Duality theorems for a nondifferentiable minimax fractional programming under generalized \((F, \alpha, \rho, d)\)-convexity

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Abstract

In this paper, we provide a dual model for a non-differentiable minimax fractional programming problem and then, the weak duality, strong duality and strict converse duality theorems are established for this dual model under the assumption of generalized \((F, \alpha, \rho, d)\)-convexity.

Keywords: Nondifferentiable fractional minimax programming; Duality; Generalized convexity.

1 Introduction

We consider the following nondifferentiable minimax fractional programming problems:

\[
\text{Primal (P):} \quad \min_{x \in \mathbb{R}^n} \left[ \max_{y \in Y} \left( \frac{f(x, y) + (x^T B x)^{1/2}}{h(x, y) - (x^T D x)^{1/2}} \right) \right]
\]

\[\text{s.t.} \quad g(x) \leq 0\]

where \(Y\) is a compact subset of \(\mathbb{R}^m\); \(f(. , .): \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}\) and \(h(. , .): \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}\) are \(C^1\) on \(\mathbb{R}^n \times \mathbb{R}^m\); \(g(.): \mathbb{R}^n \rightarrow \mathbb{R}^r\) is \(C^1\) on \(\mathbb{R}^n\); \(B\) and \(D\) are \(n \times n\) symmetric positive semidefinite matrices. Throughout this paper, we assume that \(h(x, y) - (x^T D x)^{1/2} > 0\) and \(f(x, y) + (x^T B x)^{1/2} \geq 0\) for each \((x, y)\) in \(X \times Y\) where \(X = \{x \in \mathbb{R}^n \mid g(x) \leq 0\}\).

Minimax mathematical programming and deriving the optimality conditions for them have been of much interest in the recent past. Many scholars have studied this matter in the presence of various assumptions. See, e.g., ([8]-[18]) and the references therein. Schmitendorf [15] established necessary and sufficient optimality conditions for minimax problem. Husain et al. [5] derived necessary and sufficient optimality conditions for minimax fractional programming problems under concept of pseudoconvexity and quasiconvexity. Tanimoto [19] applied these optimality conditions to define a dual problem and derived duality theorems, which were extended for the fractional analogue of generalized minimax problem by Yadav and Mukherjee [17]. Liang et al. [20, 21] introduced a unified formulation of generalized convexity, which was

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called \((F, \alpha, \rho, d)\)-convexity, by relaxing the definition of \((F, \rho)\)-convexity in [22] and \(V\)-invex function in [10]; and then obtained some corresponding sufficient optimality conditions and duality results for the single objective fractional problems and multiobjective problems. The need of \(\alpha\) parameter in definition of \((F, \alpha, \rho, d)\)-convexity may be motivated from [25]-[32]. Lai et al. [24] derived necessary and sufficient conditions for nondifferentiable minimax fractional problem with generalized convexity and applied these optimality conditions to construct one parametric dual model and also discussed duality theorems. Lai and Lee [23] obtained duality theorems for two parametric-free dual models of nondifferentiable minimax fractional problem involving generalized convexity assumptions. Ahmad and Husain [9] established sufficient conditions and duality theorems for two parameter-free dual models of (P) under \((F, \alpha, \rho, d)\) assumptions. In fact, they extended the results of [23, 24].

In this paper, motivated by Lai and Lee [23] and Liang et al. [20, 21] and Ahmad et al. [9], we first introduce the new notions of \((F, \alpha, \rho, d)\)-concavity, \((F, \alpha, \rho, d)\)-quasiconcavity, \((F, \alpha, \rho, d)\)-quasiconvexity and strictly \((F, \alpha, \rho, d)\)-quasiconvexity and then, we consider a Wolf dual model for problem \((P)\) and establish weak, strong and strict converse duality theorems under the assumption of generalized \((F, \alpha, \rho, d)\)-convexity. In fact, in view of the generalized \((F, \alpha, \rho, d)\)-convexity, we extend the results of [23].

2 Notations and preliminary results

Let \(\mathbb{R}^n\) be the \(n\)-dimensional Euclidean space and \(X\) an open set in \(\mathbb{R}^n\).

