\Lambda(U)$. So the latter assertion follows from the former one.

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(2) We proceed by using induction on k. The case $k = 1$ is just (1).

Now suppose $k \geq 2$. By induction assumption, we have that U-lim.dim$_\Lambda(E_i) \leq i + 1$ for any $0 \leq i \leq m - 1$ and U-lim.dim$_\Lambda(E_{m+j}) \leq m + j$ for any $0 \leq j \leq k - 2$. By Corollaries 17.3.5 and 17.3.3, for any $N \in$mod Γ^{op}, we have that s.grade$_U \mathrm{Ext}_\Gamma^{i+1}(N, U) \geq i$ for any $1 \leq i \leq m$ and s.grade$_U \mathrm{Ext}_\Gamma^{m+j}(N, U) \geq m + j$ for any $1 \leq j \leq k - 1$. By Corollary 17.3.3, it suffices to prove that s.grade$_U \mathrm{Ext}_\Gamma^{m+k}(N, U) \geq m + k$.

Suppose that N has a projective resolution as (8). By Lemma 17.3.4, we have that $\mathcal{T}_U^i(\mathrm{mod}\ \Gamma^{op}) = \Omega_U^i(\mathrm{mod}\ \Gamma^{op})$ for any $1 \leq i \leq m+k$. Notice that Coker$(Q_{m+k} \to Q_{m+k-1})$ is $(m+k-1)$-syzygy, so, by Lemma 17.3.11, it is in $\Omega_U^{m+k-1}(\mathrm{mod}\ \Gamma^{op})$ and hence is in $\mathcal{T}_U^{m+k-1}(\mathrm{mod}\ \Gamma^{op})$, which implies that $\mathrm{Ext}_\Lambda^i(\mathrm{Tr}_U \Omega_\Gamma^{m+k-1}(N), U) = 0$ for any $1 \leq i \leq m + k - 1$.

By assumption, U-lim.dim$_\Gamma(\bigoplus_{i=0}^m E_i') \leq m$ and U-lim.dim$_\Gamma(E'_{m+j}) \leq m + j$ for any $1 \leq j \leq k - 1$. Then, by Corollary 17.3.3, we have that s.grade$_U \mathrm{Ext}_\Lambda^{m+k}(M, U) \geq m + k$ for any $M \in$mod Λ.

Let X be a submodule of $\mathrm{Ext}_\Gamma^{m+k}(N, U)$. Then grade$_U X \geq m + k - 1$. By [9, Lemma 2], we have an exact sequence:

$$0 \to X \xrightarrow{f} \mathrm{Tr}_U \Omega_\Gamma^{m+k-1}(N) \to \mathrm{Coker} f \to 0.$$

We then get an embedding $0 \to \mathrm{Ext}_\Lambda^{m+k-1}(X, U) \to \mathrm{Ext}_\Lambda^{m+k}(\mathrm{Coker} f, U)$. Since s.grade$_U$ $\mathrm{Ext}_\Lambda^{m+k}(\mathrm{Coker} f, U) \geq m + k$, grade$_U \mathrm{Ext}_\Lambda^{m+k-1}(X, U) \geq m + k$. It follows from Lemma 16.3.9 that grade$_U X \geq m + k$ and s.grade$_U \mathrm{Ext}_\Gamma^{m+k}(N, U) \geq m + k$. □

Putting $m = 0$, by Proposition 17.5.7(2), U-lim.dim$_\Lambda(E_i) \leq i$ for any $0 \leq i \leq k - 1$ if U-lim.dim$_\Gamma(E_i') \leq i$ for any $0 \leq i \leq k - 1$. Combining this result with Corollary 17.3.3(2) and their dual statements, we then get the symmetry of k-Gorenstein modules (see [21, Theorem 7.5]).

Putting ${}_\Lambda U_\Gamma = {}_\Lambda \Lambda_\Lambda$, the following corollary is an immediate consequence of Proposition 17.5.7, which is a generalization of the result of Auslander on the symmetry of k-Gorenstein rings.

Corollary 17.5.8 *Assume that m is a non-negative integer and l.fd*$_\Lambda(I_i) \leq i + 1$ *for any* $0 \leq i \leq m - 1$.

(1) If r.fd$_\Lambda(\bigoplus_{i=0}^m I_i') \leq m$*, then l.fd*$_\Lambda(I_m) \leq m$*; Especially, if l.id*$_\Lambda(\Lambda) \leq m$*, then l.fd*$_\Lambda(I_m) \leq m$.

(2) For a positive integer k, if r.fd$_\Lambda(\bigoplus_{i=0}^m I_i') \leq m$ *and r.fd*$_\Lambda(I'_{m+j}) \leq m + j$ *for any* $1 \leq j \leq k - 1$*, then l.fd*$_\Lambda(I_{m+j}) \leq m + j$ *for any* $0 \leq j \leq k - 1$.

When $m = 0$, the result in Corollary 17.5.8(2) is equivalent to the symmetry of k-Gorenstein rings (see [6, Theorem 3.7]). In the following, we give an example satisfying the conditions in Corollary 17.5.8 for the case $m = 1$ and $k = 2$ as follows.

Example 17.5.9 Let K be a field and Λ a finite dimensional K-algebra which is given by the quiver:

$$\begin{tikzcd}
1 \arrow[dr, "\alpha"] & & 4 \\
& 3 \arrow[ur, "\beta"] \arrow[dr] & \\
2 \arrow[ur] & & 5
\end{tikzcd}$$

modulo the ideal $\beta\alpha$. Then $l.\mathrm{fd}_\Lambda(I_0)$= $l.\mathrm{fd}_\Lambda(I_1)$=$r.\mathrm{fd}_\Lambda(I_0')$=$r.\mathrm{fd}_\Lambda(I_1')$=1, $l.\mathrm{fd}_\Lambda(I_2)$=$r.\mathrm{fd}_\Lambda(I_2')$=2 and $l.\mathrm{id}_\Lambda(\Lambda)$ =$r.\mathrm{id}_\Lambda(\Lambda)$=2.

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Chapter 18

Γ-Separated Covers

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18.1 Introduction

Rings with a complex module category can often be studied by considering covers of their modules in a subcategory related to an overring with a much simpler structure. For example, by a classical result of Enochs [10], all modules over a commutative domain R have torsion free covers, that is, covers by R-submodules of Q-vector spaces where Q is the quotient field of R. A general theory of covers was developed by Enochs' school, proving the Flat Cover Conjecture (FCC) in [4] and other interesting results, cf. [11, 19].

On the other hand, a structure theory of finitely generated modules over a class of commutative rings called "Dedekind-like" was recently introduced by Klingler and Levy [14, 15]. We postpone the somewhat technical definition of these rings [Definition 18.4.1], and the reason for this definition [Remark 18.4.2], except to say that they are commutative, reduced (no nonzero nilpotent elements), noetherian rings. Some interesting examples of these rings are (see [15, Examples 2.2]):

Naturally occurring examples of Dedekind-like rings.
(E-1) $\mathbb{Z}[\sqrt{n}\,]$ when n is squarefree.
(E-2) Integral group ring $\mathbb{Z}G_n$ (cyclic order n) when n is squarefree.
(E-3) All subrings of squarefree index in $\mathbb{Z} \oplus \dots \oplus \mathbb{Z}$.
(E-4) $\mathbb{R} + x\mathbb{C}[x]$ and $\mathbb{R} + x\mathbb{C}[[x]]$.
(E-5) $k[x, y]/(xy)$ for any field k.

In connection with (E-1) we note that Dedekind-like rings of algebraic integers seem to be the only non integrally-closed rings of algebraic integers whose finitely generated module category has been

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described since Steinitz did the integrally closed case in 1911 [18], in his description of modules over (what are now called) Dedekind domains.

The relevance of these rings to the present note is the following. The *normalization* Γ of an arbitrary Dedekind-like ring Λ is a direct product of Dedekind domains, and hence the structure of mod-Γ is known by Steinitz's work. Klingler and Levy call Λ-modules "Γ-separated" if they are Λ-submodules of Γ-modules. Their approach to the description of mod-Λ is to make use of what they call "Γ-separated covers" of Λ-modules [Definition 18.3.1 below]. These reduce the description of mod-Λ to the much simpler (and known) structure of mod-Γ. These covers are similar to — but not exactly the same as — covers in the sense of Enochs. For example (unlike torsionfree modules over integral domains or flat modules over any ring), the class of Γ-separated Λ-modules is not closed under extensions when Λ is Dedekind-like [Theorem 18.4.10 and Example 18.4.12].

The main purpose of this note is to clarify the precise relation of these two types of covers, and use this to improve some of the Klingler-Levy results.

Notation 18.1.1 Throughout this note Λ denotes a subring of a ring Γ. $\mathcal{G}$ denotes the class of all (say, right) Γ-*separated* Λ-modules; that is, all Λ-submodules of Γ-modules. $\mathcal{G}_0$ denotes the class of all finitely generated modules in $\mathcal{G}$. Mod-Λ and mod-Λ denote the classes, respectively, of all Λ-modules and finitely generated Λ-modules for any ring Λ — for definiteness: right modules unless the contrary is stated. Thus $\mathcal{G}_0 = \mathcal{G} \cap \text{mod-}\Lambda$. The notation $\mathcal{P}$ and $\mathcal{F}$ denotes the classes of projective and flat Λ-modules, respectively.

We say that two Λ-homomorphisms $f_1\colon H_1 \to M$ and $f_2\colon H_2 \to M$ are *isomorphic* if there is a Λ-isomorphism $\beta\colon H_1 \to H_2$ such that $f_1 = f_2\beta$. For example projective covers of a module are isomorphic.

We review the definitions of covers and covering classes in 18.2.1. In Theorem 18.2.5 we prove that $\mathcal{G}$ is a covering class. We introduce the definition of Γ-separated cover in 18.3.1, for an arbitrary pair $\Lambda \subseteq \Gamma$. In Theorem 18.3.3, we show that the (always unique) $\mathcal{G}$-cover is the largest among the (possibly nonunique) Γ-separated covers of a module.

If Λ is Dedekind-like and its normalization Γ is a finitely generated Λ-module, we show that $\mathcal{G}$-covers and Γ-separated covers of arbitrary Λ-modules coincide [Theorem 18.4.8(i)]. This answers, in the affirmative, Klingler-Levy's question [14, Remarks 4.8] of whether Γ-separated covers of infinitely generated Λ-modules exist in this 'classical' situation.

For arbitrary Dedekind-like rings (i.e., Γ_Λ not necessarily finitely generated), we show that $\mathcal{G}$-covers and Γ-separated covers of finitely generated Λ-modules coincide [Theorem 18.4.8(ii)], thus making the general theory of covers available for use here.

To deal with the fact that the class of Γ-separated Λ-modules is not closed under extensions, we make use of El Bashir's generalization [9] of FCC, providing covers in certain classes of modules not closed under extensions [Lemma 18.2.2]. In fact, for noetherian rings closure under extensions in the general setting is equivalent to the setting being a cotilting one [Theorem 18.2.5(ii)].

18.2 $\mathcal{G}$-Covers

We begin by recalling the basics of the theory of covers of modules over an arbitrary ring Λ.

18.2.1 Covers Let M be a Λ-module, $\mathcal{C}$ a class of Λ-modules, and $f : C \to M$ a Λ-homomorphism with $C \in \mathcal{C}$. Then f is a $\mathcal{C}$-*precover* of M provided that for each $C' \in \mathcal{C}$ and each Λ-homomorphism $f' : C' \to M$, f' factorizes through f (that is, there is a Λ-homomorphism $g : C' \to C$ such that $f' = fg$).

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The $\mathcal{C}$-precover f is *special* if f is surjective and $\mathrm{Ext}^1_\Lambda(\mathcal{C}, \ker(f)) = 0$. The $\mathcal{C}$-precover f is a *$\mathcal{C}$-cover* of M if f is *right minimal* (that is, each endomorphism g of C satisfying $fg = f$ is an automorphism). If $\mathcal{C}$ is closed under extensions and contains all projective modules then any $\mathcal{C}$-cover is special by the Wakamatsu Lemma [19, 2.1.1].

$\mathcal{C}$ is a *precovering (special precovering, covering)* class provided that each module $M \in$ Mod-Λ has a $\mathcal{C}$-precover (a special $\mathcal{C}$-precover, a $\mathcal{C}$-cover).

In general, $\mathcal{C}$-precovers need not exist, and the existence of a $\mathcal{C}$-precover of a module M does not imply existence of a $\mathcal{C}$-cover of M. However, any $\mathcal{C}$-cover is easily seen to be unique up to isomorphism of maps, as defined in Notation 18.1.1.

We call a class $\mathcal{C} \subseteq$ mod-Λ *contravariantly finite* provided that each $M \in$ mod-Λ has a $\mathcal{C}$-cover.

For example, the class $\mathcal{P}$ of all projective modules is a precovering class for any ring Λ. By a classical result of Bass, $\mathcal{P}$ is a covering class iff Λ is right perfect. The solution of the Flat Cover Conjecture (FCC) in [4] says that the class $\mathcal{F}$ of all flat modules is a covering class for any ring. In fact, both proofs of FCC in [4] have generalizations showing that covers are rather frequent, as the next lemma shows. For a class of modules $\mathcal{C}$, we denote by $\varinjlim \mathcal{C}$ the class of all modules that are direct limits of direct systems of modules from $\mathcal{C}$; for example, $\mathcal{F} = \varinjlim \mathcal{P}$.

Lemma 18.2.2 *Let $\mathcal{C}$ be a class of Λ-modules closed under finite direct sums and direct limits. Assume there is a subset $\mathcal{S} \subseteq \mathcal{C}$ such that $\mathcal{C} = \varinjlim \mathcal{S}$. Then $\mathcal{C}$ is a covering class.*

Proof The lemma is a particular case of [9, Theorem 3.2] which proves the same result for arbitrary Grothendieck categories. □

18.2.3 Cotilting classes For a module M, denote by $\mathrm{Cog}(M)$ the class of all modules cogenerated by M, that is, of all modules isomorphic to submodules of arbitrary direct products of copies of M. For a class of modules $\mathcal{C}$, put ${}^\perp\mathcal{C} = \mathrm{Ker}\,\mathrm{Ext}^1_\Lambda(-, \mathcal{C}) = \{M \in \text{Mod-}\Lambda \mid \mathrm{Ext}^1_\Lambda(M, C) = 0 \text{ for all } C \in \mathcal{C}\}$.

A Λ-module C is a *cotilting module* provided that $\mathrm{Cog}(C) = {}^\perp C$. Equivalently, C is cotilting iff C has injective dimension ≤ 1, $\mathrm{Ext}^1_\Lambda(C^I, C) = 0$ for any set I, and there are an injective cogenerator W for Mod-Λ and an exact sequence $0 \to C_1 \to C_0 \to W \to 0$ where C_0 and C_1 are direct summands in a (possibly infinite) direct product of copies of C. The latter definition is much longer, but shows that cotilting modules are just the category-theoretic duals of the better known (infinitely generated) tilting modules. Indeed, cotilting modules are close to being "dual": each cotilting module is pure-injective, [3, Theorem 2.8].

A class of modules $\mathcal{C}$ is a *cotilting class* provided there is a cotilting module C such that $\mathcal{C} = \mathrm{Cog}(C)$. By [8, Corollary 10], each cotilting class is a covering class in the sense of 18.2.1. In fact, cotilting classes are exactly the special precovering classes closed under products and submodules, [1, Theorem 2.5].

Lemma 18.2.4 *[5] Let Λ be a right noetherian ring. Let $\mathcal{S}$ be a class of finitely presented Λ-modules such that $\Lambda \in \mathcal{S}$, $\mathcal{S}$ is closed under finite direct sums, submodules, and extensions. Let $\mathcal{C} = \varinjlim \mathcal{S}$. Then $\mathcal{C}$ is a cotilting class.*

Proof By [7, Lemma 4.4], $\mathcal{C}$ is a torsion-free class in Mod-Λ. By [2, Lemma 2.1(iii) and Theorem 2.3], $\mathcal{C} = {}^\perp\mathcal{I}$ for a class of pure-injective modules $\mathcal{I}$, so $\mathcal{C}$ is a covering class by [8, Corollary 10]. By the Wakamatsu lemma and [1, Theorem 2.5], $\mathcal{C}$ is a cotilting class. □

A module M is *cotorsion* if $\mathrm{Ext}^1_\Lambda(\mathcal{F}, M) = 0$. For example, any pure-injective module is cotorsion.

Theorem 18.2.5 *Let Λ be a ring. Then (notation as in 18.1.1):*

(i) $\mathcal{G} = \varinjlim \mathcal{G}_0$, and $\mathcal{G}$ is a covering class containing $\mathcal{F}$ and closed under $\varinjlim$. Each $\mathcal{G}$-cover is a Λ-epimorphism.

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(ii) Assume Λ is right noetherian. Then $\mathcal{G}$ is a cotilting class if and only if $\mathcal{G}_0$ is closed under extensions. In this case, for any Λ-module M, the (unique) $\mathcal{G}$-cover of M, $g : G \to M$, is special, and $\ker(g)$ *is a cotorsion Λ-module of injective dimension ≤ 1.*

Proof (i) Clearly $\mathcal{G}$ is closed under submodules and products. Since $\Lambda \in \mathcal{G}$, we have $\mathcal{P} \subseteq \mathcal{G}$. *Caution:* But $\mathcal{G}$ need not be closed under extensions, as we show in Theorem 18.4.10 and Example 18.4.12.

Since $\mathcal{G}$ is closed under submodules, we have $\mathcal{G} \subseteq \varinjlim \mathcal{G}_0$.

Let $M \in \mathcal{G}$. Then there is $N \in$ Mod-Γ such that $M \subseteq N$. Consider the canonical maps $\nu_M : M \to M \otimes_\Lambda \Gamma$ and $\eta_M : M \otimes_\Lambda \Gamma \to M{\cdot}\Gamma\ (\subseteq N)$. Since $\eta_M \nu_M$ equals the identity on M, ν_M is monic.

Let $M = \varinjlim_{i \in I} G_i$ where $(I, \leq)$ is an upper directed set and $(G_i, g_{ij} \mid i \leq j \in I)$ a direct system of elements of $\mathcal{G}$; in particular, for each $i \in I$, the map ν_{G_i} is monic. The induced system $(G_i \otimes_\Lambda \Gamma, g_{ij} \otimes_\Lambda 1_\Gamma \mid i \leq j \in I)$ of Γ-modules is also direct, and for all $i \leq j \in I$, there is a commutative diagram

$$\begin{array}{ccc} G_i & \xrightarrow{\nu_{G_i}} & G_i \otimes_\Lambda \Gamma \\ {\scriptstyle g_{ij}}\downarrow & & \downarrow{\scriptstyle g_{ij}\otimes_\Lambda 1_\Gamma} \\ G_j & \xrightarrow{\nu_{G_j}} & G_j \otimes_\Lambda \Gamma \end{array}$$

Since $\varinjlim$ is a left exact functor, we infer that the canonical Λ-homomorphism $M \to \varinjlim_{i \in I}(G_i \otimes_\Lambda \Gamma)$ is monic. Since the functor $- \otimes_\Lambda \Gamma$ commutes with direct limits, we also have the canonical Γ-isomorphism $\varinjlim_{i \in I}(G_i \otimes_\Lambda \Gamma) \cong M \otimes_\Lambda \Gamma$. It follows that $M \in \mathcal{G}$, so $\varinjlim \mathcal{G}_0 \subseteq \varinjlim \mathcal{G} \subseteq \mathcal{G}$, and hence $\mathcal{G} = \varinjlim \mathcal{G}_0$ is closed under direct limits.

By Lemma 18.2.2, $\mathcal{G}$ is a covering class of right Λ-modules. Since $\mathcal{P} \in \mathcal{G}$, each $\mathcal{G}$-cover is a Λ-epimorphism.

(ii) If $\mathcal{G}$ is a cotilting class then $\mathcal{G}$, and hence also $\mathcal{G}_0$, is closed under extensions. Conversely, since $\mathcal{G}_0$ consists of finitely presented modules, and $\mathcal{G} = \varinjlim \mathcal{G}_0$ by part (i), $\mathcal{G}$ is a cotilting class by Lemma 18.2.4.

Finally, since $\mathcal{P} \subseteq \mathcal{G}$ and $\mathcal{G}$ is closed under extensions, $\mathcal{G}$-covers are special by the Wakamatsu lemma. In particular, if K is the kernel of a $\mathcal{G}$-cover then $\mathrm{Ext}^1_\Lambda(\mathcal{F}, K) = 0$ by part (i), that is, K is a cotorsion module. Since $\mathcal{G} = {}^\perp\{C\}$ where C has injective dimension ≤ 1, the condition $\mathrm{Ext}^1_R(\mathcal{G}, K) = 0$ implies that K has injective dimension ≤ 1 by the Baer Test of Injectivity and dimension shifting. □

Remark 18.2.6 Though $\mathcal{G}$ is a covering class closed under submodules and products, it is not cotilting in general: $\mathcal{G}_0$ is not closed under extensions for any Dedekind-like ring $\Lambda \neq \Gamma$ [Theorem 18.4.10 and Example 18.4.12]. In that case, $\mathcal{G}$ is not special precovering; in fact, if W is an injective cogenerator for Mod-Λ then the $\mathcal{G}$-cover of W is not special by [1, Theorem 2.5].

Example 18.2.7 Let Λ be a commutative domain, $\Gamma = E(\Lambda)$ its quotient field, and $K = \Gamma/\Lambda$. To avoid trivialities, we assume $K \neq 0$. We show:

(i) $\mathcal{G}$ (= the class of all torsionfree modules) is a cotilting class.

(ii) Assume that Λ is noetherian and is not a complete local ring. Then the $\mathcal{G}$-cover (= torsionfree cover) of every nonzero Λ-module of finite length is infinitely generated.

Proof (i) $\mathcal{G}$ is a covering class by [10]. So by the Wakamatsu lemma and [1, Theorem 2.5], $\mathcal{G}$ is a cotilting class. In fact, it is easy to construct a cotilting module C cogenerating the class $\mathcal{G}$ as follows.

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We have $\mathcal{G} = \mathrm{Cog}_\Lambda(\Gamma)$. By [6, VII.2.2], $\mathcal{G} = \{M \mid \mathrm{Tor}_1^R(M, K) = 0\}$, and the Ext-Tor relations [6, VI.5.1] yield that $\mathcal{G} = {}^\perp\{K^*\}$ where $K^* = \mathrm{Hom}_{\mathbb{Z}}(K, \mathbb{Q}/\mathbb{Z})$. Moreover, K^* is a torsion-free Λ-module by [6, VII.1.5]. So $C = \Gamma \oplus K^*$ is a cotilting module such that $\mathcal{G} = \mathrm{Cog}(C) = {}^\perp\{C\}$.

(ii) Let $0 \neq M \in$ mod-Λ have finite length, and assume that its torsionfree cover $f\colon F \to M$ is finitely generated. Since M has finite length and is nonzero, it has a simple submodule, necessarily isomorphic to $k(\mathfrak{m}) = \Lambda/\mathfrak{m} = \Lambda_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} = \hat{\Lambda}_\mathfrak{m}/\hat{\mathfrak{m}}_\mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of Λ, and $\hat{\Lambda}_\mathfrak{m}$ the $\mathfrak{m}$-adic completion of $\Lambda_\mathfrak{m}$. Therefore there exists a nonzero Λ-homomorphism $g\colon \Lambda_\mathfrak{m} \to M$, and g factors through the torsionfree cover of M, say $g\colon \Lambda_\mathfrak{m} \xrightarrow{h} F \xrightarrow{f} M$. We consider two cases.

Case 1: Λ is not a local ring. We claim that $h(1) \neq 0$. Otherwise $h(\Lambda) = \Lambda h(1) = 0$. Then, for any $x/d \in \Lambda_\mathfrak{m}$ we have $d{\cdot}h(x/d) = h(x) = 0$. Since F is torsionfree, we have $h(x/d) = 0$; that is, $h(\Lambda_\mathfrak{m}) = 0$, a contradiction.

Next we claim that h is monic. Suppose not. Then $h(x/d) = 0$ for some $x, d \in \Lambda$ with $x \neq 0$ and $d \notin \mathfrak{m}$. Then $h(x) = d{\cdot}h(x/d) = 0$, and hence $x{\cdot}h(1) = 0$. Since F is torsionfree, this yields the contradiction $h(1) = 0$. Thus h is monic.

Since F is finitely generated and Λ is noetherian, the image of the monic map h is finitely generated; and hence $\Lambda_\mathfrak{m}$ is a finitely generated Λ-module. Let $0 \neq d \in R$ be a common denominator for some finite set of generators. Then $d\Lambda_\mathfrak{m} \subseteq \Lambda$.

Since Λ is not local, it has a maximal ideal $\mathfrak{n} \neq \mathfrak{m}$. Choose an element $x \in \mathfrak{n} - \mathfrak{m}$. Then x is a unit in $\Lambda_\mathfrak{m}$, and hence $dx^{-i} \in \Lambda$ for every positive integer i. But then $d \in \cap_{i=1}^\infty \mathfrak{n}^i$. Since Λ is a noetherian domain this intersection equals zero, by the Krull intersection theorem. Thus we have the contradiction that $d = 0$.

Case 2: Λ is local with maximal ideal $\mathfrak{m}$, and $\Lambda \neq \hat{\Lambda}_\mathfrak{m}$. First we prove a simple lemma, for which we do not know a reference: *If $\hat{\Lambda}_\mathfrak{m}$ is a finitely generated Λ-module, then $\Lambda = \hat{\Lambda}_\mathfrak{m}$.* We want to show that the natural map $\nu\colon \Lambda \to \hat{\Lambda}$ is an isomorphism. Since $\hat{\Lambda}_\mathfrak{m}$ is a faithfully flat Λ-module, it suffices to show that the induced map $\hat{\nu}\colon \hat{\Lambda}_\mathfrak{m} \otimes_\Lambda \Lambda \to \hat{\Lambda}_\mathfrak{m} \otimes_\Lambda \hat{\Lambda}_\mathfrak{m}$ is an isomorphism. Since, moreover, both Λ and $\hat{\Lambda}_\mathfrak{m}$ are finitely generated Λ-modules, their $\mathfrak{m}$-adic completions are given by tensoring with $\hat{\Lambda}_\mathfrak{m}$. Therefore $\hat{\nu}$ can be identified with the identity map on $\hat{\Lambda}_\mathfrak{m}$. In particular, it is an isomorphism, completing the proof of the lemma.

As in the paragraph before Case 1, there is a nonzero map $g'\colon \hat{\Lambda}_\mathfrak{m} \to M$, and g' factors through the torsionfree cover of M, say $g'\colon \hat{\Lambda}_\mathfrak{m} \xrightarrow{h'} F \xrightarrow{f} M$. We claim that the restriction $h = h' \restriction \Lambda$ is nonzero.

Suppose that $h = 0$, and choose any $\hat{x} \in \hat{\Lambda}$. Say $\hat{x} = \lim_{n=1}^\infty x_n$ with each $x_n \in \Lambda$. By passing to a subsequence, we may assume that $\hat{x} - x_n \in \hat{\mathfrak{m}}^n = \mathfrak{m}^n\hat{\Lambda}$ for all $\mathfrak{n}$. Then $h'(\hat{x}) \in \cap_n \mathfrak{m}^n F$ which equals zero by the Krull intersection theorem since F is finitely generated. Thus we have the contradiction that $h' = 0$, proving the claim.

Next note that $h\colon \Lambda \to F$ is monic because Λ/B is a torsion module for every nonzero ideal B and F is torsionfree. Therefore we may assume that $\Lambda \subseteq F$ and h equals the identity on Λ. We claim that h' is monic.

Take any $\hat{x} \in \mathrm{Ker}(h')$ and, as above, write $\hat{x} = \lim_{n=1}^\infty x_n$, the limit of a Cauchy sequence in Λ with $\hat{x} - x_n \in \mathfrak{m}^n\hat{\Lambda}_\mathfrak{m}$ for every n. Since h' equals the identity on Λ, applying h' to the previous "$\in$" statement yields $x_n \in \mathfrak{m}^n F$ for all n. Therefore the sequence $x_1, x_2, \ldots$ is a Cauchy sequence in F converging to 0. Since Λ is a submodule of the finitely generated Λ-module F, the $\mathfrak{m}$-adic topology on Λ coincides with the topology induced by the $\mathfrak{m}$-adic topology on F [17, Theorem 8.6]. Therefore the sequence $x_1, x_2, \ldots$ is also a Cauchy sequence in Λ converging to 0. Therefore the limit $\hat{x}$ of this sequence equals 0, completing the proof of the claim.

Since F_Λ is finitely generated and h' is monic, we see that $\hat{\Lambda}_\mathfrak{m}$ is a finitely generated Λ-module. Therefore the lemma at the beginning of Case 2 of this proof yields the contradiction $\Lambda = \hat{\Lambda}$. $\square$

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18.3 Γ-Separated Covers

In this section we define Γ-separated covers, and compare them with $\mathcal{G}$-covers and $\mathcal{G}_0$-covers. We do this in the context of arbitrary rings — a much more general context than that considered by Klingler and Levy. In this generality, Γ-separated covers are easily seen not to be unique [Example 18.3.5], but they always exist [Theorem 18.3.2].

Definition 18.3.1 Let Λ and Γ be given (as in Notation 18.1.1), which determines $\mathcal{G}$ and $\mathcal{G}_0$.

We call a homomorphism $g : G \to M$ of Λ-modules a Γ-*separated cover* of M provided that:

(i) g is surjective;

(ii) $G \in \mathcal{G}$; and

(iii) In every factorization $G \xrightarrow{h} G' \xrightarrow{g'} M$ of g, with h surjective and $G' \in \mathcal{G}$, h must be an isomorphism. (Intuitively: G' is no closer to M than G is.)

Notice that g is close to being right minimal: If $h : G \to G$ is a Λ-homomorphism such that $g = gh$ then h is a monomorphism. However, h need not be an isomorphism in the present generality: let $\Lambda = \mathbb{Z}$, $\Gamma = \mathbb{Q}$, and $g : \mathbb{Z} \to \mathbb{Z}_2$ be the Γ-separated cover of $\mathbb{Z}_2$ given by the projection modulo 2. Then $g = gh$ where (the nonsurjective map) $h : \mathbb{Z} \to \mathbb{Z}$ maps 1 to 3. However (in less generality) see Theorem 18.4.8.

If $g : G \to M$ is a Γ-separated cover of a finitely generated module M such that $G \in \mathcal{G}_0$ then g is called a *finitely generated* Γ-*separated cover* of M.

For Λ right noetherian and M_Λ finitely generated, Klingler and Levy observed that (for any Γ) it is trivial that M has a Γ-separated cover [14, Proposition 4.7]. In [14, 4.8] they cited several instances in which the finite generation hypothesis can be dropped, and asked whethether it is always unnecessary. Our first result shows that this is indeed the case.

Theorem 18.3.2 *Every right Λ-module has a Γ-separated cover. In more detail:*

(i) Let $f\colon H \to M$ be a Λ-epimorphism with $H \in \mathcal{G}$. Then there exists a factorization $H \xrightarrow{h} G \xrightarrow{g} M$ of f such that h is surjective and g is a Γ-separated cover of M.

(ii) If M is a κ-generated Λ-module (κ any finite or infinite cardinal) then M has a Γ-separated cover $g\colon G \to M$ where G is κ-generated.

Proof (i) Let $\mathcal{K}$ be the set of all submodules $K \subseteq \ker(f)$ such that $H/K \in \mathcal{G}$. It suffices to show that $\mathcal{K}$ has a maximal submodule K_0 (with respect to $\subseteq$). For then the factorization $H \xrightarrow{h} G = H/K_0 \xrightarrow{g} M$ of f satisfies the desired conditions. Therefore, by a simple application of Zorn's Lemma, it suffices to show that the union U of every totally ordered subset $\mathcal{T}$ of $\mathcal{K}$ is again in $\mathcal{K}$; that is, $H/U \in \mathcal{G}$.

Every $T' \subseteq T'' \in \mathcal{T}$ induces a natural map $H/T' \to H/T''$, and these maps make the set of modules $\{H/T \mid T \in \mathcal{T}\}$ into a direct (in fact, totally ordered) system whose direct limit is H/U (because the maps in the system are surjective). Since $\mathcal{G}$ is closed under direct limits [Theorem 18.2.5(i)], we see that $H/U \in \mathcal{G}$, completing the proof.

(ii) If M is κ-generated then applying part (i) to an epimorphism $f\colon H = \Lambda^{(\kappa)} \to M$, we get a Γ-separated cover $g\colon G \to M$ with $f = gh$ for an epimorphism $h : \Lambda^{(\kappa)} \to G$, so G is κ-generated. □

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Although Γ-separated covers are not always unique, in the generality considered in this section, there is a unique largest such cover of any M, namely the $\mathcal{G}$-cover f of M; and all other Γ-separated covers of M are isomorphic to restrictions of f, as described in the next theorem.

Theorem 18.3.3 *Let Λ be a ring, M a Λ-module, $f\colon H \to M$ the $\mathcal{G}$-cover of M, and $g\colon G \to M$ a Γ-separated cover of M. Then:*

(i) f is a Γ-separated cover of M.

(ii) There is a submodule $H' \subseteq H$ such that the restriction $f \restriction H'$ is a Γ-separated cover of M isomorphic to g.

(iii) If g is a $\mathcal{G}$-precover of M then g is the $\mathcal{G}$-cover of M (necessarily isomorphic to f).

Proof (ii) Since f is a $\mathcal{G}$-precover, there is a factorization $g\colon G \xrightarrow{h} H \xrightarrow{f} M$. Put $H' = \mathrm{Im}(h)$. Since $H' \in \mathcal{G}$ and g is a Γ-separated cover of M, we have $\mathrm{Ker}(h) = 0$, as desired.

(i) Since $\mathcal{G}$-covers are surjective [Theorem 18.2.5], Theorem 18.3.2 yields a factorization $f\colon H \xrightarrow{h} G' \xrightarrow{g'} M$, with h and g' surjective, such that g' is a Γ-separated cover of M. Part (ii) yields a submodule $H' \subseteq H$ such that $f \restriction H'$ is isomorphic to g'. Thus there is an isomorphism $\theta\colon G' \cong H'$ such that $g' = f\theta$.

We also have $f = g'h$, and therefore $f = f(\theta h)$. Since f is right minimal, we have $\mathrm{Ker}(h) = 0$ and $H' = H$; that is, f and g' are isomorphic Γ-separated covers of M.

(iii) Since g is a precover there is a factorization $f = g\alpha$ for some $\alpha\colon H \to G$. Since f is a $\mathcal{G}$-cover, there is a factorization $g = f\beta$ for some $\beta\colon G \to H$. Therefore $f = f(\beta\alpha)$. Right minimality of f implies that $\beta\alpha$ is an automorphism of H, and hence β is surjective. Since g is a Γ-separated cover, β is an isomorphism $G \cong H$; and this shows that g is isomorphic to the $\mathcal{G}$-cover f, and hence is itself a $\mathcal{G}$-cover. □

There is a similar result for finitely generated modules:

Theorem 18.3.4 *Let Λ be a ring, M a finitely generated Λ-module, $f\colon H \to M$ the $\mathcal{G}$-cover of M. Assume there exists a $\mathcal{G}_0$-cover $f_0\colon H_0 \to M$. Then:*

(i) f_0 is a finitely generated Γ-separated cover of M.

(ii) Every finitely generated Γ-separated cover of M is isomorphic to a restriction of f_0.

(iii) Let g be a finitely generated Γ-separated cover of M. If g is a $\mathcal{G}_0$-precover of M, then g is a $\mathcal{G}_0$-cover of M (necessarily isomorphic to f_0).

(iv) There is a finitely generated pure submodule $H' \subseteq H$ such that $f \restriction H'$ is a $\mathcal{G}_0$-cover of M isomorphic to f_0.

Proof (i) We claim that f_0 is surjective. There is a surjective map $\phi\colon F \to M$ with F free of finite rank. The claim holds since ϕ factors through f_0.

Now choose a factorization $f_0 = \beta\alpha$ with both factors surjective, $\alpha\colon H_0 \to K_0$, and $K_0 \in \mathcal{G}_0$. We need to show tht α is monic. Since f_0 is a $\mathcal{G}_0$-cover, there is a factorization $\beta = f_0\gamma$. Then right minimality of $f_0 = f_0(\gamma\alpha)$ shows that $\gamma\alpha$ is an automorphism of H_0, and hence α is monic.

(ii) and (iii) We omit the details which are the same as in the proof of Theorem 18.3.3, (ii) and (iii), with $\mathcal{G}_0$ replacing $\mathcal{G}$.

(iv) By Theorem 18.3.3(ii) and by part (i), there is a finitely generated submodule $H' \subseteq H$ such that $f \restriction H'$ is a Γ-separated cover isomorphic to f_0. Since $f \restriction H'$ is isomorphic to the $\mathcal{G}_0$-cover f_0, we see that $f \restriction H'$ is a $\mathcal{G}_0$-cover of M, as desired.

It now suffices to prove that H' is pure in H. H is the directed union of all finitely generated Λ-modules L such that $H' \subseteq L \subseteq H$. Therefore if we can show that H' is a direct summand of

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every such L — and hence pure in L — we will know that H' is pure in the directed union H of these submodules L. Fix such an L and, for brevity, write $f_L = f \restriction L$ and $f_H = f \restriction H$.

Since $f(H') = M$ and $H' \subseteq L$ we also have $f(L) = M$, and therefore we have $H' + \ker(f_L) = L$. It therefore suffices to show that $H' \cap \ker(f_L) = 0$.

Since $f_{H'}$ is a $\mathcal{G}_0$-cover of M there is a factorization $f\colon L \xrightarrow{\pi} H' \xrightarrow{f} M$. Thus $f = f\pi$ on L and hence on H'. Then right minimality of $f_{H'}$ shows that π is an automorphism on H'. In particular, $H' \cap \ker(f_L) = 0$, as desired. □

Example 18.3.5 (Non-uniqueness of Γ-separated covers) Let Λ be a commutative domain, Γ the quotient field of Λ, and assume that $\Lambda \neq \Gamma$. Thus $\mathcal{G}$ is the class of all torsion-free Λ-modules. We show:

(i) If Λ has a nonprincipal finitely generated ideal, then some simple Λ-module k has nonisomorphic finitely generated Γ-separated covers.

(ii) If Λ is noetherian but not local and complete, then k also has an infinitely generated Γ-separated cover.

Proof (i) Let $A \neq 0$ be a finitely generated ideal of Λ. Then A has a maximal Λ-submodule, and hence A maps onto a simple Λ-module k.

We claim that every epimorphism $g\colon A \to k$ is a Γ-separated cover. We need to show that there is no surjective factorization $g\colon A \to A/B \to k$ with A/B torsionfree and $B \neq 0$. But since any nonzero element of B annihilates A/B, this is obvious.

Thus, choosing A to be finitely generated and nonprincipal we get one Γ-separated cover $A \to k$. A second such cover, not isomorphic to the first, is (the case $A = \Lambda$:) any surjection $\Lambda \to k$.

(ii) Let $g\colon G \to k$ be the $\mathcal{G}$ (= torsionfree) cover of k, and hence a Γ-separated cover of k [Theorem 18.3.3(i)]. Since k_Λ has finite length, Example 18.2.7(ii) shows that G_Λ is not finitely generated. □

18.4 The Dedekind-Like Case

In this section we define Dedekind-like rings, and give the reason for this rather technical definition. Then we compare Γ-separated covers with $\mathcal{G}$-covers and $\mathcal{G}_0$-covers in the context of these rings.

Definition 18.4.1 Let Λ be a reduced (no nonzero nilpotent elements) commutative noetherian ring with normalization Γ. Following [15, 10.1], we call Λ *Dedekind-like* provided that the following conditions hold:

(i) Γ is a direct sum of Dedekind domains.

(ii) $(\Gamma/\Lambda)_{\mathfrak{m}}$ is either a simple $\Lambda_{\mathfrak{m}}$-module or 0 for all maximal ideals $\mathfrak{m}$ of Λ.

(iii) $\mathfrak{m}_{\mathfrak{m}} = \operatorname{rad}(\Gamma_{\mathfrak{m}})$ in $\Gamma_{\mathfrak{m}}$ (the Jacobson radical) for all maximal ideals $\mathfrak{m}$ of Λ.

We do not consider fields to be Dedekind domains. Therefore Dedekind-like rings have Krull dimension 1.

We call a Dedekind-like ring *classical* if Γ is a finitely generated Λ-module. All of examples (E-1)–(E-5) in the Introduction to this note are *classical Dedekind-like* rings. An example of a nonclassical Dedekind-like ring is constructed in [12].

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Remark 18.4.2 (Reason for the name "Dedekind-like") Let Ω be a commutative noetherian ring. For the purpose of discussing Ω-modules we assume, without loss of generality, that the ring Ω is indecomposable.

In [15, Theorem 14.5] it is proved that if the category of Ω-modules of finite length does not have wild representation type, then Ω is either a homomorphic image of a Dedekind-like ring or else is an artinian local ring of (composition) length 4, called a "Klein ring".

Then [15] describes the detailed structure of finitely generated Λ-modules when Λ is Dedekind-like, extending Steinitz's well-known results for Dedekind domains [18]. There is a possible slight exception to this new structure theory, involving characteristic 2 [15, Additional Hypothesis 10.2]. But this possible exception does not apply to the results in the present paper.

We note tbat the structure of finitely generated modules over Klein rings can also be described [14, §11].

There is a formal relation between the classical and nonclassical cases that we need:

Lemma 18.4.3 *Let $\mathfrak{m}$ be a maximal ideal of a Dedekind-like ring Λ with normalization Γ. Then the local ring $\Lambda_{\mathfrak{m}}$ is a classical Dedekind-like ring with normalization $\Gamma_{\mathfrak{m}}$. Moreover, if $g\colon G \to M_\Lambda$ is a finitely generated Γ-separated cover, then $g_{\mathfrak{m}}\colon G_{\mathfrak{m}} \to M_{\mathfrak{m}}$ is a finitely generated $\Gamma_{\mathfrak{m}}$-separated cover.*

Proof First note that all local Dedekind-like rings are classical. In fact, by Definition 18.4.1(ii), Γ_Λ is generated by 2 elements. For the statement about localizing Λ and Γ see [14, Proposition 10.6 and Remarks 5.3(i)]. For the statement about separated covers see [15, Theorem 18.13]. □

Our results about classical Dedekind-like rings are more complete than those about nonclassical ones. Also, our results relating $\mathcal{G}$-covers to Γ-separated covers apply to a class of (commutative and noncommutative) rings much wider than classical Dedekind-like rings. The next lemma identifies these rings.

Lemma 18.4.4 *Let $\rho\colon \Gamma \to \bar{\Gamma}$ be a surjective ring homomorphism, where Γ is right noetherian and $\bar{\Gamma}$ is semisimple artinian. Let $\bar{\Lambda}$ be a subring of $\bar{\Gamma}$ such that $\bar{\Gamma}$ is a finitely generated $\bar{\Lambda}$-module, and let*

$$\Lambda = \{x \in \Gamma \mid \rho(x) \in \bar{\Lambda}\}$$

Then Λ is a right noetherian ring, and every classical Dedekind-like ring with normalization Γ is of this form.

For the proof that Λ is right noetherian, see [14, Lemma 4.2]. The proof that all classical Dedekind-like rings have this form is the case $\Omega = \Gamma$ of [15, Proposition 18.3(ii)] (because Γ_Λ is finitely generated in the classical case).

The main property of Γ-separated covers proved in [14, 15] is:

Theorem 18.4.5 (Almost functorial property) *Let Λ and Γ be as in Lemma 18.4.4 (e.g., any classical Dedekind-like ring with normalization Γ). Let $f\colon M_1 \to M_2$ be a Λ-homomorphism, and let $\phi_i\colon G_i \to M_i$ $(i = 1, 2)$ be Γ-separated covers. Then:*

(i) f can be lifted to a Λ-homomorphism $\theta\colon G_1 \to G_2$ such that $f\phi_1 = \phi_2\theta$.

(ii) If f is monic or epic, so is any such θ.

If Λ is a nonclassical Dedekind-like ring and M_1, M_2 are finitely generated, the same conclusions hold.

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See [15, Theorem 18.10] for the case of Dedekind-like rings (classical or not), and [14, Remarks 4.8(ii) and Theorem 4.12] for the situation in Lemma 18.4.4.

An immediate consequence of the almost functorial property is:

Corollary 18.4.6 *Let M be a right Λ-module.*

(i) If Λ and Γ are as in Lemma 18.4.4, then M has a unique Γ-separated cover $g\colon G \to M$ (up to isomorphism of maps), and if M is finitely generated, so is G.

(ii) If Λ is a nonclassical Dedekind-like ring with normalization Γ, and M is finitely generated, then M has a finitely generated Γ-separated cover g, and every Γ-separated cover of M is isomorphic to g.

Proof By Theorem 18.3.2, M has a Γ-separated cover, which can be chosen to be finitely generated if M is. To complete the proof, apply the almost functorial property with $M_1 = M_2 = M$ and f the identity map on M. □

We note the following property of Γ-separated covers:

Theorem 18.4.7 *Let $g\colon G \to M$ be a Γ-separated cover. If either of the following conditions holds, then g is a "minimal epimorphism" (no submodule properly smaller than G is mapped by g onto M).*

(i) Λ and Γ are as in Lemma 18.4.4; or

(ii) Λ is a nonclassical Dedekind-like ring with normalization Γ, and M (hence G) is finitely generated.

Proof In situation (i) this is proved in [14, Lemma 4.10 and Remarks 4.8(ii)]. For Dedekind-like rings (classical or not) see [15, Theorem 18.15]. Note that, in part (ii), finite generation in G results from the uniqueness statement in Corollary 18.4.6(ii). □

Theorem 18.4.8 *Let Λ be a ring and M a right Λ-module. (Thus M has at least one Γ-separated cover, say $g\colon G \to M$ [Theorem 18.3.2].)*

(i) If Λ and Γ are as in Lemma 18.4.4, then g is the $\mathcal{G}$-cover of M. If, in addition, M is finitely generated, then g is also the $\mathcal{G}_0$-cover of M.

(ii) If Λ is a nonclassical Dedekind-like ring with normalization Γ, and M is finitely generated, then g is the $\mathcal{G}$-cover and the $\mathcal{G}_0$-cover of M.

Thus, in either situation, $\mathcal{G}_0$ is contravariantly finite.

Proof Let f be the $\mathcal{G}$-cover of M (which exists by Theorem 18.2.5). Then, by Theorem 18.3.3(i), f is also a Γ-separated cover of M. Parts (i) and (ii) of Corollary 18.4.6 give the uniqueness of Γ-separated covers, hence an isomorphism of f to g in the cases (i) and (ii), respectively.

If M is finitely generated then, by Corollary 18.4.6, G is also finitely generated, and hence the $\mathcal{G}$-cover g is also the $\mathcal{G}_0$-cover of M. □

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We now begin working toward the proof that $\mathcal{G}_0$ (and hence $\mathcal{G}$) is far from being closed under extensions, when Λ is Dedekind-like.

Lemma 18.4.9 *Let ϕ: $S \to M$ be a Γ-separated cover of a finitely generated Λ-module, where Λ is Dedekind-like with normalization Γ, let $K = \ker(\phi)$, and let X be a Γ-module containing S. Then:*

(i) $\Gamma K \subseteq S$ (where ΓK denotes the Γ-submodule of X generated by K).

(ii) ΓK is semisimple as a Γ-module and as a Λ-module.

(iii) Every semisimple Λ-module is Γ-separated.

Proof (i) It suffices to show that $\Gamma_{\mathfrak{m}} K_{\mathfrak{m}} \subseteq S_{\mathfrak{m}}$ (in $X_{\mathfrak{m}}$) for every maximal ideal $\mathfrak{m}$ of Λ. Therefore, after a change of notation, we may assume that Λ is a local ring with maximal ideal $\mathfrak{m}$. Moreover, after this change of notation, Λ remains Dedekind-like with normalization Γ, and ϕ remains a Γ-separated cover with kernel K [Lemma 18.4.3].

What is gained by this reduction to the local case is that $\mathfrak{m}$ is now an ideal of Γ [Definition 18.4.1(iii)] and $K \subseteq \mathfrak{m}S$ [14, Lemma 4.9]. But then $\Gamma K \subseteq \Gamma \mathfrak{m} S = \mathfrak{m} S \subseteq S$, as desired.

(ii) Γ-semisimplicity of ΓK is proved in [15, Corollary 18.9]. Thus it suffices to prove that every simple Γ-module Y is Λ-semisimple. In fact, Y is the direct sum of at most two simple Λ-modules, by [15, Theorem and Definition 11.3] together with the "lying over" theorem for integral extensions of commutative rings.

(iii) It suffices to show that every *simple* Λ-module N is Γ-separated. Recall that over any commutative noetherian ring, every module of finite length is the direct sum of its (finitely many) nonzero localizations at maximal ideals. (See, e.g., [15, Lemma 6.3].) Since N is simple, this implies that there is a maximal ideal $\mathfrak{m}$ of Λ such that $N = N_{\mathfrak{m}}$. Thus N is isomorphic to the unique simple $\Lambda_{\mathfrak{m}}$-module $\Lambda_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}$. But, by the definition of "Dedekind-like ring", $\mathfrak{m}_{\mathfrak{m}}$ is an ideal of $\Gamma_{\mathfrak{m}}$. Therefore the inclusion $\Lambda_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}} \subseteq \Gamma_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}$ shows that $\Lambda_{\mathfrak{m}}$ is $\Gamma_{\mathfrak{m}}$-separated, and hence Γ-separated. □

Theorem 18.4.10 *Let Λ be a Dedekind-like ring with normalization Γ. Then every finitely generated Λ-module is an extension of a Γ-separated module by a Γ-separated module.*

Proof Let M be a finitely generated Λ-module and ϕ: $S \to M$ a Γ-separated cover. Let $K = \ker(\phi)$ so that we may assume that ϕ is the natural homomorphism $S \to S/K = M$. Since S is Γ-separated, there is a Γ-module X such that $S \subseteq \Gamma S = X$.

We have $K \subseteq \Gamma K \subseteq S$ by Lemma 18.4.9(i). Hence we have the following short exact sequence of Λ-modules.

$$0 \to (\Gamma K)/K \to S/K \to S/(\Gamma K) \to 0$$

It therefore suffices to prove that the Λ-modules $(\Gamma K)/K$ and $S/(\Gamma K)$ are Γ-separated. This holds for $S/(\Gamma K)$ since $S/(\Gamma K) \subseteq (\Gamma S)/(\Gamma K)$, a Γ-module.

ΓK is semisimple as a Λ-module by Lemma 18.4.9(ii), and hence so is its homomorphic image $(\Gamma K)/K$. Therefore, by Lemma 18.4.9(iii), $(\Gamma K)/K$ is a Γ-separated Λ-module. □

Lemma 18.4.11 *Let Λ be a local Dedekind-like ring with maximal ideal $\mathfrak{m}$ and normalization Γ, and let S be a Λ-module.*

Suppose that S is Γ-separated, $S \neq \mathfrak{m}S$, and $\mathfrak{m}S$ has a simple Λ-submodule A that is not a Γ-submodule of $\mathfrak{m}S$. Then the Λ-module S/A is not Γ-separated, and the natural map $S \to S/A$ is a Γ-separated cover.

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Proof Since Λ is local, its maximal ideal $\mathfrak{m}$ is an ideal of Γ [Definition 18.4.1]. Since S is Γ-separated, this implies that $\mathfrak{m}S$ is a Γ-submodule of S. (*Caution:* If S is not Γ-separated, $\mathfrak{m}S$ can fail to be a Γ-module. The difficulty is that the left-hand side of the relation $\gamma(ms) = (\gamma m)s$ does not make sense if S is not Γ-separated.)

Note that we can consider S to be a Λ-submodule of $X = \Gamma \otimes_\Lambda S$. For, since S is Γ-separated, the composite map $S \to \Gamma \otimes_\Lambda S \to \Gamma S$ makes sense and equals 1_S. When this is done, we have $X = \Gamma S$.

We claim that S/A is not Γ-separated. It suffices to show that the canonical map τ': $S/A \to \Gamma \otimes_\Lambda (S/A)$ is not an injection. By right-exactness of $\otimes$, applied to the short exact sequence $A \to S \to S/A$ we obtain the identification (i.e., Γ-isomorphism)

$$\Gamma \otimes_\Lambda S/A = X/\Gamma A \quad \text{via} \quad \gamma \otimes (s + A) \to \gamma s + \Gamma A$$

In terms of this identification we can identify the map τ' with the map ν: $S/A \to X/\Gamma A$ given by $\nu(s + A) = s + \Gamma A$. Since the Λ-submodule A of $\mathfrak{m}S$ is not a Γ-submodule of $\mathfrak{m}S$, there exist $\gamma \in \Gamma$ and $a \in A$ such that $\gamma a \notin A$ (but $\gamma a \in \mathfrak{m}S \subseteq S$), and hence $0 \neq \gamma a + A \in \ker(\nu)$, proving the claim.

Since A is a simple Λ-module, the natural surjection $S \to S/A$ has no proper surjective factorizations, where "proper" means that neither factor has nonzero kernel. Since S is Γ-separated and (by the above claim) S/A is not, we conclude that $S \to S/A$ is a Γ-separated cover. □

To complete the proof that $\mathcal{G}_0$ is not closed under extensions we need to know that non-Γ-separated modules actually exist over some Dedekind-like ring Λ.

Example 18.4.12 Suppose $\Lambda \neq \Gamma$. We show that there exists a cyclic non-Γ-separated Λ-module M of finite length, and display its Γ-separated cover ϕ: $S \to M$.

Proof Suppose first that Λ is local with maximal ideal $\mathfrak{m}$ and residue field $k = \Lambda/\mathfrak{m}$. Then Γ_Λ is finitely generated and $\mathfrak{m} = \mathrm{rad}(\Gamma)$ [Definition 18.4.1].

We claim that, in this situation, Γ is a direct product of semilocal principal ideal domains. By the previous paragraph, the ring $\Gamma/\mathfrak{m}$ is a finite dimensional algebra over the field $k = \Lambda/\mathfrak{m}$, and therefore has only finitely many maximal ideals. Since $\mathfrak{m} = \mathrm{rad}(\Gamma)$, every maximal ideal of Γ contains $\mathfrak{m}$, and hence Γ has only finitely many maximal ideals. By the definition of "Dedekind-like", Γ is a direct product of Dedekind domains; and since Γ is semilocal, so are all of these Dedekind domains. Thus the claim follows from the well-known (and easily proved) fact that every semilocal Dedekind domain is a principal ideal domain.

Next we claim that the Γ-module $\mathfrak{m}/\mathfrak{m}^2$ has a simple Λ-submodule A that is not a Γ-submodule.

Since Γ is a direct product of semilocal principal ideal domains, $\mathfrak{m} = \mathrm{rad}(\Gamma)$ is a principal ideal of Γ (but not of Λ), say $\mathfrak{m} = \Gamma p$ where p is a non-zero-divisor of Γ. Therefore $\mathfrak{m}/\mathfrak{m}^2 \cong \Gamma/\mathfrak{m}$ as Γ-modules. Therefore it suffices to show that $\Gamma/\mathfrak{m}$ has a simple Λ-submodule that is not a Γ-submodule. The simple Λ-submodule $\Lambda/\mathfrak{m}$ of $\Gamma/\mathfrak{m}$ satisfies the required conditions since $\Lambda \neq \Gamma$ and $\Gamma\Lambda = \Gamma$.

Let $S = \Lambda/\mathfrak{m}^2$. By the previous claim, there is a Λ-submodule A of $\mathfrak{m}S = \mathfrak{m}/\mathfrak{m}^2$ that is not a Γ-submodule of $\mathfrak{m}S$. Then the natural map ϕ: $S \to M = S/A$ is a Γ-separated cover of the non-Γ-separated cyclic Λ-module M [Lemma 18.4.11]. Moreover, S_Λ and M_Λ have finite length because $\Lambda/\mathfrak{m} \cong k$ and $\mathfrak{m}/\mathfrak{m}^2 \cong \Gamma/\mathfrak{m}$ which (as shown above) is a finite dimensional k-algebra.

Now consider a general (nonlocal) Λ. Since $\Lambda \neq \Gamma$ there is a maximal ideal $\mathfrak{m}$ of Λ such that $\Lambda_\mathfrak{m} \neq \Gamma_\mathfrak{m}$ (in $\Gamma_\mathfrak{m}$). Recall that $\Lambda_\mathfrak{m}$ is again Dedekind-like with normalization $\Gamma_\mathfrak{m}$ [Lemma 18.4.3], and let ϕ: $S \to M$ be the $\Gamma_\mathfrak{m}$-separated cover of the nonseparated $\Lambda_\mathfrak{m}$-module M obtained above.

To complete the proof it suffices to note that every $\Lambda_\mathfrak{m}$-module of finite length is a Λ-module whose Λ-submodules are all $\Lambda_\mathfrak{m}$-submodules [15, Lemma 6.2]. □

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18.5 Open Problems

1. Let M be a module over a nonclassical Dedekind-like ring. If M is not finitely generated, then M has Γ-separated covers [Theorem 18.3.2], but we do not know whether these covers satisify the almost functorial property [Theorem 18.4.5], are unique [Corollary 18.4.6], or have the minimal epimorphism property [Theorem 18.4.7].

2. In the general setting where Λ is an arbitrary right noetherian ring, and Γ is arbitrary, does contravariant finiteness of $\mathcal{G}_0$ imply that finitely generated Γ-separated covers of all finitely generated modules M are isomorphic (hence isomorphic to the $\mathcal{G}_0$-cover of M)?

We have affirmative answers for the rings in Theorem 18.4.8, in particular, for all Dedekind-like rings. (Moreover, by Theorem 18.4.8, $\mathcal{G}_0$-covers coincide with the $\mathcal{G}$-covers in this case.)

Also, the answer is affirmative if Λ is a DVR with the quotient field Γ. Then $\mathcal{G}_0$ ($\mathcal{G}$) is the class of all finitely generated projective modules (flat modules), so any finitely generated Γ-separated cover g of M is isomorphic to a (surjective) restriction of the projective cover f of M, hence g is isomorphic to f. (However, if Λ is not complete and M is a nonzero module of finite length, then f is not isomorphic to the $\mathcal{G}$-cover of M by Example 18.2.7(ii).)

3. Can the semisimplicity condition in Lemma 18.4.4 be weakened in any reasonable way that allows the theorems about these rings in Section 18.4 — especially the almost functorial property, uniqueness of separated covers, and minimal epimorphism properties — to remain true?

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Chapter 19

The Cotorsion Dimension of Modules and Rings

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Abstract In this paper, we introduce a dimension, called the cotorsion dimension, for modules and rings. The relations between the cotorsion dimension and other dimensions are discussed. Various results are developed, some extending known results.

Keywords: Cotorsion dimension; Cotorsion envelope; Flat cover; Perfect ring.

19.1 Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary.

Let R be a ring and M a right R-module. Recall that M is called *cotorsion* [7] if $\mathrm{Ext}^1_R(F, M) = 0$ for any flat right R-module F. The class of cotorsion modules contains all pure-injective (hence, injective) modules. A homomorphism $\phi : M \to C$ with C cotorsion is called a *cotorsion preenvelope* of M [6, 27] if for any homomorphism $f\colon M \to C'$ where C' is cotorsion, there is a homomorphism $g : C \to C'$ such that $g\phi = f$. Moreover if the only such g are automorphisms of C when $C' = C$ and $f = \phi$, the cotorsion preenvelope ϕ is called a *cotorsion envelope* of M. A homomorphism $\phi : F \to M$ with F flat is called a *flat cover* of M if for any homomorphism $f : F' \to M$ where F' is flat, there is a homomorphism $g : F' \to F$ such that $\phi g = f$, moreover when $F' = F$ and $f = \phi$, the only such g are automorphisms of F. It is now well known that all R-modules have flat covers for any ring R [2], and it has been proven that every R-module has a cotorsion envelope if and only if every R-module has a flat cover [27]. Thus all R-modules have cotorsion envelopes for arbitrary ring R. Note that cotorsion envelopes or flat covers are unique up to isomorphism.

In what follows, for an R-module M, $E(M)$, $C(M)$ and $F(M)$ stand for the injective envelope, cotorsion envelope and flat cover respectively. We write M_R to indicate a right R-module. The projective (resp. injective) dimension of M is denoted by $\mathrm{pd}(M)$ (resp. $\mathrm{id}(M)$). We denote by

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rD(R) (resp. wD(R)) the right (resp. the weak) global dimension of a ring R. General background material can be found in [1], [9], [23], [27].

We are going to define a dimension, called the cotorsion dimension, for modules and rings. It measures how far away a module is from being cotorsion, and how far away a ring is from being perfect.

Let R be a ring. For any right R-module M, the *cotorsion dimension* cd(M) of M is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}_R^{n+1}(F, M) = 0$ for any flat right R-module F. If there is no such n, set $\mathrm{cd}(M) = \infty$. The right *global cotorsion dimension* r.cot.D(R) of R is defined as the supremum of the cotorsion dimensions of right R-modules. The aim of this paper is to investigate these dimensions.

In Section 19.2, we give the definition and show some of the general results. Let R be a ring. First we prove that r.cot.D(R) = sup{pd(F): F is a flat right R-module} = sup{cd(F): F is a flat right R-module} (part of Theorem 19.2.5), which gives rise to some characterizations of right perfect rings (Corollary 19.2.9) and extends [27, Proposition 3.3.1]. Then it is shown that r.cot.D(R) ≤ 1 if and only if every quotient module of any cotorsion (or injective) right R-module is cotorsion if and only if every pure submodule of any projective right R-module is projective (Theorem 19.2.11). This removes the unnecessary hypothesis that R is a commutative domain from [15, Theorem 3.2]. For a ring R such that the cotorsion envelope of any projective right R-module is projective, we have that r.cot.D(R) ≤ 1 if and only if the projectivity of $C(M)$ implies the projectivity of M for any right R-module M (Theorem 19.2.13). The relation rD(R) $\leq$ wD(R)+ r.cot.D(R) is proven to be true for any ring R (Theorem 19.2.14). Finally, for a left coherent ring R, it is shown that R is right perfect if and only if every flat cotorsion right R-module is projective (Proposition 19.2.20).

Section 19.3 is devoted to the cotorsion dimension under change of rings. We first get that if $\varphi : R \to S$ is a surjective ring homomorphism and S_R a flat right R-module, then r.cot.D(S) $\leq$ r.cot.D(R) (Corollary 19.3.2). Then we prove that if S is an almost excellent extension of R, then r.cot.D(S) $\leq$ r.cot.D(R), and the equality holds in case r.cot.D(R) $< \infty$ (Corollary 19.3.4 and Theorem 19.3.5).

In Section 19.4, some applications in commutative rings are discussed. We start by showing that for a ring R with cot.D(R) ≤ 1, $\mathrm{Ext}_R^1(F, M)$ is cotorsion for any flat R-module F and any R-module M (Proposition 19.4.3), which is motivated by [11, Problem 48, p.462]. Then, for a surjective ring homomorphism $\varphi : R \to S$ with $K = \mathrm{Ker}(\varphi)$ and S_R projective, it is shown that, for any R-module M, either $\mathrm{cd}(M_R) \leq \sup\{\mathrm{pd}(R/I)_R : I \subseteq K\}$, or $\mathrm{cd}(M_R) = \mathrm{cd}(\mathrm{Hom}_R(S, M))$, where $\mathrm{Hom}_R(S, M)$ may be regarded as an R-module or S-module (Theorem 19.4.5). As a corollary, we get that a ring R is perfect if and only if there is a quotient ring $S = R/K$ of R such that S is a perfect ring and R/I is a projective R-module for any $I \subseteq K$ (Corollary 19.4.7). In the last part of this section, we prove that a ring R is von Neumann regular if and only if $\mathrm{Hom}_R(A, B)$ is injective (or flat) for all cotorsion R-modules A and B (Proposition 19.4.10).

19.2 General Results

We start with the following.

Proposition 19.2.1 *For any right R-module M and integer $n \geq 0$, the following are equivalent:*

1. *$\mathrm{cd}(M) \leq n$.*
2. $\mathrm{Ext}_R^{n+1}(F, M) = 0$ *for any flat right R-module F.*
3. $\mathrm{Ext}_R^{n+j}(F, M) = 0$ *for any flat right R-module F and $j \geq 1$.*

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4. *If the sequence* $0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0$ *is exact with* $C^0, C^1, \ldots, C^{n-1}$ *cotorsion, then* C^n *is also cotorsion.*

5. $cd(F(M)) \leq n$.

Proof The proof of (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) is standard homological algebra fare.

(1) $\Leftrightarrow$ (5). Let K be the kernel of the flat cover $F(M) \to M$, then we have the exact sequence $0 \to K \to F(M) \to M \to 0$ with K cotorsion. Note that $\mathrm{Ext}^n_R(F, K) = 0$ for all $n \geq 1$ and flat modules F by the proof of [27, Proposition 3.1.2], so the result follows. □

Corollary 19.2.2 *Let* M *be any right* R*-module. Then the following are identical:*

1. $cd(M)$.

2. $inf\{k$*: there exists an exact sequence* $0 \to M \to C^0 \to C^1 \to \cdots \to C^k \to 0$*, where each* C^i *is a cotorsion right* R*-module,* $i = 0, 1, \ldots, k\}$.

3. *The integer* n *such that* M *admits a minimal cotorsion resolution, i.e., an exact sequence* $0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0$*, where each* C^i *is cotorsion,* $L^i = Coker(C^{i-2} \to C^{i-1}) \to C^i$ *is a cotorsion envelope of* L^i*,* $C^i \neq 0$*,* $i = 0, 1, \ldots, n$*,* $C^{-2} = 0$*,* $C^{-1} = M$.

Proof (1) = (2) is straightforward.

(1) $\leq$ (3) is trivial. Assume (1) $<$ (3) $= n$. Let (1) $= k < \infty$. By Proposition 19.2.1, L^k is a cotorsion right R-module. Consider the exact sequence $0 \to L^k \to C^k \to L^{k+1} \to 0$, since $L^k \to C^k$ is a cotorsion envelope of L^k, it follows that $L^{k+1} = 0$, and hence $C^{k+1} = 0$, a contradiction. Therefore (1) = (3). □

Proposition 19.2.3 *Let* R *be a ring,* $0 \to A \to B \to C \to 0$ *an exact sequence of right* R*-modules. If two of* $cd(A)$*,* $cd(B)$*,* $cd(C)$ *are finite, so is the third. Moreover*

1. $cd(B) \leq sup\{cd(A), cd(C)\}$.

2. $cd(A) \leq sup\{cd(B), cd(C) + 1\}$.

3. $cd(C) \leq sup\{cd(B), cd(A) - 1\}$.

Proof It is a routine exercise. □

The next corollary is an immediate consequence of Proposition 19.2.3.

Corollary 19.2.4 *Let* R *be a ring,* $0 \to A \to B \to C \to 0$ *an exact sequence of right* R*-modules. If* B *is cotorsion,* $cd(A) > 0$*, then* $cd(A) = cd(C) + 1$.

Theorem 19.2.5 *Let* R *be a ring. Then*

1. $r.cot.D(R)$
   $= sup\{pd(F)$*:* F *is a flat right* R*-module*$\}$
   $= sup\{cd(F)$*:* F *is a flat right* R*-module*$\}$.

2. $r.cot.D(R)$
   $\leq sup\{pd(M)$*:* $pd(M) < \infty\}$
   $\leq sup\{id(P)$*:* P *is a projective right* R*-module*$\}$
   $\leq rD(R)$.
   All equalities hold if R *is a von Neumann regular ring.*

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3. If r.cot.D$(R) < \infty$, then
r.cot.D(R)
= sup$\{pd(F)$: F is a flat cotorsion right R-module$\}$
= sup$\{pd(C(F))$: F is a flat right R-module$\}$
= sup$\{pd(F(M))$: M is a cotorsion right R-module$\}$
= sup$\{cd(P)$: P is a projective right R-module$\}$.

Proof (1). First, we show that r.cot.D$(R) \leq$ sup$\{$pd(F): F is a flat right R-module$\}$. We may assume sup$\{$pd(F): F is a flat right R-module$\} = m < \infty$. Let M be any right R-module. It follows that $\mathrm{Ext}_R^{m+1}(F, M) = 0$ for any flat right R-module F since pd$(F) \leq m$, so cd$(M) \leq m$. Thus r.cot.D$(R) \leq m$.

It is clear that sup$\{$cd(F): F is a flat right R-module$\} \leq$ r.cot.D(R). Next we shall show that sup$\{$pd(F): F is a flat right R-module$\} \leq$ sup$\{$cd(F): F is a flat right R-module$\}$. In fact, we may assume that sup$\{$cd(F): F is a flat right R-module$\} = n < \infty$. Let M be any flat right R-module, N any right R-module. There exists an exact sequence $0 \to K \to F(N) \to N \to 0$. By [27, Lemma 2.1.1], K is cotorsion. We have the following exact sequence

$$\mathrm{Ext}_R^{n+1}(M, F(N)) \to \mathrm{Ext}_R^{n+1}(M, N) \to \mathrm{Ext}_R^{n+2}(M, K) = 0.$$

Note that $\mathrm{Ext}_R^{n+1}(M, F(N)) = 0$ since cd$(F(N)) \leq n$. So $\mathrm{Ext}_R^{n+1}(M, N) = 0$, which implies pd$(M) \leq n$, as desired.

(2). By (1), r.cot.D$(R) \leq$ sup$\{$pd(M): pd$(M) < \infty\}$ follows from [13, Proposition 6]. The last inequality is obvious. Next we shall show sup$\{$pd(M): pd$(M) < \infty\} \leq$ sup$\{$id(P): P is a projective right R-module$\}$. In fact, we may assume sup$\{$id(P): P is a projective right R-module$\}$ $= m < \infty$. Let M be any right R-module with pd$(M) = n < \infty$. We claim that $n \leq m$. Otherwise, let $n > m$. For any right R-module N, there exists an exact sequence $0 \to K \to P \to N \to 0$ with P projective, which induces the exact sequence

$$\mathrm{Ext}_R^{n}(M, P) \to \mathrm{Ext}_R^{n}(M, N) \to \mathrm{Ext}_R^{n+1}(M, K).$$

Note that $\mathrm{Ext}_R^{n}(M, P) = 0$ since id$(P) \leq m < n$, and $\mathrm{Ext}_R^{n+1}(M, K) = 0$ since pd$(M) = n$. Thus $\mathrm{Ext}_R^{n}(M, N) = 0$, and hence pd$(M) \leq n - 1$; this is a contradiction. So $n \leq m$, as required.

The last statement is obvious.

(3). The inequalities r.cot.D$(R) \geq$ sup$\{$pd(F): F is a flat cotorsion right R-module$\}$ $\geq$ sup$\{$pd$(C(F))$: F is a flat right R-module$\}$ are clear since cotorsion envelopes of flat modules are always flat. Next we shall show that r.cot.D$(R) \leq$ sup$\{$pd$(C(F))$: F is a flat right R-module$\}$. Assume sup$\{$pd$(C(F))$: F is a flat right R-module$\} = m < \infty$. For any flat right R-module F, cd$(F) = t < \infty$ since r.cot.D$(R) < \infty$. Thus, by Corollary 19.2.2, M admits a minimal cotorsion resolution

$$0 \to F \to C^0 \to C^1 \to \cdots \to C^{t-1} \to C^t \to 0.$$

Note that each C^i is a cotorsion envelope of the flat right R-module L^i, $i = 0, 1, \ldots, t$. By hypothesis, pd$(C^i) \leq m$, $i = 0, 1, \ldots, t$. Therefore pd$(F) \leq m$. So r.cot.D$(R) =$ sup$\{$pd(F): F is a flat right R-module$\} \leq m$. Thus r.cot.D$(R) =$ sup$\{$pd(M): M is flat$\} \geq$ sup$\{$pd$(F(M))$: M is cotorsion$\}$ $\geq$ sup$\{$pd(M): M is flat cotorsion$\} =$ r.cot.D(R), and hence r.cot.D$(R) =$ sup$\{$pd$(F(M))$: M is a cotorsion right R-module$\}$ follows.

Now we prove that r.cot.D$(R) =$ sup$\{$cd(P): P is a projective right R-module$\}$. Let sup$\{$cd(P): P is a projective right R-module$\} = n < \infty$. For any flat right R-module F, pd$(F) = m < \infty$ since r.cot.D$(R) < \infty$. Thus there exists an exact sequence

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to F \to 0,$$

where P_i is projective, $i = 0, 1, \ldots, m$. Thus cd$(F) \leq n$ by hypothesis and Proposition 19.2.3. This completes the proof. □

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Remark 19.2.6 Note that pure injective modules are cotorsion, so [14, Proposition 1.1(a)] (that asserts sup{pd(F): F is a flat right R-module} $\leq$ right pure global dimension of the ring R) is an immediate consequence of Theorem 19.2.5 (1).

Corollary 19.2.7 *Let R be a ring, then the following are equivalent for an integer $n \geq 0$:*

1. *r.cot.D$(R) \leq n$.*
2. *All flat right R-modules are of projective dimension $\leq n$.*
3. *All flat right R-modules are of cotorsion dimension $\leq n$.*
4. *r.cot.D$(R) < \infty$, and all flat cotorsion right R-modules are of projective dimension $\leq n$.*
5. *r.cot.D$(R) < \infty$, and all projective right R-modules are of cotorsion dimension $\leq n$.*
6. $\mathrm{Ext}_R^{n+1}(M, N) = 0$ *for all flat right R-modules M and N.*
7. $\mathrm{Ext}_R^{n+j}(M, N) = 0$ *for all flat right R-modules M, N and $j \geq 1$.*

Remark 19.2.8 The equivalences of (2), (6) and (7) of Corollary 19.2.7 appeared in [9, Theorem 8.4.12] under the hypothesis that R is left coherent.

By [12, Corollary 10], if every projective right R-module is cotorsion, then R is right perfect. So we obtain some characterizations of right perfect rings by specializing Corollary 19.2.7 to the case $n = 0$. The equivalences of (2) through (4) in the following corollary are due to Xu [27, Proposition 3.3.1].

Corollary 19.2.9 *The following are equivalent for any ring R:*

1. *r.cot.D$(R) = 0$.*
2. *Every right R-module is cotorsion.*
3. *R is right perfect.*
4. *Every flat right R-module is cotorsion.*
5. *Every projective right R-module is cotorsion.*
6. *r.cot.D$(R) < \infty$, and every flat cotorsion right R-module is projective.*
7. $\mathrm{Ext}_R^1(M, N) = 0$ *for all flat right R-modules M and N.*

Remark 19.2.10 By Corollary 19.2.9, r.cot.D(R) measures how far away a ring is from being right perfect. It is well known that right perfect rings need not be left perfect (see [1, p.322]), so r.cot.D$(R) \neq$ l.cot.D(R) in general.

Let R be a ring. It is well known that rD$(R) \leq 1$ if and only if every quotient module of any injective right R-module is injective. Here we prove that r.cot.D$(R) \leq 1$ if and only if every quotient module of any cotorsion right R-module is cotorsion as shown in the following theorem.

Theorem 19.2.11 *Let R be a ring, then the following are equivalent:*

1. *r.cot.D$(R) \leq 1$.*
2. *All flat right R-modules are of projective dimension ≤ 1.*

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3. *All flat right R-modules are of cotorsion dimension ≤ 1.*

4. *Every quotient module of any injective right R-module is cotorsion.*

5. *Every quotient module of any cotorsion right R-module is cotorsion.*

6. *Every pure submodule module of any projective right R-module is projective.*

Proof (1) $\Rightarrow$ (4). Let E be any injective right R-module and K a submodule of E. The exactness of the sequence $0 \to K \to E \to E/K \to 0$ induces the exact sequence

$$0 = \mathrm{Ext}^1_R(F, E) \to \mathrm{Ext}^1_R(F, E/K) \to \mathrm{Ext}^2_R(F, K),$$

where F is a flat right R-module. Note that $\mathrm{Ext}^2_R(F, K) = 0$ by (1) and Proposition 19.2.1, so $\mathrm{Ext}^1_R(F, E/K) = 0$, as required.

(4) $\Rightarrow$ (1). Let M be any right R-module. Then there exists an exact sequence $0 \to M \to E \to E/M \to 0$ with E injective. Thus $\mathrm{cd}(M) \leq 1$ since E/M is cotorsion, and hence $\text{r.cot.D}(R) \leq 1$.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Corollary 19.2.7.

(2) $\Rightarrow$ (6). Let M be a projective right R-module and N a pure submodule of M. Then $0 \to N \to M \to M/N \to 0$ is exact. Note that M/N is flat and hence $\mathrm{pd}(M/N) \leq 1$ by (2). Thus N is projective.

(6) $\Rightarrow$ (2). Let M be any flat right R-module. There exists an exact sequence $0 \to N \to P \to M \to 0$ with P projective. Note that N is a pure submodule of P, so N is projective. It follows that $\mathrm{pd}(M) \leq 1$.

(5) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (5). Let M be any cotorsion right R-module and N any submodule of M. There exists an exact sequence $0 \to N \to E(N) \to L \to 0$. Consider the following pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E(N) \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & M/N \arrow[r] & 0 \\
 & L \arrow[r, equal] \arrow[d] & L \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

By (4), L is cotorsion. Since M is cotorsion, H is cotorsion by [27, Proposition 3.1.2]. Note that $E(N)$ is cotorsion, it follows that M/N is cotorsion by [27, Proposition 3.1.2] again. □

We note that the equivalences of (2), (4), (5) and (6) in the previous theorem have recently been proven for commutative domains ([15, Theorem 3.2]).

By [27, Theorem 3.3.2], a ring R is von Neumann regular if and only if every cotorsion right R-module is flat. Replacing "flat" with "projective", we have the following

Proposition 19.2.12 *Let R be a ring. Then the following are equivalent:*

1. *R is a semisimple Artinian ring.*

2. *Every cotorsion right R-module is projective.*

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3. *r.cot.D*$(R) \leq 1$ *and the cotorsion envelope of every simple right R-module is projective.*

Proof (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear.

(2) $\Rightarrow$ (1). It is easy to see that R is quasi-Frobenius and von Neumann regular, and hence (1) follows.

(3) $\Rightarrow$ (1). By (3), every simple right R-module M is a pure submodule of a projective right R-module, and hence M is projective by Theorem 19.2.11. So (1) follows. □

We know that the cotorsion envelope of any flat right R-module is always flat. Rothmaler [22] has discussed when the pure-injective envelope of any flat right R-module is flat. It is natural to consider the condition that the cotorsion (pure-injective) envelope of any projective right R-module is projective. For a ring with this condition, we have the following

Theorem 19.2.13 *Let R be a ring such that the cotorsion envelope of any projective right R-module is projective. Then the following are equivalent:*

1. *r.cot.D*$(R) \leq 1$.
2. *The projectivity of $C(M)$ implies the projectivity of M for any right R-module M.*

If "cotorsion envelope" is replaced with "pure-injective envelope", the result still holds.

Proof (1) $\Rightarrow$ (2). Assume M is a right R-module such that $C(M)$ is projective. Note that M is a pure submodule of $C(M)$, so M is projective by Theorem 19.2.11.

(2) $\Rightarrow$ (1). Let M be a pure submodule of a projective right R-module P; it is enough to show that M is projective by Theorem 19.2.11. In fact, there is an exact sequence

$$0 \longrightarrow M \xrightarrow{f} P \longrightarrow L \longrightarrow 0\,,$$

where L is flat. By the defining property of cotorsion envelope, there exists $g : C(M) \to C(P)$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\phi"'] & P \arrow[d, "\psi"] \\ C(M) \arrow[r, "g"'] & C(P) \end{tikzcd}$$

commutes, i.e., $g\phi = \psi f$. Consider the pushout diagram of f and ϕ:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "\phi"'] & P \arrow[r] \arrow[d, "\gamma"'] & L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C(M) \arrow[r, "\alpha"'] & K \arrow[r] & L \arrow[r] & 0 \end{tikzcd}$$

Note that the second row is split, so there is $\beta : K \to C(M)$ such that $\beta\alpha = 1$. It follows that $\beta\gamma : P \to C(M)$ factors through ψ. Hence there is $\sigma : C(P) \to C(M)$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d, "\gamma"'] & C(P) \arrow[d, "\sigma"] \\ K \arrow[r, "\beta"'] & C(M) \end{tikzcd}$$

commutes, i.e., $\sigma\psi = \beta\gamma$. Then $\sigma g\phi = \sigma\psi f = \beta\gamma f = \beta\alpha\phi = \phi$. The defining property of cotorsion envelope now implies that σg is an automorphism of $C(M)$. Therefore $C(M)$ is isomorphic

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to a direct summand of $C(P)$. Since $C(P)$ is projective by hypothesis, $C(M)$ is projective. So M is projective by (2), as required.

The last statement can be proven similarly. □

It is well known that rD(R) = wD(R) when R is right perfect; rD(R) = r.cot.D(R) when R is von Neumann regular by Theorem 19.2.5 (2). In general, we have the following inequality.

Theorem 19.2.14 *Let R be a ring, then rD(R) $\leq$ r.cot.D(R) + wD(R).*

Proof We may assume that both r.cot.D(R) and wD(R) are finite. Let r.cot.D(R) $= m < \infty$ and wD(R) $= n < \infty$. Suppose M is a right R-module, then M admits a flat resolution

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0.$$

Let $K_i = \mathrm{Ker}(F_i \to F_{i-1})$, $i = 0, 1, 2, \ldots, n-1$, $F_{-1} = M$, $F_n = K_{n-1}$. Then we have the following short exact sequences

$$0 \to F_n \to F_{n-1} \to K_{n-2} \to 0,$$

$$0 \to K_{n-2} \to F_{n-2} \to K_{n-3} \to 0,$$

$$\cdots\cdots,$$

$$0 \to K_0 \to F_0 \to M \to 0.$$

Note that $\mathrm{pd}(K_{n-2}) \leq 1 + \sup\{\mathrm{pd}(F_n), \mathrm{pd}(F_{n-1})\}$ by [23, Lemma 9.26]. Since $\mathrm{pd}(F_i) \leq m$, $i = 0, 1, \ldots, n$, it follows that $\mathrm{pd}(K_{n-2}) \leq 1 + m$, $\mathrm{pd}(K_{n-3}) \leq 2 + m, \cdots, \mathrm{pd}(M) \leq n + m$. This completes the proof. □

Remark 19.2.15 In general, the inequality in Theorem 19.2.14 may be strict. Indeed, if R is right Noetherian, but not right perfect (e.g., the integer ring $\mathbb{Z}$), then rD(R) $= wD(R)$ (see [23, Theorem 9.22]) and r.cot.D(R) $\neq 0$. In this case, the inequality is strict. It is easy to verify that, if R is right Noetherian, then rD(R) = r.cot.D(R) + wD(R) if and only if R is right Artinian.

Recall that a ring R is called an *n-Gorenstein ring* if R is a left and right Noetherian ring with $\mathrm{id}({}_RR) \leq n$ and $\mathrm{id}(R_R) \leq n$ for an integer $n \geq 0$. For this ring, we have the following

Proposition 19.2.16 *If R is an n-Gorenstein ring, then r.cot.D(R) $\leq n$ and l.cot.D(R) $\leq n$.*

Proof Recall that a right R-module M is called FP-injective if $\mathrm{Ext}^1_R(N, M) = 0$ for all finitely presented right R-modules N. Note that a right R-module M is FP-injective if and only if M is injective when R is right Noetherian. It follows that r.cot.D(R) $= \sup\{\mathrm{cd}(M)$: M is a flat right R-module$\} \leq \sup\{\mathrm{id}(M)$: M is a flat right R-module$\}$ = $\mathrm{id}(R_R) \leq n$ by [5, Theorem 3.8]. The inequality l.cot.D(R) $\leq n$ can be proven similarly. □

Corollary 19.2.17 *[8, Corollary 3.4]. If R is a 1-Gorenstein ring, then every quotient module of each injective right (left) R-module is cotorsion.*

Proof It follows from Proposition 19.2.16 and Theorem 19.2.11. □

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For an exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, if B and C are both cotorsion, we know $\mathrm{cd}(A) \leq 1$ by Proposition 19.2.3 (2). However A need not be cotorsion in general (see [27, p.75]). Next we discuss when A is cotorsion if B and C are.

Proposition 19.2.18 *Let R be a ring. Then the following are equivalent:*

1. *The cotorsion envelope of every flat right R-module is projective.*
2. *The flat cover of every cotorsion right R-module is projective.*
3. *Every flat cotorsion right R-module is projective.*
4. *Every flat right R-module is a pure submodule of some projective right R-module.*

Proof (1) $\Rightarrow$ (4). Let F be a flat right R-module. There exists an exact sequence $0 \to F \to C(F) \to L \to 0$. By (1), $C(F)$ is projective. Note that L is flat, so the exact sequence is pure, and (4) follows.

(4) $\Rightarrow$ (3). Let F be a flat cotorsion right R-module. By (4), there exists a projective right R-module P and a pure exact sequence $0 \to F \to P \to L \to 0$. Note that L is flat. It follows that the exact sequence is split. Thus F is projective.

(2) $\Leftrightarrow$ (3) $\Rightarrow$ (1) are easy. □

Proposition 19.2.19 *Let R be a ring satisfying the equivalent conditions in Proposition 19.2.18.*

1. *Assume $0 \to A \to B \to C \to 0$ is an exact sequence of right R-modules, then if two of A, B, C are cotorsion, so is the third.*
2. *r.cot.D$(R) = 0$ or r.cot.D$(R) = \infty$.*

Proof It is clear that (1) implies (2). We now prove (1).

It is enough to show that A is cotorsion if B and C are cotorsion by [27, Proposition 3.1.2]. Let F be any flat right R-module. By Proposition 19.2.18, there exists a pure exact sequence $0 \to F \to P \to L \to 0$ with P projective. Note that L is flat. The exact sequence $0 \to A \to B \to C \to 0$ gives rise to the following exact sequence

$$\mathrm{Ext}^1_R(L, C) \to \mathrm{Ext}^2_R(L, A) \to \mathrm{Ext}^2_R(L, B),$$

which implies $\mathrm{Ext}^2_R(L, A) = 0$ since the first term and the last term are both zero by hypothesis. In addition, the exact sequence $0 \to F \to P \to L \to 0$ yields the following exact sequence

$$\mathrm{Ext}^1_R(P, A) \to \mathrm{Ext}^1_R(F, A) \to \mathrm{Ext}^2_R(L, A).$$

Note that the first term and the last term are both zero, so $\mathrm{Ext}^1_R(F, A) = 0$. This completes the proof. □

We end this section with the following result which is of independent interest.

Recall that a ring R is called *left coherent* if every finitely generated left ideal is finitely presented.

Proposition 19.2.20 *Let R be a left coherent ring, then the following are equivalent:*

1. *R is right perfect.*
2. *R is a ring satisfying the equivalent conditions in Proposition 19.2.18.*

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Proof (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). For any family $\{R_i\}_{i\in I}$, where each $R_i \cong R$ is a right R-module, $\prod_{i\in I} R_i$ is a flat right R-module since R is left coherent. Hence we have an exact sequence

$$0 \to \prod_{i\in I} R_i \to C(\prod_{i\in I} R_i) \to L \to 0,$$

where $C(\prod_{i\in I} R_i)$ and L are flat by [27, Theorem 3.4.2]. By hypothesis, $C(\prod_{i\in I} R_i)$ is projective. Thus $\prod_{i\in I} R_i$ is a pure submodule of a projective right R-module, and hence it is a pure submodule of a free right R-module. It follows that R is a right perfect ring by [4, Theorem 3.1]. $\square$

19.3 Cotorsion Dimension under Change of Rings

We begin with the following.

Proposition 19.3.1 *Let $\varphi : R \to S$ be a surjective ring homomorphism.*

1. *If M_S is a right S-module, then $cd(M_R) \leq cd(M_S)$. Moreover, if S_R is a flat right R-module, then $cd(M_S) = cd(M_R)$.*
2. *If S_R is a flat right R-module, and M_R is a cotorsion right R-module, then* $\mathrm{Hom}_R(S, M)$ *is a cotorsion right S-module, and hence a cotorsion right R-module.*

Proof (1). We may assume $\mathrm{cd}(M_S) = n < \infty$. Then there exists an exact sequence

$$0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0,$$

where each C^i is a cotorsion right S-module, $i = 0, 1, \ldots, n$. By [27, Proposition 3.3.3], each C^i is also cotorsion as a right R-module. So $\mathrm{cd}(M_R) \leq n$.

If S_R is a flat right R-module, we claim $\mathrm{cd}(M_S) \leq \mathrm{cd}(M_R)$. In fact, we may assume $\mathrm{cd}(M_R) = n < \infty$. Let F be a flat right S-module, then F is a flat right R-module. Thus $\mathrm{Ext}_S^{n+1}(F_S, M_S) = \mathrm{Ext}_R^{n+1}(F_R, M_R) = 0$ by [23, Theorem 11.65]. Therefore $\mathrm{cd}(M_S) \leq n$, and hence $\mathrm{cd}(M_S) = \mathrm{cd}(M_R)$.

(2). By hypothesis, $\mathrm{Ext}_R^1(S, M) = 0$. Let X be a flat right S-module, then X is a flat right R-module. Thus

$$\mathrm{Ext}_S^1(X, \mathrm{Hom}_R(S, M)) = \mathrm{Ext}_R^1(X, M) = 0$$

by [24, Lemma 3.1]. Therefore $\mathrm{Hom}_R(S, M)$ is a cotorsion right S-module, and hence a cotorsion right R-module by [27, Proposition 3.3.3]. $\square$

Corollary 19.3.2 *Let $\varphi : R \to S$ be a surjective ring homomorphism and S_R a flat right R-module, then r.cot.$D(S) \leq$ r.cot.$D(R)$.*

Recall that a ring S is said to be an *almost excellent extension* of a ring R [28, 29] if the following conditions are satisfied:

1. S is a finite *normalizing extension* of a ring R [25], that is, R and S have the same identity and there are elements $s_1, \cdots, s_n \in S$ such that $S = Rs_1 + \cdots + Rs_n$ and $Rs_i = s_i R$ for all $i = 1, \cdots, n$.