Definition 1. [22] A functional \(F : X \times X \times \mathbb{R}^n \to \mathbb{R}\) is said to be sublinear if \(\forall x, \bar{x} \in X\)

\((i)\) \(F(x, \bar{x}; a_1 + a_2) \leq F(x, \bar{x}; a_1) + F(x, \bar{x}; a_2) \forall a_1, a_2 \in \mathbb{R}^n,\)

\((ii)\) \(F(x, \bar{x}; \beta a) = \beta F(x, \bar{x}; a) \forall \beta \in \mathbb{R}_+\) and \(\forall a \in \mathbb{R}^n.\)

By (i), it is clear that \(-F(x, \bar{x}; a) \leq F(x, \bar{x}; -a)\) and by (ii), \(F(x, \bar{x}; 0) = 0\)

Definition 2. Given an open set \(X \subset \mathbb{R}^n\), a number \(\rho \in \mathbb{R}\), and two functions \(\alpha : X \times X \to \mathbb{R}_+ \setminus \{0\}\) and \(d(,.) : X \times X \to \mathbb{R}\) a differentiable function \(\zeta\) over \(X\) is said to be \((F, \alpha, \rho, d)\) - concave if for any \(\bar{x}, x \in X\), \(F : X \times X \times \mathbb{R}^n \to \mathbb{R}\) is sublinear and \(\zeta(x)\) satisfies the following condition:

\[\zeta(x) - \zeta(\bar{x}) \leq F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) + \rho d^2(x, \bar{x}).\]

Further, \(\zeta\) is said to be strictly \((F, \alpha, \rho, d)\) - concave, if for any \(\bar{x}, x \in X\) and \(x \neq \bar{x}\), there exists a sublinear functional \(F : X \times X \times \mathbb{R}^n \to \mathbb{R}\) such that:

\[\zeta(x) - \zeta(\bar{x}) < F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) + \rho d^2(x, \bar{x}).\]

Definition 3. [20, 21] Given an open set \(X \subset \mathbb{R}^n\), a number \(\rho \in \mathbb{R}\), and two functions \(\alpha : X \times X \to \mathbb{R}_+ \setminus \{0\}\) and \(d(,.) : X \times X \to \mathbb{R}\) a differentiable function \(\zeta\) over \(X\) is said to be \((F, \alpha, \rho, d)\)-convex if for any \(\bar{x}, x \in X\), \(F : X \times X \times \mathbb{R}^n \to \mathbb{R}\) is sublinear and \(\zeta(x)\) satisfies the
following condition:
\[
\zeta(x) - \zeta(\bar{x}) \geq F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) + \rho d^2(x, \bar{x}).
\]

Further, \( \zeta \) is said to be strictly \((F, \alpha, \rho, d)\)-convex, if for any \( \bar{x}, x \in X \) and \( x \neq \bar{x} \), there exists a sublinear functional \( F : X \times X \times \mathbb{R}^n \rightarrow \mathbb{R} \) such that:
\[
\zeta(x) - \zeta(\bar{x}) > F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) + \rho d^2(x, \bar{x}).
\]

**Remark 1.**

(i) If \( \alpha(x, \bar{x}) = 1 \) for all \( x, \bar{x} \in X \), then \((F, \alpha, \rho, d)\)-convexity is \((F, \rho)\)-convexity in [22].

(ii) If \( F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) = \nabla \zeta(\bar{x}) \eta(x, \bar{x}) \) for a certain map \( \eta : X \times X \rightarrow \mathbb{R}^n \), then \((F, \alpha, \rho, d)\)-convexity is \( \rho \)-invexity in [11].

(iii) If \( \rho = 0 \) or \( d(x, \bar{x}) = 0 \) for all \( x, \bar{x} \in X \) and \( F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) = \nabla \zeta(\bar{x}) \eta(x, \bar{x}) \) for a certain map \( \eta : X \times X \rightarrow \mathbb{R}^n \), then \((F, \alpha, \rho, d)\)-convexity is invexity in [6].

**Remark 2.**

If \( \zeta \) is \((F, \alpha, \rho, d)\)-convex then \(-\zeta\) is \((F, \alpha, -\rho, d)\)-concave.

**Definition 4.** [9] Given an open set \( X \subset \mathbb{R}^n \), a number \( \rho \in \mathbb{R} \), and two functions \( \alpha : X \times X \rightarrow \mathbb{R}_+ \setminus \{0\} \) and \( d(., .) : X \times X \rightarrow \mathbb{R} \) a differentiable function \( \zeta \) over \( X \) is said to be \((F, \alpha, \rho, d)\)-pseudoconvex if for any \( \bar{x}, x \in X \), there exists a sublinear functional \( F : X \times X \times \mathbb{R}^n \rightarrow \mathbb{R} \) such that:
\[
F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) \geq -\rho d^2(x, \bar{x}) \implies \zeta(x) \geq \zeta(\bar{x}).
\]