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2. ${}_RS$ is flat and S_R is projective.

3. S is right R-projective, that is, if M_S is a submodule of N_S and M_R is a direct summand of N_R, then M_S is a direct summand of N_S.

Further, S is an *excellent extension* of R if S is an almost excellent extension of R and S is free with basis $s_1, \cdots, s_n$ as both a right and a left R-module with $s_1 = 1_R$. The concept of excellent extension was introduced by Passman [18] and named by Bonami [3]. Examples of excellent extensions include finite matrix rings [18], and crossed product $R * G$ where G is a finite group with $|G|^{-1} \in R$ [19]. The notion of almost excellent extensions was introduced and studied in [28] as a non-trivial generalization of excellent extensions.

Let S be a finite normalizing extension (in particular, an (almost) excellent extension) of a ring R. It is well known that R is right perfect if and only if S is right perfect [21, Corollary 7]. It seems natural to generalize descent of right perfectness to cotorsion dimension in the case when S is an (almost) excellent extension of a ring R and this is the main goal of the rest of this section.

Theorem 19.3.3 *Let S be an almost excellent extension of a ring R and M_S a right S-module. Then*

1. $cd(M_S) = cd(M_R) = cd(\mathrm{Hom}_R(S, M))$.

2. *M_S is cotorsion if and only if M_R is cotorsion if and only if $\mathrm{Hom}_R(S, M)$ is a cotorsion right S-module.*

Proof (1). We first prove that $\mathrm{cd}(M_S) \leq \mathrm{cd}(M_R)$. We may assume that $\mathrm{cd}(M_R) = n < \infty$. Let N_S be a flat right S-module. Then N_R is a flat right R-module by [29, Lemma 1.2 (3)]. Note that $\mathrm{Ext}_R^{n+1}(N, M) \cong \mathrm{Ext}_S^{n+1}(N \otimes_R S, M)$ by [23, Theorem 11.65]. Since $\mathrm{Ext}_R^{n+1}(N, M) = 0$, $\mathrm{Ext}_S^{n+1}(N \otimes_R S, M) = 0$. Thus $\mathrm{Ext}_S^{n+1}(N, M) = 0$ by [29, Lemma 1.1 (1)], and so $\mathrm{cd}(M_S) \leq n$.

Conversely, suppose $\mathrm{cd}(M_S) = n < \infty$. Let N_R be a flat right R-module. Then $N \otimes_R S$ is a flat right S-module, and so $\mathrm{Ext}_R^{n+1}(N \otimes_R S, M) = 0$. Thus, by the above isomorphism, we get $\mathrm{Ext}_R^{n+1}(N, M) = 0$, and hence $\mathrm{cd}(M_R) \leq n$.

By [16, Lemma 2.16], if E_R is a cotorsion right R-module, then $\mathrm{Hom}_R(S, E)$ is a cotorsion right S-module. Hence $\mathrm{cd}(\mathrm{Hom}_R(S, M)) \leq \mathrm{cd}(M_R)$ by Corollary 19.2.2. Since M_S is isomorphic to a direct summand of $\mathrm{Hom}_R(S, M)$ by [29, Lemma 1.1 (2)], $\mathrm{cd}(M_S) \leq \mathrm{cd}(\mathrm{Hom}_R(S, M))$. So (1) holds.

(2) follows from (1). □

Corollary 19.3.4 *Let R and S be rings.*

1. *If S is an almost excellent extension of R, then $r.cot.D(S) \leq r.cot.D(R)$.*

2. *If S is an excellent extension of R, then $r.cot.D(S) = r.cot.D(R)$.*

Proof (1) follows from Theorem 19.3.3.

(2). Since S is an excellent extension of R, R is an R-bimodule direct summand of S. Let ${}_RS_R = R \oplus T$, and M_R be any right R-module. Observe that $\mathrm{Hom}_R(S, M) \cong \mathrm{Hom}_R(R, M) \oplus \mathrm{Hom}_R(T, M)$. Therefore

$$\mathrm{cd}(M_R) \leq \mathrm{cd}(\mathrm{Hom}_R(S, M)) \leq \mathrm{r.cot.D}(S)$$

by Theorem 19.3.3 (1), and hence $\mathrm{r.cot.D}(R) \leq \mathrm{r.cot.D}(S)$. So (2) follows from (1). □

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Theorem 19.3.5 *Let S be an almost excellent extension of a ring R. If $r.cot.D(R) < \infty$, then $r.cot.D(S) = r.cot.D(R)$.*

Proof It is enough to show that r.cot.D$(R) \leq$ r.cot.D(S) by Corollary 19.3.4. Suppose r.cot.D$(R) = n < \infty$. Then there exists a right R-module M such that cd$(M_R) = n$. Define a right R-homomorphism $\alpha : \mathrm{Hom}_R(S, M) \to M$ via $\alpha(f) = f(1)$ for any $f \in \mathrm{Hom}_R(S, M)$. Since S_R is projective, the epimorphism $\pi : M \to M/\mathrm{im}(\alpha)$ induces the epimorphism $\pi_* : \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, M/\mathrm{im}(\alpha))$. Let $f \in \mathrm{Hom}_R(S, M)$ and $s \in S$. Then $\pi_*(f)(s) = \pi(f(s)) = \pi((fs)(1)) = \pi(\alpha(fs)) = 0$, and so $\ker(\pi_*) = \mathrm{Hom}_R(S, M)$. It follows that $\mathrm{Hom}_R(S, M/\mathrm{im}(\alpha)) = 0$, and hence $M/\mathrm{im}(\alpha) = 0$ by [25, Proposition 2.1]. Thus α is epic, and so we have a right R-module exact sequence $0 \to K \to \mathrm{Hom}_R(S, M) \to M \to 0$. By Propositon 19.2.3 (3), we have $n = \mathrm{cd}(M_R) \leq \sup\{\mathrm{cd}(\mathrm{Hom}_R(S, M)), \mathrm{cd}(K_R) - 1\} \leq$ r.cot.D$(R) = n$. Since $\mathrm{cd}(K_R) - 1 \leq n - 1$, then $\mathrm{cd}(\mathrm{Hom}_R(S, M)) = n$. On the other hand, $\mathrm{cd}(\mathrm{Hom}_R(S, M)) \leq$ r.cot.D(S) by Theorem 19.3.3. Therefore r.cot.D$(R) \leq$ r.cot.D(S), as desired. □

19.4 Applications in Commutative Rings

In this section, all rings are assumed to be commutative. We need the following lemma which will be frequently used in the sequel.

Lemma 19.4.1 *Let R be a ring and M an R-module, then the following are equivalent:*

1. *M is cotorsion.*
2. $\mathrm{Hom}_R(F, M)$ *is a cotorsion R-module for any flat R-module F.*
3. $\mathrm{Hom}_R(P, M)$ *is a cotorsion R-module for any projective R-module P.*

   Moreover, if the class of cotorsion R-modules is closed under direct sums, then the above conditions are also equivalent to

4. *$P \otimes_R M$ is a cotorsion R-module for any projective R-module P.*

Proof (1) ⇒ (2). Let N, F be two flat R-modules. There exists an exact sequence $0 \to K \to G \to N \to 0$ with G projective, which yields the exactness of the sequence $0 \to K \otimes_R F \to G \otimes_R F \to N \otimes_R F \to 0$. Note that $N \otimes_R F$ is flat. We have the following exact sequence

$$\mathrm{Hom}_R(G \otimes_R F, M) \to \mathrm{Hom}_R(K \otimes_R F, M) \to \mathrm{Ext}^1_R(N \otimes_R F, M) = 0,$$

which gives rise to the exactness of the sequence

$$\mathrm{Hom}_R(G, \mathrm{Hom}_R(F, M)) \to \mathrm{Hom}_R(K, \mathrm{Hom}_R(F, M)) \to 0.$$

On the other hand, the following sequence

$$\begin{aligned}\mathrm{Hom}_R(G, \mathrm{Hom}_R(F, M)) \to \mathrm{Hom}_R(K, \mathrm{Hom}_R(F, M)) \to \\ \mathrm{Ext}^1_R(N, \mathrm{Hom}_R(F, M)) \to \mathrm{Ext}^1_R(G, \mathrm{Hom}_R(F, M)) = 0\end{aligned}$$

is exact. Thus $\mathrm{Ext}^1_R(N, \mathrm{Hom}_R(F, M)) = 0$, and (2) follows.

(2) ⇒ (3) is trivial.

(3) ⇒ (1) follows by letting $P = R$.

The last statement is easy to verify. □

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Corollary 19.4.2 *Let R be a ring such that the class of cotorsion R-modules is closed under direct sums. Then the following are equivalent:*

1. *The cotorsion envelope of any projective R-module is always projective.*

2. *$C(R_R)$ is projective.*

Proof (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Consider the exact sequence $0 \to R \to C(R_R) \to N \to 0$. Let M be any projective R-module, then $0 \to R \otimes_R M \to C(R_R) \otimes_R M \to N \otimes_R M \to 0$ is also exact. Note that $C(R_R) \otimes_R M$ is projective, and cotorsion by Lemma 19.4.1. It follows that $M \to C(R_R) \otimes_R M$ is a cotorsion preenvelope of M since $N \otimes_R M$ is flat. Hence $C(M)$ is projective since it is a direct summand of $C(R_R) \otimes_R M$ by [9, Proposition 6.1.2]. □

The next proposition shows that if R is a Dedekind domain, then $\mathrm{Ext}^1_R(B, C)$ is cotorsion for all R-modules B and C, which may be viewed as an answer to [11, Problem 48, p.462].

Proposition 19.4.3 *Let R be a ring.*

1. *If $D(R) \leq 1$ (i.e., R is a hereditary ring), then $\mathrm{Ext}^1_R(B, C)$ is cotorsion for all R-modules B and C.*

2. *If cot.D$(R) \leq 1$, then $\mathrm{Ext}^1_R(F, M)$ is cotorsion for any flat R-module F and any R-module M.*

Proof (1) follows from the isomorphism

$$\mathrm{Ext}^1_R(\mathrm{Tor}^R_1(A, B), C) \cong \mathrm{Ext}^1_R(A, \mathrm{Ext}^1_R(B, C))$$

for all R-modules A, B and C (see [23, p.343]).

(2). Let M be any R-module. By hypothesis, there exists an exact sequence $0 \to M \to C^0 \to C^1 \to 0$, where C^0 and C^1 are cotorsion. So the sequence $\mathrm{Hom}_R(F, C^1) \to \mathrm{Ext}^1_R(F, M) \to \mathrm{Ext}^1_R(F, C^0) = 0$ is exact for any flat R-module F. By Lemma 19.4.1, $\mathrm{Hom}_R(F, C^1)$ is cotorsion, and hence $\mathrm{Ext}^1_R(F, M)$ is cotorsion by Theorem 19.2.11. □

We omit the proof of the next proposition which can be deduced easily from Lemma 19.4.1.

Proposition 19.4.4 *Let R be a ring and M an R-module. Then the following are equivalent:*

1. *$cd(M) \leq n$.*

2. *$cd(\mathrm{Hom}_R(P, M)) \leq n$ for any projective R-module P.*

We are now in a position to prove the following

Theorem 19.4.5 *Let $\varphi : R \to S$ be a surjective ring homomorphism with $K = Ker(\varphi)$. If S_R is projective, then, for any R-module M, either $cd(M_R) \leq sup\{pd(R/I)_R : I \subseteq K\}$, or $cd(M_R) = cd(\mathrm{Hom}_R(S, M))$, where $\mathrm{Hom}_R(S, M)$ may be regarded as an R-module or S-module.*

Proof Let $\sup\{\mathrm{pd}(R/I)_R : I \subseteq K\} = n$. We may assume $n < \infty$.

Suppose $\mathrm{cd}(M_R) > n$. We shall show that $\mathrm{cd}(M_R) = \mathrm{cd}(\mathrm{Hom}_R(S, M))$.

In fact, there exists an exact sequence

$$0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0,$$

where each C^i is a cotorsion R-module, $i = 1, 2, \ldots, n-1$. Thus

$$\mathrm{cd}(M_R) = \mathrm{cd}(C^n) + n$$

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by Corollary 19.2.4, and

$$\mathrm{Ext}_R^j(R/I, C^n) \cong \mathrm{Ext}_R^{n+j}(R/I, M) = 0$$

for all $j > 0$, and all $I \subseteq K$.

We claim that $\mathrm{cd}(C^n) = \mathrm{cd}(\mathrm{Hom}_R(S, C^n))$.

In fact, $\mathrm{cd}(\mathrm{Hom}_R(S, C^n)) \leq \mathrm{cd}(C^n)$ by Proposition 19.4.4. We only need to show that $\mathrm{cd}(C^n) \leq \mathrm{cd}(\mathrm{Hom}_R(S, C^n))$. Note that $C^n \cong \mathrm{Hom}_R(R, C^n)$ and the exactness of $0 \to K \to R \to S \to 0$ induces an exact sequence $0 \to \mathrm{Hom}_R(S, C^n) \to \mathrm{Hom}_R(R, C^n) \to \mathrm{Hom}_R(K, C^n) \to 0$. It is enough to show that $\mathrm{Hom}_R(K, C^n)$ is an injective R-module by Proposition 19.2.3 (1).

Let L be any ideal of R. The exactness of $0 \to K/LK \to R/LK \to R/K \to 0$ gives an exact sequence

$$\mathrm{Ext}_R^1(R/LK, C^n) \to \mathrm{Ext}_R^1(K/LK, C^n) \to \mathrm{Ext}_R^2(R/K, C^n).$$

Since $\mathrm{Ext}_R^1(R/LK, C^n) = \mathrm{Ext}_R^2(R/K, C^n) = 0$ by the first part of the proof, $\mathrm{Ext}_R^1(K/LK, C^n) = 0$. Hence the exact sequence $0 \to LK \to K \to K/LK \to 0$ yields the exactness of

$$\mathrm{Hom}_R(K, C^n) \to \mathrm{Hom}_R(LK, C^n) \to 0.$$

Note that

$$\mathrm{Hom}_R(R, \mathrm{Hom}_R(K, C^n)) \cong \mathrm{Hom}_R(K, C^n),$$
$$\mathrm{Hom}_R(L, \mathrm{Hom}_R(K, C^n)) \cong \mathrm{Hom}_R(L \otimes K, C^n) \cong \mathrm{Hom}_R(LK, C^n).$$

The last isomorphism holds by the flatness of K. Thus the sequence

$$\mathrm{Hom}_R(R, \mathrm{Hom}_R(K, C^n)) \to \mathrm{Hom}_R(L, \mathrm{Hom}_R(K, C^n)) \to 0$$

is exact, and so $\mathrm{Hom}_R(K, C^n)$ is R-injective, as required.

On the other hand, since S_R is projective, we have the following exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, C^0) \to \\ \mathrm{Hom}_R(S, C^1) \to \cdots \to \mathrm{Hom}_R(S, C^{n-1}) \to \mathrm{Hom}_R(S, C^n) \to 0, \end{aligned}$$

where each $\mathrm{Hom}_R(S, C^i)$, $i = 1, 2, \ldots, n-1$, is a cotorsion R-module by Proposition 19.3.1 (2). Note that

$$\mathrm{cd}(\mathrm{Hom}_R(S, C^n)) = \mathrm{cd}(C^n) = \mathrm{cd}(M_R) - n > 0.$$

Thus $\mathrm{cd}(\mathrm{Hom}_R(S, M)) > n$, and so

$$\mathrm{cd}(\mathrm{Hom}_R(S, M)) = \mathrm{cd}(\mathrm{Hom}_R(S, C^n)) + n$$

by Corollary 19.2.4. It follows that $\mathrm{cd}(M_R) = \mathrm{cd}(\mathrm{Hom}_R(S, M))$, where $\mathrm{Hom}_R(S, M)$ may be regarded as an R-module or S-module by Proposition 19.3.1 (1). □

Corollary 19.4.6 *Let $\varphi : R \to S$ be a surjective ring homomorphism with $K = Ker(\varphi)$. If S_R is projective, then either cot.D$(R) \leq$ sup$\{pd(R/I)_R : I \subseteq K\}$, or cot.D$(R)$ = cot.D(S).*

Proof Let $\sup\{\mathrm{pd}(R/I)_R : I \subseteq K\} = n$. If $\mathrm{cd}(M_R) \leq n$ for every R-module M_R, then $\mathrm{cot.D}(R) \leq n$. If there is M_R such that $\mathrm{cd}(M_R) > n$, then $\mathrm{cd}(M_R) = \mathrm{cd}(\mathrm{Hom}_R(S, M)) \leq \mathrm{cot.D}(S)$ by Theorem 19.4.5, and so $\mathrm{cot.D}(R) \leq \mathrm{cot.D}(S)$. Note that $\mathrm{cot.D}(S) \leq \mathrm{cot.D}(R)$ by Corollary 19.3.2. So $\mathrm{cot.D}(R) = \mathrm{cot.D}(S)$. □

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Corollary 19.4.7 *A ring R is perfect if and only if there is a quotient ring $S = R/K$ of R such that S is a perfect ring and R/I is a projective R-module for any $I \subseteq K$.*

Corollary 19.4.8 *Let K be a maximal ideal of a ring R such that R/K is a projective R-module, then cot.D$(R) \leq$ sup$\{pd(R/I)_R : I \subseteq K\}$.*

Proposition 19.4.9 *Let P be any prime ideal of a ring R, then cot.D$(R_P) \leq$ cot.D(R), where R_P is the localization of R at P.*

Proof We may assume cot.D$(R) = n < \infty$. Let M be any flat R_P-module. Since R_P is a flat R-module, then M is a flat R-module. Thus pd$(M_R) \leq n$. There exists a projective resolution of M_R

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0,$$

which induces an R_P-module exact sequence

$$0 \to (F_n)_P \to (F_{n-1})_P \to \cdots \to (F_1)_P \to (F_0)_P \to M_P \to 0.$$

Note that since each $(F_i)_P$ is a projective R_P-module, $i = 0, 1, \ldots, n$, it follows that pd$(M_P)_{R_P} \leq n$. Since $(M_P)_{R_P} \cong M_{R_P}$, pd$(M_{R_P}) \leq n$. Thus cot.D$(R_P) \leq n$, as required. □

It is well known that R is a coherent ring if and only if Hom$_R(A, B)$ is flat for all injective R-modules A and B ([17]). By [5, Corollary 3.22], R is an IF ring (the ring for which every injective R-module is flat) if and only if Hom$_R(A, B)$ is injective for all injective R-modules A and B. Continuing this style of characterizing rings by properties of homormophism modules of certain special R-modules, we conclude this paper with the following easy results for completeness.

Proposition 19.4.10 *Let R be a ring, then the following are equivalent:*

1. *R is a von Neumann regular ring.*
2. *For each cotorsion R-module A,* Hom$_R(A, B)$ *is injective for all cotorsion (or injective) R-modules B.*
3. *For each cotorsion R-module A,* Hom$_R(A, B)$ *is flat for all cotorsion (or injective) R-modules B.*

Proof (1) ⇒ (2). Let A and B be cotorsion, then Hom$_R(A, B)$ is cotorsion by Lemma 19.4.1 (for A is flat by (1)). Thus Hom$_R(A, B)$ is injective by [27, Theorem 3.3.2].

(2) ⇒ (1). Let A be a cotorsion R-module. (2) implies that Hom$_R(A, -)$ preserves injectives. Thus A is flat by [10, Proposition 11.35], and (1) follows from [27, Theorem 3.3.2].

(1) ⇒ (3) is trivial.

(3) ⇒ (1). Let S be any simple R-module. Then S is cotorsion by [16, Lemma 2.14]. Let $E = E(\oplus_{i \in I} S_i)$, where $\{S_i\}_{i \in I}$ is an irredundant set of representatives of the simple R-modules. Then E is an injective cogenerator by [1, Corollary 18.19]. Note that Hom$_R(S, E)$ is flat by (3) and Hom$_R(S, E) \cong S$ by the proof of [26, Lemma 2.6]. Thus S is flat, and hence R is regular by [20, 3.3]. □

Proposition 19.4.11 *Let R be a ring, then the following are equivalent:*

1. *R is a semisimple Artinian ring.*
2. *For each cotorsion R-module A,* Hom$_R(A, B)$ *is projective for all cotorsion (or injective) R-modules B.*

Proof (1) ⇒ (2) is trivial.

(2) ⇒ (1). Let S be any simple R-module. By (2) and the proof of (3) ⇒ (1) in Proposition 19.4.10, S is projective. So R is semisimple Artinian. □

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Remark 19.4.12 We wonder what kind of commutative rings is characterized by the condition that every homomorphism module of cotorsion modules is cotorsion. This kind of rings, of course, contains perfect rings and von Neumann regular rings. It is easy to verify that a ring R is of this kind if and only if $\mathrm{Hom}_R(A, B)$ is cotorsion for all R-modules A and all cotorsion R-modules B.

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Chapter 20

Maximal Subrings of Homogeneous Functions

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Abstract For an abelian group, A, the subnear-ring structure and subring structure of the near-ring of zero preserving functions on A have been the subjects of recent investigations. In this expository paper we discuss some of the questions investigated and present some of the results.

Subject classifications: Primary 20K30; Secondary 16Y30.

Keywords: Endomorphism rings; near-rings of mappings.

20.1 Introduction

Let A be an abelian group and let $M_0(A) = \{f\colon\ A \to A \mid f(0) = 0\}$ denote the near-ring of zero preserving functions on A where the operations are pointwise addition and composition of functions. It is well known that $M_0(A)$ is a simple near-ring, [11], for any group A, not necessarily abelian. On the other hand, $M_0(A)$ does contain subrings, for example, $\mathrm{End}(A)$, the ring of all endomorphisms of the abelian group A. This raises several questions:

Q1. What is the subring structure of $M_0(A)$?
Q2. What are the maximal subrings of $M_0(A)$?
Q3. When is $\mathrm{End}(A)$ maximal as a subring of $M_0(A)$?
Q4. When is $\mathrm{End}(A)$ maximal as a subnear-ring of $M_0(A)$?

We note for $|A| \in \{1, 2\}$, $M_0(A) = \mathrm{End}(A)$. For $|A| = 1$, $M_0(A) = \{0\}$ and for $|A| = 2$, $M_0(A) = \{0, id\}$, hence the questions Q1–Q4 are trivial. Thus in the sequel we take $|A| \geq 3$.

Convention. Throughout the remainder of this paper all groups are abelian and the adjective "abelian" is often omitted. We use $\mathbb{Z}, \mathbb{Z}_n, \mathbb{Q}$ to denote the additive group (or ring) of integers, integers modulo n, and the rational numbers respectively. Moreover $\mathbb{N}$ is the set of positive integers. For undefined notations and concepts regarding abelian groups we refer the reader to [4].

Recently Neumaier, [9], determined all maximal subnear-rings of $M_0(G)$, G a finite group, not necessarily abelian. For finite abelian groups, A, Neumaier found that $\mathrm{End}(A)$ is maximal as a subnear-ring of $M_0(A)$ if and only if $A \cong \bigoplus_{\text{finite}} \mathbb{Z}_2$ or $|A| = 3$. In fact, these are the only cases when one considers *all* abelian groups.

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Theorem 20.1.1 ([7]) *Let A be an abelian group with $|A| \geq 3$. Then* $\mathrm{End}(A)$ *is a maximal subnear-ring of $M_0(A)$ if and only if $A \cong \bigoplus_{\text{finite}} \mathbb{Z}_2$ or $|A| = 3$.*

We have thus answered Q4 above and found that $\mathrm{End}(A)$ is "almost never" a maximal subnear-ring of $M_0(A)$. On the other hand, we will next see that the situation is quite different as regards Q3. Moreover, as one might guess, the structure theory of abelian groups is prominent in this investigation.

For an abelian group A let $M_{\mathbb{Z}}(A) := \{f \in M_0(A) | f(na) = nf(a), \forall n \in \mathbb{Z}, \forall a \in A\}$ be the near-ring of homogeneous functions on A. We note that if R is a subring of $M_0(A)$ which contains $\mathrm{End}(A)$, then R is contained in $M_{\mathbb{Z}}(A)$. This follows from the fact that, for each $r \in R$, $r(id + id) = r + r$, so $r(2a) = 2r(a)$, for each $a \in A$. Then using induction and ring properties, one obtains $r(ka) = k(ra)$ for each $k \in \mathbb{Z}$ and $a \in A$, i.e., $r \in M_{\mathbb{Z}}(A)$. Hence in particular, if $M_{\mathbb{Z}}(A)$ is a ring, then it is maximal.

Therefore the question as to when $\mathrm{End}(A)$ is a maximal subring of $M_0(A)$ is tied to the relationship between $\mathrm{End}(A)$ and $M_{\mathbb{Z}}(A)$. This relationship has been investigated by Jutta Hausen in [5] and by Hausen and Johnson in [6].

Theorem 20.1.2 ([5],[6]) *Let A be an abelian group.*

i) If A is a torsion group then $M_{\mathbb{Z}}(A)$ is a ring if and only if $M_{\mathbb{Z}}(A) =$ $\mathrm{End}(A)$ *and this happens if and only if A is a subgroup of $\bigoplus_{p\in\Pi} \mathbb{Z}(p^\infty)$. Here we let Π denote the set of prime integers.*

ii) If A is a torsion-free abelian group then $M_{\mathbb{Z}}(A) =$ $\mathrm{End}(A)$ *if and only if $A \subseteq \mathbb{Q}$, i.e., A is a subgroup of the group of rational numbers. Further, if A is torsion-free, then $M_{\mathbb{Z}}(A)$ is a ring different from* $\mathrm{End}(A)$ *if and only if A is absolutely anisotropic of rank at least 2.*

iii) If A is a mixed abelian group then $M_{\mathbb{Z}}(A)$ is never a ring.

Corollary 20.1.3 *If $A \subseteq \bigoplus_{p\in\Pi} \mathbb{Z}(p^\infty)$, then* $\mathrm{End}(A)$ *is a maximal subring of $M_0(A)$.*

However, it is not always the case that $\mathrm{End}(A)$ is a maximal subring of $M_0(A)$.

Theorem 20.1.4 ([3]) *Let A be a torsion-free group of rank at least 2. If* $\mathrm{End}(A) \subseteq \mathbb{Q}$, *then* $\mathrm{End}(A)$ *is not a maximal subring of $M_0(A)$.*

Proof [Sketch of Proof] Let $0 \neq a \in A$, let $\langle a \rangle_*$ denote the pure subgroup of A generated by a and let $R = \{f \in M_{\mathbb{Z}}(A) | f(\langle a \rangle_*) \subseteq \langle a \rangle_*$, for all $a \in A\}$. It is clear that R is a subnear-ring of $M_{\mathbb{Z}}(A)$. Calculations show that R is also left distributive and hence a ring. Since the rank of A is at least 2, it is not hard to show $R \neq \mathrm{End}(A)$. Since $\mathbb{Q}$ is contained in R, the result follows. □

In the next section we show that for torsion groups, the answer to Q3 is "always". As the above theorem indicates, this is not the case for torsion-free groups. In Section 20.3 we consider this case.

20.2 The Case of Torsion Groups

Following the usual convention, we say an abelian group A is *E-locally cyclic* (*E-lc*) if for each $a, b \in A$, there exists $c \in A$ and $\alpha, \beta \in \mathrm{End}(A)$ such that $\alpha(c) = a$ and $\beta(c) = b$. Reid ([10],[12]), among others, has used the concept of E-cyclic but as far as the author knows there is no characterization of E-locally cyclic groups. We now show that if A is an E-locally cyclic

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abelian group then $\mathrm{End}(A)$ is indeed a maximal subring of $M_0(A)$ and we identify many classes of E-locally cyclic abelian groups.

Theorem 20.2.1 ([7]) *If A is an E-locally cyclic abelian group then* $\mathrm{End}(A)$ *is a maximal subring of $M_0(A)$.*

Proof [Sketch of Proof] Let R be a subring of $M_0(A)$ with $R \supseteq \mathrm{End}(A)$. Now let $r \in R$, $a, b \in A$. Since A is E-lc, there exists some $c \in A$ and $\alpha, \beta \in \mathrm{End}(A)$ with $\alpha(c) = a$ and $\beta(c) = b$. Calculations show that $r \in \mathrm{End}(A)$ so $R = \mathrm{End}(A)$ as desired. □

The converse of this theorem is not true. By modifying an example of Arnold and Dugas (see [2], page 158) one obtains a torsion-free group, A, of finite rank for which $\mathrm{End}(A)$ is a maximal subring of $M_0(A)$ but A is not E-lc. (See also [3], Example 1.)

We next give several classes of groups which are E-lc. Of course, as corollaries, each group, A, in one of these classes has the property that $\mathrm{End}(A)$ is a maximal subring of $M_0(A)$.

Theorem 20.2.2 ([7]) *i) A direct sum of E-lc groups is E-lc.*

ii) A direct sum of cyclic groups is E-lc.

iii) Finitely generated groups are E-lc.

iv) Divisible groups are E-lc.

As a result of iv) of the above we have the following.

Corollary 20.2.3 *An abelian group is E-lc if and only if its reduced summand is E-lc.*

We now state the major result of this section.

Theorem 20.2.4 ([7]) *Every torsion group is E-lc. Thus for every torsion group, A,* $\mathrm{End}(A)$ *is a maximal subring of $M_0(A)$.*

Proof [Sketch of Proof] From the results above, one first restricts to reduced groups and then to reduced p-groups. If A is a bounded reduced p-group then A is the direct sum of cyclics and the result follows from ii) of Theorem 20.2.2. If A is an unbounded reduced p-group, one uses the fact that A has a cyclic summand of high enough order. □

As we see in the next section and as we have seen above, the situation for torsion-free groups is not as nice.

20.3 The Case of Torsion-Free Groups

If A is a torsion-free abelian group, we write QA for $Q \otimes_{\mathbb{Z}} A$. Note that A is an $\mathrm{End}(A)$-module by setting $\varphi \cdot a = \varphi(a)$ for all $\varphi \in \mathrm{End}(A)$ and $a \in A$. The same holds for QA and $Q\mathrm{End}(A)$.

Definition 20.3.1 A torsion-free group A is $qElc$ if and only if for all elements $a, b \in A$ there exists some $c \in A$ such that $a, b \in (Q\mathrm{End}(A)) \cdot c$.

Any E-lc group is $qElc$. However, the converse is not true. In fact, the example mentioned after Theorem 20.2.1 is an example of a $qElc$ group that is not E-lc. We next collect some properties of $qElc$ groups.

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Theorem 20.3.2 ([3]) *Let A, B be torsion-free abelian groups. Then the following hold:*

(1) If A is quasi-isomorphic to B and A is $qElc$, then B is $qElc$.

(2) The following are equivalent:

(2.1) A is $qElc$.

(2.2) Any finite subset of A is contained in a cyclic $Q\mathrm{End}(A)$-submodule of QA, i.e., QA is a locally cyclic $Q\mathrm{End}(A)$-module.

(2.3) For any $a, b \in A$ there are $\alpha, \beta \in \mathrm{End}(A)$ and some $n \in \mathbb{N}$, $c \in A$ such that $\alpha(c) = na$ and $\beta(c) = nb$.

(3) If A is torsion-free of finite rank (tffr), then A is $qElc$ if and only if QA is a cyclic $Q\mathrm{End}(A)$-module.

(4) Any direct sum of $qElc$ groups is again $qElc$.

(5) If A is tffr and $qElc$, then $\mathrm{rank}(\mathrm{End}(A)) \geq \mathrm{rank}(A)$.

By using some of these properties we obtain the next result which is the motivation for introducing $qElc$ groups.

Theorem 20.3.3 ([3]) *If A is a torsion-free group which is $qElc$ then* $\mathrm{End}(A)$ *is a maximal subring of* $\mathrm{End}(A)$.

One uses the results above and Arnold's work [1] to completely determine which torsion-free groups of rank 2 are $qElc$. Such a group A is either almost completely decomposable, i.e., A is quasi-equal to a completely decomposable group, or strongly indecomposable, i.e., $Q\mathrm{End}(A)$ has only the trivial idempotents. Recall that any rank 1 group $A \subseteq Q$ is locally cyclic and thus A is E-lc and $qElc$. By Theorem 20.3.2 (4) we have that completely decomposable groups are $qElc$ and Theorem 20.3.2 (1) implies that any almost completely decomposable group is $qElc$. Therefore, we may now assume that A is strongly indecomposable.

Theorem 20.3.4 ([3]) *Let A be a torsion-free group of rank 2. Then the following are equivalent:*

(1) A is $qElc$.

(2) $Q\mathrm{End}(A)$ is not isomorphic to Q.

(3) $\dim_Q(Q\mathrm{End}(A)) \geq \mathrm{rank}(A) = 2$.

(4) $\mathrm{End}(A)$ *is a maximal subring of $M_0(A)$.*

When A is strongly indecomposable, then the inequality in (3) becomes an equality.

Corollary 20.3.5 *Let A be a torsion-free group of rank 2. Then* $\mathrm{End}(A)$ *is a maximal subring of $M_0(A)$ if and only if* $\dim_Q(Q\mathrm{End}(A)) > 1$.

No characterization of torsion-free groups, A, for which $\mathrm{End}(A)$ is a maximal subring of $M_0(A)$ is known. We conclude this section with a few scattered results about strongly indecomposable torsion-free groups of finite rank that are $qElc$. Recall that a torsion-free group of finite rank A is strongly indecomposable if and only if $Q\mathrm{End}(A)$ is a local ring.

Theorem 20.3.6 ([3]) *Let A be a strongly indecomposable tffr group such that $Q\mathrm{End}(A) = D$ is a division ring. Then the following are equivalent:*

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(1) *A is $qElc$.*

(2) $\mathrm{rank}(\mathrm{End}(A)) = \dim_Q(D) = \dim_Q(QA) = \mathrm{rank}(A)$.

(3) $\dim_D(QA) = 1$.

(4) *$Dx = Dy$ for all $0 \neq x, y \in QA$.*

(5) *$Dx \cap Dy \neq \{0\}$ for all $0 \neq x, y \in QA$.*

Theorem 20.3.7 ([3]) *Let A be a tffr strongly indecomposable group with $Q\mathrm{End}(A) = Q + J$ where J denotes the Jacobson radical of $Q\mathrm{End}(A)$. Then A is $qElc$ if and only if there exists some $c \in QA$ such that* $\dim_Q(Jc) = \mathrm{rank}(A) - 1$.

20.4 Subrings of $M_0(A)$

In this short section we make some comments about the question Q2 of Section 20.1. Suppose that A is a finite abelian group. From Theorem 20.1.2, if $A \subseteq \bigoplus_{\text{finite}} \mathbb{Z}(p^\infty)$, then $M_{\mathbb{Z}}(A) = \mathrm{End}(A)$. As we have also noted above, if R is any subring of $M_0(A)$, then $R \subseteq M_{\mathbb{Z}}(A)$. Thus, in this case, i.e., in the case of finite cyclic groups, all subrings of $M_0(A)$ are rings of endomorphisms of A. Hence there is a unique maximal subring, $\mathrm{End}(A)$. However, if $A \not\subseteq \bigoplus_{\text{finite}} \mathbb{Z}(p^\infty)$ we find there can be other maximal subrings.

Let $A = (\mathbb{Z}_p)^n$ for some prime p and let $\Pi = \{\Pi_i\}_{i=1}^s$ be a partition of A by $\mathbb{Z}_p$-subspaces Π_i. Define $\mathcal{R}(\Pi) = \{f \in M_0(A) | f_{|\Pi_i} \in \mathrm{End}(\Pi_i)$ for each $i\}$. One shows that $\mathcal{R}(\Pi)$ is a subring of $M_0(A)$ and $\mathcal{R}(\Pi) \not\subseteq \mathrm{End}(A)$. In fact $\mathcal{R}(\Pi) = \mathrm{End}(\Pi_1) \oplus \cdots \oplus \mathrm{End}(\Pi_s)$ with the pointwise operations of addition and composition. Note that for $i \neq j$, one can define $\rho \in \mathcal{R}(\Pi)$ such that for $0 \neq a \in \Pi_i$ and $0 \neq b \in \Pi_j$, $\rho(a+b) \neq \rho(a) + \rho(b)$. Further, since $\mathcal{R}(\Pi)_{|\Pi_i} = \mathrm{End}(\Pi_i)$, there exists $g_i \in \Pi_i$ such that $\mathcal{R}(\Pi)g_i = \Pi_i$. Now suppose S is a subring of $M_0(A)$ with $S \supseteq \mathcal{R}(\Pi)$. Then $Sg_i \supseteq \mathcal{R}(\Pi)g_i = \Pi_i$. Assume $Sg_i \supsetneqq \Pi_i$ for some i, say $w \in Sg_i \backslash \Pi_i$. Hence there is some $\sigma \in S$ with $\sigma(g_i) = w$. Since $\mathcal{R}(\Pi) \subseteq S$, we have, for each $\rho \in \mathcal{R}(\Pi)$, $\rho(\sigma + id) = \rho\sigma + \rho \cdot id$. Thus we have $\rho(w + g_i) = \rho(\sigma(g_i) + id(g_i)) = \rho(\sigma + id)(g_i) = (\rho\sigma + \rho)g_i = \rho(w) + \rho(g_i)$. However, since $w \notin \Pi_i$, this is a contradiction and so $Sg_i = \Pi_i$ for each i. This in turn implies $S/\Pi_i = \mathrm{End}(\Pi_i)$ and $S = \mathcal{R}(\Pi)$. Therefore $\mathcal{R}(\Pi)$ is a maximal subring of $M_0(A)$ for each partition Π of $(\mathbb{Z}_p)^n$.

This example indicates that the questions Q1 and Q2 of Section 20.1 should lead to some interesting results relating the structures of abelian groups and near-rings of mappings. Preliminary investigations, ([8]), show that this is indeed the case. It should be mentioned however that almost nothing is known about subrings of $M_0(A)$, other than $\mathrm{End}(A)$, when A is not a finite group.

Problem: Investigate the subring structure of $M_0(A)$ when A is a torsion-free group of finite rank.

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Chapter 21

Isotype Separable Subgroups of Mixed Abelian Groups

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Abstract Suppose that H is an isotype subgroup of a global mixed abelian group G and that κ is an arbitrary infinite cardinal. If H has a κ-cover of almost balanced pure subgroups, it is shown that H is an almost strongly κ-separable subgroup of G. The converse is established in the case when G is a global Warfield group. Our results generalize similar theorems for torsion-free and p-local abelian groups.

Subject classifications: 20K21.

21.1 Introduction

Throughout, G denotes an arbitrary additively written abelian group; in particular, the case where G is a nonsplit mixed group is not excluded. At times we refer to G as a *global group* to emphasize that G is not necessarily a p-local mixed group. For the most part, our notation and terminology are in agreement with [2], [6], and [7]. Two exceptions are that we write $|x|_p^G$ for the p-height of x in G, and $\|x\|^G$ denotes the height matrix of x in G.

The notion of a *separable subgroup*, introduced almost twenty-five years ago by P. Hill [4] in the context of p-local torsion groups, has played an important role in the structure theory of abelian groups. For example, separability and its various generalizations have proved to be extremely useful in the study of isotype subgroups of abelian groups and the dimension theory of such groups. As examples for its use in the study of isotype subgroups, we mention [4], [9], [10] and [12] in the p-local case, [1] in the torsion-free case, and [7], [13], [14] and [15] in the general mixed case. As examples for the dimension theory, we refer the reader to [3], [16] and [17]. These references are not intended to be exhaustive, but are representative of our emphasis here.

To exhibit some known results and to explain our generalizations, we need some definitions. First, a subgroup H of an abelian group G is called a *strongly separable* subgroup if to each $g \in G$

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there corresponds a countable subset $A \subseteq G$ such that, for each $x \in H$, $\|g + x\|^G \leq \|g + x'\|^G$ for some $x' \in A$. We hasten to mention that "strongly separable" reduces to the usual meaning of "separable" in the p-local and torsion-free settings. Here, and in the sequel, κ denotes an arbitrary infinite cardinal.

Definition 21.1.1 A collection $\mathcal{C}$ of subgroups of G is called a *κ-cover* for G if the following four conditions are satisfied.

(1) $\mathcal{C}$ contains the trivial subgroup 0.

(2) $|N| \leq \kappa$ for all $N \in \mathcal{C}$.

(3) $\mathcal{C}$ is closed under unions of ascending chains of length at most κ.

(4) If A is a subgroup of G with $|A| \leq \kappa$, then there is an $N \in \mathcal{C}$ with $A \subseteq N$.

Definition 21.1.2 A subgroup N of G is called *almost balanced* in G if the following conditions are satisfied:

(1) N is nice in G; that is, for all primes p and ordinals α, the cokernel of the inclusion map $(p^\alpha G + N)/N \rightarrowtail p^\alpha(G/N)$ contains no element of order p.

(2) To each $g \in G$ there corresponds a positive integer m such that the coset $mg + N$ contains an element x with $\|x\|^G = \|mg + N\|^{G/N}$.

As motivation for our subsequent work, we now recall some results that hold for p-local torsion groups ([10]), for p-local mixed groups ([12]), and for torsion-free groups ([1]). These results can be reformulated and combined as follows.

Theorem 21.1.3 ([10], [12], [1]) *Suppose that H has an $\aleph_0$-cover of almost balanced pure subgroups. If G is either a p-local or torsion-free group that contains H as an isotype subgroup, then H is a strongly separable subgroup of G.*

It follows from Proposition 1.7 of [7] that every knice subgroup of a global group is almost balanced. It then easily follows from [7, Theorem 3.2] that every global Warfield group has an $\aleph_0$-cover of almost balanced pure subgroups. But it was shown in [8] that a global Warfield group need not be strongly separable in a group in which it appears as an isotype subgroup. However, it follows from a result of [14] that, for every infinite cardinal κ, a global Warfield group is almost strongly κ-separable in every group in which it appears as an isotype subgroup. Our notion of "almost strong κ-separability" was introduced in [16] and, in the case where $\kappa = \aleph_0$, corresponds to "strong separability" for p-local and torsion-free groups. For the convenience of the reader, we include the following definition.

Definition 21.1.4 Let H be a subgroup of a global group G.

(1) Call H *locally κ-separable* in G if to each $g \in G$ and prime p there corresponds a subset $A \subseteq H$ where $|A| \leq \kappa$ and the following condition is satisfied: If $x \in H$, there is an $x' \in A$ such that $|g + x|_p^G \leq |g + x'|_p^G$.

(2) Call H *almost strongly κ-separable* in G if it is locally κ-separable and to each $g \in G$ there corresponds a subset $B \subseteq H$ where $|B| \leq \kappa$ and the following condition is satisfied: To each $x \in H$ there corresponds an $x' \in B$ and a positive integer m such that $\|m(g + x)\|^G \leq \|m(g + x')\|^G$.

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In this paper, we generalize Theorem 21.1.3 by showing that a global group with a κ-cover of almost balanced pure subgroups is an almost strongly κ-separable subgroup of any group in which it appears as an isotype subgroup (Theorem 21.2.3). Also, in Theorem 21.4.2, we prove a form of the converse of Theorem 21.2.3; namely, an isotype subgroup H of a global Warfield group G is almost strongly κ-separable in G if and only if H has a κ-cover consisting of almost balanced pure subgroups. This latter result can be viewed as a generalization of Theorem 4 in [10] for isotype subgroups of totally projective p-groups.

21.2 Subgroups with κ-covers of Almost Balanced Pure Subgroups

Our first result establishes a condition under which local κ-separability and almost strongly κ-separability are equivalent.

Proposition 21.2.1 *Suppose that H has a κ-cover consisting of almost balanced pure subgroups and that H is an isotype subgroup of G. If H is locally κ-separable in G, then H is almost strongly κ-separable in G.*

Proof Suppose to the contrary that that H is not almost strongly κ-separable in G. Therefore, there is an element $g \in G$ such that, for each subset $K \subseteq H$ with $|K| \leq \kappa$, there is an $h^* \in H$ such that the inequality $\|m(g+x)\|^G \geq \|m(g+h^*)\|^G$ fails for all $x \in K$ and positive integers m.

Let $\mathcal{C}$ be a κ-cover in H consisting of pure almost balanced subgroups and let τ be the first ordinal of cardinality κ. We now construct inductively a smooth ascending chain

$$0 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_\alpha \subseteq \ldots \qquad (\alpha < \tau)$$

in $\mathcal{C}$ such that, for all $\alpha < \tau$, the following condition is satisfied :

(†) If $h_\alpha \in H_\alpha$ and the inequality $|p^k m(g+h_\alpha)|_p^G \lneqq |p^k m(g+h)|_p^G$ holds for some $h \in H$, prime p nonnegative integer k and positive integer m, then there is an $h_{\alpha+1} \in H_{\alpha+1}$ such that $|p^k m(g+h_\alpha)|_p^G \lneqq |p^k m(g+h_{\alpha+1})|_p^G$.

To carry out the induction, suppose that for some ordinal $\mu < \tau$ we have obtained a smooth chain

$$0 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_\alpha \subseteq \ldots \qquad (\alpha < \mu)$$

in $\mathcal{C}$ such that (†) holds for all α with $\alpha + 1 < \mu$. It suffices to construct an $H_\mu \in \mathcal{C}$ with the desired properties. If μ is a limit ordinal, we simply set $H_\mu = \bigcup_{\alpha<\mu} H_\alpha$ as we must and observe that $H_\mu \in \mathcal{C}$ with (†) not relevant in this case. On the other hand, if $\mu = \alpha + 1$ for some α, let $\mathcal{S}$ be the set of ordered 4-tuples $s = (h_\alpha, p, k, m)$ where $h_\alpha \in H_\alpha$, p is a prime, k is a nonnegative integer, m is a positive integer and $|p^k m(g+h_\alpha)|_p^G \lneqq |p^k m(g+h)|_p^G$ for some $h \in H$. For each such s select and fix a single $h_s \in H$ such that $|p^k m(g+h_\alpha)|_p^G \lneqq |p^k m(g+h_s)|_p^G$. Since $|\mathcal{S}| \leq \kappa$ and g is fixed, the subgroup $A = \langle h_s : s \in \mathcal{S}\rangle$ has cardinality not exceeding κ. We now select an $H_{\alpha+1} \in \mathcal{C}$ that contains $H_\alpha + A$ and observe that (†) holds for all α with $\alpha + 1 \leq \mu$. This completes the induction

Now let $H_\kappa = \bigcup_{\alpha<\tau} H_\alpha$ and observe that $H_\kappa \in \mathcal{C}$. In particular, $|H_\kappa| \leq \kappa$ so there exists an $h^* \in H$ such that $\|m(g+x)\|^G \geq \||m(g+h^*)\|^G$ fails for all positive integers m and $x \in H_\kappa$. Moreover, H_κ is a pure almost balanced subgroup of H. Therefore, we may select and fix a positive integer m and an $h' \in H_\kappa$ such that $\|m(h^* - h')\|^G \geq \|m(h^* + x)\|^G$ for all $x \in H_\kappa$. With h' and m so chosen, $\|m(g+h')\|^G \geq \||m(g+h^*)\|^G$ fails. Thus, for some prime p and nonnegative integer k, we have that $|p^k m(g+h')|_p^G \lneqq |p^k m(g+h^*)|_p^G$. Since h' belongs to H_α for some α, it follows

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from condition (†) that $|p^k m(g+h')|_p^G \lneqq |p^k m(g+h'')|_p^G$ for some $h'' \in H_\kappa$. However, this leads to the contradiction

$$|p^k m(g+h')|_p^G \lneqq |p^k m(h^*-h'')|_p^G \leq |p^k m(h^*-h')|_p^G = |p^k m(g+h')|_p^G.$$

We conclude that H must be almost strongly κ-separable in G. □

Lemma 21.2.2 ([16]) *If H is an almost balanced pure subgroup of G, then*

$$p^\alpha(G/H) = (p^\alpha G + H)/H$$

for all primes p and ordinals α.

We now have the necessary ingredients to prove the main result of this section.

Theorem 21.2.3 *If H has a κ-cover consisting of almost balanced pure subgroups, then H is almost strongly κ-separable in any group G in which it appears as an isotype subgroup.*

Proof In view of Proposition 21.2.1, it suffices to show that H is locally κ-separable in any group in which it appears as an isotype subgroup. Suppose to the contrary that there is a global group G that contains H as an isotype subgroup but that H is not locally κ-separable in G. Therefore, there is a prime p and an element $g \in G$ such that, for each subset $K \subseteq H$ with $|K| \leq \kappa$, there is an $h^* \in H$ such that $|g+x|_p^G \lneqq |g+h^*|_p^G$ for all $x \in K$.

Let $\mathcal{C}$ be a κ-cover in H consisting of pure almost balanced subgroups and let τ be the first ordinal of cardinality κ. By an argument similar to that in the proof of Proposition 21.2.1, we obtain a smooth ascending chain

$$0 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_\alpha \subseteq \ldots \qquad (\alpha < \tau)$$

in $\mathcal{C}$ such that, for all $\alpha < \tau$, the following condition is satisfied:

(∗) If $h_\alpha \in H_\alpha$ and the inequality $|g+h_\alpha|_p^G \lneqq |g+h|_p^G$ holds for some $h \in H$, then there is an $h_{\alpha+1} \in H_{\alpha+1}$ such that $|g+h_\alpha|_p^G \lneqq |g+h_{\alpha+1}|_p^G$.

Let $H_\kappa = \bigcup_{\alpha<\tau} H_\alpha$ and observe that $H_\kappa \in \mathcal{C}$. In particular, $|H_\kappa| \leq \kappa$ so there exists an $h^* \in H$ such that $|g+x|_p^G \lneqq |g+h^*|_p^G$ for all $x \in H_\kappa$. Moreover, H_κ is a pure almost balanced subgroup of H. Therefore, by Lemma 21.2.2 we may select an $h' \in H_\kappa$ such that $|h^*-h'|_p^G \geq |h^*+x|_p^G$ for all $x \in H_\kappa$. With h' so chosen, $|g+h'|_p^G \lneqq |g+h^*|_p^G$ and since h' belongs to H_α for some α, it follows from condition (∗) that $|g+h'|_p^G \lneqq |g+h''|_p^G$ for some $h'' \in H_\kappa$. However, this leads to the contradiction

$$|g+h'|_p^G \lneqq |h^*-h''|_p^G \leq |h^*-h'|_p^G = |g+h'|_p^G.$$

We conclude that H must be locally κ-separable in G and hence almost strongly κ-separable by Proposition 21.2.1. □

21.3 Intersection Closure of Global Warfield Groups

Recall that a family $\mathcal{C}$ of subgroups of G is called an $H(\aleph_0)$-*family* if the following three conditions are satisfied:

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(1) $\mathcal{C}$ contains the trivial subgroup 0.

(2) If $\{N_i\}_{i\in I} \subseteq \mathcal{C}$, then $\sum_{i\in I} N_i \in \mathcal{C}$.

(3) If $N \in \mathcal{C}$ and if A is any countable subgroup of G, then there is an $M \in \mathcal{C}$ such that $N + A \subseteq M$ and $|M/N| \leq \aleph_0$.

If $\mathcal{C}$ satisfies the condition

(4) If $\{N_i\}_{i\in I} \subseteq \mathcal{C}$, then $\bigcap_{i\in I} N_i \in \mathcal{C}$,

we call $\mathcal{C}$ *intersection closed*.

In Theorem 21.3.4 below, we establish a result that is crucial for our proof of Theorem 21.4.2. Namely, we show that any global Warfield group has an intersection closed $H(\aleph_0)$-family consisting of pure knice subgroups. This generalizes the corresponding (but much weaker) result established in [11] for totally projective p-groups.

The *closed set method*, introduced for torsion and torsion-free groups by P. Hill in [5], provides a means for converting $F(\aleph_0)$-families of nice or pure subgroups into $H(\aleph_0)$-families of the same sorts of subgroups. For our proof of Theorem 21.3.4, we require a generalized global version of the method that is inspired by the application in [1] to torsion-free groups. Although we did things much more generally in [16] and [17] to deal with $F(\kappa)$-families for arbitrary infinite cardinals κ, here we only need to deal with $F(\aleph_0)$ -families. Recall that an $F(\aleph_0)$-family in G is a smooth ascending chain $\{G_\alpha\}_{\alpha<\tau}$ of subgroups of G such that $G_0 = 0$, $G = \bigcup_{\alpha<\tau} G_\alpha$, and $|G_{\alpha+1}/G_\alpha| \leq \aleph_0$ for all α.

We now establish some notation and terminology that will remain in force throughout this section. Given a global group G, we select and fix an $F(\aleph_0)$-family $\{G_\alpha\}_{\alpha<\tau}$ consisting of pure subgroups. Associated with this $F(\aleph_0)$-family, there is a set $\{B_\alpha\}_{\alpha<\tau}$ of countable subgroups where $G_{\alpha+1} = G_\alpha + B_\alpha$ for each $\alpha < \tau$. Note that $G_\alpha = \sum_{\beta<\alpha} B_\beta$ for all $\alpha < \tau$. With the G_α and B_α in place, call a subset S of τ *closed* if, for each $\lambda \in S$,

$$B_\lambda \cap G_\lambda \subseteq \sum\{B_\alpha : \alpha \in S \text{ and } \alpha < \lambda\}.$$

For each subset $S \subseteq \tau$, we define $G(S) = \sum\{B_\alpha : \alpha \in S\}$. For a given $S \subseteq \tau$, each nonzero element $x \in G(S)$ has a *standard representation*

$$x = b_{\mu(1)} + b_{\mu(2)} + \cdots + b_{\mu(m)}$$

where $\mu(i) \in S$ and $b_{\mu(i)}$ is a nonzero element of $B_{\mu(i)}$ for $i = 1, 2, \ldots, m$,

$$\mu(1) < \mu(2) < \cdots < \mu(m),$$

and $\mu(m)$ is minimal. In the case where S is a closed subset of τ, [15, Lemma 4.1] shows that $\mu(m) = \nu(x)$, where $\nu(x)$ denotes the least ordinal with $x \in G_{\nu(x)+1}$.

Lemma 21.3.1 *Let $\{S_\alpha\}_{\alpha<\gamma}$ be an arbitrary family of closed subsets of τ and set $S = \bigcap_{\alpha<\gamma} S_\alpha$. If x is a nonzero element of $\bigcap_{\alpha<\gamma} G(S_\alpha)$, then there is a representation*

$$x = b_{\mu(1)} + b_{\mu(2)} + \cdots + b_{\mu(m)} \text{ with } \mu(1) < \mu(2) < \cdots < \mu(m) = \nu(x),$$

where $\mu(i) \in S$ and $b_{\mu(i)}$ is a nonzero element of $B_{\mu(i)}$ for $i = 1, 2, \ldots, m$.

Proof Proceeding by induction, we assume that the conclusion of the lemma holds whenever y is a nonzero element of $\bigcap_{\alpha<\gamma} G(S_\alpha)$ and $\nu(y) < \nu(x)$.

Since $x \in \bigcap_{\alpha<\gamma} G(S_\alpha)$, we have for each $\alpha < \gamma$ a *standard representaion*

$$x = b_{\mu_\alpha(1)} + b_{\mu_\alpha(2)} + \cdots + b_{\mu_\alpha(m_\alpha)}$$

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where the $\mu_\alpha(i) \in S$. Then, for each $\alpha < \gamma$, $\nu(x) = \mu_\alpha(m_\alpha)$ and hence $\nu(x) \in \bigcap_{\alpha<\gamma} S_\alpha = S$. Thus, if $b = b_{\mu_\alpha(m_\alpha)}$,

$$b \in G(S) = \sum\{B_\alpha : \alpha \in S\} \subseteq \bigcap_{\alpha<\gamma} G(S_\alpha).$$

Notice then that $y = x - b \in \bigcap_{\alpha<\gamma} G(S_\alpha)$ and also $y = b_{\mu_\alpha(1)} + \cdots + b_{\mu_\alpha(m_\alpha-1)} \in G_{\nu(x)}$ since

$$\mu_\alpha(1) < \cdots < \mu_\alpha(m_\alpha - 1) < \mu_\alpha(m_\alpha) = \nu(x).$$

Therefore $\nu(y) < \nu(x)$ and our induction hypothesis yields

$$y = b'_{\mu(1)} + \cdots + b'_{\mu(k)}$$

with $\mu(1) < \cdots < \mu(k) = \nu(y) < \nu(x)$, $\mu(i) \in S$, and $0 \neq b_{\mu(i)} \in B_{\mu(i)}$ for $i = 1, 2, \ldots, k$. Finally,

$$x = b'_{\mu(1)} + \cdots + b'_{\mu(k)} + b$$

with $b \in B_{\nu(x)}$ and $\nu(x) \in S$. □

Lemma 21.3.2 *Suppose S is a closed subset of τ and let $\delta \in S$. If $0 \neq x \in B_\delta \cap G_\delta$, then $\nu(x) < \delta$.*

Proof Note that $x \in B_\delta \subseteq G(S)$ and therefore we have a standard representation

$$x = b_{\mu(1)} + b_{\mu(2)} + \cdots + b_{\mu(m)} \text{ and } \mu(1) < \cdots < \mu(m),$$

where $\mu(i) \in S$, $0 \neq b_{\mu(i)} \in B_{\mu(i)}$, and $\mu(m)$ is minimal. As we know, however, $\mu(m) = \nu(x)$. Also since S is a closed subset of τ, we have

$$x \in B_\delta \cap G_\delta \subseteq \sum\{B_\alpha : \alpha \in S \text{ and } \alpha < \delta\}.$$

Consequently, $\nu(x) \geq \delta$ would contradict the minimality of $\mu(m)$. □

Proposition 21.3.3 *If $\{S_\alpha\}_{\alpha<\gamma}$ is a family of closed subsets of τ, then $S = \bigcap_{\alpha<\gamma} S_\alpha$ is a closed subset of τ and $G(S) = \bigcap_{\alpha<\gamma} G(S_\alpha)$.*

Proof First suppose that $x \in B_\delta \cap G_\delta$, where $\delta \in S$. As each S_α is a closed subset of τ,

$$x \in \sum\{B_\mu : \mu \in S_\alpha \text{ and } \mu < \delta\}$$

for all $\alpha < \gamma$. Thus Lemma 21.3.2 implies that $\nu(x) < \delta$ and, by Lemma 21.3.1,

$$x \in \sum\{B_\mu : \mu \in S \text{ and } \mu \leq \nu(x) < \delta\}.$$

Therefore $S = \bigcap_{\alpha<\gamma} S_\alpha$ is indeed a closed subset of τ. Since the inclusion $G(S) \subseteq \bigcap_{\alpha<\gamma} G(S_\alpha)$ is trivial and the reverse inclusion follows from Lemma 3.1, the proof is complete. □

Theorem 21.3.4 *If G is a global Warfield group, then G has an intersection closed $H(\aleph_0)$-family consisting of pure knice subgroups.*

Proof By Proposition 4.2 of [15], every global group has an $H(\aleph_0)$-family of pure subgroups. Since G has an $H(\aleph_0)$-family of knice subgroups by Theorem 3.2 of [7], and since the intersection of two $H(\aleph_0)$-families is an $H(\aleph_0)$-family, G has an $H(\aleph_0)$-family $\mathcal{C}$ consisting of pure knice subgroups. Extract from $\mathcal{C}$ an $F(\aleph_0)$-family $\{G_\alpha\}_{\alpha<\tau}$ and, as in the proof of Theorem 4.9 of [15], select an associated family $\{B_\alpha\}_{\alpha<\tau}$ of countable pure subgroups such that $B_\alpha \in \mathcal{C}$ and $G_{\alpha+1} = G_\alpha + B_\alpha$ for all $\alpha < \tau$. Now choose $\mathcal{C}_G$ to consist of all subgroups of the form $G(S) = \sum\{B_\alpha : \alpha \in S\}$ with S a closed subset of τ. It is easily verified that $\mathcal{C}_G$ is an $H(\aleph_0)$-family in G and, by Proposition 21.3.3, $\mathcal{C}_G$ is intersection closed. Moreover, each $G(S) \in \mathcal{C}_G$ is both pure and knice in G since $\mathcal{C}$ is an $H(\aleph_0)$-family. □

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21.4 Isotype Separable Subgroups of Global Warfield Groups

Suppose that H and N are subgroups of G. We call H and N *locally compatible* if to each pair $(h, x) \in H \times N$ and prime p there corresponds an $x' \in H \cap N$ such that $|h + x|_p^G \leq |h + x'|_p^G$. In addition, we say that H and N are *almost strongly compatible,* and write $H \parallel N$, if H and N are locally compatible and satisfy the following property: to each pair $(h, x) \in H \times N$, there corresponds an $x' \in H \cap N$ and a positive integer m such that $\|m(h + x)\|^G \leq \|mh + x'\|^G$.

Lemma 21.4.1 *Suppose that H and N are subgroups of G where H is isotype in G and N is almost balanced in G. If $H \parallel N$, then $H \cap N$ is almost balanced in H.*

Proof To see that $H \cap N$ is a nice subgroup of H, supose that $h + (H \cap N) \in p^\alpha(H/(H \cap N))$ and $ph \in p^\alpha H + (H \cap N)$ for some prime p and ordinal α. We need to show that $h \in p^\alpha H + (H \cap N)$. First observe that $h + N \in p^\alpha(G/N)$ and $ph \in p^\alpha G + N$. But N is nice in G so that $h \in p^\alpha G + N$. Write $h = z + x$ where $z \in p^\alpha G$ and $x \in N$. Then, $h - x \in p^\alpha G$, and by local compatibility, there is an $x' \in H \cap N$ such that

$$\alpha \leq |h - x|_p^G \leq |h - x'|_p^G = |h - x'|_p^H.$$

Hence, $h = (h - x') + x' \in p^\alpha H + (H \cap N)$, as desired. Therefore, $H \cap N$ is a nice subgroup of H.

To complete the proof, we need to show that for a given $h \in H$ there is a positive integer m and an $h' \in H \cap N$ such that

$$\|mh + (H \cap N)\|^{H/(H\cap N)} = \|mh + h'\|^H. \tag{21.1}$$

In order to do this, we first use the hypothesis that N is almost balanced in G to obtain a positive integer k and $y \in N$ with

$$\|kh + N\|^{G/N} = \|kh + y\|^G.$$

Then, since $H \parallel N$, there is a positive integer l and $h' \in H \cap N$ such that

$$\|l(kh + y)\|^G \leq \|lkh + h'\|^G.$$

Therefore,

$$\begin{aligned}\|lkh + (H \cap N)\|^{H/(H\cap N)} &\leq \|lkh + N\|^{G/N} = \|l(kh + y)\|^G \\ &\leq \|lkh + h'\|^G = \|lkh + h'\|^H \leq \|lkh + (H \cap N)\|^{H/(H\cap N)}\end{aligned}$$

and we obtain (21.1) by taking $m = lk$. □

In conclusion, we can now prove a form of the converse of Theorem 21.2.3.

Theorem 21.4.2 *Suppose that H is an isotype subgroup of a global Warfield group G. Then H is almost strongly κ-separable in G if and only if H has a κ-cover consisting of almost balanced pure subgroups.*

Proof In view of Theorem 21.2.3, we may assume that H is almost strongly κ-separable in G and show that H has κ-cover of almost balanced pure subgroups.

By Theorem 21.3.4, G has an intersection closed $H(\aleph_0)$-family $\mathcal{C}$ of pure knice subgroups, and take $\mathcal{C}'$ to be the collection of all those subgroups N of G that satisfy the following four properties.

(a) $N \in \mathcal{C}$.

(b) $|N| \leq \kappa$.

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(c) $H \parallel N$.

(d) $H \cap N$ is pure in H.

We claim that $\mathcal{C}_H = \{H \cap N : N \in \mathcal{C}'\}$ is a κ-cover for H consisting of almost balanced pure subgroups. Certainly $\mathcal{C}_H$ contains the trival subgroup 0 since $0 \in \mathcal{C}$ and hence in C'. Also, by condition (b), each element of $\mathcal{C}_H$ has cardinality not exceeding κ. Moreover, since knice subgroups are almost balanced, Lemma 4.1 together with conditions (c) and (d) imply that $\mathcal{C}_H$ consists of almost balanced pure subgroups of H. Therefore, to establish the claim, and thereby complete the proof of the theorem, it remains to show that any subgroup of H of cardinality not exceeding κ is contained in a member of $\mathcal{C}_H$, and that $\mathcal{C}_H$ is closed under unions of ascending chains of length at most κ.

To verify the former property, assume that K is a subgroup of H of cardinality not exceeding κ. To verify that K is contained in a member of $\mathcal{C}_H$, we consruct inductively three sequences $\{N_i\}_{i<\omega_0}$, $\{A_i\}_{i<\omega_0}$, and $\{B_i\}_{i<\omega_0}$ of subgroups of G such that

$$K \subseteq N_0 \subseteq A_0 \subseteq B_0 \subseteq \cdots \subseteq N_i \subseteq A_i \subseteq B_i \subseteq \ldots \quad (i < \omega_0)$$

is an ascending sequence that satifies the following conditions for all $i < \omega_0$.

(i) $|N_i| \leq \kappa$, $|A_i| \leq \kappa$, and $|B_i| \leq \kappa$.

(ii) $N_i \in \mathcal{C}$.

(iii) $H \parallel A_i$.

(iv) $H \cap B_i$ is pure in H.

Once this is accomplished,

$$N = \bigcup_{i<\omega_0} N_i = \bigcup_{i<\omega_0} A_i = \bigcup_{i<\omega_0} B_i$$

is in $\mathcal{C}'$ since $|N| \leq \kappa$ and conditions (ii), (iii), and (iv) are all inductive. We would then have $K \subseteq H \cap N$ and $H \cap N \in \mathcal{C}_H$. To carry out the induction, we use the fact that $\mathcal{C}$ is an $H(\aleph_0)$-family to select $N_0 \in \mathcal{C}$ such that $K \subseteq N_0$ and $|N_0| \leq \kappa$. Then, since H is almost strongly κ-separable in G, by [16, Lemma 3.1] there is a subgroup A_0 of G such that $N_0 \subseteq A_0$, $|A_0| \leq \kappa$ and $H \parallel A_0$. To obtain a suitable B_0, select a pure subgroup P_0 of H such that $H \cap A_0 \subseteq P_0$ and $|P_0| \leq \kappa$. Then set $B_0 = A_0 + P_0$ and observe that $|B_0| \leq \kappa$ and $H \cap B_0 = H \cap (A_0 + P_0) = (H \cap A_0) + P_0 = P_0$ is pure in H. Finally, if suitable N_k, A_k, and B_k have been constructed for some integer $k \geq 0$, we obtain N_{k+1}, A_{k+1}, and B_{k+1} by simply repeating the preceding argument with the subgroup K replaced by B_k.

It remains to show that $\mathcal{C}_H$ is closed under unions of ascending chains of length at most κ. To this end, suppose that

$$H \cap N_0 \subseteq H \cap N_1 \subseteq \cdots \subseteq H \cap N_\alpha \subseteq \ldots \qquad (\alpha < \mu)$$

is an ascending chain in $\mathcal{C}_H$, where each $N_\alpha \in \mathcal{C}'$ and μ is some ordinal of cardinality not exceeding κ. The difficulty here is that the N_α need not ascend; however, each N_α is a member of $\mathcal{C}$, an intersection closed $H(\aleph_0)$-family of pure knice subgroups of G. Consequently, if we proceed as in the proof of [10, Lemma 2] and set $M_\alpha = \bigcap_{\beta \geq \alpha} N_\beta$ for each $\alpha < \mu$, then each $M_\alpha \in \mathcal{C}$ with $|M_\alpha| \leq \kappa$. Moreover,

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq \ldots \qquad (\alpha < \mu)$$

is an ascending chain. Hence, $M = \bigcup_{\alpha<\mu} M_\alpha \in \mathcal{C}$, $|M| \leq \kappa$, and $H \cap M = \bigcap_{\alpha<\mu}(H \cap N_\alpha)$. It remains to show that $M \in \mathcal{C}'$. Note that we already have that M satisifies conditions (a) and (b).

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Also, since each $H \cap N_\alpha$ is pure in H and purity in H is an inductive property, $H \cap M$ is pure in H. Thus, M satisfies condition (d). Finally, because $H \cap M_\alpha = H \cap N_\alpha$ and $M_\alpha \subseteq N_\alpha$ for each α, $H \parallel N_\alpha$ implies that $H \parallel M_\alpha$. Therefore, M satisfies condition (c) since almost strong compatibility with H is an inductive property, and we conclude that $M \in \mathcal{C}'$. $\square$

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Chapter 22

Note on the Generalized Derivation Tower Theorem for Lie Algebras

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Abstract We provide an algorithm for decomposing a finite-dimensional Lie algebra over a field of characteristic 0 permitting to generalize the derivation tower theorem for Lie algebras proved by E. Schenkman [4].

22.1 Introduction

A Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ of characteristic 0 is called complete if the center of $\mathfrak{g}$ is trivial and all derivations of $\mathfrak{g}$ are inner. Let $(\mathfrak{g}_n)$ be the series of algebras defined by $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_{n+1} = \operatorname{Der}\mathfrak{g}_n$; it is called the tower of derivation algebras of $\mathfrak{g}$. E. Schenkman proved that the derivation tower theorem which asserts that if the center of $\mathfrak{g}$ is trivial, then the derivation tower of $\mathfrak{g}$ terminates with a complete Lie algebras $\widehat{\mathfrak{g}}$, [4]. In this note we revisit this theorem aiming to provide an explicit construction of the limit Lie algebra $\widehat{\mathfrak{g}}$. The method is based on Γ-decomposition in terms of so-called Γ-triples which are essentially unique, cf Theorem 22.2.4.2. This allows to characterize completeness of $\mathfrak{g}$ in terms of the representation μ associated to a Γ-triple, cf Theorem 22.2.10.3. Both theorems cited provide us with a technique of decomposing Lie algebras that allows to construct the limit $\widehat{\mathfrak{g}}$ when it exists: In Section 22.2 we carry out this explicit construction under the assumption that the center of $\mathfrak{g}_n$ is trivial, i.e., the center of $\operatorname{Der}(\mathfrak{g}_{n+1})$ is zero. The form of $\widehat{\mathfrak{g}}$ given in (22.55) follows from the main result of Theorem 22.3.1 concerning Lie algebras with trivial center.

In Section 22.3 the general case is considered allowing the bad case when the sequence of the dimension increases divergently. The remaining cases are classified in two classes, the first case dealt with by Theorem 22.3.1 and a second class allowing to describe $\widehat{\mathfrak{g}}$ as $\mathbb{K} \times [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]$, the latter

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being a perfect complete Lie algebra. So, excluding the bad case we obtain concrete structure results for the limit Lie algebra $\widehat{\mathfrak{g}}$, cf Theorem 22.4.3.

22.2 Γ-Decomposition

Throughout this paper, $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field $\mathbb{K}$ of characteristic 0, $Z(\mathfrak{g})$ its center, $\mathfrak{r}$ its radical, $\mathfrak{n}$ its largest nilpotent ideal, $C^{\infty}(\mathfrak{g})$ is the intersection of the ideals $C^{p}(\mathfrak{g})$ of the central descending sequence of $\mathfrak{g}$ and $\operatorname{Der}\mathfrak{g}$ its Lie algebra of derivations. The Lie algebra $\operatorname{Der}\mathfrak{g}$ is then algebraic [1, p. 179]. The Lie algebra $\operatorname{ad}\mathfrak{g}$ is an ideal of $\operatorname{Der}\mathfrak{g}$. Let $e(\operatorname{ad}\mathfrak{g})$ be the smallest Lie algebra which is algebraic in $\mathfrak{g}$ and contains $\operatorname{ad}\mathfrak{g}$ [1, p. 173]. Then we have

$$\operatorname{ad}\mathfrak{g} \subset e(\operatorname{ad}\mathfrak{g}) \subset \operatorname{Der}\mathfrak{g}. \tag{22.1}$$

A Lie subalgebra Γ of $\mathfrak{gl}(\mathfrak{g})$ is said to be *completely reducible* or c.r. if its natural action on $\mathfrak{g}$ is semi-simple. This means that this Lie algebra is reductive and its center consists of linear maps which are all semi-simple. A Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ is said to be *maximal completely reducible* or m.c.r. if it is maximal among the c.r. Lie subalgebras. Two m.c.r. Lie subalgebras of $\mathfrak{gl}(\mathfrak{g})$ are isomorphic [3]. Let $u \in \mathfrak{gl}(\mathfrak{g})$ and let $u = u\,|_{S} + u\,|_{N}$ be its Jordan decomposition with $\mathfrak{u}\,|_{S}$ ($u\,|_{N}$ resp.) its semi-simple (nilpotent resp.) component. If $\mathfrak{u} \subset \mathfrak{gl}(\mathfrak{g})$ is a subspace, we will denote by $\mathfrak{u}\,|_{S}$ ($\mathfrak{u}\,|_{N}$ resp.) the set of semi-simple (nilpotent resp.) components of the Jordan decomposition.

Lemma 22.2.1 *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and Γ be a c.r Lie subalgebra of* $\operatorname{Der}\mathfrak{g}$. *Then $\mathfrak{g}$ satisfies*

1. $\mathfrak{g} = \mathfrak{g}^{\Gamma} \oplus \Gamma\cdot\mathfrak{g}$, $\left[\mathfrak{g}^{\Gamma}, \Gamma\cdot\mathfrak{g}\right] \subset \Gamma\cdot\mathfrak{g}$, *where* $\mathfrak{g}^{\Gamma} := \{x \in \mathfrak{g} : \alpha\cdot x = 0, \forall\alpha \in \Gamma\}$, $\Gamma\cdot\mathfrak{g} := \{\alpha\cdot\mathfrak{g}, \forall\alpha \in \Gamma\}$,
2. *If we set* $\mathfrak{p} := \Gamma\cdot\mathfrak{g} + [\Gamma\cdot\mathfrak{g}, \Gamma\cdot\mathfrak{g}]$, *then $\mathfrak{p}$ is an ideal of $\mathfrak{g}$ generated by $\Gamma\cdot\mathfrak{g}$ such that* $C^{p}(\mathfrak{g}) = \mathfrak{p} + C^{p}\left(\mathfrak{g}^{\Gamma}\right)\ \forall p \in \mathbb{N}\cup\{\infty\}$,
3. *There exists a Levi subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\Gamma\cdot\mathfrak{s} \subset \mathfrak{s}$,*
4. *If we set* $\mathfrak{a} := \left(\operatorname{ad}_{\mathfrak{g}}\right)^{-1}(\operatorname{ad}\mathfrak{g}\cap\Gamma)$ *then $\mathfrak{a}$ is a reductive Lie subalgebra of $\mathfrak{g}$ satisfying $\Gamma\cdot\mathfrak{a} \subset \mathfrak{a}$.*

Proof The natural action of Γ on $\mathfrak{g}$ being semi-simple, we then have $\mathfrak{g} = \mathfrak{g}^{\Gamma} \oplus \Gamma\cdot\mathfrak{g}$. For all $x \in \mathfrak{g}^{\Gamma}$, $y \in \mathfrak{g}$ and $\alpha \in \Gamma \subset \operatorname{Der}\mathfrak{g}$ then

$$\alpha\cdot[x, y] = [\alpha\cdot x, y] + [x, \alpha\cdot y] = [x, \alpha\cdot y] \tag{22.2}$$

and

$$\left[\mathfrak{g}^{\Gamma}, \Gamma\cdot\mathfrak{g}\right] \subset \Gamma\cdot\mathfrak{g}. \tag{22.3}$$

The statement 1. holds. Since $\left[\mathfrak{g}^{\Gamma}, \Gamma\cdot\mathfrak{g}\right] \subset \Gamma\cdot\mathfrak{g}$ we deduce that $\mathfrak{p}$ is an ideal of $\mathfrak{g}$ and generated by $\Gamma\cdot\mathfrak{g}$. The rest of the statement 2. is obvious. Γ being reductive, there exists a Levi subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that

$$\Gamma := \operatorname{ad}\mathfrak{s} \oplus Z(\Gamma) \quad \text{and} \quad [\Gamma, \Gamma] = \operatorname{ad}\mathfrak{s}. \tag{22.4}$$

Since Γ is not maximal then there exists a m.c.r subalgebra Γ_{max} of $\operatorname{Der}\mathfrak{g}$ containing Γ such that $\Gamma_{max}\cdot\mathfrak{s} \subset \mathfrak{s}$ and a fortiori $\Gamma\cdot\mathfrak{s} \subset \mathfrak{s}$. □

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Lemma 22.2.2 *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. Then there exists a nilpotent Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + C^{\infty}\left(\mathfrak{g}^{\Gamma}\right)$.*

Proof This is obvious if $\dim \mathfrak{g} = 0$. We reason by induction on $n := \dim \mathfrak{g}$. We may assume that $\mathfrak{g}$ is not nilpotent else we set $\mathfrak{g} := \mathfrak{h}$. There exists $x \in \mathfrak{g}$ such that $\delta := \operatorname{ad} x \mid_S$ is different from zero and $\mathfrak{g} = \delta \cdot \mathfrak{g} \oplus \mathfrak{g}^{\delta}$. We have $\dim \mathfrak{g}^{\delta} < n$ and $\mathfrak{g}^{\delta} = \mathfrak{h} + C^{\infty}\left(\mathfrak{g}^{\delta}\right)$ by the induction hypothesis. The inclusion $\delta \cdot \mathfrak{g} \subset C^{\infty}(\mathfrak{g})$ yields $\delta(\mathfrak{g}) \subset C^{\infty}\left(\mathfrak{g}^{\Gamma}\right)$ and $\mathfrak{g} = \mathfrak{h} + C^{\infty}\left(\mathfrak{g}^{\Gamma}\right)$. □

Corollary 22.2.3 *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. There exists a Levi subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ and a nilpotent Lie subalgebra $\mathfrak{h}$ of $\mathfrak{r}$ such that*

1. $\mathfrak{g} = \mathfrak{s} + \mathfrak{h} + \mathfrak{n}$ *and* $[\mathfrak{s}, \mathfrak{h}] = 0$,
2. $e(\operatorname{ad}\mathfrak{g}) = \Gamma \oplus \Delta$ *with* $\Gamma := \operatorname{ad}\mathfrak{s} + \operatorname{ad}\mathfrak{h} \mid_S$, $\Delta := \operatorname{ad}\mathfrak{h} \mid_N + \operatorname{ad}\mathfrak{n}$ *and* $\Gamma \cdot \mathfrak{h} = 0$. *We will say that Γ is a m.c.r Lie subalgebra of $e(\operatorname{ad}\mathfrak{g})$ associated to $(\mathfrak{s}, \mathfrak{h})$.*

Proof By Lemma 22.2.2 there exists a nilpotent Lie subalgebra $\mathfrak{h}$ of $\mathfrak{r}$ such that

$$\mathfrak{r}^{\operatorname{ad}\mathfrak{s}} = \mathfrak{h} + C^{\infty}\left(\mathfrak{r}^{\operatorname{ad}\mathfrak{s}}\right) \subset \mathfrak{h} + \mathfrak{n} \quad \text{and} \quad \mathfrak{r} = \mathfrak{r}^{\operatorname{ad}\mathfrak{s}} \oplus [\mathfrak{s}, \mathfrak{r}]. \tag{22.5}$$

We have $[\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{n}$, hence $\mathfrak{g} = \mathfrak{s} + \mathfrak{h} + \mathfrak{n}$ and $[\mathfrak{s}, \mathfrak{h}] = 0$ and $\mathfrak{h}$ is nilpotent. It follows that

$$\operatorname{ad}\mathfrak{g} = \operatorname{ad}\mathfrak{s} + \operatorname{ad}\mathfrak{h} + \operatorname{ad}\mathfrak{n} \quad \text{and} \quad e(\operatorname{ad}\mathfrak{g}) = \operatorname{ad}\mathfrak{s} + e(\operatorname{ad}\mathfrak{h}) + \operatorname{ad}\mathfrak{n}. \tag{22.6}$$

The Lie algebra $e(\operatorname{ad}\mathfrak{h})$ is nilpotent and admits a *Chevalley decomposition*

$$\operatorname{ad}\mathfrak{h} \mid_N \oplus \operatorname{ad}\mathfrak{h} \mid_S, \quad [1]. \tag{22.7}$$

It follows that

$$e(\operatorname{ad}\mathfrak{g}) = \Gamma \oplus \Delta \quad \text{and} \quad \Gamma \cdot \mathfrak{h} = 0 \tag{22.8}$$

with

$$\Gamma := \operatorname{ad}\mathfrak{s} + \operatorname{ad}\mathfrak{h} \mid_S \quad \text{and} \quad \Delta := \operatorname{ad}\mathfrak{h} \mid_N + \operatorname{ad}\mathfrak{n}. \tag{22.9}$$

Hence Γ is maximal by construction. □

We introduce the notion of a Γ-decomposition in *iv)* hereafter.

Theorem 22.2.4 *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and $\mathfrak{r}$ its radical.*

1. *There is a bijection of the set of c.r.m subalgebras of $e(\operatorname{ad}\mathfrak{g})$ into the set of sequences of vector spaces $(\mathfrak{s}, \mathfrak{k}, \mathfrak{m})$ of $\mathfrak{g}$ such that:*
   - *i)* $\mathfrak{s}$ *is a Levi subalgebra of* $\mathfrak{g}$,
   - *ii)* $\mathfrak{k}$ *is an ideal nilpotent of* $\mathfrak{g}$ *such that* $[\mathfrak{s}, \mathfrak{k}] = 0$,
   - *iii)* $\mathfrak{m}$ *is a subspace of* $\mathfrak{r}$ *such that* $\mathfrak{r} = \mathfrak{m} \oplus \mathfrak{k}$ *and* $[\mathfrak{s} \oplus \mathfrak{k}, \mathfrak{m}] = \mathfrak{m}$, *and*
   - *iv)* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{k} \oplus \mathfrak{m}$. *Will call* $(\mathfrak{s}, \mathfrak{k}, \mathfrak{m})$ *a* Γ-triple *and* $\mathfrak{s} \oplus \mathfrak{k} \oplus \mathfrak{m}$ *a* Γ-decomposition *of* $\mathfrak{g}$.
2. *Let $(\mathfrak{s}_i, \mathfrak{k}_i, \mathfrak{m}_i)$ be two Γ_i-triple of $\mathfrak{g}$ with $i = 1, 2$, then there exists an inner automorphism γ of $\mathfrak{g}$ such that*

$$\Gamma_2 = \gamma \circ \Gamma_1 \circ \gamma^{-1}, \quad \mathfrak{s}_2 = \gamma(\mathfrak{s}_1), \quad \mathfrak{m}_2 = \gamma(\mathfrak{m}_1), \quad \mathfrak{k}_2 = \gamma(\mathfrak{k}_1). \tag{22.10}$$

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Proof If we assume that there exists such a decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{k} \oplus \mathfrak{m}$, we construct Γ by setting

$$\Gamma := \mathrm{ad}\,\mathfrak{s} \oplus \mathrm{ad}\,\mathfrak{k}\,|_{\mathrm{S}}, \tag{22.11}$$

i.e., Corollary 22.2.3.2. Conversely, let Γ be a m.c.r Lie subalgebra of $e\,(\mathrm{ad}\,\mathfrak{g})$, and we set

$$\mathfrak{k} := \mathfrak{g}^{\Gamma} \quad \mathfrak{m} := \Gamma \cdot \mathfrak{r} \quad \text{and} \quad \mathfrak{s} := \left(\mathrm{ad}_{\mathfrak{g}}\right)^{-1}(\mathrm{ad}\,\mathfrak{g} \cap \Gamma)\,. \tag{22.12}$$

It is obvious that $\mathfrak{s}$ is a Levi subalgebra of $\mathfrak{g}$. We have

$$\Gamma \cdot \mathfrak{g} = \Gamma \cdot \mathfrak{m} \oplus \Gamma \cdot \mathfrak{k} \oplus \Gamma \cdot \mathfrak{s} = \mathfrak{m} \oplus \mathfrak{s} \tag{22.13}$$

since

$$\Gamma \cdot \mathfrak{m} = \mathfrak{m}, \quad \Gamma \cdot \mathfrak{s} = \mathfrak{s} \quad \text{and} \quad \Gamma \cdot \mathfrak{k} = 0. \tag{22.14}$$

By Lemma 22.2.1, we have

$$\mathfrak{g} = \mathfrak{g}^{\Gamma} \oplus \Gamma \cdot \mathfrak{g} = \mathfrak{s} \oplus \mathfrak{k} \oplus \mathfrak{m}. \tag{22.15}$$

Since

$$[\Gamma, \mathrm{ad}\,\mathfrak{k}] = \mathrm{ad}\,\left(\Gamma \cdot \mathfrak{k}\right) = 0 \tag{22.16}$$

it follows that

$$[\Gamma, \mathrm{ad}\,\mathfrak{k}\,|_{\mathrm{S}}] = [\Gamma, \mathrm{ad}\,\mathfrak{k}\,|_{\mathrm{N}}] = 0. \tag{22.17}$$

Hence

$$\mathrm{ad}\,\mathfrak{k}\,|_{\mathrm{S}} \subset Z\,(\Gamma) \subset \Gamma \tag{22.18}$$

because Γ is m.c.r Lie subalgebra. The Lie algebra

$$Z\,(\Gamma) + \mathrm{ad}\,\mathfrak{k} = Z\,(\Gamma) \oplus \mathrm{ad}\,\mathfrak{k}\,|_{\mathrm{N}} \tag{22.19}$$

is thus nilpotent. Then $\mathrm{ad}\,\mathfrak{k}$ is nilpotent and we conclude that $\mathfrak{k}$ is nilpotent. We have $[\mathfrak{s} \oplus \mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ since $\mathfrak{m} \subset \mathfrak{r}$. If $[\mathfrak{s} \oplus \mathfrak{k}, \mathfrak{m}] \neq \mathfrak{m}$ then

$$(\mathrm{ad}\,\mathfrak{s} + e(\mathrm{ad}\,\mathfrak{k}))\mathfrak{m} \neq \mathfrak{m} \quad \text{and} \quad \Gamma \cdot \mathfrak{m} \neq \mathfrak{m}. \tag{22.20}$$

This is a contradiction. The statement 2. is a consequence of G. D. Mostow's theorem [3] applied to $e\,(\mathrm{ad}\,\mathfrak{g})$. □

Corollary 22.2.5 *Let* $(\mathfrak{s}, \mathfrak{k}, \mathfrak{m})$ *be a* Γ*-triple of* $\mathfrak{g}$ *with* Γ *a c.r.m Lie subalgebra* $e\,(\mathrm{ad}\,\mathfrak{g})$. *Then*

$$\mathrm{Der}\,\mathfrak{g} = \mathrm{ad}\,\mathfrak{s} \oplus \mathrm{ad}\,\mathfrak{m} \oplus (\mathrm{Der}\,\mathfrak{g})^{\Gamma}$$

where $(\mathrm{Der}\,\mathfrak{g})^{\Gamma}$ *is the centralizer of* Γ *in* $\mathrm{Der}\,\mathfrak{g}$.

Proof The adjoint representation of Γ in $\mathrm{Der}\,\mathfrak{g}$ being semi-simple, then

$$\mathrm{Der}\,\mathfrak{g} = [\mathrm{Der}\,\mathfrak{g}, \Gamma] \oplus (\mathrm{Der}\,\mathfrak{g})^{\Gamma}\,. \tag{22.21}$$

Since

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$$[\Gamma, \mathrm{Der}\,\mathfrak{g}] \subset [e\,(\mathrm{ad}\,\mathfrak{g})\,, \mathrm{Der}\,\mathfrak{g}] \subset \mathrm{ad}\,\mathfrak{g} \tag{22.22}$$

and

$$(\mathrm{Der}\,\mathfrak{g})^{\Gamma} \cap \mathrm{ad}\,\mathfrak{g} = \mathrm{ad}\,\mathfrak{k} \tag{22.23}$$

hence

$$\mathrm{Der}\,\mathfrak{g} = \mathrm{ad}\,\mathfrak{s} \oplus \mathrm{ad}\,\mathfrak{m} \oplus (\mathrm{Der}\,\mathfrak{g})^{\Gamma}\,. \tag{22.24}$$

□

Remark The vector space $\mathfrak{m}$ is not a subalgebra of $\mathfrak{g}$ in general. If we consider the solvable Lie algebra generated by $\{x_1, x_2, x_3, x_4, x_5\}$ with its multiplication defined by

$$[x_1, x_2] = x_5, \quad [x_1, x_3] = x_3, \quad [x_1, x_4] = -x_4, \quad [x_3, x_4] = x_5. \tag{22.25}$$

Then

$$\Gamma = \mathbb{K} \cdot \mathrm{ad}\,x_1\,|_{\mathrm{S}}, \quad \mathfrak{k} = \mathbb{K} \cdot x_1 + \mathbb{K} \cdot x_2 + \mathbb{K} \cdot x_5, \quad \mathfrak{m} = \mathbb{K} \cdot x_3 + \mathbb{K} \cdot x_4. \tag{22.26}$$

We observe that $\mathfrak{m}$ is not an ideal.