Further, \( \zeta \) is said to be strictly \((F, \alpha, \rho, d)\)-pseudoconvex, if for any \( \bar{x}, x \in X \) and \( x \neq \bar{x} \) there exists a sublinear functional \( F : X \times X \times \mathbb{R}^n \rightarrow \mathbb{R} \) such that:
\[
F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) \geq -\rho d^2(x, \bar{x}) \implies \zeta(x) > \zeta(\bar{x})
\]
or equivalently,
\[
\zeta(x) \leq \zeta(\bar{x}) \implies F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) < -\rho d^2(x, \bar{x})
\]

**Definition 5.** Given an open set \( X \subset \mathbb{R}^n \), a number \( \rho \in \mathbb{R} \), and two functions \( \alpha : X \times X \rightarrow \mathbb{R}_+ \setminus \{0\} \) and \( d(., .) : X \times X \rightarrow \mathbb{R} \) a differentiable function \( \zeta \) over \( X \) is said to be \((F, \alpha, \rho, d)\)-quasiconvex if for any \( \bar{x}, x \in X \), \( F : X \times X \times \mathbb{R}^n \rightarrow \mathbb{R} \) is sublinear and \( \zeta(.) \) satisfies the following condition:
\[
F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) > -\rho d^2(x, \bar{x}) \implies \zeta(x) > \zeta(\bar{x})
\]
or equivalently,
\[
\zeta(x) \leq \zeta(\bar{x}) \implies F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) \leq -\rho d^2(x, \bar{x})
\]

Further, \( \zeta \) is said to be strictly \((F, \alpha, \rho, d)\)-quasiconvex if for any \( \bar{x}, x \in X \) and \( x \neq \bar{x} \) there exists a sublinear functional \( F : X \times X \times \mathbb{R}^n \rightarrow \mathbb{R} \) such that:
\[
F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) \geq -\rho d^2(x, \bar{x}) \implies \zeta(x) > \zeta(\bar{x})
\]

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or equivalently,
\[ \zeta(x) \leq \zeta(\bar{x}) \Rightarrow F(x, \bar{x}; \alpha(x, \bar{x})\nabla \zeta(\bar{x})) < -\rho d^2(x, \bar{x}). \]

**Example 1.** If \( X = \mathbb{R} \), \( \alpha(x, \bar{x}) = 1 \), \( \rho = 0 \) or \( d(x, \bar{x}) = 0 \) and also \( F(x, \bar{x}; \alpha(x, \bar{x})\nabla \zeta(\bar{x})) = \nabla \zeta(\bar{x})(x - \bar{x}) \) then the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^3 + x \) is \((F, \alpha, \rho, d)\)-quasiconvex.

**Remark 3.**

If \( F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) = \alpha(x, \bar{x})\nabla \zeta(\bar{x})\eta(x, \bar{x}) \) for a certain map \( \eta : X \times X \to \mathbb{R}^n \), then \((F, \alpha, \rho, d)\)-quasiconvexity is quasi-\(\alpha\)-invexity in [30]. Also if \( X \) be a nonempty closed set in a real Hilbert space \( H \) and \( \zeta : X \to H \) and \( \rho > 0 \), \( d(x, \bar{x}) = ||\eta(x, \bar{x})|| \) in which \( \eta : X \times X \to \mathbb{R}^n \) is continuous function and also \( F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) = \alpha(x, \bar{x})\nabla \zeta(\bar{x})\eta(x, \bar{x}) \) then \((F, \alpha, \rho, d)\)-quasiconvex functions reduce to strongly quasi \( \alpha \)-invex functions in [28].

**Definition 6.** Given an open set \( X \subset \mathbb{R}^n \), a number \( \rho \in \mathbb{R} \), and two functions \( \alpha : X \times X \to \mathbb{R}_+ \setminus \{0\} \) and \( d(., .) : X \times X \to \mathbb{R} \) a differentiable function \( \zeta \) over \( X \) is said to be \((F, \alpha, \rho, d)\)-quasiconcave if for any \( \bar{x}, x \in X \), \( F : X \times X \times \mathbb{R}^n \to \mathbb{R} \) is sublinear and \( \zeta(\cdot) \) satisfies the following condition:

\[ \zeta(x) \geq \zeta(\bar{x}) \Rightarrow F(x, \bar{x}; \alpha(x, \bar{x})\nabla \zeta(\bar{x})) \geq -\rho d^2(x, \bar{x}) \]

or equivalently,

\[ F(x, \bar{x}; \alpha(x, \bar{x})\nabla \zeta(\bar{x})) < -\rho d^2(x, \bar{x}) \Rightarrow \zeta(x) < \zeta(\bar{x}). \]

**Remark 4.**

If \( f \) is \((F, \alpha, -\rho, d)\)-quasiconvex then \(-f \) is \((F, \alpha, \rho, d)\)-quasiconcave.