Proposition 22.2.6 *Let $(\mathfrak{s}_i, \mathfrak{k}_i, \mathfrak{m}_i)$ be a Γ_i-triple of the Lie algebra $\mathfrak{g}_i$ with each Γ_i a c.r.m Lie subalgebra of $e\,(\mathrm{ad}\,\mathfrak{g}_i)$ with $i = 1, 2$. Then $(\mathfrak{s}_1 \times \mathfrak{s}_2, \mathfrak{k}_1 \times \mathfrak{k}_2, \mathfrak{m}_1 \times \mathfrak{m}_2)$ is a $\Gamma_1 \times \Gamma_2$-triple of the Lie algebra $\mathfrak{g}_1 \times \mathfrak{g}_2$.*

Proposition 22.2.7 *Let $f : \mathfrak{g} \to \mathfrak{g}'$ be a Lie algebra epimorphism and Γ a c.r.m subalgebra of $e\,(\mathrm{ad}\,\mathfrak{g})$. Then*

1. *The subalgebra Γ' induced by Γ on the quotient $\mathfrak{g}'$ satisfying $\Gamma' \circ f = f \circ \Gamma$ is a c.r.m subalgebra of $e\left(\mathrm{ad}\,\mathfrak{g}'\right)$.*

2. *If $(\mathfrak{s}, \mathfrak{k}, \mathfrak{m})$ is a Γ-triple of $\mathfrak{g}$ with Γ a c.r.m Lie subalgebra of $e\,(\mathrm{ad}\,\mathfrak{g})$, then $(f\,(\mathfrak{s})\,, f\,(\mathfrak{k})\,, f\,(\mathfrak{m}))$ is a Γ'-triple of $\mathfrak{g}'$.*

Proof It is easy to check property 1. We have

$$\mathfrak{g}' = f\,(\mathfrak{s}) \oplus f\,(\mathfrak{k}) \oplus f\,(\mathfrak{m}) \tag{22.27}$$

where $f\,(\mathfrak{k}) \oplus f\,(\mathfrak{m})$ is the radical of $\mathfrak{g}'$. Since

$$[f\,(\mathfrak{s}) \oplus f\,(\mathfrak{m})\,, f\,(\mathfrak{m})] = f\,(\mathfrak{m})\,, \tag{22.28}$$

then $(f\,(\mathfrak{s})\,, f\,(\mathfrak{k})\,, f\,(\mathfrak{m}))$ is a Γ'-triple of $\mathfrak{g}'$. □

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Lemma 22.2.8 *Let* $(\mathfrak{s}, \mathfrak{k}, \mathfrak{m})$ *be a* Γ*-triple of* $\mathfrak{g}$ *with* Γ *a c.r.m Lie subalgebra of* $e\,(\mathrm{ad}\,\mathfrak{g})$*. The inclusion* $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ *defines a representation*

$$\mu : \mathfrak{k} \to \mathfrak{gl}\,(\mathfrak{m})\,, \quad x \mapsto \mathrm{ad}\, x \mid_m .$$

We then have :

1. $Z\,(\mathfrak{g}) = Z\,(\mathfrak{k}) \cap \ker \mu$.
2. μ *is injective if and only if* $Z\,(\mathfrak{g}) = 0$.
3. *The Lie subalgebra* $\hat{\mathfrak{n}}$ *of* $\mathfrak{g}$ *generated by* $\mathfrak{m}$ *is a nilpotent ideal of* $\mathfrak{g}$ *such that*

$$\hat{\mathfrak{n}} = C^{\infty}\,(\mathfrak{g}) \cap \mathfrak{n} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]\,, \quad C^{p}\,(\mathfrak{g}) = \mathfrak{s} + C^{p}\,(\mathfrak{k}) + \hat{\mathfrak{n}}, \quad \forall p \in \mathbb{N}.$$

Proof Let $x \in Z\,(\mathfrak{k}) \cap \ker \mu$ then

$$[s \oplus \mathfrak{k} \oplus \mathfrak{m}, x] = [\mathfrak{k}, x] = 0 \tag{22.29}$$

and $x \in Z\,(g)$. Conversely, if $x \in Z\,(\mathfrak{g})$ we have $[\mathfrak{g}, x] = \{0\}$, hence

$$\Gamma \cdot x \subset \mathfrak{k} \quad \text{and} \quad \Gamma^2 \cdot x = 0. \tag{22.30}$$

This means that $x \in \mathfrak{k}$ so $x \in Z\,(\mathfrak{k}) \cap \ker \mu$. Hence statement 1 holds. The assumption that μ is not injective is equivalent to

$$Z\,(\mathfrak{g}) = Z\,(\mathfrak{k}) \cap \ker \mu \neq 0 \tag{22.31}$$

since every non-null nilpotent ideal intersects the center. Hence statement 2 holds. Using Lemma 22.2.1.2 and the Γ-decomposition of $\mathfrak{g}$, we deduce statement 3. □

Lemma 22.2.9 *Let* V *be a vector space over* $\mathbb{K}$*. Let* $\mathfrak{b}, \mathfrak{a}$ *be two Lie subalgebras of* $\mathfrak{gl}\,(\mathrm{V})$ *such that* $\mathfrak{a} \subset \mathfrak{b}$*. Let* Φ *be a map of* V^2 *into* $\mathfrak{g} := \mathfrak{a} \oplus \mathrm{V}$*. We define a bracket* $[,]$ *on* $\mathfrak{g}$ *by*

$$\begin{aligned} [x, v] &= x \cdot v \\ [v_1, v_2] &= \Phi(v_1, v_2) \\ [x, y] &= x \cdot y - y \cdot x \end{aligned}$$

with $x, y \in \mathfrak{a}$, $v_1, v_2 \in \mathrm{V}$*. If this bracket defines a Lie structure on* $\mathfrak{g}$*, then it can be lifted to* $\mathfrak{g}' := \mathfrak{b} \oplus \mathrm{V}$ *by setting :*

$$[x, \Phi(v_1, v_2)] = \Phi\,(x \cdot v_1, v_2) + \Phi\,(v, x \cdot v_2)$$

for all $(x, v_1, v_2) \in \mathfrak{b} \times \mathrm{V}^2$.

Let $(\mathfrak{s}, \mathfrak{k}, \mathfrak{m})$ be a Γ-triple of $\mathfrak{g}$ with Γ a m.c.r Lie subalgebra $e\,(\mathrm{ad}\,\mathfrak{g})$. Let $\delta \in (\mathrm{Der}\,\mathfrak{g})^{\Gamma}$. Then it is easy to check that

$$\delta\,(\mathfrak{m}) = \mathfrak{m}, \quad \delta\,(\mathfrak{k}) = \mathfrak{k} \tag{22.32}$$

and

$$\delta\,(\mathfrak{s}) = 0. \tag{22.33}$$

This means that δ defines a linear map

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$$\Theta : (\operatorname{Der}\mathfrak{g})^{\Gamma} \to \mathfrak{gl}(\mathfrak{m}) \quad \delta \mapsto \delta\mid_{\mathfrak{m}} . \tag{22.34}$$

The set of derivations $(\operatorname{Der}\hat{\mathfrak{n}})^{\Gamma}$ of $\hat{\mathfrak{n}}$ which commute with the restriction $\Gamma\mid_{\hat{\mathfrak{n}}}$ of Γ to $\hat{\mathfrak{n}}$ stabilizes $\mathfrak{k}\cap\hat{\mathfrak{n}}$ and $\mathfrak{m}$. There exists an isomorphism of Lie algebras of $(\operatorname{Der}\hat{\mathfrak{n}})^{\Gamma}$ into $(\operatorname{Der}\hat{\mathfrak{n}})^{\Gamma}\mid_{\mathfrak{m}}$. The image of Θ is contained in $\mathfrak{B} := (\operatorname{Der}\hat{\mathfrak{n}})^{\Gamma}\mid_{\mathfrak{m}}$.

Lemma 22.2.8 and Lemma 22.2.9 entail the following:

Theorem 22.2.10 *Let $(\mathfrak{s}, \mathfrak{k}, \mathfrak{m})$ be a Γ-triple of $\mathfrak{g}$ with Γ a c.r.m subalgebra of $e(\operatorname{ad}\mathfrak{g})$. Let $\mu : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{m})$ be the representation defined by $\mu(x) = \operatorname{ad} x\mid_{m}$. Assume μ is injective, then:*

1. *The map Θ defines an isomorphism of $\operatorname{Der}(\mathfrak{g})^{\Gamma}$ into the normalizer $\mathrm{N}_{\mathcal{B}}(\mu(\mathfrak{k}))$ of $\mu(\mathfrak{k})$ in $\mathfrak{B}$.*
2. *We may identify the Lie algebra $\operatorname{Der}\mathfrak{g}$ with the Lie algebra $s \oplus \mathrm{N}_{\mathcal{B}}(\mu(\mathfrak{k})) \oplus \mathfrak{m}$ with its law Φ defined from the bracket $[,]$ of $\mathfrak{g}$ in the following way, for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathfrak{s} \times \mathrm{N}_{\mathfrak{B}}(\mu(\mathfrak{k})) \times \mathfrak{m}$:*

$$\Phi(x_1, x_2 + y_2 + z_2) = [x_1, x_2] + [x_1, z_2]$$
$$\Phi(y_1, y_2 + z_2) = y_1 \cdot y_2 - y_2 \cdot y_1 + y_1 \cdot z_2$$
$$\Phi(z_1, z_2) = \mu(k) + m$$

   with k and m the projections of $[z_1, z_2]$ into $\mathfrak{k}$ and $\mathfrak{m}$, respectively.
3. *In particular $\mathfrak{g}$ is complete if and only if $\mu(\mathfrak{k})$ is equal to its normalizer in $\mathfrak{B}$.*

Proof Let $\delta \in (\operatorname{Der}\mathfrak{g})^{\Gamma}$ such that $\Theta(\delta) = 0$, hence:

$$0 = \delta([x, y]) = [\delta(x), y] + [x, \delta(y)] = \mu(\delta(x)) \cdot y \quad \forall \quad (x, y) \in \mathfrak{k} \times \mathfrak{m}. \tag{22.35}$$

It follows that

$$\delta(x) \in \ker\mu = 0, \quad \text{and} \quad \delta(\mathfrak{g}) = \delta(\mathfrak{k}) = 0. \tag{22.36}$$

Then injectivity of Θ follows. Let δ_1 be an element of $\mathrm{N}_{\mathcal{B}}(\mu(\mathfrak{k}))$. For all $x_1 \in \mathfrak{k}$, there exists $x_2 = \mu^{-1}([\delta_1, \mu(x_1)])$ since μ is injective. It follows that

$$[\delta_1, \mu(x_1)] = \mu(x_2), \tag{22.37}$$

so we may define a derivation δ_2 of $\mathfrak{k}$ by $\delta_2(x_1) = x_2$.
The surjectivity derives from the linear application δ defined by

$$\delta\mid_{\mathfrak{s}} = 0, \quad \delta\mid_{\mathfrak{k}} = \delta_2, \quad \delta\mid_{\mathfrak{m}} = \delta_1 \tag{22.38}$$

which belongs to $(\operatorname{Der}\mathfrak{g})^{\Gamma}$. Let δ_3 be the derivation of $\hat{\mathfrak{n}}$ which extends δ. Then δ_3 is equal to δ_2 on $\mathfrak{k}\cap\hat{\mathfrak{n}}$. Indeed for $x \in \mathfrak{k}\cap\hat{\mathfrak{n}}$ we have

$$\left[\delta_3, \operatorname{ad}_{\hat{\mathfrak{n}}}(x)\right] = \operatorname{ad}_{\hat{\mathfrak{n}}}(\delta_3(x)) \tag{22.39}$$

and by restriction to $\mathfrak{m}$ this yields

$$[\delta_1, \mu(x)] = \mu(\delta_3(x)) \tag{22.40}$$

which is also equal to $\mu(\delta_2(x))$ by definition of δ_2. Then $\delta_3(x) = \delta_2(x)$ because of the injectivity of μ.
It follows that $\delta_1 \in (\operatorname{Der}\mathfrak{g})^{\Gamma}$ since the equality

$$\delta_1([x, y]) = [\delta_2(x), y] + [x, \delta_1(y)] \quad \forall \quad (x, y) \in \mathfrak{k} \times \mathfrak{m} \tag{22.41}$$

is equivalent to

$$[\delta_1, \mu(x)] = \mu(\delta_2(x)). \tag{22.42}$$

Thus statement 1 holds. By Corollary 22.2.5, Lemma 22.2.9 and statement 1, we arrive at statement 2. The statement 3 follows from statements 1 and 2. □

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Theorem 22.2.10 allows to identify the Lie algebra $\mathfrak{g}$ with the Lie subalgebra $\mathfrak{s} \oplus \mu(\mathfrak{k}) \oplus \mathfrak{m}$ of $\mathfrak{s} \oplus \mathrm{N}_{\mathfrak{B}}(\mu(\mathfrak{k})) \oplus \mathfrak{m}$ via the isomorphism defined by

$$s + k + m \mapsto s + \mu(k) + m \quad \forall \quad (s, k, m) \in \mathfrak{s} \times \mathfrak{k} \times \mathfrak{m}. \tag{22.43}$$

We define a Lie algebra $\mathfrak{s} \oplus \mathfrak{B} \oplus \mathfrak{m}$ with the same law Φ by taking y_1 and y_2 in $\mathfrak{B}$ instead of $\mathrm{N}_{\mathfrak{B}}(\mu(\mathfrak{k}))$. We verify that this is really a Lie algebra by using Lemma 22.2.9. If $\mathfrak{g}$ is identified with $\mathfrak{s} \oplus \mu(\mathfrak{k}) \oplus \mathfrak{m}$, we have the Lie algebra inclusions:

$$\mathfrak{g} \subset \mathfrak{s} \oplus \mathrm{N}_{\mathfrak{B}}(\mu(\mathfrak{k})) \oplus \mathfrak{m} \subset \mathfrak{s} \oplus \mathfrak{B} \oplus \mathfrak{m}. \tag{22.44}$$

Corollary 22.2.11 *Let* $(\mathfrak{s}, \mathfrak{k}, \mathfrak{m})$ *be a* Γ*-triple of* $\mathfrak{g}$ *with* Γ *a c.r.m subalgebra of* $e\,(\mathrm{ad}\,\mathfrak{g})$*. The Lie algebra* $\mathfrak{g}' = \mathfrak{s} \oplus \mathfrak{B} \oplus \mathfrak{m}$ *is complete.*

Proof The triple $(\mathfrak{s}, \mathcal{B}, \mathfrak{m})$ of $\mathfrak{g}'$ is a Γ'-triple with Γ' obtained by the extension of Γ to $\mathfrak{g}'$ which is trivial on $\mathfrak{B}$. Using Theorem 22.2.10.3 for the natural representation μ of $\mathfrak{B}$ in $\mathfrak{m}$, we deduce that $\mathfrak{s} \oplus \mathcal{B} \oplus \mathfrak{m}$ is complete. □

22.3 Derivation Tower of Lie Algebras: Case with Trivial Center

We recall that the *derivation tower* of a Lie algebra $\mathfrak{g}$ is the sequence of Lie algebras $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ such that

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \text{and} \quad \mathfrak{g}_{n+1} = \mathrm{Der}\ (\mathfrak{g}_n)\,. \tag{22.45}$$

If the center of $\mathfrak{g}_n$ is trivial, hence the center of Der $(\mathfrak{g}_{n+1})$ is zero. Then we can identify $\mathfrak{g}_n$ with ad $\mathfrak{g}_n$, and we have a sequence of ideals:

$$\mathfrak{g}_n \lhd \mathfrak{g}_{n+1} \lhd \ldots \lhd \mathfrak{g}_{n+k} \lhd \ldots. \tag{22.46}$$

In this case, Schenkman proved that this sequence has a limit $\widehat{\mathfrak{g}}$, [4]. If $\mathcal{A}$ is a Lie subalgebra of a Lie algebra $\mathfrak{B}$, we consider the sequence of normalizers in $\mathfrak{B}$:

$$\mathrm{N}_{\mathfrak{B}}^{p+1}\mathcal{A} = \mathrm{N}_{\mathfrak{B}}(\mathrm{N}_{\mathfrak{B}}^{p}\mathcal{A}), \quad \mathrm{N}_{\mathfrak{B}}^{0}\mathcal{A} = \mathcal{A}. \tag{22.47}$$

The following sequence of ideals

$$\mathcal{A} \lhd \ldots \lhd \mathrm{N}_{\mathfrak{B}}^{p}\mathcal{A} \lhd \mathrm{N}_{\mathfrak{B}}^{p+1}\mathcal{A} \lhd \ldots \lhd \mathrm{N}_{\mathfrak{B}}^{\infty}\mathcal{A} \subset \mathfrak{B} \tag{22.48}$$

terminates for an integer p since the dimension is finite, say $\mathrm{N}_{\mathfrak{B}}^{q+1}\mathcal{A} = \mathrm{N}_{\mathfrak{B}}^{q}\mathcal{A}$ and the Lie algebra $\mathrm{N}_{\mathfrak{B}}^{q}\mathcal{A}$ denoted $\mathrm{N}_{\mathfrak{B}}^{\infty}\mathcal{A}$ will be equal to its normalizer in $\mathfrak{B}$. With this notation we state:

Theorem 22.3.1 *Let* $\mathfrak{s} \oplus \mathfrak{k} \oplus \mathfrak{m}$ *be a* Γ*-decomposition of a Lie algebra* $\mathfrak{g}$ *with trivial center and* Γ *being a m.c.r. Lie subalgebra of* $e(\mathrm{ad}\,\mathfrak{g})$*. The sequence of normalizers*

$$\mu(\mathfrak{k}) \lhd \mathrm{N}_{\mathfrak{B}}^{1}(\mu(\mathfrak{k})) \lhd \ldots \lhd \mathrm{N}_{\mathcal{B}}^{n}(\mu(\mathfrak{k})) \lhd \ldots \lhd \widehat{\mathrm{N}_{\mathcal{B}}}(\mu(\mathfrak{k}))$$

contained in $\mathfrak{B} = (\mathrm{Der}\,\mathfrak{n})^{\Gamma} \mid \mathfrak{m}$ *terminates at* $\widehat{\mathrm{N}_{\mathfrak{B}}}(\mu(\mathfrak{k})) = \mathrm{N}_{\mathfrak{B}}^{q}(\mu(\mathfrak{k}))$ *for some integer* q *such that* $\mathrm{N}_{\mathfrak{B}}^{q+1}(\mu(\mathfrak{k})) = \mathrm{N}_{\mathfrak{B}}^{q}(\mu(\mathfrak{k}))$*. The derivation tower of a Lie algebra* $\mathfrak{g}$ *is given by the Lie subalgebras* $\mathfrak{g}_n = \mathfrak{s} \oplus \mathrm{N}_{\mathfrak{B}}^{n}(\mu(\mathfrak{k})) \oplus \mathfrak{m}$ *of* $\mathfrak{s} \oplus \mathfrak{k} \oplus \mathfrak{m}$ *and terminates at* $\widehat{\mathfrak{g}} := \mathfrak{s} \oplus \widehat{\mathrm{N}_{\mathfrak{B}}}(\mu(\mathfrak{k})) \oplus \mathfrak{m}$*, thus* $\widehat{\mathfrak{g}}$ *is complete.*

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Proof For $\dim \mathfrak{g} = 0$, $\mathfrak{g}$ always admits a decomposition $\mathfrak{s} \oplus \mathfrak{k} \oplus \mathfrak{m}$ associated with Γ, cf. Theorem 22.2.4. We reason by induction on $n := \dim \mathfrak{g}$. Let us assume that for an integer $n \geq 0$ we have

$$\mathfrak{g}_n = \mathfrak{s} \oplus \mathrm{N}^n_{\mathfrak{B}}\left(\mu\left(\mathfrak{k}\right)\right) \oplus \mathfrak{m} \tag{22.49}$$

associated with the m.c.r. algebra Γ_n obtained by the extension of Γ to $\mathfrak{g}_n$ which is trivial on $\mathrm{N}^n_{\mathfrak{B}}(\mu(\mathfrak{k}))$ using the inclusion given by (1). The Lie algebra Γ_n satisfies

$$\Gamma_n \cdot \mathfrak{r} = \mathfrak{m} \tag{22.50}$$

and μ_n is injective since it is given by the natural representation of $\mathfrak{B}$ in $\mathfrak{m}$. Because of Theorem 22.2.10, it is possible to identify $\mathfrak{g}_{n+1}$ with

$$\mathfrak{s} \oplus \mathrm{N}^{n+1}_{\mathcal{B}}(\mu(\mathfrak{k})) \oplus \mathfrak{m}. \tag{22.51}$$

The Lie algebra Γ_n acts by the adjoint representation on $\mathfrak{g}_{n+1}$ which is equivalent to the extension of Γ_n to $\mathfrak{g}_{n+1}$ trivial on $\mathrm{N}^{n+1}_{\mathfrak{B}}(\mu(\mathfrak{k}))$. We then set

$$\Gamma_{n+1} := \mathrm{ad}\,\Gamma_n \tag{22.52}$$

and we have

$$\Gamma_{n+1} \mid \mathfrak{g} = \Gamma_n \mid \mathfrak{g} = \Gamma. \tag{22.53}$$

The rest of the proof is obvious. □

We then have

$$\widehat{\mathrm{N}_{\mathfrak{B}}}(\mu(\mathfrak{k})) = (\widehat{\mathrm{N}}_{(\mathrm{Der}\,\mathfrak{n})^{\Gamma}}(\mathrm{ad}\,\mathfrak{k})) \mid_{\mathfrak{m}} \tag{22.54}$$

and the Lie algebra $\widehat{\mathfrak{g}}$ is also given by

$$\widehat{\mathfrak{g}} = \mathfrak{s} \oplus (\widehat{\mathrm{N}}_{(\mathrm{Der}\,\mathfrak{n})^{\Gamma}}(\mathrm{ad}\,\mathfrak{k})) \mid_{\mathfrak{m}} \oplus \mathfrak{m} \tag{22.55}$$

Example 22.3.2 Let $\mathfrak{a}$ be a nilpotent Lie subalgebra of $\mathfrak{gl}(\mathrm{V})$ such that $\mathfrak{a} \cdot \mathrm{V} = \mathrm{V}$. Let $\mathfrak{g} = \mathfrak{a} \oplus V$ be a semi-direct product for the natural representation of $\mathfrak{a}$ by the abelian Lie algebra V. This decomposition is associated to Γ with

$$\mathfrak{k} = \mathfrak{a}, \quad \mathfrak{m} = \mathrm{V}, \quad \Gamma = \mathrm{ad}\,A \tag{22.56}$$

where A is the set of semi-simple components of elements of $\mathfrak{a}$. With notation as in Theorem 22.3.1, $\mathfrak{B}$ is the centralizer of A in $\mathfrak{gl}(\mathrm{V})$ and we have

$$\widehat{\mathfrak{g}} = \widehat{\mathrm{N}_{\mathfrak{B}}}(\mathfrak{a}) \oplus \mathrm{V}. \tag{22.57}$$

Corollary 22.3.3 *We rediscover the major result given in [4]:*

$$\dim \widehat{\mathfrak{g}} \leq \dim(\mathfrak{s} \oplus \mathfrak{B} \oplus \mathfrak{m}) \leq \dim \mathrm{Der}\left(C^{\infty}\left(\mathfrak{g}\right)\right) + \dim Z\left(C^{\infty}\left(\mathfrak{g}\right)\right). \tag{22.58}$$

Proof Let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{k} \oplus \mathfrak{m}$ be the decomposition of $\mathfrak{g}$ associated with Γ with

$$C^{\infty}\left(\mathfrak{g}\right) = \mathfrak{s} \oplus \mathfrak{n} \tag{22.59}$$

cf. Corollary 22.2.5. The Lie subalgebra

$$\mathrm{ad}\left(C^{\infty}\left(\mathfrak{g}\right)\right) + \left(\mathrm{Der}\left(C^{\infty}\left(\mathfrak{g}\right)\right)\right)^{\Gamma} \tag{22.60}$$

of $\mathrm{Der}\,(C^{\infty}\,(\mathfrak{g}))$, where $(\mathrm{Der}\,(C^{\infty}\,(\mathfrak{g})))^{\Gamma}$ is the centralizer of the restriction of Γ to $C^{\infty}\,(\mathfrak{g})$ in $\mathrm{Der}\,(C^{\infty}\,(\mathfrak{g}))$, can be written as

$$\mathrm{ad}\,\mathfrak{s} \oplus \left(\mathrm{Der}\left(C^{\infty}\left(\mathfrak{g}\right)\right)\right)^{\Gamma} \oplus \mathrm{ad}\,\mathfrak{m}. \tag{22.61}$$

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The isomorphisms

$$\left(\mathrm{Der}\,\left(C^{\infty}\left(\mathfrak{g}\right)\right)\right)^{\Gamma}\cong\left(\mathrm{Der}\,\hat{\mathfrak{n}}\right)^{\Gamma}\cong\left(\mathrm{Der}\,\hat{\mathfrak{n}}\right)^{\Gamma}\,|_{\mathfrak{m}} \tag{22.62}$$

show that its dimension is always greater than

$$\dim\mathfrak{s}+\dim(\mathrm{Der}\,\hat{\mathfrak{n}})^{\Gamma}+\dim\mathfrak{m}+\dim Z(C^{\infty}\left(\mathfrak{g}\right)) \tag{22.63}$$

which is equal to

$$\dim(\mathfrak{s}\oplus\mathfrak{B}\oplus\mathfrak{m})-\dim Z(C^{\infty}\left(\mathfrak{g}\right)) \tag{22.64}$$

and the result follows. □

The case where the sequence terminates at the first degree is descibed in the following.

Proposition 22.3.4 *If* $\mathfrak{g}$ *is a Lie algebra with trivial center, then* $\mathrm{Der}\,\mathfrak{g}$ *is complete if and only if the ideal* $\mathrm{ad}\,\mathfrak{g}$ *of* $\mathrm{Der}\,\mathfrak{g}$ *is characteristic.*

Proof The necessity is trivial. Now if $\mathrm{ad}\,\mathfrak{g}$ is a characteristic ideal of $\mathrm{Der}\,\mathfrak{g}$, it is stable under $\mathrm{Der}\,(\mathrm{Der}\,\mathfrak{g})$ denoted by $\mathrm{Der}^{2}\,(\mathfrak{g})$. The image of the restriction morphism ρ of $\mathrm{Der}^{2}\,(\mathfrak{g})$ to $\mathrm{ad}\,\mathfrak{g}$ is equal to

$$\mathrm{Der}\,\left(\mathrm{ad}\,\mathfrak{g}\right)\cong\mathrm{ad}\,\left(\mathrm{Der}\,\mathfrak{g}\right). \tag{22.65}$$

The kernel $\mathfrak{J}$ of ρ has zero intersection with $\mathrm{ad}\,(\mathrm{Der}\,\mathfrak{g})$, and $\mathrm{Der}^{2}\,(\mathfrak{g})$ is the direct sum of the ideal $\mathfrak{J}$ and $\mathrm{ad}\,(\mathrm{Der}\,\mathfrak{g})$. We have

$$\left[\mathrm{ad}\,\left(\mathrm{Der}\,\mathfrak{g}\right),\mathfrak{J}\right]=-\mathrm{ad}\,\left(\mathfrak{J}\cdot\mathrm{Der}\,\mathfrak{g}\right)]=0 \tag{22.66}$$

and $\mathfrak{J}\cdot\mathrm{Der}\,\mathfrak{g}$ is zero since the center of $\mathrm{Der}\,\mathfrak{g}$ is zero, so $\mathfrak{J}=0$. Thus

$$\mathrm{Der}^{2}\,(\mathfrak{g})=\mathrm{ad}\,\left(\mathrm{Der}\,\mathfrak{g}\right). \tag{22.67}$$

□

Corollary 22.3.5 *If* $Z\,(\mathfrak{g})=0$ *and* $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ *then* $\mathrm{Der}\,\mathfrak{g}$ *is complete.*

22.4 The Derivation Tower of Lie Algebras: General Case

Let $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ be the derivation tower of a Lie algebra $\mathfrak{g}$. We now consider the general case. The ideal I of $\mathrm{Der}\,\mathfrak{g}$ of derivations which commute with $\mathrm{ad}\,\mathfrak{g}$ is the set of derivations of images contained in $Z\,(\mathfrak{g})$. Hence I vanishes on $[\mathfrak{g},\mathfrak{g}]$ and contains the center of $\mathrm{Der}\,\mathfrak{g}$. If $Z\,(\mathfrak{g})$ or $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ then $\mathrm{I}=0$. If $\mathfrak{g}$ is the direct product $\mathfrak{g}_1\times\mathfrak{g}_2$, we denote I_{ij} where i is different to j, for the set of derivations vanishing on $[\mathfrak{g}_i,\mathfrak{g}_i]\times\mathfrak{g}_j$ and their images contained in $Z\,(\mathfrak{g})$.

Lemma 22.4.1 *We set*

$$\overline{\mathrm{Der}}\,\mathfrak{g}_k=\{f\in\mathrm{Der}\,\left(\mathfrak{g}_1\times\mathfrak{g}_2\right):f\mid 0\times\mathfrak{g}_k=0\} \tag{22.68}$$

for $k=1,2$. *Then*

$$\mathrm{Der}\,\left(\mathfrak{g}_1\times\mathfrak{g}_2\right)=\overline{\mathrm{Der}}\,\mathfrak{g}_1\oplus\overline{\mathrm{Der}}\,\mathfrak{g}_2\oplus\mathrm{I}_{12}\oplus\mathrm{I}_{21}. \tag{22.69}$$

Proof We decompose each derivation into four linear maps $\mathfrak{g}_i\to\mathfrak{g}_j$ for $i,j\in\{1,2\}$ by expressing the derivation property. □

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Lemma 22.4.2 *The sequence* $(\mathfrak{g}_n)$ *defined by a Lie algebra* $\mathfrak{g} = \mathbb{K} \times \mathfrak{a}$ *such that* $\dim(\operatorname{Der}\mathfrak{g}) = \dim\mathfrak{g}$ *belongs to case* 1 *or* 2 *of Theorem 22.4.5 for* $\mathfrak{a} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]$.

Proof Lemma 22.4.1 shows that

$$\operatorname{Der}\mathfrak{g} = \mathbb{K}\epsilon \oplus \overline{\operatorname{Der}}\,\mathfrak{a} \oplus \mathrm{I}_{12} \oplus \mathrm{I}_{21} \tag{22.70}$$

where ϵ is the identity on $\mathbb{K}$ and 0 on $\mathfrak{a}$. So

$$\dim(\operatorname{Der}\mathfrak{g}) = \dim\mathfrak{g} = 1 + \dim(\operatorname{Der}\mathfrak{a}) + \dim Z(\mathfrak{a}) + \dim(\mathfrak{a}/[\mathfrak{a},\mathfrak{a}])\,. \tag{22.71}$$

We necessarily have

$$[\mathfrak{a},\mathfrak{a}] = \mathfrak{a} \quad \text{and} \quad \operatorname{Der}\mathfrak{a} = \operatorname{ad}\mathfrak{a}. \tag{22.72}$$

If $Z(\mathfrak{a}) = 0$ then $\mathfrak{a}$ is perfect, complete and $\operatorname{Der}\mathfrak{g} \cong \mathbb{K} \times \mathfrak{a}$: the sequence terminates at $\mathbb{K} \times \mathfrak{a}$, case 2.
If $Z(\mathfrak{a}) \neq 0$ then we have

$$\operatorname{Der}\mathfrak{g} \cong \operatorname{ad}\mathfrak{a} \oplus \mathrm{I} \tag{22.73}$$

(the center of $\operatorname{ad}\mathfrak{a}$ is trivial when $[\mathfrak{a},\mathfrak{a}] = \mathfrak{a}$) with $\mathrm{I} = \mathbb{K}\epsilon \oplus \mathrm{I}_{12}$.
We have $[\epsilon, f] = -f$ for all $f \in \mathrm{I}_{12}$ such that the center of $\operatorname{Der}\mathfrak{g}$ is zero and the sequence belongs to case 1. □

Proposition 22.4.3 *Let* $\mathfrak{a}$ *be a characteristic ideal of codimension 1 of* $\mathfrak{g}$. *Then*

1. $\dim(\operatorname{Der}\mathfrak{g}) - \dim(\mathfrak{g}) = \dim(\operatorname{Der}\mathfrak{g}\mid_{\mathfrak{a}}) - \dim(\operatorname{ad}\mathfrak{g}\mid_{\mathfrak{a}})$.

2. If $\dim(\operatorname{Der}\mathfrak{g}) = \dim(\mathfrak{g})$, *then*

i) $\operatorname{Der}\mathfrak{g}\mid_{\mathfrak{b}} = \operatorname{ad}\mathfrak{g}\mid_{\mathfrak{b}}$ *for any ideal* $\mathfrak{b}$ *contained in* $\mathfrak{a}$,

ii) $\mathfrak{g}$ *is algebraic,*

iii) any ideal of codimension 1 *of* $\mathfrak{g}$ *is characteristic.*

Proof Let δ be a derivation of $\mathfrak{g}$ vanishing on $\mathfrak{a}$, then it vanishes on $[\mathfrak{g},\mathfrak{g}]$ and $[\delta\cdot\mathfrak{g},\mathfrak{a}] = 0$. If $\mathfrak{a}$ is not a direct factor of $\mathfrak{g}$ $\delta\cdot\mathfrak{g} \subset Z(\mathfrak{a})$. We check that all morphisms of $\mathfrak{g}$ vanishing on $\mathfrak{a}$ and with image contained in $Z(\mathfrak{a})$ are derivations of $\mathfrak{g}$. Hence we have:

$$\dim(\operatorname{Der}\mathfrak{g}\mid_{\mathfrak{a}}) = \dim(\operatorname{Der}\mathfrak{g}) - \dim(Z(\mathfrak{a}))\,. \tag{22.74}$$

Since

$$\dim(\mathfrak{g}) = \dim(ad\mathfrak{g}\mid_{\mathfrak{a}}) + \dim(Z(\mathfrak{a})) \tag{22.75}$$

from (22.74) and (22.75) the statement follows. If the ideal $\mathfrak{a}$ is direct factor of $\mathfrak{g}$ then it is perfect since $\mathfrak{a}$ is characteristic and we directly verify the equality 1.
If $\dim\operatorname{Der}\mathfrak{g} = \dim\mathfrak{g}$ then

$$\operatorname{Der}\mathfrak{g}\mid_{\mathfrak{b}} = \operatorname{ad}\mathfrak{g}\mid_{\mathfrak{b}} \tag{22.76}$$

by 1. If the radical of $\mathfrak{g}$ is not nilpotent then

$$\operatorname{Der}\mathfrak{g} = \operatorname{ad}\mathfrak{g} + \mathrm{J} \tag{22.77}$$

where J is the ideal of derivations vanishing on $\mathfrak{b} = [\mathfrak{g},\mathfrak{g}] + \mathfrak{n}$ and we have $\mathrm{J}^2 = 0$. Hence

$$\operatorname{Der}\mathfrak{g} = \Gamma \oplus N \tag{22.78}$$

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where Γ is a m.c.r Lie subalgebra of e (ad $\mathfrak{g}$) and

$$N := \mathrm{J} + ad\mathfrak{n} \tag{22.79}$$

is the largest nilpotent Lie subalgebra. We may check that

$$Z(\mathfrak{g}) = Z(\mathfrak{n})^{\Gamma}. \tag{22.80}$$

Let I be the ideal of morphisms of $\mathfrak{g}$ which vanish on $\mathfrak{b}$ and have their images contained in $Z(\mathfrak{g})$. Then

$$\mathrm{I} \cap \mathrm{ad}\,\mathfrak{g} = 0 \tag{22.81}$$

indeed, if $x \in \mathfrak{g}$ such that

$$[x, \mathfrak{g}] \subset Z(\mathfrak{g}) \tag{22.82}$$

then $\Gamma \cdot x = 0$ and if $[x, \mathfrak{b}] = 0$ then $x \in Z(\mathfrak{g})$. Hence

$$\dim(\mathrm{ad}\,\mathfrak{g} + \mathrm{I}) = \dim(\mathfrak{g}) + (\mathrm{codim}(\mathfrak{b}) - 1) \cdot \dim(Z(\mathfrak{g})) \tag{22.83}$$

If $\mathrm{codim}(\mathfrak{b}) > 1$ then the center of $\mathfrak{g}$ is trivial and $\mathfrak{g}$ is complete. Hence it is algebraic. If $\mathrm{codim}(\mathfrak{b}) = 1$ then

$$\mathfrak{g} = \mathfrak{s} + \mathbb{K} \cdot x + n. \tag{22.84}$$

By (22.74), then

$$\mathrm{Der}\,\mathfrak{g} = \mathrm{ad}\,\mathfrak{g} \oplus \mathrm{I} \tag{22.85}$$

and there exists an element $y \in \mathfrak{g}$ such that $\mathrm{ad}\,y \mid \mathfrak{a}$ is the semi-simple component of $\mathrm{ad}\,x \mid \mathfrak{a}$. The decomposition $\mathfrak{s} \oplus \mathbb{K} \cdot \mathrm{ad}\,y \oplus \mathfrak{n}$ of $\mathfrak{g}$ satisfies the following property

$$[\mathfrak{s} \oplus \mathbb{K} \cdot \mathrm{ad}\,y, \mathbb{K} \cdot \mathrm{ad}\,y] = 0. \tag{22.86}$$

The torus $\mathbb{K} \cdot \mathrm{ad}\,y$ is maximal in the centralizer of $\mathrm{ad}\,\mathfrak{s}$ in $\mathrm{Der}\,\mathfrak{g}$ which is algebraic. Hence $\mathfrak{g}$ is algebraic, cf. [1]. □

Corollary 22.4.4 *If* $\dim(\mathrm{Der}\,\mathfrak{g}) = \dim(\mathfrak{g})$ *then the condition* $Z(\mathfrak{g}) \neq 0$ *means that the codimension of* $[\mathfrak{g}, \mathfrak{g}]$ *in* $\mathfrak{g}$ *is* 1 *or* 0.

Theorem 22.4.5 *Let* $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ *be the derivation tower of a Lie algebra* $\mathfrak{g}$*. Then it belongs to one of the following distinct cases:*

1. $Z(\mathfrak{g}_n) = 0$ *for n sufficiently large and the sequence terminates at a complete Lie algebra given by Theorem 22.3.1.*

2. $Z(\mathfrak{g}_n) \neq 0$ *for all n and the sequence terminates at a Lie algebra* $\widehat{\mathfrak{g}}$ *equal to* $\mathbb{K} \times [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]$ *where* $[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]$ *is a perfect (i.e., equal to its derived ideal) and complete Lie algebra.*

3. *The sequence of dimensions of* $\mathfrak{g}_n$ *increases and diverges.*

Proof First let us assume that if $(\mathfrak{g}_n)$ does not satisfy 1., i.e. if $Z(\mathfrak{g}_n) \neq 0$ for all n, then the sequence of dimensions $\dim \mathfrak{g}_n$ increases in the large sense. If there would exist an integer n such that $\dim(\mathrm{Der}\,\mathfrak{g}) < \dim \mathfrak{g}_n$ then $\mathfrak{g}_n$ should be perfect, cf. [2], and the center of $\mathrm{Der}\,\mathfrak{g}_n$ should be zero since $\mathrm{I} = 0$, a contradiction. Let us assume that $(\mathfrak{g}_n)$ does not satisfy 3., it just remains to study the sequences $\mathfrak{g}_n$ such that $Z(\mathfrak{g}_n) \neq 0$, with $\dim \mathfrak{g}_n = \dim \mathfrak{g}_p$ for all $n \geq p$ and we set $\mathfrak{g} = \mathfrak{g}_p$. From Corollary 22.4.4, a Lie algebra satisfying

$$\dim(\mathrm{Der}\,\mathfrak{g}) = \dim \mathfrak{g}, \quad Z(\mathfrak{g}) \neq 0, \quad [\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g} \tag{22.87}$$

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admits an ideal $\mathfrak{a}$ of codimension one equal to $[\mathfrak{g}, \mathfrak{g}]$. We may assume that $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$ since a sequence associated with $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ belongs to case 1.
If $Z(\mathfrak{g})$ is not included in $\mathfrak{a}$, then we have

$$\mathfrak{g} \cong \mathbb{K} \times \mathfrak{a} \tag{22.88}$$

and we may conclude in view of Lemma 22.4.2. Suppose now that $Z(\mathfrak{g}) \subset \mathfrak{a}$. By Proposition 22.4.3.2. we derive

$$\mathrm{Der}\,\mathfrak{g} \mid_{\mathfrak{a}} = \mathrm{ad}\,\mathfrak{g} \mid_{\mathfrak{a}} \tag{22.89}$$

thus we may write:

$$\mathrm{Der}\,\mathfrak{g} = \mathrm{ad}\,\mathfrak{g} + \mathfrak{J} \tag{22.90}$$

where $\mathfrak{J}$ is the ideal of derivations vanishing on $\mathfrak{a}$.
If $\mathfrak{g} = \mathbb{K}x \oplus \mathfrak{a}$ for $x \notin \mathfrak{a}$, $\mathfrak{J}$ is the set of morphisms δ of $\mathfrak{g}$ such that

$$\delta(\mathfrak{a}) = 0 \quad \text{and} \quad \delta(x) \in Z_{\mathfrak{g}}(\mathfrak{a}), \tag{22.91}$$

the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. We have $Z_{\mathfrak{g}}(\mathfrak{a}) \subset \mathfrak{a}$ because

$$Z(\mathfrak{g}) \subset \mathfrak{a}, \quad [\mathrm{Der}\,\mathfrak{g}, \mathrm{Der}\,\mathfrak{g}] = \mathrm{ad}\,_{\mathfrak{g}}\mathfrak{a} \tag{22.92}$$

since $\mathfrak{J}^2 = 0$ and $\mathfrak{J} \cdot \mathfrak{g} \subset \mathfrak{a}$. If $\mathrm{Der}\,\mathfrak{g}$ has non-zero center, its derived ideal is also codimension 1 and the dimension of $\mathrm{ad}\,_{\mathfrak{g}}(\mathfrak{a})$ is the same as $\mathfrak{a}$, so $Z(\mathfrak{g}) = 0$, a contradiction. $\square$

Corollary 22.4.6 *If the sequence $(\mathfrak{g}_n)$ has an element $\mathfrak{g}_p$ with non-nilpotent radical, then all g_n, $n \geq p$, have this property and the sequence is of type* 1 *or* 2. *The sequence of dimensions of type* 3 *increases strictly from $\mathfrak{g}_p$ on.*

Proof We will show that the sequence $(\mathfrak{g}_n)$ associated with $\mathfrak{g}$, with $\mathfrak{r}(\mathfrak{g}) \neq \mathfrak{n}(\mathfrak{g})$, $Z(\mathfrak{g}) \neq 0$ and $\dim \mathrm{Der}\,\mathfrak{g} = \dim \mathfrak{g}$, is of type 1. The algebraic Lie algebra $\mathfrak{g}$ has a decomposition $\mathfrak{s} \oplus \mathfrak{u} \oplus \mathfrak{n}$ such that $[\mathfrak{s} \oplus \mathfrak{u}, \mathfrak{u}] = 0$ and the ideal $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{s} \oplus \mathfrak{n}$ is of codimension 1, by proof of Theorem 22.4.5. The ideal $C_2\mathfrak{g}$ of $x \in \mathfrak{g}$ such that $[x, \mathfrak{g}] \subset Z(\mathfrak{g})$ is equal to $Z(\mathfrak{g})$ because we have $[x, \mathfrak{s} \oplus \mathfrak{n}] = 0$ and that $[x, \mathfrak{u}] \subset Z(\mathfrak{g})$ means $[x, \mathfrak{u}] = 0$, $\mathfrak{u}$ consisting of semi-simple elements. So we have $\mathrm{ad}\,\mathfrak{g} \cap \mathrm{I} = \mathrm{ad}\,(C_2\mathfrak{g}) = 0$ and $\mathrm{Der}\,\mathfrak{g}$ is a direct product of the ideal $\mathrm{ad}\,\mathfrak{g}$ with its center is trivial by the ideal I which satisfies $\mathrm{I}^2 = 0$. We conclude by using Lemma 22.4.1 showing that the center of $\mathrm{Der}\,(\mathrm{Der}\,\mathfrak{g})$ is trivial. $\square$

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Chapter 23

Quotient Divisible Groups, ω-Groups, and an Example of Fuchs

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23.1 Introduction

Groups here will generally be torsion free and always abelian. In a classical and seminal paper [2] Beaumont and Pierce studied, among other things, what they called quotient divisible groups. These are the torsion free groups (for them of finite rank) that contain a free subgroup modulo which they are divisible. Without loss of generality the free subgroup can be taken to be *full*, that is, with a torsion quotient, so the group G in question fits into an exact sequence of the form

$$0 \to X \to G \to D \to 0$$

with X free and D torsion divisible. These groups are of interest because, for example, the additive groups of full subrings of semisimple rational algebras have such structure.

Various authors (e.g., [4, 5, 7, 8, 9, 10]) have examined the question of the cardinality of the set of all subgroups, or of all pure subgroups, or of all basic subgroups, etc. of a given group G. Certainly a group cannot have more subgroups than it has subsets. In an investigation of certain topological questions ([3]), those groups for which the cardinality of the set of subgroups matched that of the group itself were called ω-groups because they were of necessity countable with countably many subgroups. It turned out that the torsion free ω-groups were characterized by the fact that they had a full free subgroup for which the corresponding quotient group was a product of finitely many groups $Z(p^\infty)$ with distinct primes p occurring. Thus these groups are special examples of quotient divisible groups.

It is well known that torsion free groups, even of finite rank, tend to be quite complicated. A number of ingenious examples have been constructed to exhibit the sometimes bizarre behavior of these groups. Among these, an example due to Fuchs [7] is a long time favorite of this author. Namely, there is, for each rank n, a torsion free group of rank n which is itself indecomposable, but in which every proper pure subgroup is free. This group in fact is found in the middle of an exact

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sequence

$$0 \to X \to G \to Z(p^\infty) \to 0$$

with X free of rank n, and p an arbitrarily chosen prime. Thus these groups are ω-groups and of course quotient divisible as well.

23.2 On ω-groups

The symbol $S(G)$ will denote the set of all subgroups of G.

Definition 23.2.1 An *ω-group* is a group G for which

$$|S(G)| < 2^{|G|}.$$

To set the stage, we recall some basic results on ω-groups as follows (see the references, especially [3]):

Theorem 23.2.2 *A torsion group G is an ω-group if and only if*

$$G = H \oplus \prod_{p \in F(G)} Z(p^\infty)$$

with H finite and $F(G)$ a finite set of primes.

Theorem 23.2.3 *A torsion free group G is an ω-group if and only if there is an exact sequence*

$$0 \to X \to G \to \prod_{p \in F(G)} Z(p^\infty) \to 0$$

with X free of finite rank and $F(G)$ a finite set of primes.

Theorem 23.2.4 *An abelian G is an ω-group if and only if $G = t(G) \oplus G_0$ with*

- $t(G) = H \oplus \prod_{p \in F(t(G))} Z(p^\infty)$, *with H finite, $F(t(G))$ finite;*
- *G_0 is a finite rank torsion free ω- group,*
- *and $F(G_0) \cap F(t(G)) = \varnothing$.*

These results reduce consideration of ω-groups to the torsion free case.

23.3 Three Remarks

In the following, the group G is to be torsion free of finite rank.

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Remark 23.3.1 If
$$0 \to X \to G \xrightarrow{\mu} \overline{G} \to 0$$
is exact with X free and $\overline{G}$ torsion with p-components $\overline{G}_p$, define $G_p = \mu^{-1}(\overline{G}_p)$ and let
$$\Phi = \Phi_X = \{\alpha \in \mathrm{End}(G) | \alpha(X) \subseteq X\}.$$
Then Φ is a full subring of End(G) and G_p is Φ-invariant for each p.

Corollary 23.3.2 *In particular, if some G_p is strongly indecomposable, then G is also strongly indecomposable.*

This all follows since, for any endomorphism α, there is an integer $m = m_\alpha \in \mathbb{Z}$, $m \neq 0$, with $m\alpha(X) \subseteq X$ so $m\alpha \in \Phi$.

Remark 23.3.3 Suppose that
$$0 \to X \to G \to \prod_{p \in F(G)} Z(p^\infty) \to 0$$
is exact with X free of finite rank and $F(G)$ a finite set of primes. Then for any subgroup B of G,
$$\frac{B+X}{X} = finite \oplus \prod_{\text{certain } p} Z(p^\infty)$$
so that $B + X$ is quasi-equal[1] to a subsum $\sum_{\text{certain } p} G_p$. Then
$$\frac{B+X}{B} \cong \frac{X}{B \cap X}$$
is finitely generated, so B is a quasi summand of $B + X$, that is, $B + X$ is quasi-equal to $B \oplus C$ for some subgroup C .

This, together with Corollary 23.3.2, yields the following:

Corollary 23.3.4 *If G/X consists of just one copy of $Z(p^\infty)$, and G is indecomposable, then every subgroup B of G is either free, or is quasi-equal to G itself.*

Proof For $B \subseteq G$, from the sequence
$$0 \to X \to G \to Z(p^\infty) \to 0$$
we conclude that either $\frac{B+X}{X}$ is finite, whence $mB \subseteq X$ for some integer $m \neq 0$ so B is free; or $\frac{B+X}{X} = Z(p^\infty)$ whence $B + X = G$. In this latter case
$$\frac{G}{B} = \frac{B+X}{B} \cong \frac{X}{B \cap X}$$
is finitely generated. Therefore the pure subgroup B^* generated by B is a summand of G. The indecomposability of G now forces $B^* = G$ so G/B is finitely generated torsion, hence finite. Thus B is quasi-equal to G as asserted. $\square$

[1] Two subgroups A, B of a torsion free group of finite rank are *quasi-equal* if each has finite index in their sum $A + B$.

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Corollary 23.3.5 *There are only two quasi-equality classes of full*[2] *subgroups of the indecomposable G, namely the class determined by X and that determined by G itself.*

We may rephrase this a bit more colorfully: If

$$0 \to X \to G \to Z(p^{\infty}) \to 0$$

and G is indecomposable, then every subgroup B of G is either finitely generated or is co-finitely generated.

In any event, we would like to find indecomposable groups to which to apply the Corollaries. One such is given in the following example.

Example 23.3.6 (Fuchs, see [7]) For any $1 < n < \omega$ there exists a torsion free group G of rank n, indecomposable, and in which every subgroup of rank less than n is free.

Construction ([7], see also [1]): The group G lives in an exact sequence

$$0 \to X \to G \to Z(p^{\infty}) \to 0$$

with X free of rank n, and is constructed from an algebraically independent set $\{1, \sigma_2, \sigma_3, \ldots, \sigma_n\}$ of p-adic integers. Algebraic independence is used to show that End(G)=$\mathbb{Z}$, thence the indecomposability.

Notice however that these groups have a property stronger than described in the references, namely that expressed in Corollary 23.3.5 and the remark following it which covers *all* subgroups, not just those of smaller rank.

The important thing for our purposes though is this: Applying the proof of Corollary 23.3.4 to a group G with

$$0 \to X \to G \to Z(p^{\infty}) \to 0$$

we get our basic

Indecomposability Criterion: A group G with

$$0 \to X \to G \to Z(p^{\infty}) \to 0$$

exact is indecomposable if and only if $\hom(G, \mathbb{Z}) = 0$

Proof Clearly indecomposability of G implies that $\hom(G, \mathbb{Z}) = 0$. Conversely suppose for our group G that $\hom(G, \mathbb{Z}) = 0$, yet $G = B \oplus C$ for some subgroups B and C with $B \neq 0$. Certainly B cannot be free, so, as in the proof of Corollary 23.3.4, $B + X = G$. Then

$$C \cong \frac{G}{B} = \frac{B + X}{B} \cong \frac{X}{B \cap X}$$

is finitely generated torsion free, hence free. The hypothesis $\hom(G, \mathbb{Z}) = 0$ now forces $C = 0$ as required. □

[2] A subgroup H is *full* in the torsion free G provided that G/H is torsion.

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Remark 23.3.7 For any finite rank torsion free group G, let

$$W = \cap\{\ker f \mid f \in \hom(G, \mathbb{Z})\}.$$

Then

$$G = W \oplus V$$

with V free abelian, rank V=rank $\hom(G, \mathbb{Z})$ and W fully invariant in G.

Proof Choose $f_1, f_2, \ldots, f_k$ maximal independent in $\hom(G, Z)$. Then $W = \cap \ker f_i$ and the sequence

$$0 \to W \to G \to \prod_{i=1}^{k} f_i(G)$$

is exact with image G free, so W splits out. □

This remark seems to give a more or less immediate proof, new to the author, of a classical result, namely: *Rings of integers in algebraic number fields are Z-orders* (i.e., they have finitely generated additive groups.) Indeed if G is the additive group of such a ring, then W, an ideal, would have finite index if non-zero whence $W = G$. But in fact $\hom(G, Z) \neq 0$ in this case (witness the trace map), so we must have $W = 0$. Observe that an analogous argument gives the general result on the integral closure of a principal ideal domain in a finite dimensional separable extension of its quotient field.

For us here though, we can use this remark to obtain a structure theorem for groups G with

$$0 \to X \to G \to Z(p^\infty) \to 0$$

exact. Call such a group *hollow* if it is indecomposable.[3] Write

$$G = W \oplus V$$

as in the remark. Recalling once again the beginning of the proof of Corollary 23.3.5, we see that we must have $W + X = G$ since there are no maps from W to Z. Therefore

$$Z(p^\infty) \cong \frac{W + X}{X} \cong \frac{W}{W \cap X}$$

and $\hom(W, Z) = 0$, so by the Indecomposability Criterion, W is indecomposable, hence hollow. We summarize this in

Proposition 23.3.8 *Any torsion free group G with*

$$0 \to X \to G \to Z(p^\infty) \to 0$$

exact and X free of finite rank is the direct sum of a free abelian group of finite rank and a hollow group.

[3] "Hollow," since every *really* proper subgroup (that is, a subgroup of infinite index) is a lattice, so it has, intuitively speaking, a discrete interior.

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23.4 Parameters

Some Notation:

- Write
$$X = \mathbb{Z}^n,$$
so X is free of rank n with chosen basis.

- Define
$$X_p = \mathbb{Z}[\frac{1}{p}] \otimes X,$$
free over the ring $\mathbb{Z}[\frac{1}{p}]$ of integral polynomials in $\frac{1}{p}$.

- Put
$$\Delta_p = X_p/X = Z(p^\infty)^n.$$

- Denote by $z_1, z_2, \ldots$ the canonical generators for $Z(p^\infty)$:
$$pz_1 = 0,\ pz_{k+1} = z_k,\ k \geqslant 1.$$

- Recall that $\hom(Z(p^\infty), Z(p^\infty)) = J_p$ where J_p is the ring of p-adic integers.

Our intention here is to determine all sequences
$$0 \to X \to G \to Z(p^\infty) \to 0.$$
To do this, consider the sequence
$$0 \to X \to X_p \to \Delta_p \to 0.$$
This yields[4]
$$0 \to Hom(Z(p^\infty), \Delta_p) \to Ext(Z(p^\infty), X) \to 0$$
so the sequences
$$0 \to X \to G \to Z(p^\infty) \to 0$$
of interest are paramatrized by J_p^n, i.e.,
$$J_p^n = Ext(Z(p^\infty), X) \cong Hom(Z(p^\infty), \Delta_p).$$

Explicitly, the exact sequence given by $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is defined by the pullback
$$\begin{array}{ccccccccc} 0 & \to & X & \to & G_h & \to & Z(p^\infty) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow h & & \\ 0 & \to & X & \to & X_p & \xrightarrow{\nu} & \Delta_p & \to & 0 \end{array}$$
where h is the map defined by $h(z_k) = (\sigma_1 z_k, \sigma_2 z_k, \ldots, \sigma_n z_k) \in \Delta_p$. That is, G_h is the pullback
$$G_h = \{(y, z) \in X_p \times Z(p^\infty) | \nu(y) = h(z)\}.$$

[4]We use the fact that $Ext(Z(p^\infty), Z[\frac{1}{p}]) = 0$ here. To see this, apply $\hom(Z(p^\infty), -)$ to the sequence $0 \to Z[\frac{1}{p}] \to Q \to Q/Z[\frac{1}{p}] \to 0$.

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23.5 Main Results

Given h, and the corresponding sequence

$$0 \to X \to G_h \to Z(p^\infty) \to 0,$$

we get

$$0 \to Hom(G_h, \mathbb{Z}) \to Hom(X, \mathbb{Z}) \to Ext(Z(p^\infty), \mathbb{Z}) \cong J_p.$$

Theorem 23.5.1 *The group $G = G_h$ is torsion free if and only if at least one of the σ_i is a p-adic unit. The image, in the sequence above, of the j^{th} projection $\pi_j \in Hom(X, \mathbb{Z})$ is the j^{th} p-adic integer σ_j belonging to h.*

The proof, while not difficult, is somewhat technical and will be presented elsewhere. Suffice it to say that our Indecomposability Criterion plays an important role. The Theorem does however make obvious the following, which is one of the main points we are trying to make here.

Corollary 23.5.2 *The group G_h is indecomposable if and only if the corresponding p-adic integers σ_j are rationally independent.*

Proof By the Indecomposability Criterion, G_h is indecomposable if and only if $Hom(G_h, \mathbb{Z}) = 0$. From the exact sequence above, this is equivalent to the map $Hom(X, \mathbb{Z}) \to Ext(Z(p^\infty), \mathbb{Z}) \cong J_p$ being monic, and this is equivalent in turn to the independence of the σ_j. □

Notice that we do not require the σ_i to be algebraically independent. In order to get an indecomposable group G they need merely be rationally independent.

23.6 Endomorphisms

Given h, or the sequence $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, what can we say about the ring End(G) of endomorphisms of the group $G = G_h$?

Notation:

$$\Phi = \{\varphi \in End(G) | \varphi(X) \subseteq X\}.$$
$$U = \mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2 \ldots + \mathbb{Z}\sigma_n \subseteq J_p.$$
$$\Lambda = \{\lambda \in J_p | \lambda U \subseteq U\}.$$

Here is our second main result:

Theorem 23.6.1 *For indecomposable G, Φ is a full subring of $End(G)$, and the rings Φ and Λ are isomorphic.*

Since U is finitely generated the ring Λ has finite rank, and of course is a subring of the p-adic integers J_p. Therefore we have the following

Corollary 23.6.2 *For indecomposable G, End(G) is isomorphic to a subring of an algebraic number field. Therefore indecomposability implies strong indecomposability for the groups under discussion.*

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The proof of the theorem above is also deferred to a later date. However we note that the ring Λ clearly consists of algebraic integers. We can then recapture Fuchs' example:

Corollary 23.6.3 *If the σ_i are algebraically independent, $\sigma_1 = 1$, and $n > 1$, then End(G)= $\mathbb{Z}$.*

For an elegant construction, due to Pierce, of myriad examples in rank 2 (and for the prime $p = 2$) see [1, Exercise 2.5].

Here is an example of a more or less generic nature.

Example 23.6.4 Let $n = 2$ and $p \equiv 1 \pmod 4$. Then there is $\sigma \in J_p$ with $\sigma^2 = -1$. Take $(\sigma_1, \sigma_2) = (1, \sigma)$. These are independent over $\mathbb{Q}$ so the corresponding group is indecomposable. In this case the ring Φ is isomorphic to the gaussian integers $\mathbb{Z}[i]$.

Our results facilitate the construction of examples of considerable generality. For example, the ring of algebraic integers in any number field may be realized as the ring of endomorphisms of a hollow group, since any such is contained in the ring of p-adic integers for any prime p unramified and of degree 1 in the field.

Finally, a remark on the classification problem for the groups corresponding to the sequences $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of p-adic integers. Since they are quotient divisible groups, they have been assigned a *q-d invariant* by Beaumont and Pierce [2, Section 5], that is, a sequence (δ_q) indexed by the primes q, in which δ_q is a vector space over the corresponding field $\widehat{\mathbb{Q}_q}$ of q-adic numbers. Then two quotient divisible groups G and G' with invariants (δ_q) and (δ'_q) are quasi-isomorphic if and only if there is a $\mathbb{Q}$-linear map φ whose q-adic extension maps δ_q onto δ'_q for each q. We refer to [2] for details in general, but in our case the situation is quite simple and transparent. The q-d invariant for the group G corresponding to the sequence $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of p-adic integers has $\delta_q = 0$ for $q \neq p$, while δ_p is the one dimensional space generated by $y = \sigma_1 x_1 + \sigma_2 x_2 + \cdots + \sigma_n x_n$. The criterion amounts to the statement that two such groups G and G' are quasi-isomorphic if and only if there is an $n \times n$ non-singular matrix over $\mathbb{Q}$ which takes the vector $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of parameters for G onto the vector $(\sigma'_1, \sigma'_2, \ldots, \sigma'_n)$ for G'. Thus there are continuum many quasi-isomorphism classes of these groups.

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Chapter 24

When are Almost Perfect Domains Noetherian?

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Abstract Characterizations of Noetherian almost perfect domains are given, in particular for P-stable domains and $\mathcal{E}$-closed domains.

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24.1 Introduction

Almost perfect domains have been introduced recently by Bazzoni and the author [1] in the investigation of the existence of strongly flat covers over commutative integral domains. This study was made possible after the solution of the Flat Cover Conjecture by Bican-El Bashir-Enochs [3]. In the local case, almost perfect domains have been studied a long time ago by J. R. Smith under the name of domains with TTN (that is, with topologically transfinite nilpotent radical). Smith published a paper in 1969 [13] on these domains, containing the major part of his dissertation at the University of South Carolina under the direction of Edgar Enochs. More recent papers by Bazzoni, Zanardo and the author ([2], [15], [9], [10]) started a more systematic investigation of almost perfect domains and their modules.

All 1-dimensional Noetherian domains are almost perfect, so the natural problem arises of looking for conditions, both of ring theoretic and module theoretic type, ensuring that an almost perfect domain is Noetherian. Since almost perfect domains are h-local, and an h-local domain is Noetherian if and only if it is such locally, the problem reduces to local almost perfect domains R. Under this standard assumption, we will denote the maximal ideal of R by P, its field of quotients by Q, and by R_1 the endomorphism ring of P, which can be identified with the fractional ideal $P : P = \{x \in Q | xP \leq P\}$.

It is easy to see that P is a principal ideal if and only if R is a DVR (see (E1) in Section 24.2). Hence, we will always assume that P is not principal, so that R_1 can be identified with $R : P$. Notice that in this case R_1/R coincides with the socle of Q/R.

A very useful characterization of the Noetherian almost perfect domains R can be derived from a result by Shores [12]: R is Noetherian if and only if P/P^2 is finitely generated.

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This is the only significant available characterization of Noetherian almost perfect domains, as far as we know. There is also a relevant sufficient condition in [2], stating that an almost perfect domain R is Noetherian provided that Q/R is injective or, more generally, if R is divisorial. On the other hand, there are many conditions necessarily satisfied by a Noetherian 1-dimensional domain that we will illustrate in Section 24.2.

The first goal of this paper is to provide another characterization of the Noetherian local almost perfect domains, by proving the converse of a necessary conditions quoted in Section 24.2, which involves the endomorphism ring of the maximal ideal P. An application to those domains such that the maximal ideal P is stable (in the sense of Eakin-Sathaye [5]) is given.

The second goal is to prove that certain necessary conditions, like that of being a classical ring or a SISI ring (see [14]), are also sufficient to ensure the Noetherian condition, if one restricts to those almost perfect domains whose completion with respect to the Prüfer topology is still a domain. These domains are called $\mathcal{E}$-closed and remind the analytically irreducible domains of the Noetherian setting (see [7, p.68]).

For all unexplained notation and terminology we refer to the monograph [6] and (for topological notions) to [9].

24.2 Known Results on the Noetherian Condition

In this section we survey some known results on the Noetherian condition for almost perfect local domains. If R is such a domain with maximal ideal P, let us denote by $E = E(R/P)$ the minimal injective cogenerator of the category of the R-modules. We recall that, by the first Proposition in [13, p. 239], every torsion R-module has essential socle. R_1 will always denote the overring $P : P$ of R.

We mark the next results, valid for an arbitrary local almost perfect domain R, depending whether they give conditions equivalent to the Noetherian one (denoted by (E)), or necessary conditions (denoted by (N)).

(E1) P is principal if and only if R is a DVR.

Let us assume that $P = pR$ for a $p \in P$. Then the socle $(Q/R)[P]$ of (Q/R) satisfies the equality $(Q/R)[P] = p^{-1}R/R$, and it is essential in Q/R. Now we imitate the proof of [6, I.2.11]; if $x \in Q \setminus R$, then $xR + R \geq p^{-1}R$ or, equivalently, $p(R + x^{-1}R) \geq x^{-1}R$; therefore $x^{-1} = pr + x^{-1}ps$ for suitable $r, s \in R$. Since $1 - ps$ is a unit of R, we conclude that $x^{-1} \in R$, hence R is a valuation domain, necessarily a DVR, being 1-dimensional and discrete.

(E2) R is Noetherian if and only if P/P^2 is finitely generated.

It is an immediate consequence of [12, Theorem 5.2], just recalling that all torsion R-modules are semiartinian.

(N1) If R is Noetherian and $P/P^2 = \bigoplus_{1\leq i\leq n} < p_i + P^2 >$, then $P = \sum_{1\leq i\leq n} p_i R$.

This is a well known consequence of the Nakayama's lemma.

(N2) If R is Noetherian, then $(Q/R)[P]$ is finitely generated and $R_1 = \bigcap_{1\leq i\leq n} p_i^{-1}P$ for suitable elements $p_i \in P$.

The statement is trivial if P is a principal ideal. Otherwise, $(Q/R)[P] = R_1/R$ and R_1 is a fractional ideal of R, hence finitely generated. If $P = p_1R + \ldots + p_nR$, then $R_1 = P : P = \bigcap_{1\leq i\leq n} P : p_iR = \bigcap_{1\leq i\leq n} p_i^{-1}P$.

(N3) If R is Noetherian, then the R-topology on R coincides both with the P-adic topology and the Prüfer topology (or f.e. topology).

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See [9, p. 3] and [14, p. 115]. So, when we speak of the completion of R in the Noetherian case, we intend with respect to each one of these topologies. The converse is not true, as the example furnished in [9] of non-Noetherian P-chained domains shows.

(N4) If R is Noetherian, then E is linearly compact (in the discrete topology), hence it is injective as a module over the completion of R, and R is a SISI ring.

See [8] and [14]. In the terminlogy of Vámos [14], this implication says that a Noetherian almost perfect domain is a classical ring. We will deal with the converse of this implication in Section 24.4.

(N5) If R is Noetherian, then $\mathrm{End}_R(E)$ is isomorphic to the completion of R, hence it is a commutative local ring.

See [7, p. 15]. We will deal with the converse of this implication in Section 24.4 in the restricted setting of $\mathcal{E}$-closed almost perfect domains.

24.3 A Characterization of Noetherian Almost Perfect Domains

First we try to extend the implication in (N1) to non-Noetherian almost perfect domains. The role of Nakayama's lemma is played by the hypothesis that the Loewy length of Q/R is ω.

Proposition 24.3.1 *Let R be a local almost perfect domain with maximal ideal P and field of quotients Q. If $l(Q/R) = \omega$, then the following facts hold:*

1) if $P/P^2 = \bigoplus_{\sigma\in\Sigma} < p_\sigma + P^2 >$, then $P = \sum_{\sigma\in\Sigma} p_\sigma R$;

2) $R_1 = \bigcap_{\sigma\in\Sigma} p_\sigma^{-1} P$.

Proof 1) Let $N = \sum_{\sigma\in\Sigma} p_\sigma R$; then $N + P^2 = P$. Setting $M = P/N$, this implies that $M = P^n M$ for all $n \geq 1$. The R-topology and the P-adic topology coincide, by [9] 24.3.1, hence there exists a $k \geq 1$ such that $P^k \leq N$. There follows that $M = P^k M = 0$, that is, $P = N$.

2) From 1) we get

$$R_1 = P : P = P : \sum_{\sigma\in\Sigma} p_\sigma R = \bigcap_{\sigma\in\Sigma} P : p_\sigma R = \bigcap_{\sigma\in\Sigma} p_\sigma^{-1} P.$$

□

Fact 2) in 24.3.1 amounts to say that $(Q/R)[P] = (\bigcap_\sigma p_\sigma^{-1} P)/R$. This fact can be generalized to the case of $l(Q/R)$ arbitrary in the following way; recall that $(Q/R)[P^\omega]$ denotes the union $\bigcup_{n\in\omega}(Q/R)[P^n]$.

Proposition 24.3.2 *Let R be a local almost perfect domain with maximal ideal P and field of quotients Q. Then $(Q/R)[P] = (\bigcap_{p\in P\setminus P^2} p^{-1} P)/R \cap (Q/R)[P^\omega]$.*

Proof We can assume P not principal. Clearly

$$(Q/R)[P] = (\bigcap_{p\in P} p^{-1} P)/R \leq (\bigcap_{p\in P\setminus P^2} p^{-1} P)/R \cap (Q/R)[P^\omega].$$

Assume, by way of contradiction, that there exists an element $x + R \in (Q/R)[P^\omega]$ ($x \in Q$) such that $px \in P$ for all $p \in P \setminus P^2$ and $x + R \notin (Q/R)[P]$. Then there exists an element $r_2 \in P^2$ such that $r_2 x \notin P$. Since $r_2 = \sum_{1\leq i\leq k} p_i q_i$, where $p_i, q_i \in P$, there exists at least one index i such that $p_i q_i x \notin P$, thus $p_i, q_i \in P^2$. Then $r_3 = p_i q_i \in P^4$. Repeating this argument for r_3, we see that for each $n \geq 2$ there exists an element $r_n \in P^n$ such that $r_n x \notin R$. But this contradicts the hypothesis that $x + R \in (Q/R)[P^\omega]$, hence we are done. □

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We give now a characterization of local almost perfect Noetherian domains obtained reversing the implication in (N2).

Theorem 24.3.3 *A local almost perfect domain R with maximal ideal P and field of quotients Q is Noetherian if and only if $(Q/R)[P]$ is finitely generated and $R_1 = \bigcap_{1\leq i\leq n} p_i^{-1}P$ for suitable $p_1, \dots, p_n \in P$.*

Proof Only the proof of the sufficiency in the case of P not principal is needed. Let us assume that $(Q/R)[P] = R_1/P = \bigcap_{1\leq i\leq n} p_i^{-1}P/R$ is finite dimensional and let $K = Q/R$. Then

$$K[P^2]/K[P] = (K/K[P])[P] \cong (Q/\bigcap_{1\leq i\leq n} p_i^{-1}P)[P].$$

Since $(Q/\bigcap_{1\leq i\leq n} p_i^{-1}P)$ embeds into

$$\bigoplus_{1\leq i\leq n} Q/p_i^{-1}P \cong \bigoplus_{1\leq i\leq n} Q/P,$$

its socle is finite dimensional, because $(Q/P)[P] \cong R/P \oplus (Q/R)[P]$ is finite dimensional. There follows that $K[P^2]/K[P]$ is finite dimensional. By Shores' result [12, Theorem 38], $K[P^{n+1}]/K[P^n]$ is finite dimensional for all $n \geq 1$ and the Loewy length of K is ω. Therefore, fixed a non-zero element $a \in P$, $a^{-1}R/R \cong R/aR$ has finite length, thus P/aR is finitely generated; hence P is finitely generated too. The Cohn's theorem gives the conclusion. □

Theorem 24.3.3 generalizes the obvious fact that, if $R_1 = \mathrm{End}_R(P)$ is finitely generated as an R-module, and P is a finitely generated R_1-module, then P is a finitely generated R-module, so that R is Noetherian. In fact, if $P = p_1R_1 + \dots + p_nR_1$, then obviously $R_1 = \bigcap_{1\leq i\leq n} p_i^{-1}P$, so the hypotheses of Theorem 24.3.3 are satisfied.

There are simple example of non-Noetherian almost perfect local domains R such that $R_1 = p^{-1}P$ for some $p \in P$ and $(Q/R)[P]$ is not finitely generated. For instance, $R = F + XK[[X]]$, where K is a field which is an extension of infinite degree of the field F.

The main question concerning Theorem 24.3.3 is whether $(Q/R)[P]$ finitely generated implies that $R_1 = \bigcap_{1\leq i\leq n} p_i^{-1}P$ for suitable $p_1, \dots, p_n \in P$ or, equivalently, if the only hypothesis that $(Q/R)[P]$ is finitely generated ensures that R is Noetherian.

We would like to analyze in more detail the almost perfect local domains satisfying the property that $R_1 = p^{-1}P$ for some $p \in P$.

In [9] we defined a local domain R with maximal ideal P a P-chained domain if, given any non-zero ideal I, there exists an $n \in \omega$ such that $P^n \geq I \geq P^{n+1}$. This is equivalent to say that the ideal P is almost nilpotent and $aP = P^2$ for every $a \in P \setminus P^2$ (see [9, Proposition 12]). It is easy to prove (see [9, Cor. 13]) that a P-chained domain is an almost perfect pseudo-valuation domain.

If we require that the above equality $aP = P^2$ holds not for all $a \in P \setminus P^2$, but only for a selected element $p \in P \setminus P^2$, then we get a larger class of domains.

Definition 24.3.4 A local domain R with maximal ideal P is *P-stable* if there exists an element $p \in P$ such that $pP = P^2$.

The next lemma should explain the term *stable* used in the definition.

Lemma 24.3.5 *Let R be a local domain with maximal ideal P, and let $R_1 = P : P$. Then R is P-stable if and only if P is a principal ideal of R_1. If this happens, then for all $n \geq 0$, $P^n/P^{n+1} \cong R_1/P$, where $P^0 = R_1$.*

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Proof Assume $pP = P^2$ for a $p \in P$. The map from R_1/P into P/P^2 induced by the multiplication by p is well defined and injective. It is also surjective; for, if $r \in P$, then $r + P^2$ is the image of $=p^{-1}r+P$, where $p^{-1}r \in R_1$, since $Pr \leq P^2 = pP$ implies $Pp^{-1}r \leq P$. Thus the desired isomorphism holds for $n = 1$, and for $n > 1$ it follows from $P^n/P^{n+1} = p^{n-1}P/p^nP \cong P/pP = P/P^2$. Coversely, if $P = pR_1$, then it is obvious that $P^2 = p^2R_1 = pP$. □

The notion of P-stable domain agrees with the notion of stability for the ideal P defined by Eakin-Sathaye [5], and it is stronger than the stability notion introduced by Sally-Vasconcelos [11], who define P to be stable if it is projective over its endomorphism ring.

Note that every valuation domain with principal maximal ideal P is P-stable, hence these domains can have arbitrary Krull dimension. The next proposition shows the relationship between P-stable domains and almost perfect domains; it is essentially proved in [2, Proposition 2.5 (3)], and we give here a slightly different proof.

Proposition 24.3.6 *Let R be a P-stable domain. Then R is almost perfect with P almost nilpotent if and only if it is* 1*-dimensional.*

Proof The necessity is obvious. Assume R of Krull dimension 1 and let $pP = P^2$ for an element $p \in P$. Then $pP^n = P^{n+1} = p^nP$ for all $n \geq 1$. Pick any $0 \neq a \in P$ and consider the multiplicative set $S = \{p^n\}_{n \geq 1}$. If $S \cap aR = \emptyset$, let J be an ideal of R which is maximal with respect to the properties: $J \geq aR$ and $S \cap J = \emptyset$. Then J is a prime ideal, hence $J = P$, which is absurd, since $p \in P$. Therefore $p^n \in aR$ for some $n \geq 1$. Thus $P^{n+1} = p^nP \leq p^nR \leq aR$, hence P is almost nilpotent and consequently R is almost perfect. □

From Theorem 24.3.3 and Lemma 24.3.5 we immediately derive the following result, which generalizes the similar result for P-chained domains [9, Proposition 20].

Corollary 24.3.7 *An almost perfect P-stable domain R is Noetherian if and only if $(Q/R)[P]$ is finitely generated.*

We exhibit now a typical example of almost perfect P-stable domain.

Example 24.3.8 Fix an integer $n \geq 1$. Let F be a field and K an extension of F, with $\{\alpha_i\}_{i \in I}$ as F-basis. Let
$$S_n = F + FX + FX^2 + \ldots + FX^{n-1} + X^nK[[X]].$$
We set also $S_0 = K[[X]]$. We know that S_0 is a DVR and that S_1 is a P_1-chained almost perfect domain, where $P_1 = XK[[X]]$ is its maximal ideal. All the domains S_n $(n \geq 0)$ have the same field of quotients $Q = K((X))$.
S_n is local 1-dimensional with maximal ideal
$$P_n = FX + FX^2 + \ldots + FX^{n-1} + X^nK[[X]].$$
Notice that $P_n = XS_{n-1}$ and that $P_n : P_n = S_{n-1}$, hence P_n is a principal ideal over its endomorphism ring. By Proposition 3.6 S_n is almost perfect. It is immediate to check that
$$P_n =< X, \alpha_iX^n | i \in I > \ ; \ \ P_n^2 =< X^2, \alpha_iX^{n+1} | i \in I >$$
hence $XP_n = P_n^2$ and S_n is P_n-stable. Furthermore, S_n is Noetherian if and only if the degree $[K : F]$ is finite.

If $n > 1$, then S_n is not P_n-chained, since $X^n \in P_n \setminus P_n^2$ but $X^nP_n \neq P_n^2$.

For all $n \geq 2$, S_{n-1} is integral over S_n and S_1 is integrally closed exactly if F is algebraically closed in K. In any case, S_n contains an ideal of S_0, hence S_0 is the unique archimedean valuation domain dominating S_n, by [15, Theorem 32].

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24.4 $\mathcal{E}$-Closed Domains

Also in this Section R always denotes a local commutative domain, P its maximal ideal, Q its field of quotients, and $E = E(R/P)$ the minimal injective cogenerator. $\widehat{R}$ denotes the completion of R in the Prüfer (that is, f.e.) topology, which is denoted by $\mathcal{E}$. A basis of neighborhoods of 0 for $\mathcal{E}$ is given by the ideals I of R such that R/I has finitely generated socle.

$\widehat{R}$ is a commutative local ring, whose maximal ideal is $\widehat{P}$, the closure of P in $\widehat{R}$, and $\widehat{R}/\widehat{P} \cong R/P$. $\widehat{R}$ naturally embeds into the endomorphism ring $A = \mathrm{End}_R(E)$ and its image coincides with its center $Z(A)$ (see [14, Proposition 1.6]); $\widehat{R}$ contains both the completion $\widetilde{R}$ of R in the R-topology, and the completion $\bar{R}$ of R in the P-adic topology. A has a unique maximal left ideal M, which is a two-sided ideal, consisting of the endomorphisms with non-zero kernel. M intersects $\widehat{R}$ in $\widehat{P}$ and R in P. Since E is an A-module, it is in a natural way an $\widehat{R}$-module (for all these facts we refer to [14] and [9]).

We are interested in those local almost perfect domains such that the local commutative ring $\widehat{R}$ is a domain. Recall that a 1-dimensional local Cohen-Macaulay ring R is *analytically irreducible* if its completion $\widetilde{R}$ in the R-topology is a domain. Matlis provided several conditions equivalent to analytic irreducibility in [7, Theorem 71]. One of these conditions says that R is *closed*, which means that every non-zero ideal of $\widetilde{R}$ intersects R non-trivially. Inspired by Matlis, we give the following

Definition 24.4.1 A local domain R is *$\mathcal{E}$-closed* if, for every non-zero ideal J of $\widehat{R}$, $J \cap R \neq 0$.

We shall need the following

Lemma 24.4.2 *Let R be a local domain. The completion topology of $\widehat{R}$ is contained in the Prüfer topology of $\widehat{R}$. If R is $\mathcal{E}$-closed, then the two topologies coincide.*

Proof A base of neighborhoods of 0 for the completion topology is the family of ideals $\widehat{I} = \mathrm{Ann}_A(E[I]) \cap \widehat{R}$, ranging I over the set of ideals such that R/I has finitely generated socle (see [9, Theorem 5]). Since $I = \mathrm{Ann}_A(E[I]) \cap R$, $\widehat{I} \cap R = I$. From $\widehat{R}/\widehat{I} \cong R/I$ and from $\widehat{R}/\widehat{P} \cong R/P$ there follows that $\widehat{I}$ is open in the Prüfer topology of $\widehat{R}$.

Assume now that R is $\mathcal{E}$-closed. If $0 \neq J$ is an ideal of $\widehat{R}$ such that $\widehat{R}/J$ has finitely generated socle as $\widehat{R}$-module, hence also as R-module, then J contains $\widehat{I}$, where $0 \neq I = J \cap R$. Then $(R/I)[P]$ is finitely generated, since $R/I \cong R + J/J \leq \widehat{R}/J$, hence I is open in the Prüfer topology of R, so J, containing $\widehat{I}$, is open in the completion topology of $\widehat{R}$. □

We can now prove the following result.

Theorem 24.4.3 *For a local almost perfect domain R, the following properties are equivalent:*

1) *R is $\mathcal{E}$-closed;*

2) *$\widehat{R}$ is a local almost perfect domain.*

If the above conditions are satisfied, then E is an injective $\widehat{R}$-module.

Proof 1) $\Rightarrow$ 2) Let $0 \neq x \in \widehat{R}$. First we show that $xE = E$. Let us assume, by way of contradiction, that $xE \neq E$. Then $C = \mathrm{Hom}_R(E, xE)$ is a right ideal of $A = \mathrm{End}_R(E)$ containing x, hence $C \cap \widehat{R}$ is a non-zero ideal of $\widehat{R}$. By hypothesis, there exists an element $0 \neq r \in R$ belonging to C, which is absurd, since $rE = E$. Let now x, y be two non-zero elements of $\widehat{R}$; then $xyE = xE = E$, hence $xy \neq 0$ and $\widehat{R}$ is a domain. Now we prove that $\widehat{R}$ is almost perfect. Let $0 \neq J$ be an ideal of $\widehat{R}$; then $I = J \cap R \neq 0$. Since J is closed in the Prüfer topology of $\widehat{R}$ (see [14], (iv) p. 115]), and this topology is contained in the completion topology, by Lemma 24.4.2, J contains $\widehat{I}$, the closure of

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I in $\widehat{R}$. Being R dense in $\widehat{R}$, we have that $R + \widehat{I} = \widehat{R}$; thus we get that $\widehat{R}/\widehat{I} = R + \widehat{I}/\widehat{I} \cong R/\widehat{I} \cap R$, and this is a ring isomorphism. Since $\widehat{I} \cap R \geq I \neq 0$, $\widehat{R}/\widehat{I}$ is a perfect ring, and such is $\widehat{R}/J$, which is a factor ring of $\widehat{R}/\widehat{I}$.

2) $\Rightarrow$ 1) Let J be an ideal of $\widehat{R}$ such that $J \cap R = 0$; let L be an ideal of $\widehat{R}$ which is maximal with respect to the properties of containing J and intersecting R trivially. Then L is a prime ideal of $\widehat{R}$ strictly contained in $\widehat{P}$, because $\widehat{P} \cap R = P$. Since $\widehat{R}$ is 1-dimensional, $L = 0$ and consequently $J = 0$. So R is closed.

Let us assume now that $\widehat{R}$ is a local almost perfect domain. By [6, VI.2.5], E is an injective $\widehat{R}$-module if and only if $\mathrm{Ext}^1_{\widehat{R}}(\widehat{R}/\widehat{P}, E) = 0$. Every exact sequence of $\widehat{R}$-modules of the form $0 \to E \to X \to \widehat{R}/\widehat{P} \to 0$ splits as a sequence of R-modules. But every R-homomorphism $\widehat{R}/\widehat{P} \cong R/P \to X$ is an $\widehat{R}$-homomorphism, hence the above exact sequence of $\widehat{R}$-modules splits too. Thus E is an injective $\widehat{R}$-module. □

The last claim in Theorem 24.4.3 should be compared with Lemma 4 in [8], which states that E is injective as an A-module exactly if it is a linearly compact R-module; in the terminology introduced in [14], this amounts to say that R is a classical ring. We examine now this property for almost perfect domains.

Recall that a commutative ring R is a *SISI* ring if every subdirectly irreducible factor ring of R is self-injective; as proved in [14, Theorem. 2.1], for R local with $E = E(R/P)$, this amounts to say that every endomorphism of E is locally a multiplication by a ring element. Vámos noted in [14] that this fact implies that the ring $A = \mathrm{End}_R(E)$ is a commutative. The converse also holds.

Proposition 24.4.4 *A local ring R with minimal injective cogenerator E is SISI if and only if $A = \mathrm{End}_R(E)$ is commutative.*

Proof Only the proof of the sufficiency is needed. Assume, by way of contradiction, that there exists an endomorphism ϕ of E which is not locally a multiplication by a ring element. Hence there exists an element $x \in E$ such that $\phi(x) \notin xR$. Let $I = \mathrm{Ann}_R(x)$. Then A strictly contains $R + \mathrm{Ann}_A(E[I])$, otherwise $\phi = r + \psi$ for some $r \in R$ and $\psi \in A$ such that $\psi(x) = 0$. But this implies that $\phi(x) = rx \in xR$, a contradiction. Therefore A strictly contains $R + \mathrm{Ann}_{\widehat{R}}(E[I]) = \widehat{R}$, hence A is noncommutative, since $\widehat{R}$ is the center of A. □

Vámos proved [14], Proposition 32 that a classical ring is a SISI ring and provided a counterexample to the converse implication, which actually holds if R is complete in the Prüfer topology.

The next Corollary 24.4.6 shows that the two notions are equivalent for $\mathcal{E}$-closed local almost perfect domains, and both are equivalent to the Noetherian condition and to the commutativity of $\mathrm{End}_R(E)$. So the implications (N4) and (N5) in Section 24.2 can be reversed for $\mathcal{E}$-closed domains. First we need the following result.

Proposition 24.4.5 *For a local almost perfect domain R, the following conditions are equivalent:*

1) R is a classical ring;

2) R is Noetherian;

3) R is a SISI ring and E is an injective $\widehat{R}$-module;

4) $A = \mathrm{End}_R(E)$ is a commutative ring (isomorphic to $\widehat{R}$) and E is an injective $\widehat{R}$-module.

Proof 1) $\Leftrightarrow$ 2) If E is linearly compact, then $E[P^2]$ is finitely generated (see [14, p.116]). Applying the functor $\mathrm{Hom}_R(-, E)$ to the exact sequence $0 \to P/P^2 \to R/P^2 \to R/P \to 0$ one obtains that $\mathrm{Hom}_R(P/P^2, E) \cong E[P^2]/E[P]$ is finitely generated. There follows that P/P^2 is finitely generated, hence R is Noetherian, by (E2) in Section 24.2. The converse is in [14, Proposition 4.1].

1) $\Leftrightarrow$ 3) See [14, Proposition 3.2] and [8, Lemma 4].

3) $\Rightarrow$ 4) See [14, Propositions 2.2 and 1.6]

4) $\Rightarrow$ 1) See [8, Lemma 4]. □

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We immediately derive from Proposition 24.4.5 and Theorem 24.4.3 the next

Corollary 24.4.6 *For a local almost perfect $\mathcal{E}$-closed domain R, the following conditions are equivalent:*

1) R is a classical ring;

2) R is Noetherian;

3) R is a SISI ring;

4) $A = \mathrm{End}_R(E)$ is a commutative ring (isomorphic to $\widehat{R}$).

In particular, 24.4.6 applies to local almost perfect domains R which are complete in the Prüfer topology.

From 24.4.6 we see that, if R is a non-Noetherian local almost perfect $\mathcal{E}$-closed domain, then $A = \mathrm{End}_R(E)$ is a non-commutative ring. We obtained the same result in [9, Theorem 10] replacing the condition "$\mathcal{E}$-closed" by the condition "P^2 is open in the Prüfer topology". Actually, looking at the proof of that Theorem, the crucial condition in order to get the non-commutativity of A is that A strictly contains $R + \mathrm{Ann}_A(E[I])$ for some basis ideal in the Prüfer topology.

We conclude by posing the two main questions that are left open in this paper.

QUESTION 1. Is an almost perfect SISI domain necessarily Noetherian?

QUESTION 2. Is a local almost perfect domain necessarily Noetherian, provided that Q/R has finitely generated socle?

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Chapter 25

Pure Invariance in Torsion-free Abelian Groups

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Abstract We describe several classes of pure fully invariant subgroups of torsion–free abelian groups.

Subject classifications: Primary: 16S50, Secondary: 20K20, 20K30.

Keywords: Endomorphism ring, pure fully invariant subgroup, torsion–free abelian group .

25.1 Introduction

Pure fully invariant subgroups are useful tools for investigating the structure of torsion–free abelian groups. They were apparently introduced by Reid, [7], who proved that they generated the invariant submodules of the group in the quasi–homomorphism category. In particular, Reid pointed out the importance of irreducible groups, that is, those with no proper pure fully invariant subgroups.

More recently, in attempting to apply Galois theoretic methods for the study of modules and their endomorphism rings ([1]) to torsion–free abelian groups, we needed to develop general techniques to identify pure fully invariant subgroups of a group.

The purpose of this paper is to describe a general class of pure fully invariant subgroups of a torsion–free abelian group, and to find criteria which imply that all such subgroups belong to the given class.

In Section 25.2 we define pure traces and kernels and show that they include the well–known constructions of socles and radicals ([5, Section 2.3]). We study their algebraic and homological properties, and show that they are quasi–homomorphism invariants.

In Section 25.3 we consider the special case of pure traces and kernels determined by completely decomposable groups. One major result is a necessary and sufficient condition on sets S and T of types for $G(S) = G(T)$ and $G[S] = G[T]$ for any torsion–free abelian group G. We adopt the notation of Fuchs, [3], for torsion–free abelian groups in general, and of Mader, [5], for completely decomposable groups. In particular, $\mathbb{Z}$, $\mathbb{Q}$, $\mathcal{I}_p$, $\mathbb{Z}_p$ and $\hat{\mathbb{Z}}$ represent the groups of integers, rationals, p–adic integers, p–adic rationals and the n–adic closure of $\mathbb{Z}$ respectively.

$G \dot{\cong} H$ means that G and H are quasi–isomorphic.

$\mathrm{Hom}(H, G)$ is the group of homomorphisms of H into G and $\mathcal{E}(G)$ the ring of endomorphisms of G.

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For any subset H of a torsion–free abelian group G, H_* is the pure subgroup of G generated by H.

A type is a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ and a ring type is a subring of $\mathbb{Q}$.

Henceforth torsion–free abelian groups are just denoted 'groups'and their pure fully invariant subgroups are denoted '*pfi subgroups* '.

25.2 Pure Fully Invariant Subgroups

Most of the results in this section can be extended *mutatis mutandis* to categories of modules closed under kernels and cokernels, but we state them for groups because that is where the major applications can be found.

There is a useful class of pfi subgroups of a group which contains many of the best known examples:

Definition 25.2.1 For any groups G and H, the *pure trace of H in G* is

$$\mathrm{tr}(H, G) = \left(\sum\{Hf : f \in \mathrm{Hom}(H, G)\}\right)_*,$$

and the *kernel of H in G* is

$$\ker(G, H) = \bigcap\{\ker f : f \in \mathrm{Hom}(G, H)\}.$$

It is routine to verify that $\mathrm{tr}(H, G)$ and $\ker(G, H)$ are pfi subgroups of G. Well–known examples include $\mathrm{tr}(\mathbb{Q}, G)$, the divisible subgroup of G, $\mathrm{tr}(\hat{\mathbb{Z}}, G)$, the maximal cotorsion summand of G, $\ker(G, \mathcal{I}_p) = p^\omega G$, the p–divisible subgroup of G, and of course the type subgroups $\mathrm{tr}(\sigma, G) = G(\sigma)$ and $\ker(G, \sigma) = G[\sigma]$ for any type σ ([5, Section 1.2]).

In the literature, for example [5, Section 2.3], $\mathrm{tr}(H, G)$ is sometimes called the H–socle of G, and $\ker(G, H)$ the H–radical of G. However, these terms often refer to different concepts, so we prefer to avoid them.

Pure trace and kernel have some obvious but useful properties.

Lemma 25.2.2

1. *If H is a subgroup of G then $H \subseteq \mathrm{tr}(H, G)$ with equality if and only if H is pure and every $f \in \mathrm{Hom}(H, G)$ maps H into H.*
2. *If $H = G/K$ is a torsion–free factor group of G then $\ker(G, H) \subseteq K$ with equality if and only if the kernel of every $f \in \mathrm{Hom}(G, H)$ contains K.*

Lemma 25.2.3 *Let G be a group and $\{H_i : i \in I\}$ a collection of groups. Then*

1. $\mathrm{tr}(\oplus_{i\in I} H_i, G) = \left(\sum_{i\in I} \mathrm{tr}(H_i, G)\right)_*$ *and*
2. $\ker(G, \oplus_{i\in I} H_i) = \cap_{i\in I} \ker(G, H_i)$.

Lemma 25.2.4 *Let $H \rightarrowtail G \twoheadrightarrow J$ be a short exact sequence of groups. Then for any group K,*

1. $\mathrm{tr}(K, H) \rightarrowtail \mathrm{tr}(K, G) \to \mathrm{tr}(K, J)$*, with the obvious induced homomorphisms, is an exact sequence, and*
2. $\dfrac{H}{\ker(H, K)} \to \dfrac{G}{\ker(G, K)} \twoheadrightarrow \dfrac{J}{\ker(J, K)}$*, with the obvious induced homomorphisms, is an exact sequence.*

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The following Lemmas show that pure trace and kernel are idempotent operators.

Lemma 25.2.5 *Let G and H be groups. Then*

(a) $\mathrm{tr}\,(\mathrm{tr}(H,\ G),\ G)) = \mathrm{tr}(H,\ G)$ *and*

(b) $\ker\left(G,\ \dfrac{G}{\ker(G,\ H)}\right) = \ker(G,\ H).$

Proof (1) Let $a \in \mathrm{tr}(H,\ G)$ so for some integer n, some finite set $b_i \in H$ and some finite set $f_i \in \mathrm{Hom}(H,\ G)$, $na = \sum b_i f_i$. Let $\iota \in \mathrm{Hom}(\mathrm{tr}(H,\ G),\ G)$ be the inclusion map. Then $na = \sum b_i f_i \iota \in \mathrm{tr}(\mathrm{tr}(H,\ G),\ G)$ and by purity, $a \in \mathrm{tr}(\mathrm{tr}(H,\ G),\ G)$.

Conversely, let $a \in \mathrm{tr}(\mathrm{tr}(H,\ G),\ G)$, say $ma = \sum c_i g_i$ for some integer m, finite set $c_i \in \mathrm{tr}(H,\ G)$ and finite set $g_i \in \mathrm{Hom}(\mathrm{tr}(H,\ G),\ G)$. For each c_i there is an integer k_i and a finite sum $\sum k_i c_i d_{ij} h_j$ with $d_{ij} \in H$ and $h_j \in \mathrm{Hom}(H,\ G)$. Hence for some integer m, $ma \in \mathrm{tr}(H,\ G)$ and by purity $a \in \mathrm{tr}(H,\ G)$.

(2) Let K denote $\ker(G,\ H)$ and let $a \in \ker\,(G,\ G/K)$. Then in particular, a is in the kernel of the natural map of G onto G/K, so $a \in K$.

Conversely, let $a \in K$ and suppose by way of contradiction that $a \notin \ker\,(G,\ G/K)$. Then there exist $g \in \mathrm{Hom}(G,\ G/K)$ such that $ag = b + K$ with $b \notin K$. Thus there exists $f \in \mathrm{Hom}(G,\ H)$ with $bh \neq 0$. Such an h induces $\bar{h} \in \mathrm{Hom}(G/K,\ H)$ with $(b + K)\bar{h} \neq 0$. But then $g\bar{h} \in \mathrm{Hom}(G,\ H)$ and $ag\bar{h} \neq 0$, contradicting the fact that $a \in K = \ker(G,\ H)$. This contradiction implies that $a \in \ker\,(G,\ G/K)$. □

Corollary 25.2.6 *1. H is a pure trace in G if and only if* $H = \mathrm{tr}(H,\ G)$.

2. K is a kernel in G if and only if $K = \ker(G,\ G/K)$.

A crucial property of pfi groups H is that every homomorphism from H to G which extends to $\mathcal{E}(G)$ has image in H. Similarly, every homomorphism from G to G/H which lifts to $\mathcal{E}(G)$ has kernel containing H. These properties are related to those of Lemma 25.2.2.

Proposition 25.2.7 *Let H be a pfi subgroup of a group G and let* $E_H : H \rightarrowtail G \twoheadrightarrow G/H$ *be the natural short exact sequence.*

1. *If every homomorphism from H to G extends to* $\mathcal{E}(G)$ *then* $H = \mathrm{tr}(H,\ G)$ *and the sequence* E_H *induces a short exact sequence of rings*

$$\mathrm{Hom}(G/H,\ G) \rightarrowtail \mathcal{E}(G) \twoheadrightarrow \mathcal{E}(H).$$

2. *If every homomorphism from G to G/H lifts to* $\mathcal{E}(G)$, *then* $H = \ker(G,\ G/H)$ *and the sequence* E_H *induces a short exact sequence of rings*

$$\mathrm{Hom}(G,\ H) \rightarrowtail \mathcal{E}(G) \twoheadrightarrow \mathcal{E}(G/H).$$

Proof (1) Let $f \in \mathrm{Hom}(H,\ G)$. Since f extends to $\mathcal{E}(G)$ and H is fully invariant in G, f maps H into H. Since H is pure, H satisfies the hypothesis of Lemma 25.2.2 (1) and hence $H = \mathrm{tr}(H,\ G)$.

The sequence E_H induces the derived sequence

$$\mathrm{Hom}(G/H,\ G) \rightarrowtail \mathrm{Hom}(G,\ G) \overset{res}{\to} \mathrm{Hom}(H,\ G).$$

Since every homomorphism of H into G extends to G, the restriction map *res* is epic and maps $\mathrm{Hom}(G,\ G)$ onto $\mathrm{Hom}(H,\ H)$. Since H is fully invariant in G, *res* preserves multiplication and hence is a ring epimorphism. Thus the kernel $\mathrm{Hom}(G/H,\ G)$ can be considered as an ideal in $\mathcal{E}(G)$.

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(2) Let $f \in \mathrm{Hom}(G, G/H)$. Since f lifts to $\mathcal{E}(G)$, f has H in its kernel, so induces an endomorphism of G/H. Thus G/H satisfies the hypothesis of Lemma 25.2.2 (2) and hence $H = \ker(H, G/H)$.

The sequence E_H induces the derived sequence

$$\mathrm{Hom}(G, H) \overset{f}{\rightarrowtail} \mathrm{Hom}(G, G) \overset{g}{\to} \mathrm{Hom}(G, G/H).$$

Since every homomorphism of G into G/H lifts to $\mathcal{E}(G)$, g is an epimorphism. As in (1), g is a ring epimorphism. □

There is a simple sufficient condition for a pfi subgroup to be both a pure trace and a kernel.

Lemma 25.2.8 *Let H be a pfi subgroup of G.*

1. $H = \mathrm{tr}(H, G)$ *if and only if* 0 *is the only homomorphism in* $\mathrm{Hom}(H, G/H)$ *which lifts to* $\mathrm{Hom}(H, G)$.

2. $H = \ker(G, G/H)$ *if and only if* 0 *is the only homomorphism in* $\mathrm{Hom}(H, G/H)$ *which extends to* $\mathrm{Hom}(G, G/H)$.

Proof (1) The short exact sequence E_H of Proposition 25.2.7 induces the exact sequence

$$\mathrm{Hom}(H, H) \rightarrowtail \mathrm{Hom}(H, G) \overset{\alpha}{\to} \mathrm{Hom}(H, G/H)$$

the image of α being the group of homomorphisms of H into G/H which lift to $\mathrm{Hom}(H, G)$ and the kernel of α being the group of homomorphisms of H into G which map H into H. Hence by Lemma 25.2.2 (1), $H = \mathrm{tr}(H, G)$ if and only if the kernel of $\alpha = \mathrm{Hom}(H, G)$, if and only if 0 is the only element of $\mathrm{Hom}(H, G/H)$ which lifts.

(2) The short exact sequence E_H induces the exact sequence

$$\mathrm{Hom}(G/H, G/H) \rightarrowtail \mathrm{Hom}(G, G/H) \overset{\beta}{\to} \mathrm{Hom}(H, G/H)$$

the image of β being the group of homomorphisms of H into G/H which extend to $\mathrm{Hom}(G, G/H)$ and the kernel of β being the group of homomorphisms of G into G/H whose kernel contains H. Hence by Lemma 25.2.2 (2), $H = \ker(G, G/H)$ if and only if the kernel of every $f \in \mathrm{Hom}(G, G/H)$ contains H, if and only if 0 is the only element of $\mathrm{Hom}(H, G/H)$ which extends. □

Corollary 25.2.9 *Let H be a pfi subgroup of G. If* $\mathrm{Hom}(H, G/H) = 0$ *then H is a pure trace and a kernel.*

Proposition 25.2.7 suggests consideration of the following classes of groups:

Definition 25.2.10
1. A group G is *pfi–injective* if for every pfi subgroup H of G, every $f \in \mathrm{Hom}(H, G)$ extends to $\mathcal{E}(G)$ and G is *pfi–projective* if for every pfi subgroup H of G, every $f \in \mathrm{Hom}(G, G/H)$ lifts to $\mathcal{E}(G)$.

2. A group G is a *trace group* if for every pfi subgroup H of G, $H = \mathrm{tr}(H, G)$, and G is a *kernel group* if for every pfi subgroup H of G, $H = \ker(G, G/H)$.

   Note that in particular, if G is a trace [kernel] group then every kernel [pure trace] is a pure trace [kernel].

Proposition 25.2.7 (1) shows that pfi–injective groups are trace groups and Proposition 25.2.7 (2) shows that pfi-projective groups are kernel groups.

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Example 25.2.11 1. Let G be a completely decomposable group, say $G = \oplus_{\sigma \in T} G_\sigma$, where T is a set of types and each G_σ is a homogeneous completely decomposable group of type σ. For each subset $S \subseteq T$, let $S' = \{\tau \in T : \exists \sigma \in S \text{ such that } \sigma \le \tau\}$ and let $T' = T \setminus S'$. Let $G_{S'} = \oplus_{\tau \in S'} G_\tau$ and $G_{T'} = \oplus_{\rho \in T'} G_\rho$.

Then $G = G_{S'} \oplus G_{T'}$, $G_{S'} = \mathrm{tr}(G_{S'}, G)$, $G_{T'} = \ker(G, G_{S'})$ and every pfi subgroup of G has this form for some choice of $S \subseteq T$. Furthermore, $\mathrm{Hom}(G_{S'}, G_{T'}) = \mathrm{Hom}(G_{T'}, G_{S'}) = 0$.

It follows that completely decomposable groups are all pfi–injective and pfi–projective, and every pfi subgroup is a pure trace and a kernel.

2. For an example of a trace and kernel group which is not pfi–injective or pfi–projective, let σ be the ring type which is divisible by a prime p if and only if $p \equiv 1 \pmod 4$, τ the ring type which is divisible by a prime p if and only if $p \equiv 3 \pmod 4$ and ρ the ring type which is divisible by a prime p if and only if $p = 2$. Let G be the rank 2 group generated by a of type σ, b of type τ such that $a + b$ has type ρ. Then $\mathcal{E}(G) \cong \mathbb{Z}$ so every pure subgroup is pfi. The pfi subgroup $\langle a \rangle_*$ has endomorphisms that do not extend to G and $G/\langle a \rangle_*$ has endomorphisms that do not lift to G, so G is neither pfi–injective nor pfi–projective.

   On the other hand, every proper pfi subgroup of G has rank 1 so G is a trace group. Similarly, every proper factor group by a pfi subgroup is rank 1 so G is a kernel group.

3. For examples of groups G containing pfi subgroups which are not pure traces or kernels, let G be an $\aleph_1$–free group of cardinality $\aleph_1$ with endomorphism ring $\mathbb{Z}$. Such groups are constructed for example in [4]. Clearly every subgroup of G is fully invariant. Let H be a pure subgroup of finite rank, so H is free, but not every $f \in \mathrm{Hom}(H, G)$ maps H into H, so H is not a pure trace. Similarly, not every $f \in \mathrm{Hom}(G, G/H)$ has H in its kernel, so H is not a kernel.

We consider now the relationship between groups H and K for which $\mathrm{tr}(H, G) = \ker(G, K)$ for some group G.

Proposition 25.2.12 *Let G be a group.*

1. *For any group K, let*

$$\mathcal{L}_K = \{L : L \text{ is a pfi subgroup of } G \text{ and } \mathrm{Hom}(L, K) = 0\}$$

   and let $H = \oplus_{L \in \mathcal{L}_K} L$. Then $\mathrm{tr}(H, G) \subseteq \ker(G, K)$ with equality if and only if $\ker(G, K)$ is a pure trace.

2. *For any group H, let*

$$\mathcal{M}_H = \{G/M : M \text{ is a pfi subgroup of } G \text{ and } \mathrm{Hom}(H, M) = 0\}.$$

   Let $K = \oplus_{M \in \mathcal{M}_H} M$. Then $\ker(G, K) \subseteq \mathrm{tr}(H, G)$ with equality if and only if $\mathrm{tr}(H, G)$ is a kernel.

Proof (1) Let $a \in \mathrm{tr}(H, G)$, say $na = \sum b_i f_i$ for some integer n, some finite set $b_i \in H$ and some finite set $f_i \in \mathrm{Hom}(H, G)$. For all $g \in \mathrm{Hom}(G, K)$, $f_i g \in \mathrm{Hom}(\oplus_{L \in \mathcal{L}_K} L, K) = 0$, so each $b_i f_i g = 0$. Hence $nag = 0$ so $ag = 0$ and hence $a \in \ker(G, K)$.

Now suppose $\ker(G, K) = \mathrm{tr}(M, G)$ for some group M. By Lemma 25.2.2, we may assume that M is a pfi subgroup of G, and we have just shown that $\mathrm{tr}(H, G) \subseteq \mathrm{tr}(M, G)$. Since $\mathrm{tr}(M, G) = \ker(G, K)$, for all $m \in M$ and $f \in \mathrm{Hom}(G, K)$, $mf = 0$. Hence $M \in \mathcal{L}_K$ so $\mathrm{tr}(M, G) \subseteq \mathrm{tr}(H, G)$.

(2) The proof is similar. □

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Next, we show that quasi–isomorphism preserves pure trace and kernel.

Proposition 25.2.13 *Let G and G' be groups and n an integer with $nG \subseteq G' \subseteq G$. Let H be pfi in G and let $H' = H \cap G'$. Then*

1. *H' is pfi in G'.*
2. *If $H = \mathrm{tr}(H,\ G)$ then $H' = \mathrm{tr}(H',\ G')$.*
3. *If $H = \ker(G,\ G/H)$ then $H' = \ker(G',\ G'/H')$.*
4. *If every $f \in \mathrm{Hom}(H,\ G)$ extends to G, then every $f \in \mathrm{Hom}(H',\ G')$ extends to G'.*
5. *If every $f \in \mathrm{Hom}(G,\ G/H)$ lifts to G, then every $f \in \mathrm{Hom}(G',\ G'/H')$ lifts to G'.*

Proof (1) Let $mx \in H'$ for some integer m and some $x \in G'$. Then $x \in G$ and $mx \in H$ so $x \in H \cap G' = H'$. Hence H' is pure in G'.

Let $f \in \mathcal{E}(G')$, so $f|_{nG} \in \mathrm{Hom}(nG,\ G)$. Let $x \in H'$ so $nx \in nG \cap H$. Then $x(nf) = n(xf) \in H$, so $xf \in H \cap G' = H'$. Hence H' is fully invariant in G'.

(2) We know $H' \subseteq \mathrm{tr}(H',\ G')$, so let $a \in \mathrm{tr}(H',\ G')$, say $ma = \sum b_i f_i$ for some integer m and some finite sets $b_i \in H'$ and $f_i \in \mathrm{Hom}(H',\ G')$. Then for all i, $nf_i \in \mathrm{Hom}(H',\ G)$ and nf_i maps H into H. Thus nf_i maps H' into H' so $ma \in H'$. By purity, $a \in H'$ as required.

(3) We know $\ker(G',\ G'/H') \subseteq H'$. Let $f \in \mathrm{Hom}(G',\ G'/H')$. Then $nf \in \mathrm{Hom}(G,\ G'/H')$ so has H in its kernel. Hence f has H' in its kernel, so $H' \subseteq \ker(G',\ G'/H')$.

(4) Let $f \in \mathrm{Hom}(H',\ G')$. Then $nf \in \mathrm{Hom}(H,\ G)$ extends to G, so by purity, f extends to G'.

(5) Let $f \in \mathrm{Hom}(G',\ G'/H')$. Then $nf \in \mathrm{Hom}(G,\ G/H)$ lifts to G, so by purity, f lifts to G'. □

Corollary 25.2.14 *If $G \doteqdot G'$, then G is pfi–injective if and only if G' is pfi–injective; G is pfi–projective if and only if G' is pfi–projective; G is a trace group if and only if G' is a trace group; and G is a kernel group if and only if G' is a kernel group.*

In particular, almost completely decomposable groups ([5]) are pfi–injective and pfi–projective. But a stronger result is also true.

Recall from [6] that a *bcd group X* is a group containing a completely decomposable subgroup A such that $eX \subseteq A$ for some positive integer e. This implies of course that A and X are quasi–isomorphic. Hence it follows that

Corollary 25.2.15 *bcd groups are pfi–injective, pfi–projective, trace groups and kernel groups.*

A number of questions arise from the results of this section, for which we have only incomplete answers.

Problems

1. Characterise trace groups, and kernel groups, and their intersection. In particular, are Butler groups trace groups and kernel groups?
2. Characterise groups which are projective [injective] with respect to short exact sequences $H \rightarrowtail G \twoheadrightarrow J$ where H is pfi.
3. Find the homological properties of short exact sequences $H \rightarrowtail G \twoheadrightarrow G/H$ for which H is a pure trace and G/H a kernel.

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4. Let $E : H \rightarrowtail G \overset{\eta}{\twoheadrightarrow} J$ be a short exact sequence with H a pure trace in G. Then E is *trace balanced* if for all groups K, $\mathrm{tr}(K, G)\eta = \mathrm{tr}(K, J)$ and E is *kernel cobalanced* if H is a kernel in G and for all groups K, the natural map

$$a + \ker(H, K) \mapsto a + \ker(G, K) : \frac{H}{\ker(H, K)} \to \frac{G}{\ker(G, K)}$$

is monic. (The map is well defined since $\ker(H, K) \subseteq \ker(G, K)$.)

Find the homological properties of trace balanced and kernel co–balanced sequences.

25.3 Traces and Kernels of cd Groups

Most of the results in this section can be extended to modules over an integral domain. Once again it is convenient to state them only for groups.

Important classes of pfi subgroups are those which are traces and kernels of completely decomposable groups. Let G be a group. Every minimal pure subgroup of G and every factor group of G by a maximal pure subgroup is isomorphic to a unique type.

For any $0 \neq a \in G$, $\langle a \rangle_*$, the pure subgroup of G generated by a, is isomorphic to $\mathrm{type}(a) = \{r \in \mathbb{Q} : ra \in G\}$, [5, Definition 1.2.17]. The *typeset* $\mathcal{T}_G$ *of* G is the set of types of elements of G.

Similarly, for any maximal pure subgroup A of G, A is the kernel of an epimorphism $f : G \to \tau$ for some type τ. We define $\mathrm{cotype}(A) := \tau = \mathrm{type}(G/A)$. The *cotypeset* $\mathrm{co}{-}\mathcal{T}_G$ *of* G is the set of cotypes of maximal pure subgroups of G.

For each non–empty set S of types, let $\bigoplus S$ (pronounced "Oplus S") denote $\oplus_{\sigma \in S}\sigma$. $\bigoplus S$ is called a *basic completely decomposable group*.

The following well–known examples of pfi subgroups of a group G are defined in and their properties studied in [5, Section 2.3].

Definition 25.3.1 For each type σ, and for each subset S of types,

1. $G(\sigma) = \mathrm{tr}(\sigma, G) = \{x \in G : \mathrm{type}\, x \geq \sigma\}$.

2. $G[\sigma] = \ker(G, \sigma) = \cap\{A : A \text{ a maximal pure subgroup of } G \text{ with } \mathrm{type}(G/A) \leq \sigma\}$.

3. $G(S) = \mathrm{tr}(\bigoplus S, G)$, with $G(\emptyset) = 0$.

4. $G[S] = \ker(G, \bigoplus S)$ with $G[\emptyset] = G$.

The following lemma follows immediately.

Lemma 25.3.2 *Let G be a group, $S \subseteq S'$ sets of types. Then*

1. *$G(S)$ and $G[S]$ are pfi subgroups of G.*

2. *$G(S) \subseteq G(S')$ and $G[S] \supseteq G[S']$.*

I now consider the question of when $G(S) = G(T)$ or $G[S] = G[T]$ for distinct sets S and T of types.

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Lemma 25.3.3 *Let $\bigoplus S$ and $\bigoplus T$ be basic completely decomposable groups. Let*

$$S' = \{\sigma \in S : \exists \tau \in T \text{ such that } \tau \leq \sigma\},$$
$$T' = \{\tau \in T : \exists \sigma \in S \text{ such that } \sigma \leq \tau\}.$$

Then $\bigoplus S(T) = \bigoplus S'$ and $\bigoplus S / \bigoplus S[T] = \bigoplus T'$.

Proof Let $a \in \bigoplus S(T)$ so that $na = \sum 1_i f_i$ for some positive integer n and some finite set $f_i : \tau_i \to \sigma_i$ where 1_i is the identity of $\tau_i \in T$, $\sigma_i \in S$, the τ_i are distinct, and each $1_i f_i \neq 0$. The last condition implies that $\tau_i \leq \sigma_i$, so that $a \in \bigoplus S(T) \subseteq \bigoplus S'$.

Conversely, let $\sigma \in S'$. Then some $\tau \in T \leq \sigma$. Hence $\sigma \in \bigoplus S(\tau)$, so that $\bigoplus S' \subseteq \bigoplus S(T)$.

Now let $b \in \bigoplus S[T]$, say $b = \sum a_i$ where $a_i \in \sigma_i \in S$ and the sum is finite. Since b is in the kernel of every homomorphism $f : \bigoplus S \to \bigoplus T$, each $\sigma_i \not\leq \tau$ for all $\tau \in T$. Hence $\bigoplus S / \bigoplus S[T] \subseteq \bigoplus T'$. Conversely, if $a \in \bigoplus T'$, then each $\sigma \in T'$ is in the kernel of every homomorphism into $\bigoplus T$. Hence $\bigoplus T' \subseteq S / \bigoplus S[T]$. □

The statement of Lemma 25.3.3 suggests defining some relations on sets of types that make sense for arbitrary posets.

Let $\langle P, \leq \rangle$ be a poset and let $\mathcal{P}$ be the set of subsets of P. For all $S, T \in \mathcal{P}$, we say that

1. S is *initial* in T, denoted $S \preccurlyeq T$ if for all $t \in T$ there exists $s \in S$ such that $s \leq t$;
2. S is *final* in T, denoted $S \succcurlyeq T$ if for all $t \in T$ there exists $s \in S$ such that $t \leq s$.

It is easy to check that these relations are partial orders on $\mathcal{P}$ and hence induce equivalence relations on $\mathcal{P}$. We say that S is *initially equivalent* to T, denoted $S \equiv_i T$, if $S \preccurlyeq T$ and $T \preccurlyeq S$; and we say that S is *finally equivalent* to T, denoted $S \equiv_f T$ if $S \succcurlyeq T$ and $T \succcurlyeq S$.

An obvious example, which is the reason for the definition, is to take as $\langle P, \leq \rangle$ the poset $\mathcal{T}$ of all types with the usual order relation. Then it is clear that for any group G with typeset $\mathcal{T}_G$ and cotypeset co$-\mathcal{T}_G$, $\mathcal{T}_G \preccurlyeq$ co$-\mathcal{T}_G$ and co$-\mathcal{T}_G \succcurlyeq \mathcal{T}_G$.

The relations can be used to describe conditions on typesets S and T which ensure that for any group G, $G(S) = G(T)$ or $G[S] = G[T]$.

Lemma 25.3.4 *Let $S, S' \in \mathcal{T}_G$ and $T, T' \in$ co$-\mathcal{T}_G$. Then*

1. *If $S' \preccurlyeq S$ then $G(S) \subseteq G(S')$.*
2. *If $T \succcurlyeq T'$ then $G[T] \subseteq G[T']$.*

Proof (1) If $S' \preccurlyeq S$ then for all $\sigma \in S$, there exists $\tau \in S'$ such that $\tau \leq \sigma$. Hence $G(\sigma) \subseteq G(\tau)$ and consequently $G(S) \subseteq G(S')$.

(2) Suppose $T \preccurlyeq T'$ and let $a \in G[T]$, so for all $\lambda \in T$, a is in the kernel of all $f \in \mathrm{Hom}(G, \lambda)$. Let $\rho \in T'$, so there exists $\lambda \in T$ with $\rho \leq \lambda$. Hence a is in the kernel of all $g \in \mathrm{Hom}(G, \rho)$, so $a \in G[T']$. Thus $G[T] \subseteq G[T']$. □

Corollary 25.3.5 *Let $S, S' \in \mathcal{T}_G$ and $T, T' \in$ co$-\mathcal{T}_G$. Then*

1. *If $S \equiv_i S'$ then $G(S) = G(S')$.*
2. *If $T \equiv_f T'$ then $G[T] = G[T']$.*

To see that the converse of Lemma 25.3.4 is false, let σ be the ring type which is divisible by a prime p if and only if $p \equiv 1 \pmod 4$, τ the ring type which is divisible by a prime p if and only if $p \equiv 3 \pmod 4$ and ρ the ring type which is divisible by a prime p if and only if $p = 2$. Let G be the rank 2 group generated by a of type σ, b of type τ with $a + b$ of type ρ. Then $G = G(\sigma, \tau) = G(\sigma, \tau, \rho)$, but $\{\sigma, \tau, \rho\}$ is not initial in $\{\sigma, \tau\}$.

The following Proposition settles the question of when $G(S) = G[T]$ for some $S \subseteq \mathcal{T}_G$ and some $T \subseteq$ co$-\mathcal{T}_G$. The proof is similar to that of Proposition 25.2.12 so is omitted.

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Proposition 25.3.6 1. *Let $T \subseteq \text{co}-\mathcal{T}_G$ and let*

$$S = \{\sigma \in \mathcal{T}_G : \textit{ for all } \tau \in T,\ \sigma \not\leq \tau\}.$$

Then $G(S) \subseteq G[T]$ and if there exists $S' \subseteq \mathcal{T}_G$ with $G[T] = G(S')$ then $G[T] = G(S)$.

2. *Let $S \subseteq \mathcal{T}_G$ and let*

$$T = \{\tau \in \text{co}-\mathcal{T}_G : \textit{ for all } \sigma \not\leq \tau\}.$$

Then $G[T] \subseteq G(S)$ and if there exists $T' \subseteq \text{co}-\mathcal{T}_G$ with $G(S) = G[T']$, then $G(S) = G[T]$. □

We now consider the analogues in the context of this Section of Definition 25.2.10.

Definition 25.3.7 1. A group G is called *cd–injective* if for every set S of types, every $f \in \text{Hom}(G(S),\ G)$ extends to $\mathcal{E}(G)$; and G is *cd–projective* if every $f \in \text{Hom}(G,\ G/G[S])$ lifts to $\mathcal{E}(G)$;

2. A group G is *flexible* if every pfi subgroup is $G(S)$ for some set S of types; and *co–flexible* if every pfi subgroup is $G[S]$.

Remark 1. The name ‘flexible’ was chosen to indicate that these groups are far from rigid, in the sense that they have enough endomorphisms to ensure that their only pfi subgroups are those they are forced to have by reason of their elementary invariants.

2. Not every trace group is flexible or co–flexible, for example $\hat{\mathbb{Z}}_p \oplus \mathbb{Z}_p$ is a trace group that is not flexible. On the other hand, it was shown in the Examples of Section 25.2 that completely decomposable groups are flexible and co–flexible.

3. In particular, a flexible and co–flexible group G must satisfy the conditions of Proposition 25.3.6.

The classes of flexible and co–flexible groups have properties similar to those of trace and kernel groups. For example, arguments similar to those of Proposition 25.2.13 show that quasi–isomorphism preserves [co–]flexibility.

Proposition 25.3.8 *If $G \doteq G'$, then G is cd–injective, cd–projective, flexible or co–flexible if and only if G' has the same properties.*

In particular, it follows that:

Corollary 25.3.9 *bcd groups are flexible and co–flexible.*

As in Section 25.2, several unanswered questions arise from the results of Section 25.3.

Problems

1. For which groups is every pure trace a cd–trace. or every kernel a cd–kernel? In particular, is this true for Butler groups?

2. Characterise the flexible and the co–flexible groups, and their intersection.

3. Characterise the groups which are cd–projective [cd–injective] with respect to short exact sequences $H \rightarrowtail G \twoheadrightarrow J$ where H is pfi.

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4. Find the homological properties of short exact sequences $H \rightarrowtail G \twoheadrightarrow G/H$ for which H is a cd–trace and G/H a cd–kernel.

5. Let $E: \ H \rightarrowtail G \overset{\eta}{\twoheadrightarrow} J$ be a short exact sequence with H a cd–trace in G. Then E is *cd–balanced* if for all sets S of types $G(S)\eta = J(S)$ and E is *cd–cobalanced* if H is a cd–kernel in G and for all sets of types T, the natural map

$$a + H[T] \mapsto a + G[T] : \ \frac{H}{H[T])} \to \frac{G}{G[T]}$$

is monic. In the context of Butler groups, these sequences have been extensively studied under the names balanced and cobalanced, [2, 8]. Find the homological properties of cd–balanced and cd–cobalanced sequences.

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Chapter 26

Compressible and Related Modules

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Abstract Let R be a ring with identity. A non-zero unital right R-module M is compressible if M embeds in each of its non-zero submodules and is monoform if every non-zero homomorphism from a submodule N of M to M is a monomorphism. Compressible and monoform modules are investigated in the case of general rings and in the case of fully bounded rings. Relationships between the classes of compressible and monoform modules and other module classes are obtained.

26.1 Introduction

Let R be a ring with identity. (In this note all rings will have an identity element and all modules will be unital right modules.) A non-zero R-module M is called *compressible* provided for each non-zero submodule N of M there exists a monomorphism $f : M \to N$. (Note that Jategaonkar [17] calls a module M compressible if for each *essential submodule* N of M there exists a monomorphism $f : M \to N$.) For example, if R is a (not necessarily commutative) domain then the right R-module R is compressible. In [9, Proposition 1.10], Goldie proves that if R is a semiprime right Noetherian ring then any uniform right ideal of R is a compressible right R-module. Earlier, in 1964 in fact, Goldie [8] defines, for a right Noetherian ring R and a finitely generated right R-module M, M is a *basic* module if M is a nonsingular compressible module and he proves that, for each right R-module M, either M is singular or M contains a basic right R-module (see [8, Theorem 3.6]).

The other key notion of interest here is that of a monoform module. A non-zero R-module M is called *monoform* if, for each submodule N of M, every non-zero homomorphism $f : N \to M$ is a monomorphism. Monoform modules arise in the study of rings and modules with Krull dimension (see [14]).

Compressible and monoform modules also feature in the work of Zelmanowitz and others in their study of generalizations of the Density Theorem of Jacobson (see [32], [33], [34] and references therein). Note that some authors call a compressible monoform module *quasi-simple* (see, for example, [18]). Let R be a ring and let M be a right R-module. The injective hull of M will be denoted by $E(M)$. Let M^* denote the sum of all submodules of $E(M)$ of the form $f(M)$, where f is an endomorphism of $E(M)$. Note that clearly $g(M^*) \subseteq M^*$ for every endomorphism g of $E(M)$

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and hence M^* is a quasi-injective module, i.e., for every submodule N of M^* and homomorphism $h : N \to M^*$, h can be lifted to M^* (see, for example, [3, Section 2.1]). Zelmanowitz proves that the module M is monoform if and only if the endomorphism ring D of M^* is a division ring. Moreover, if M is compressible and monoform then the endomorphism ring S of M is a right Ore domain with right quotient division ring D (see [33, Proposition 1.2] or [14, Corollary 2.7]). He further proves that a ring R has a faithful compressible monoform right module if and only if there exist a division ring D, a left D-, right R-bimodule V and a faithful right R-module M such that $V = DM$ and for all positive integers t and elements $v_j (1 \le j \le t)$ of V which are linearly independent over D, there exists a non-zero element a in D with the property that for all $m_j (1 \le j \le t)$ in M, there exists an element r in R such that $av_j = m_j r (1 \le j \le t)$.

In this survey we shall be interested in investigating compressible, monoform and some related modules over general rings and over particular classes of rings. If R is a commutative ring then an R-module M is compressible if and only if M is isomorphic to an R-module of the form A/P, where $P \subset A$ are ideals of R with P prime, and in this case M is monoform (see Theorem 26.2.10 and Corollary 26.3.5). This result extends to the case of a nonsingular module M over a right Noetherian ring R (or, more generally, a ring R with right Krull dimension) (Corollary 26.4.11). For any ring R, a nonsingular module is uniform if and only if it is monoform (Theorem 26.3.2). If R is a semiprime right Goldie ring then every finitely generated nonsingular monoform right R-module is compressible if and only if R is a left Goldie ring (Theorem 26.4.14). In the last section we consider modules over right fully bounded rings. We prove that if R is a right fully bounded ring such that R/P is a right and left Goldie ring for every prime ideal P then a right R-module M is compressible if and only if M is isomorphic to a non-zero submodule of a finitely generated uniform (equivalently, monoform) prime right R-module (Theorem 26.5.9).

26.2 Prime and Compressible Modules

Let R be any ring. A non-zero right R-module M is called *prime* if, whenever N is a non-zero submodule of M and A is an ideal of R such that $NA = 0$, then $MA = 0$. For example, an ideal P of a ring R is prime if and only if the right (or left) R-module R/P is prime. For any ring R and simple right R-module U any direct sum of isomorphic copies of U is a prime right R-module. If R is a simple ring then every non-zero right (or left) R-module is clearly prime. We shall show first that many rings, beside simple rings, have a good supply of prime modules. We shall be interested in the following two properties of a ring R:

$(P1)$ For every proper ideal A of R there exist a positive integer n and prime ideals $P_i (1 \le i \le n)$ of R, each containing A, such that $P_1 \dots P_n \subseteq A$.

$(P2)$ R satisfies the ascending chain condition on prime ideals.

Clearly simple rings satisfy $(P1)$ and $(P2)$. These properties are of interest here because of the following result.

Proposition 26.2.1 *(See [26, Lemma 2].) Let R be a ring such that R satisfies $(P1)$ and $(P2)$. Then every non-zero right (or left) R-module contains a submodule which is a prime module.*

It is natural to ask which rings satisfy the hypotheses, and hence the conclusion, of Proposition 26.2.1. First we prove a simple lemma.

Lemma 26.2.2 *Let A and B be ideals of a ring R which are both finitely generated right ideals. Then the ideal AB is a finitely generated right ideal.*

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Proof Suppose that $A = a_1R + \ldots + a_mR$ and $B = b_1R + \ldots + b_nR$ for some positive integers m, n and elements $a_i(1 \le i \le m)$ of A and $b_j(1 \le j \le n)$ of B. Note that $Rb_j \subseteq b_1R + \ldots + b_nR$ for each $1 \le j \le n$ and hence AB is generated as a right ideal by the finite set of elements $a_ib_j(1 \le i \le m, 1 \le j \le n)$. □

Proposition 26.2.3 *Let R be a ring such that either*

(a) R satisfies the ascending chain condition on (two-sided) ideals, or

(b) every prime ideal of R is a finitely generated right ideal of R, or

(c) R has right Krull dimension.

Then R satisfies $(P1)$ and $(P2)$ and every non-zero right (or left) R-module contains a submodule which is a prime module.

Proof Suppose first that (a) holds. Then R satisfies $(P1)$ and $(P2)$ by [26, Lemma 1].

Now suppose that R satisfies (b). Suppose that there exists a proper ideal A of R for which $(P1)$ fails. Let $\bigwedge$ denote the collection of such ideals A. Let $B_i(i \in I)$ be any chain in $\bigwedge$ and let $B = \bigcup_i B_i$. Suppose that B does not belong to $\bigwedge$. Clearly B is a proper ideal of R. Thus there exist a positive integer n and prime ideals $P_j(1 \le j \le n)$ of R, each containing B, such that $P_1 \ldots P_n$ is contained in B. By Lemma 26.2.2 $P_1 \ldots P_n$ is a finitely generated right ideal of R and hence $P_1 \ldots P_n$ is contained in B_i for some i in I, a contradiction. It follows that B belongs to $\bigwedge$. By Zorn's Lemma, $\bigwedge$ contains a maximal member C. Clearly C is not a prime ideal of R and hence there exist proper ideals G and H of R, each properly containing C, such that $GH \subseteq C$. Since G and H both contain a finite product of prime ideals, each containing C, then so too does C, a contradiction. Thus R satisfies $(P1)$. Now let $Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \ldots$ be any ascending chain of prime ideals of R. Let Q denote the union of the ideals $Q_k(k \ge 1)$. Since R satisfies $(P1)$ it follows that there exist a positive integer t and prime ideals $V_j(1 \le j \le t)$, each containing Q, such that $V_1 \ldots V_t \subseteq Q$. By Lemma 26.2.2, $V_1 \ldots V_t$ is a finitely generated right ideal of R and hence is contained in Q_k for some $k > 1$. It is easy to check now that $Q_k = V_j$ for some $1 \le j \le t$ and hence $Q_k = Q_{k+1} = \ldots$. Thus R satisfies $(P2)$.

If (c) holds then R satisfies $(P1)$ and $(P2)$ by [14, Theorems 7.1 and 7.4]. In any case Proposition 1.1 applies and the result is proved. □

Note that we are not clear about the relationship (if any) between (a) and (b) in Proposition 26.2.3. Note further that it is quite easy to give examples of rings that satisfy $(P1)$ and $(P2)$ but not (a), (b) or (c) in Proposition 26.2.3. For example, let S denote any commutative Noetherian domain and let Q denote the field of fractions of S such that the S-module Q does not have Krull dimension (see [22] or [28]). Let R denote the subring of the ring of 2×2 upper triangular matrices over Q whose $(1, 1)$ and $(2, 2)$ entries belong to S. Let N denote the prime radical of R. Note that N consists of all matrices in R with zero $(1, 1)$ and $(2, 2)$ entries. Let P denote the prime ideal of R consisting of all matrices with zero $(1, 1)$ entry. Because Q is not a finitely generated S-module, R does not satisfy the ascending chain condition on ideals and the prime ideal P is not a finitely generated right ideal of R. Moreover, R does not have right Krull dimension. On the other hand, because Q is a divisible S-module, if A is any proper ideal of R then either $N \subseteq A$ or $A \subseteq N$. In either case there is a finite collection of prime ideals containing A whose product is contained in A.

For any ring R and right R-module M, the singular submodule of M, denoted by $Z(M)$, is the collection of elements m in M such that $mE = 0$ for some essential right ideal E of R. The module M is called *singular* if $Z(M) = M$ and is called *nonsingular* if $Z(M) = 0$.

Proposition 26.2.4 *Let R be a prime ring. Then*
(i) every non-zero free right R-module is prime, and
(ii) every non-zero nonsingular right R-module is prime.

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Proof *(i)* Let F be any non-zero free right R-module with basis $\{x_i : i \in I\}$. Let N be a non-zero submodule of F and let A be an ideal of R such that $NA = 0$. Let m be any non-zero element of N. Then the ith component m_i of m is non-zero for some i in I. It follows that $m_i A = 0$ in the ring R and hence $A = 0$. In this case $FA = 0$.

(ii) Let M be any nonsingular right R-module. Suppose that $xB = 0$ for some element x of M and non-zero ideal B of R. Note that, for any right ideal C of R, $C \cap B = 0$ implies that $CB = 0$ and hence $C = 0$. Thus B is an essential right ideal of R. It follows that $x = 0$. Therefore M is a prime module. □

Let R be any ring and let M be an R-module. If N is a submodule of M then the collection of submodules L of M such that $N \cap L = 0$ has a maximal member K (say) by Zorn's Lemma, and in this case we call K a *complement* of N (in M). A submodule G of M is called a *complement* (in M) if there exists a submodule H of M such that G is a complement of H in M.

Basic properties of prime modules can be found in [24]. Note in particular that any non-zero submodule of a prime module is also prime.

Proposition 26.2.5 *Let R be any ring. Then the following statements are equivalent for a non-zero right R-module M.*

(i) M is prime.

(ii) Every non-zero 2-generated submodule of M is prime.

(iii) M/K is a prime module for every proper complement K in M.

Proof *(i) ⇒ (ii)* Clear.

(ii) ⇒ (i) Let N be a non-zero submodule of M and let A be an ideal of R such that $NA = 0$. Let m be any non-zero element of N. Let x be any element of M and let $L = mR + xR$. By hypothesis, L is a prime module. It follows that because $mA = 0$ we have $LA = 0$ and in particular $xA = 0$. This implies that $MA = 0$. Hence M is prime.

(i) ⇒ (iii) Let H be a submodule of M properly containing K and let B be an ideal of R such that $(H/K)B = 0$, i.e., $HB \subseteq K$. There exists a submodule G of M such that K is a complement of G in M. Then $H \cap G$ is non-zero and $(H \cap G)B \subseteq K \cap G = 0$. By hypothesis, $MB = 0$ and hence $(M/K)B = 0$. It follows that M/K is a prime module.

(iii) ⇒ (i) Clear because 0 is a complement in M. □

For any right R-module X and non-empty subset Y of X, the *annihilator* of Y in R will be denoted by $\operatorname{ann}_R(Y)$, i.e., $\operatorname{ann}_R(Y)$ is the set of elements r in R such that $yr = 0$ for all y in Y. In particular, if $Y = \{y\}$ then $\operatorname{ann}_R(Y)$ will be denoted by $\operatorname{ann}_R(y)$. Note that a non-zero right R-module M is prime if and only if $\operatorname{ann}_R(N) = \operatorname{ann}_R(M)$ for every non-zero submodule N of M. A right R-module M is called *fully faithful* if every non-zero submodule of M is faithful, i.e., $\operatorname{ann}_R(N) = 0$ for every non-zero submodule N of M. The next result is [24, Proposition 1.1].

Proposition 26.2.6 *Let R be any ring and let M be a non-zero right R-module with annihilator P. Then M is a prime right R-module if and only if M is a fully faithful right (R/P)-module, and in this case P is a prime ideal of R.*

Proof Straightforward. □

Corollary 26.2.7 *Let R be a commutative ring. A non-zero R-module M is prime if and only if the annihilator of M is a prime ideal P of R and M is a torsion-free module over the domain R/P.*

Proof By Proposition 26.2.6. □

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Now we consider compressible modules. For any ring R, note that any non-zero submodule of a compressible right R-module is also a compressible module. Note further that a right R-module M is simple if and only if it is a compressible module with non-zero socle. It follows that for any ring R, non-zero homomorphic images of compressible modules need not be compressible and direct sums of compressible modules need not be compressible. The relationship between compressible and prime modules is given in the next result.

Proposition 26.2.8 *For any ring R, every compressible right R-module is prime.*

Proof Let N be a non-zero submodule of a compressible module M and let A be an ideal of R such that $NA = 0$. Then there exists a monomorphism $f : M \to N$, so that $f(MA) = f(M)A = 0$ and hence $MA = 0$. Thus M is a prime module. □

If R is any ring and U is a simple right R-module then the module $U \oplus U$ is a prime module which is not compressible. For commutative rings we have a partial converse of Proposition 26.2.8 which is contained in the next result. Recall that, for any ring R, a right R-module M is called *uniform* provided M is non-zero and the intersection of any two non-zero submodules of M is non-zero.

Lemma 26.2.9 *Let R be a commutative ring. Then a finitely generated non-zero R-module M is compressible if and only if M is a uniform prime module.*

Proof Without loss of generality, M is a faithful R-module. Suppose first that M is compressible. By Proposition 26.2.8, M is prime and, by Corollary 26.2.7, R is a domain. Let m be any non-zero element of M. Because M is prime, the submodule mR is isomorphic to R and hence is a uniform R-module. But M embeds in mR, so that M is also a uniform R-module.

Conversely, suppose that M is a uniform prime R-module. In this case, M is a torsion-free module over the commutative domain R. There exist a positive integer k and elements $m_i (1 \leq i \leq k)$ of M such that $M = m_1R + \ldots + m_kR$. Let N be any non-zero submodule of M. Because M is uniform it follows that M/N is a torsion R-module. For each $1 \leq i \leq k$, there exists a non-zero element c_i in R such that m_ic_i belongs to N. Let $c = c_1 \ldots c_k$. Then c is a non-zero element of R and Mc is contained in N. Define the mapping $f : M \to N$ by $f(m) = mc$ for all m in M. It is easy to check that f is a monomorphism. □

Theorem 26.2.10 *Let R be a commutative ring. Then the following statements are equivalent for an R-module M.*

(i) M is compressible.

(ii) M is isomorphic to an R-module of the form A/P for some prime ideal P of R and ideal A of R properly containing P.

(iii) M is isomorphic to a non-zero submodule of a finitely generated uniform prime R-module.

Proof *(i) ⇒ (ii)* Let m be any non-zero element of M. Then mR is compressible and, by Lemma 1.9, mR is a uniform prime R-module. Because M is compressible, M embeds in mR. But $mR \cong R/P$ for some prime ideal P of R. Thus M embeds in the R-module R/P, as required.

(ii) ⇒ (iii) Note that A/P is a submodule of the finitely generated uniform prime R-module R/P.

(iii) ⇒ (i) Suppose that M is isomorphic to a non-zero submodule of a finitely generated uniform prime module M'. By Lemma 26.2.9, M' is compressible and hence M is also compressible. □

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It is clear that if M is a compressible module which contains a uniform submodule then M is uniform. However, in general compressible modules need not be uniform and in fact need not have finite uniform dimension. Recall that a module has *finite uniform dimension* if it does not contain an infinite direct sum of non-zero submodules. Note the following well known result which is proved for completeness.

Lemma 26.2.11 *Let R be any ring and let M be a uniform submodule of a free right R-module. Then M is isomorphic to a right ideal of R.*

Proof Suppose that M is a submodule of a free module F. There exist free submodules F' and F'' of F such that F' is finitely generated, $F = F' \oplus F''$ and $M \cap F'$ is non-zero. It follows that $M \cap F'' = 0$ and hence M embeds in F'. Thus without loss of generality we can suppose that $F = R^n$ for some positive integer n. Clearly, there exist projections $p_i : F \to R (1 \leq i \leq n)$ such that $M \cap \ker p_1 \cap \ldots \cap \ker p_n = 0$. Since M is uniform it follows that $M \cap \ker p_i = 0$ for some i and hence M is isomorphic to a right ideal of R. □

Theorem 26.2.12 *Let R be a domain which is not right Ore. Then every non-zero countably generated free right R-module is compressible.*

Proof Let F be a free right R-module with countable basis $\{x_n : n \geq 1\}$. Let N be any non-zero submodule of F. If N has finite uniform dimension then F contains a uniform submodule X by [23, Lemma 2.2.7]. Now X embeds in R by Lemma 26.2.11 and it follows that R is a right Ore domain, a contradiction. Thus N contains an infinite direct sum $Y_1 \oplus Y_2 \oplus Y_3 \oplus \ldots$ of non-zero submodules $Y_n (n \geq 1)$. For each $n \geq 1$, choose any non-zero element y_n in Y_n. Define a mapping $f : F \to N$ by $f(x_n) = y_n$ for each $n \geq 1$. It is easy to check that f is a monomorphism. Thus F is compressible. □

26.3 Monoform Modules

Let R be any ring. Recall that a non-zero R-module M is monoform provided every non-zero homomorphism from a non-zero submodule N to M is a monomorphism. It is clear that every non-zero submodule of a monoform module is also monoform. Note the following result which should be compared with Proposition 26.2.5.

Proposition 26.3.1 *Let R be any ring. Then a non-zero right R-module M is monoform if and only if every non-zero 3-generated submodule of M is monoform.*

Proof The necessity is clear. Conversely, suppose that every non-zero 3-generated submodule of M is monoform. Let N be any non-zero submodule of M and let $f : N \to M$ be a homomorphism with non-zero kernel. Let x be any non-zero element of N such that $f(x) = 0$. Let y be any element of N and let $z = f(y)$. Consider the submodule $L = xR + yR + zR$ of M. Let $H = xR + yR$. Then the restriction g of f to H is a homomorphism from H to L. Moreover, x belongs to the kernel of g. By hypothesis L is monoform and hence $g = 0$. In particular, $f(y) = g(y) = 0$. It follows that $f = 0$. Thus M is monoform. □

It is also clear that every monoform module is uniform. However, the converse is false. Let R be any commutative ring which contains a maximal ideal P which is a principal but not idempotent ideal and let U denote the R-module R/P^2. Then U is a uniform R-module such that there exists an isomorphism $U/PU \to U$, so that U is not monoform. For nonsingular modules we have the following result.

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Theorem 26.3.2 *Let R be any ring. Then a nonsingular right R-module M is monoform if and only if M is uniform.*

Proof The necessity is clear. Conversely, let M be a uniform module. Let N be any non-zero submodule of M and let $f : N \to M$ be a non-zero homomorphism with kernel K. Then N/K is isomorphic to the non-zero submodule $f(N)$ of M, so that N/K is a nonsingular module. Since M is uniform it follows that $K = 0$ and hence that f is a monomorphism. □

If R is a commutative ring then we have the following fact.

Theorem 26.3.3 *Let R be a commutative ring. Then an R-module M is monoform if and only if M is a uniform prime module.*

Proof Let M be any monoform R-module. Let N be a non-zero submodule of M and let a be an element of R such that $Na = 0$. Define the mapping $f : M \to M$ by $f(m) = ma$ for all m in M. Clearly f is a homomorphism such that $f(N) = 0$. By hypothesis, $f = 0$, i.e., $Ma = 0$. It follows that M is a prime module. Also it is clear that M is uniform. Conversely, suppose that M is a uniform prime R-module. Without loss of generality we can suppose that M is faithful. By Corollary 26.2.7, R is a domain and M is a torsion-free (i.e., nonsingular) R-module. Then M is monoform by Theorem 26.3.2. □

Corollary 26.3.4 *Let R be a commutative ring. An R-module M is compressible if and only if M is isomorphic to a non-zero submodule of a finitely generated monoform R-module.*

Proof By Theorems 26.2.10 and 26.3.3. □

Corollary 26.3.5 *Let R be a commutative ring. Then every compressible R-module is monoform.*

Proof By Corollary 26.3.4. □

Note that the converses of Corollaries 26.3.4 and 26.3.5 are false in general. If R is a commutative domain which is not a field and if Q is the field of fractions of R then the R-module Q is a uniform prime (and hence monoform) R-module which is not compressible.

Let R be any ring. The category of right R-modules will be denoted by Mod-R. Recall that, for any right R-module M, the injective hull of M will be denoted by $E(M)$. Following [6], we shall call the right R-module M *cocritical* provided that M is non-zero and that there exists an hereditary torsion theory τ on Mod-R such that M is τ-torsion-free but M/N is τ-torsion for every non-zero submodule N of M. Cocritical modules are discussed in [6, Section 18]. In [20] a right ideal A of R is called *critical* if the cyclic R-module R/A is cocritical.

Theorem 26.3.6 *(See [6, Proposition 18.2] or [14, Theorem 2.9].) For any ring R, the following statements are equivalent for a right R-module M.*

(i) M is monoform.

(ii) $\mathrm{Hom}_R(M/N, E(M)) = 0$ for every non-zero submodule N of M.

(iii) M is cocritical.

Proof *(i) ⇒ (ii)* Suppose that M is monoform. Let N be submodule of M such that there exists a non-zero homomorphism $f : M/N \to E(M)$. Let $p : M \to M/N$ denote the canonical projection. Let L denote the set of elements m in M such that $fp(m)$ belongs to M. Then L is a submodule of M and the restriction $g : L \to M$ of fp is a homomorphism. Note that $fp(M) = f(M/N)$ is a non-zero submodule of $E(M)$ and hence $M \cap fp(M)$ is non-zero. It follows that g is a non-zero homomorphism. By hypothesis, g is a monomorphism. But $fp(N) = 0$ implies that N is contained in L. Moreover, $g(N) = fp(N) = 0$. Thus $N = 0$.

(ii) ⇒ (iii) Let τ denote the hereditary torsion theory cogenerated by $E(M)$ (see [29, p. 139]). By (ii), M/N is τ-torsion for every non-zero submodule N of M and M is τ-torsion-free.

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(iii) ⇒ *(i)* Suppose that M is cocritical with respect to an hereditary torsion theory τ. Let H be a submodule of M such that there exists a non-zero homomorphism $h : H \to M$. If G denotes the kernel of H then H/G is τ-torsion-free. It follows that $G = 0$. Hence M is monoform. □

Combining Theorems 26.3.2 and 26.3.6 we see that any nonsingular uniform module is cocritical and this gives [20, Lemma 3.2] as a special case. Given any ring R, the Krull dimension of any right R-module X, if it exists, will be denoted by $k(X)$. A right R-module M with Krull dimension will be called *k-critical* if M is non-zero and $k(M/N) < k(M)$ for every non-zero submodule N of M. Note the following facts about Krull dimension taken from [23, Lemmas 6.2.4, 6.2.6, 6.2.10, 6.2.11 and 6.2.12].

Lemma 26.3.7 *Let R be any ring.*

(i) For any submodule N of an R-module M, M has Krull dimension if and only if N and M/N both have Krull dimension, and in this case $k(M) = \sup\{k(N), k(M/N)\}$.

(ii) An R-module with Krull dimension has finite uniform dimension.

(iii) Any non-zero R-module with Krull dimension contains a k-critical submodule.

(iv) Any non-zero submodule of a k-critical R-module is k-critical.

(v) Any k-critical module is uniform.

The next result is taken from [14, Corollary 2.5] (see also [20, Lemma 4.2]).

Theorem 26.3.8 *For any ring R, every k-critical right R-module is cocritical.*

Proof Let M be a k-critical right R-module. Let N be a non-zero submodule of M. Suppose there exists a non-zero homomorphism $f : M/N \to E(M)$. Then $L = f(M/N)$ is a non-zero submodule of $E(M)$. By Lemma 26.3.7, the module L has Krull dimension and $k(L) < k(M)$. On the other hand, $L \cap M$ is a non-zero submodule of M and hence $k(L \cap M) = k(M)$ by Lemma 26.3.7 again. But this gives $k(M) < k(M)$, a contradiction. Thus $\mathrm{Hom}_R(M/N, E(M)) = 0$. Apply Theorem 26.3.6. □

Corollary 26.3.9 *Let R be any ring. Then the following implications hold for a right R-module M: M is k-critical ⇒ M is cocritical ⇔ M is monoform ⇒ M is uniform.*

Proof By Theorems 26.3.6 and 26.3.8. □

The next result shows that compressible modules with Krull dimension are k-critical.

Proposition 26.3.10 *Let R be any ring and let M be a compressible right R-module which contains a non-zero submodule which has Krull dimension. Then M is k-critical.*

Proof Let N be a non-zero submodule of M such that N has Krull dimension. By Lemma 26.3.7, N contains a (non-zero) k-critical submodule K. By hypothesis, M embeds in K and, by Lemma 26.3.7 again, M is k-critical. □

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From Proposition 26.3.10 it easily follows that if R is a ring with right Krull dimension then every compressible right R-module is k-critical. Goldie [10, p.166] asks if the converse is true for rings R which are right and left Noetherian. Goodearl [12] and Musson [25] both give examples of right and left Noetherian domains R for which there is a k-critical module which does not have a compressible submodule. Here we shall content ourselves with a simple example of a k-critical module over a right Noetherian PI ring which is not compressible (see also [14, Example 6.9]).

Example 26.3.11 For every field F and ordinal $\alpha > 0$ there exists an F-algebra R such that R is a right Noetherian right nonsingular PI ring with right Krull dimension α and a right ideal A of R such that A is a k-critical right R-module with Krull dimension α but A is not compressible.

Proof By [14, Theorem 9.8] (or see [15]) there exists an F-algebra S such that S is a commutative Noetherian domain with Krull dimension α. Let R denote the subring of the ring of 2×2 upper triangular matrices with entries in S such that the (1,1) entry is in F. Then it is routine to check that R is a right Noetherian right nonsingular PI ring with right Krull dimension α. Let A denote the ideal of R consisting of all matrices in R with (2,2) entry 0. In addition, let N denote the ideal of R consisting of all matrices in R with (1,1) and (2,2) entries 0. Note that N is the prime (i.e., nilpotent) radical of R, N is contained in A and A/N is a simple R-module. Next note that the Krull dimension $k(A_R)$ of the right R-module A is given by $k(A_R) = k(N_R) = k(S_S) = \alpha$. Let a be any non-zero element of A. Then $B = aR \cap N$ is a non-zero right ideal of R and $k((A/(aR \cap N))_R) = k((N/(aR \cap N))_R) = k((N/(aR \cap N))_S) < \alpha$, because every proper homomorphic image of S has Krull dimension less than α. It follows that the R-module A satisfies $k((A/aR)_R) < \alpha$ for every non-zero element a in A. Thus A is a k-critical R-module. Let $f : A \to N$ be any R-homomorphism. Let e denote the element of A with (1,1) entry 1 and all other entries 0. Then $f(e) = f(e^2) = f(e)e \in Ne = 0$. It follows that A is not a compressible right R-module. □

Note that the ring R in Example 26.3.11 is not a semiprime ring. If R is a semiprime ring then every k-critical right ideal of R is a compressible right R-module. This is a consequence of the next result.

Proposition 26.3.12 *Let R be a semiprime ring. Then every monoform submodule of a free right R-module is compressible.*

Proof Suppose that M is a monoform submodule of a free right R-module F. Note that M is uniform and hence, by Lemma 26.2.11, M is isomorphic to a right ideal A of R. Let B be any non-zero submodule of the R-module A. Then B is a non-zero right ideal of R and hence B^2 is non-zero. It follows that BA is non-zero. Let b be any element of B such that $bA \neq 0$. Define a mapping $f : A \to B$ by $f(a) = ba$ for all a in A. Then f is a non-zero homomorphism. Because A is monoform, f is a monomorphism. It follows that A, and hence M, is compressible. □

Corollary 26.3.13 *Let R be a semiprime ring. Then every nonsingular uniform submodule of a free right R-module is compressible.*

Proof By Theorem 26.3.2 and Proposition 26.3.12. □

Corollary 26.3.14 *Let R be a semiprime right Goldie ring. Then every uniform submodule of a free right R-module is compressible.*

Proof Note first that the ring R is right nonsingular (see, for example, [13, Corollary 5.4]) and hence any free right R-module is nonsingular. Apply Corollary 26.3.13. □

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Corollary 26.3.13 gives the following generalization of [9, Proposition 1.10]: For any semiprime ring R, every nonsingular uniform right ideal of R is a compressible right R-module. Under certain circumstances, Proposition 26.3.12 can be improved somewhat. For any element a of a ring R, we set $\mathbf{r}(a) = \operatorname{ann}_R(\{a\})$.

Proposition 26.3.15 *A ring R is a right Ore domain if and only if the right R-module R is monoform. In this case, the right R-module R is compressible.*

Proof Let R be a right Ore domain. By Theorem 26.3.2 the right R-module R is monoform. Conversely, suppose that the right R-module R is monoform. Let a be any non-zero element of R. Define a mapping $f : R \to R$ by $f(a) = ar$ for all r in R. Clearly f is a non-zero homomorphism and hence a monomorphism. Thus $\mathbf{r}(a) = 0$. It follows that R is a domain and hence, because R is a uniform right R-module, a right Ore domain. The last part follows easily. □

A ring R will be called *right compressible* if the right R-module R is compressible.

Proposition 26.3.16 *A ring R is right compressible if and only if for each non-zero element a in R there exists an element b in R such that $\mathbf{r}(ab) = 0$. In this case, R is a prime right nonsingular ring.*

Proof Suppose that R is right compressible. Let a be a non-zero element of R. There exists a monomorphism $f : R \to aR$. If $f(1) = ab$, for some b in R, then $\mathbf{r}(ab) = 0$. Conversely, suppose that R has the stated condition. Let E be any non-zero right ideal of R and let c be a non-zero element of E. There exists d in R such that $\mathbf{r}(cd) = 0$. Define a mapping $g : R \to E$ by $g(s) = cds$ for all s in R. Then g is a monomorphism.

Now suppose that the ring R is right compressible. Then R is a prime ring by Propositions 26.2.6 and 26.2.8. Suppose that R is not right nonsingular. By the first part of the proof, there exists an element z in $Z(R)$ such that $\mathbf{r}(z) = 0$, a contradiction. Thus R is right nonsingular. □

Let R be a right compressible ring and let Q denote the maximal (i.e., complete) right ring of quotients of R (see [29, p. 200]). Because R is right nonsingular (Proposition 26.3.16), the right R-module Q is the injective envelope of the right R-module R and Q is a right self-injective von Neumann regular ring (see [29, Chapter XII Section 2]). The next result shows how to produce examples of right compressible rings.

Proposition 26.3.17 *Let R be a right compressible ring with maximal right ring of quotients Q. Let S be a ring such that either*
(a) S is a subring of Q containing R, or
(b) $S = eRe$ for some idempotent e in R, or
(c) $S = R[x]$ is the polynomial ring in an indeterminate x over R.
Then S is a right compressible ring.

Proof *(a)* Let s be any non-zero element of S. Then $R \cap sR$ is non-zero and hence st is a non-zero element of R for some element t in S. By Proposition 26.3.16, there exists r in R such that $\mathbf{r}(str) = 0$. Because R is an essential submodule of the R-module S it is easy to see that str has zero right annihilator in S. By Proposition 26.3.16 it follows that S is right compressible.

(b) Let a be any non-zero element of S. Then $a = ebe$ for some non-zero element b in R. By Proposition 26.3.16 again, there exists an element c in R such that $\mathbf{r}(bc) = 0$. Note that $bce = b(ece)$ and the right annihilator of bce in S is zero. It follows by Proposition 26.3.16 that S is right compressible.

(c) Let $f(x)$ be any non-zero element of S. Let u be the leading coefficient of $f(x)$. By hypothesis, there exists an element v in R such that $\mathbf{r}(uv) = 0$. Then $f(x)v$ has leading coefficient uv and hence $\mathbf{r}(f(x)v) = 0$ in S. By Proposition 26.3.16, S is a right compressible ring. □

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Finally in this section we give an example of a monoform module which is not k-critical. Let R be a commutative Noetherian integral domain with field of fractions Q. Suppose that either R is not semilocal or R is not one-dimensional. By [22, Theorem 1] (or see [28, Theorem 2.7]) the R-module Q does not have Krull dimension and hence cannot be k-critical. However, by Theorem 26.3.2 the R-module Q is monoform. To give another example, let S be any (non-zero) commutative domain with Krull dimension and let R denote the ring of all upper triangular matrices with entries in S. Then R is a right nonsingular PI ring with right Krull dimension. Let A denote the ideal of R consisting of all matrices with zero (2,2) entry. Then A is a nonsingular uniform right R-module. By Theorem 26.3.2, A is a monoform R-module but it is easy to check that A is not k-critical.

26.4 Nonsingular Modules

Let R be a ring and let M be a right R-module with annihilator A in R. Then, of course, M is a right (R/A)-module. Suppose that M is a nonsingular R-module. Now suppose that $m(E/A) = 0$ for some right ideal E of R containing A such that E/A is an essential right ideal of the ring R/A. It is easy to check that E is an essential right ideal of R and $mE = 0$. Thus if M is a nonsingular R-module then M is a nonsingular (R/A)-module. We shall call the module M *ann-nonsingular* if M is nonsingular as an (R/A)-module. For example, if R is a commutative ring then every prime R-module is ann-nonsingular by Corollary 26.2.7. In particular, if R is a commutative domain and P is a non-zero prime ideal of R then the R-module R/P is singular but ann-nonsingular. Note that, for a general ring R, if M is a compressible R-module then M is a compressible (R/A)-module and hence M is ann-nonsingular or the (R/A)-module M is singular. The first result in this section is immediate from Theorem 26.3.2 and the second is immediate from Propositions 26.2.4 and 26.2.6.

Theorem 26.4.1 *Let R be any ring. Then an ann-nonsingular right R-module M is monoform if and only if M is uniform.*

Proposition 26.4.2 *Let R be any ring. Then an ann-nonsingular right R-module M is prime if and only if the annihilator of M is a prime ideal of R.*

Now we shall consider ann-nonsingular compressible modules. First we prove a result for non-singular modules.

Proposition 26.4.3 *Let R be any ring. Then every nonsingular compressible right R-module is isomorphic to a non-zero right ideal of R.*

Proof Suppose that M is a non-zero nonsingular compressible right R-module. Let m be any non-zero element of M. Let $B = \mathrm{ann}_R(m)$. By hypothesis, the right ideal B is not an essential right ideal of R. Thus there exists a non-zero right ideal C of R such that $B \cap C = 0$. Define a mapping $f : C \to M$ by $f(c) = mc$ for all c in C. Then f is a monomorphism. Thus mC is isomorphic to C. Because M is compressible, there exists a monomorphism $g : M \to mC$ and hence there exists a monomorphism $h : M \to R$. □

Corollary 26.4.4 *Let R be any ring and let M be any ann-nonsingular compressible right R-module with (prime) annihilator P. Then M is isomorphic to a right R-module of the form A/P for some right ideal A of R properly containing P.*

Proof By Proposition 26.4.3. □

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Corollary 26.4.5 *Let R be any ring and let M be a compressible right R-module with annihilator P such that the ring R/P is right nonsingular. Then the following statements are equivalent.*
(i) M is ann-nonsingular.
(ii) M can be embedded in the right R-module R/P.
(iii) M can be embedded in a free right (R/P)-module.

Proof *(i) $\Rightarrow$ (ii)* By Corollary 26.4.4.
(ii)$\Rightarrow$ (iii) $\Rightarrow$ (i) Clear. □

The next result is essentially taken from [9, Proposition 1.10] (or see [7, Lemma 3.3]). We shall give its proof for completeness.

Lemma 26.4.6 *Let A be a uniform right ideal of a ring R and let B be a nonsingular right ideal of R such that BA is non-zero. Then A embeds in B.*

Proof Because BA is non-zero, there exists b in A such that bA is non-zero. Define a mapping $f : A \to B$ by $f(a) = ba$ for all a in A. Clearly, f is a homomorphism. Suppose that f is not a monomorphism. Then there exists a non-zero element c in A such that $bc = f(c) = 0$. Let a be any element of A. Because cR is an essential submodule of A, there exists an essential right ideal E of R such that aE is contained in cR and hence $baE \subseteq bcR = 0$. Since B is nonsingular it follows that $ba = 0$. Hence $bA = 0$, a contradiction. Thus f is a monomorphism. □

Corollary 26.4.7 *Let R be any ring and let A be a nonsingular uniform right ideal of R such that BA is non-zero for every non-zero right ideal B of R contained in A. Then A is a compressible right R-module.*

Proof By Lemma 26.4.6. □

Lemma 26.4.8 *Let R be a ring such that either (a) R is a semiprime right Goldie ring or (b) R is a prime ring which contains a uniform right ideal. Then every nonsingular compressible right R-module is monoform.*

Proof Let M be any nonsingular compressible right R-module. By Proposition 26.4.3, M is isomorphic to a non-zero right ideal A of R. It follows (using Lemma 26.4.6 in case *(b)*) that M contains a uniform submodule and hence M is uniform. By Theorem 26.3.2, M is monoform. □

Corollary 26.4.9 *Let R be any ring and let M be any ann-nonsingular compressible right R-module with annihilator P. Then M is a monoform module if and only if the ring R/P contains a uniform right ideal.*

Proof By Proposition 26.2.8 P is a prime ideal of R. Suppose that M is monoform. Then M is uniform and the ring R/P has a uniform right ideal by Corollary 26.4.4. Conversely, if the prime ring R/P has a uniform right ideal then M is monoform by Lemma 26.4.8. □

Note that Corollary 26.4.9 generalizes Corollary 26.3.5.

Theorem 26.4.10 *Let R be any ring and let M be a non-zero right R-module with annihilator P such that the ring R/P contains a uniform right ideal. Then M is an ann-nonsingular compressible right R-module if and only if P is a semiprime ideal of R and M is isomorphic to a nonsingular uniform right ideal of the ring R/P. In this case P is a prime ideal of R.*

Proof The necessity follows by Propositions 26.2.8 and 26.4.3 and Corollary 26.4.9. Conversely, the sufficiency follows by Corollary 26.4.7. □

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Theorem 26.4.10 has several consequences which we record next.

Corollary 26.4.11 *Let R be any ring and let M be a non-zero right R-module with annihilator P such that the ring R/P is right Goldie. Then M is an ann-nonsingular compressible right R-module if and only if P is a prime ideal of R and M is isomorphic to a uniform right ideal of the ring R/P.*

Proof By Theorem 26.4.10 because semiprime right Goldie rings are right nonsingular (see, for example, [13, Corollary 5.4] or [23, Lemma 2.3.4]). □

Corollary 26.4.12 *Let R be a ring such that either R has right Krull dimension or R satisfies a polynomial identity. Then a right R-module M is ann-nonsingular and compressible if and only if the annihilator P of M in R is a prime ideal of R and M is isomorphic to a uniform right ideal of R/P.*

Proof By Corollary 26.4.11 and [23, Proposition 6.3.5 and Corollary 13.6.6]. □

Let R be a semiprime right Goldie ring. Then every nonsingular compressible right R-module is monoform by Lemma 26.4.8. The converse is not true in general, for if R is a commutative domain which is not a field and Q is the field of fractions of R, then the R-module Q is uniform but not compressible. We next investigate when nonsingular uniform (i.e., monoform) modules are compressible. Note that this is the case for submodules of free modules over semiprime rings by Corollary 26.3.13.

Lemma 26.4.13 *Let R be a semiprime ring. Then a nonsingular uniform (monoform) right R-module M is compressible if and only if M is isomorphic to a right ideal of R.*

Proof The necessity follows by Proposition 26.4.3 and the sufficiency by Corollary 26.3.13. □

Theorem 26.4.14 *Let R be a semiprime right Goldie ring. Then the following statements are equivalent.*
(i) Every finitely generated nonsingular uniform (monoform) right R-module is compressible.
(ii) Every finitely generated nonsingular right R-module embeds in a free right R-module.
(iii) R is a left Goldie ring.

Proof *(i) ⇒ (ii)* Let M be any finitely generated nonsingular right R-module. There exist a free right R-module F of finite rank and a submodule K of F such that M is isomorphic to F/K. Note that F has finite uniform dimension. Then M has finite uniform dimension by [3, Section 5.10]. It follows that $E(M) = E_1 \oplus \ldots \oplus E_n$ for some positive integer n and nonsingular indecomposable injective R-modules $E_i (1 \leq i \leq n)$. For each $1 \leq i \leq n$, if $p_i : E(M) \to E_i$ denotes the canonical projection then $p_i(M)$ is a finitely generated nonsingular uniform R-module. By *(i)* each $p_i(M)$ is compressible so that $p_i(M)$ embeds in a free right R-module (Lemma 26.4.13). It follows that M embeds in a free right R-module.

(ii) ⇒ (i) By Proposition 26.3.12.

(ii) ⇔ (iii) By [21, Theorem 5.3] (see also [5]). □

Corollary 26.4.15 *Let R be a ring such that R/P is a right Goldie ring for every prime ideal P of R. Then the following statements are equivalent.*

(i) Every finitely generated ann-nonsingular uniform (monoform) prime right R-module is compressible.

(ii) R/P is a left Goldie ring for every prime ideal P of R.

Proof *(i) ⇒ (ii)* Let P be any prime ideal of R. Let M be any finitely generated nonsingular uniform right (R/P)-module. By Proposition 26.2.4, M is a prime (R/P)-module and hence also a prime R-module. Moreover $P = \mathrm{ann}_R(M)$. Thus M is a finitely generated ann-nonsingular

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uniform prime right R-module and hence is compressible as an R-module and also as an (R/P)-module. Thus every finitely generated nonsingular uniform right (R/P)-module is compressible. By Theorem 26.4.14, R/P is a left Goldie ring.

(ii) $\Rightarrow$ *(i)* Let U be a finitely generated ann-nonsingular uniform prime right R-module. Let $Q = \text{ann}_R(U)$. Then Q is a prime ideal of R by Proposition 26.2.6. By Theorem 26.4.14 U is a compressible (R/Q)-module and hence is also a compressible R-module. □

This leads us to the next result which should be compared with [16, Theorem 2.5].

Theorem 26.4.16 *Let R be any ring and let M be a non-zero finitely generated ann-nonsingular right R-module with annihilator P such that the ring R/P is right and left Goldie. Then the following statements are equivalent.*

(i) M is a uniform (monoform) prime module.

(ii) M is a uniform (monoform) module and P is a semiprime ideal of R.

(iii) M is compressible.

If, in addition, R has right Krull dimension then (i) is equivalent to

(iv) M is k-critical and P is a semiprime ideal of R.

Proof *(i)* $\Rightarrow$ *(ii)* Clear.
(ii) $\Rightarrow$ *(iii)* Without loss of generality, M is a faithful R-module. Hence M is a finitely generated nonsingular uniform module over the semiprime right and left Goldie ring R. By Theorem 26.4.14, M is compressible.
(iii) $\Rightarrow$ *(i)* By Corollary 26.4.11.

Now suppose that R has right Krull dimension.
(iii) $\Rightarrow$ *(iv)* By Propositions 26.2.8 and 26.3.10.
(iv) $\Rightarrow$ *(ii)* Clear. □

Corollary 26.4.17 *Let R be any ring and let M be a non-zero ann-nonsingular right R-module with annihilator P such that the ring R/P is right and left Goldie. Then M is compressible if and only if M is isomorphic to a submodule of a finitely generated uniform (monoform) prime module.*

Proof Suppose that M is compressible. Then M is a prime module by Proposition 26.2.8. Let m be any non-zero element of M. Then mR is finitely generated compressible and $\text{ann}_R(mR) = P$. By Lemma 26.4.8, mR is a uniform prime module. Moreover, M embeds in mR because M is compressible. Conversely, suppose that M embeds in a finitely generated uniform prime module M'. By Theorem 26.4.16, M' is a compressible module and hence so also is M.

Note that in Theorem 26.4.16 the condition that P be a semiprime ideal in *(ii)* and *(iv)* is required because of the examples in [12] and [25]. Note that in Example 26.3.11, the k-critical (and hence monoform) R-module A does not have semiprime annihilator and hence cannot be prime. If R is a ring with right Krull dimension α, for some ordinal $\alpha > 0$, several authors have investigated when every k-critical right R-module M with $k(M) = \alpha$ has a prime annihilator. Recall that if R is a ring with right Krull dimension then an ideal I of R is said to be *weakly ideal invariant* if $k(I/AI) < k(R/I)$ for every right ideal A of R such that $k(R/A) < k(R/I)$. In [4, Theorem 2.9], Feller proves that if R is a ring with right Krull dimension α such that the prime radical of R is weakly ideal invariant and M is a k-critical right R-module with $k(M) = \alpha$ then $\text{ann}_R(M)$ is a prime ideal of R. Conversely, Brown, Lenagan and Stafford [1, Theorem 2.5] prove that if R is a right Noetherian ring with right Krull dimension α such that every k-critical right R-module with Krull dimension α has prime annihilator then the prime radical of R is weakly ideal invariant. □

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26.5 Fully Bounded Rings

We shall call a ring R *right bounded* if every essential right ideal of R contains a non-zero ideal I of R such that I is an essential right ideal of R. A ring R is called *right fully bounded* if every prime homomorphic image of R is right bounded. Also an R-module M is called *bounded* if $R/\operatorname{ann}_R(M)$ is a right bounded ring. We begin this section with an observation that shows the relevance of right fully bounded rings to our study. The following result holds for right Noetherian rings in particular.

Theorem 26.5.1 *Let R be a ring such that every prime ideal is a finitely generated right ideal and every finitely generated prime right R-module is ann-nonsingular. Then R is right fully bounded.*

Proof Suppose first that R is prime. Suppose that R is not right bounded. Let $\bigwedge$ denote the collection of essential right ideals of R which do not contain a non-zero ideal. Let $E_i (i \in I)$ denote any chain in $\bigwedge$ and let $E = \bigcup_I E_i$. Suppose that E contains a non-zero ideal A of R. By Proposition 26.2.3, A contains a finite product B of non-zero prime ideals of R. By Lemma 26.2.2, B is a finitely generated right ideal of R and hence the non-zero ideal B is contained in E_i for some i in I, a contradiction. Thus $\bigwedge$ contains a maximal member H. Clearly R/H is a non-zero R-module. By Proposition 26.2.3, there exists a right ideal G of R containing H such that G/H is a finitely generated prime right R-module. By hypothesis G/H is ann-nonsingular. Because G/H is a singular R-module, we conclude that $Q = \operatorname{ann}_R(G/H)$ is a non-zero (prime) ideal of R. Moreover, by the choice of H there exists a non-zero ideal C of R such that $C \subseteq G$. Then CQ is a non-zero ideal of R and $CQ \subseteq H$, a contradiction. Thus R is right bounded.

In general, let P be any prime ideal of R. The ring R/P inherits the properties of the ring R and hence by the above argument R/P is right bounded. It follows that the ring R is right fully bounded. □

Note that in Theorem 26.5.1 the ring R is not only right fully bounded but it is also right Noetherian by [19, Proposition 3.3] (see also [19, p. 95 Remark]). This fact is also proved in [27, Corollary 5]. Now we investigate the converse of Theorem 26.5.1.

Lemma 26.5.2 *Let R be any ring. Then every bounded prime right R-module is ann-nonsingular.*

Proof Suppose that M is a non-zero bounded prime right R-module. Let $P = \operatorname{ann}_R(M)$. Note that P is a prime ideal of R by Proposition 26.2.6. Let m be an element of M such that $m(E/P) = 0$ for some right ideal E of R containing P such that E/P is an essential right ideal of the ring R/P. By hypothesis, there exists an ideal A of R such that A properly contains P and A is contained in E. It follows that $mA = 0$. Since MA is non-zero and M is prime it follows that $m = 0$. Hence M is a nonsingular (R/P)-module and hence also an ann-nonsingular R-module. □

Lemma 26.5.2 has a number of consequences. The first should be compared with Proposition 26.4.2.

Corollary 26.5.3 *Let R be any ring. Then a finitely generated bounded uniform right R-module M is prime if and only if M has prime annihilator.*

Proof The necessity follows by Proposition 26.2.6. Conversely, suppose that M has prime annihilator P. Without loss of generality $P = 0$ and R is a right bounded prime ring. Suppose that $mA = 0$ for some non-zero element m in M and ideal A in R. There exist a positive integer n and elements $m_i (1 \leq i \leq n)$ in M such that $M = m_1R + \ldots + m_nR$. Since M is uniform it follows that for each $1 \leq i \leq n$ there exists an essential right ideal E_i of R such that $m_iE_i \subseteq mR$. By hypothesis there exists a non-zero ideal B of R such that $B \subseteq E_1 \cap \ldots \cap E_n$. Then $MB \subseteq mR$ and hence $MBA = 0$. This implies that $BA = 0$ and hence $A = 0$. It follows that M is a prime module. □

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Corollary 26.5.4 *Let R be any ring. Then every bounded uniform prime right R-module is monoform.*

Proof By Theorem 26.4.1 and Lemma 26.5.2. □

Next we combine Theorem 26.5.1 and Lemma 26.5.2 to prove the following result which should be compared with Corollary 26.4.5.

Theorem 26.5.5 *The following statements are equivalent for a right Noetherian ring R.*
(i) Every finitely generated prime right R-module M embeds in a free right $(R/ann_R(M))$-module.
(ii) R is a right fully bounded ring such that R/P is a left Goldie ring for every prime ideal P of R.

Proof *(i) ⇒ (ii)* Let M be a finitely generated prime right R-module and let $P = \mathrm{ann}_R(M)$. By Proposition 26.2.6, P is a prime ideal of R and hence R/P is a right Noetherian prime ring. By [13, Corollary 5.4] (or see [23, Lemma 2.3.4]), the ring R/P is right nonsingular and hence, by *(i)*, M is a nonsingular (R/P)-module. Thus every finitely generated prime right R-module is ann-nonsingular. By Theorem 26.5.1, R is right fully bounded. Next let Q be a prime ideal of R. Let X be a finitely generated nonsingular right (R/Q)-module. By Proposition 26.2.4, X is a finitely generated ann-nonsingular prime right R-module so that, by *(i)*, X embeds in a free right (R/Q)-module. Applying Theorem 26.4.14, it follows that the ring R/Q is left Goldie.

(ii) ⇒ (i) Let Y be any finitely generated prime right R-module. Let $V = \mathrm{ann}_R(Y)$. Then Y is a nonsingular right (R/V)-module by Lemma 26.5.2. By Theorem 26.4.14, Y embeds in a free right (R/V)-module. This completes the proof. □

Lemma 26.5.2 allows us to apply our earlier results.

Theorem 26.5.6 *Let R be any ring and let M be a bounded compressible right R-module with annihilator P. Then M is isomorphic to a right R-module of the form A/P for some right ideal A of R properly containing P. Moreover, M is monoform if and only if R/P contains a uniform right ideal.*

Proof By Proposition 26.2.8, Lemma 26.5.2 and Corollaries 26.4.4 and 26.4.9. □

A ring R will be called a *right FBG* ring if every prime homomorphic image of R is a right bounded right Goldie ring. Also a ring R is called a *right FBN* ring provided R is a right fully bounded right Noetherian ring. Clearly right FBN rings are right FBG. However, rings satisfying a polynomial identity, in particular commutative rings or subrings of matrix rings over commutative rings, are right FBG rings by [23, Corollary 13.6.6].

Corollary 26.5.7 *Let R be a right FBG ring. Then a right R-module M is compressible if and only if the annihilator P of M in R is a prime ideal of R and M is isomorphic to a uniform right ideal of R/P.*

Proof Suppose first that M is compressible. By Proposition 26.2.8 M is a prime module and, by Proposition 26.2.6, P is a prime ideal of R. Next M is ann-nonsingular by Lemma 26.5.2. Applying Corollary 26.4.11, we deduce that M is isomorphic to a uniform right ideal of R/P. Conversely, if M is isomorphic to a uniform right ideal of the ring R/P then M is compressible by Corollary 26.4.11. □

In particular, Corollary 26.5.7 shows that if R is a right FBG ring then every compressible right R-module is a uniform prime module. Now note the following result.

Proposition 26.5.8 *Let R be a right FBG ring. Then the following statements are equivalent.*
(i) Every finitely generated uniform prime right R-module is compressible.
(ii) R/P is a left Goldie ring for every prime ideal P of R.

Proof By Corollary 26.4.15 and Lemma 26.5.2. □

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The next result generalizes Theorem 26.2.10.

Theorem 26.5.9 *Let R be a right FBG ring such that R/P is a left Goldie ring for every prime ideal P of R. Then a right R-module M is compressible if and only if M is isomorphic to a non-zero submodule of a finitely generated uniform (monoform) prime right R-module.*

Proof The necessity follows by Corollary 26.5.7 and the sufficiency by Proposition 26.5.8. □

Another consequence of Lemma 26.5.2 is the following result.

Proposition 26.5.10 *Let R be a right fully bounded ring with right Krull dimension. Then every non-zero right R-module contains a compressible monoform submodule.*

Proof Let M be any non-zero R-module. By Proposition 26.2.3, M contains a submodule K which is a prime R-module. Without loss of generality we can suppose that K is cyclic. Thus K has Krull dimension and hence K contains a uniform submodule U. Note that U is a uniform prime module. Let $P = \mathrm{ann}_R(U)$. By Lemma 26.5.2 U is a nonsingular module over the prime ring R/P. From the proof of Proposition 26.4.3 we see that U contains a submodule V which is isomorphic to an R-module of the form E/P for some right ideal E of R properly containing P. By Theorem 26.3.2 and Corollary 26.4.7, V is a compressible monoform module. □

As we noted above, from Proposition 26.3.10 it easily follows that if R is a ring with right Krull dimension then every compressible right R-module is k-critical. We now investigate when k-critical modules are compressible. Because compressible modules are prime (Proposition 26.2.8) as a first step we consider when k-critical modules are prime. Note that Example 26.3.11 gives an example of a k-critical right ideal A of a right FBN ring R such that A is not a prime module. In fact, the annihilator of A in R is the zero ideal which is not semiprime. Recall the following result.

Lemma 26.5.11 *(See [13, Theorem 8.9] or [23, Proposition 6.4.12].) Let R be a right FBN ring and let M be a faithful finitely generated right R-module. Then $k(M) = k(R)$.*

Corollary 26.5.12 *Let R be a prime right FBN ring. Then every faithful finitely generated k-critical right R-module is prime.*

Proof Let M be any faithful finitely generated k-critical R-module. By Proposition 26.2.3 there exists a submodule K of M such that K is a prime module. Let $P = \mathrm{ann}_R(K)$. By Lemma 26.5.11, $k(R) = k(M) = k(K) = k(R/P)$. Thus, by [23, Proposition 6.3.11], $P = 0$. Now let m be any non-zero element of M such that $mJ = 0$ for some ideal J of R. Note that M is a uniform module (Lemma 26.3.7) and hence $mR \cap K \neq 0$. Then $(mR \cap K)J = 0$ which implies that $J = 0$. It follows that M is prime. □

Compare the next result with [16, Theorem 2.5].

Theorem 26.5.13 *Let R be a right FBN ring. Then the following statements are equivalent for a finitely generated right R-module M.*
(i) M is a k-critical module with prime annihilator.
(ii) M is a uniform prime module.

Proof *(i) ⇒ (ii)* By Lemma 26.3.7 and Corollary 26.5.12.

(ii) ⇒ (i) By Proposition 26.2.6, M has prime annihilator P. Without loss of generality $P = 0$. Lemma 26.5.11 gives that $k(M) = k(R)$. Since M is uniform it follows that, for every non-zero submodule N of M, M/N is a singular module over the prime ring R and hence $k(M/N) < k(R)$ by [13, Proposition 13.7] (or see [23, Proposition 6.3.11 *(iii)*]). Thus $k(M/N) < k(M)$ for every non-zero submodule N of M and we have proved that M is k-critical. □

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Corollary 26.5.14 *Let R be a right FBN ring. Then the following statements are equivalent.*
(i) Every k-critical right R-module with prime annihilator is compressible.
(ii) R/P is a left Goldie ring for every prime ideal P of R.

Proof By Proposition 26.5.8 and Theorem 26.5.13. □

Note that Jategaonkar [16, Theorem 2.5] proves that if R is a right and left FBN ring then every finitely generated k-critical right R-module is compressible and, independently, Chamarie and Hudry [2, Proposition 1.4] and Wangneo and Tewari [31, Theorem 3.6] prove that this is also the case if R is a right Noetherian ring which is integral over its centre (in which case R is right FBN but need not be left FBN). In [30, Theorem 2.6] it is proved that if R is a right and left Noetherian ring which is integral over its centre and $S = R[x]$ is the polynomial ring in an indeterminate x over R then every finitely generated k-critical right S-module is compressible.

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