**Example 2.** If \( X = \mathbb{R} \), \( \alpha(x, \bar{x}) = 1 \), \( \rho = 0 \) or \( d(x, \bar{x}) = 0 \) and also \( F(x, \bar{x}; \alpha(x, \bar{x})\nabla \zeta(\bar{x})) = \nabla \zeta(\bar{x})(x - \bar{x}) \) then the function \((-f) : \mathbb{R} \to \mathbb{R} \) defined by \((-f)(x) = x^3 \) is \((F, \alpha, \rho, d)\)-quasiconcave but \( f(x) \) is not \((F, \alpha, -\rho, d)\)-quasiconvex.

We often use the **generalized Schwartz inequality** as follows [4]:

\[ x^TBv \leq (x^TBx)^{1/2}(v^TBv)^{1/2} \]  \hspace{1cm} (1)

for each \( x \) and \( v \). In which \( B \) is an \( n \times n \) symmetric positive semidefinite matrix.

We observe that equality holds if \( Bx = \lambda Bv \) for some \( \lambda \geq 0 \).

Evidently, if \( v^TBv \leq 1 \), we have

\[ x^TBv \leq (x^TBx)^{1/2}. \]  \hspace{1cm} (2)

Further, we define for an \( x^0 \in X \),

\[ ... \]
\[
Y(x^0) = \left\{ y \in Y \left| \frac{f(x^0, y) + (x^{0T} Bx^0)^{1/2}}{h(x^0, y) - (x^{0T} Dx^0)^{1/2}} = \sup_{z \in Y} \left( \frac{f(x^0, z) + (x^{0T} Bx^0)^{1/2}}{h(x^0, z) - (x^{0T} Dx^0)^{1/2}} \right) \right. \right\}.
\]

\[
K(x^0) = \left\{ (s, t, \bar{y}) | s \text{ is a positive integer with } 1 \leq s \leq n + 1, t \in \mathbb{R}^+_* \right\},
\]

\[
\sum_{i=1}^{r} t_i = 1, \quad \bar{y} = (y_1, \ldots, y_s) \in \mathbb{R}^{ms} \text{ with } y_i \in Y(x^0) \text{ for } i = 1, \ldots, s \right\},
\]

and

\[
J(x^0) = \{ j \in J \mid h_j(x^0) = 0 \},
\]

where \( J = \{1, \ldots, p\} \).

If the functions \( f, g \) and \( h \) in problem (P) are continuous differentiable with respect to \( x \in \mathbb{R}^n \) then [24] established the necessary sufficient optimality conditions as follows.

**Theorem A (Necessary conditions):** Let \( x^0 \) is an optimal solution of the problem (P) satisfying \( x^{0T} Bx > 0 \) and \( x^{0T} Dx > 0 \) and \( \nabla_x g_j(x^0) \), \( j \in J(x^0) \) being linear independent. Then there exist \( (s, t^*, \bar{y}) \in K(x^0) \), \( u, v \in \mathbb{R}^n \) and \( \mu^* \in \mathbb{R}^r_+ \) such that the Kuhn-Tucker type conditions hold. That is:

\[
\sum_{i=1}^{s} t_i^* \left( \nabla_x f(x^0, \bar{y}_i) + Bu \right) - k_0 \left( \nabla_x h(x^0, \bar{y}_i) - Dv \right) + \sum_{j=1}^{r} \mu_j^* \nabla_x g_j(x^0) = 0,
\]

\[
f(x^0, \bar{y}_i) + (x^{0T} Bx^0)^{1/2} - k_0 (h(x^0, \bar{y}_i) - (x^{0T} Dx^0)^{1/2}) = 0, \quad i = 1, \ldots, s
\]

\[
\sum_{j=1}^{r} \mu_j^* g_j(x^0) = 0,
\]

\[
u^T Bu \leq 1, \quad v^T Dv \leq 1,
\]

\[
x^{0T} Bu = (x^{0T} Bx^0)^{1/2}, \quad x^{0T} Dv = (x^{0T} Dx^0)^{1/2}.
\]

**Remark 5.**

In Theorem A; it is assumed that both the matrices \( A \) and \( B \) are positive definite at the solution \( x^0 \). If one of \( x^{0T} Bx^0 \) and \( x^{0T} Dx^0 \) is zero or both of them are zero; then for \( (s, t^*, \bar{y}) \in K(x^0) \), we define a set \( Z_\bar{y}(x^0) \) by

\[
Z_\bar{y}(x^0) = \left\{ z \in \mathbb{R}^n \mid \sum_{j=1}^{r} z_j \nabla_x g_j(x^0) \leq 0, j \in J(x^0) \text{ with any one of conditions (i)-(iii) holds} \right\}
\]

where

\[
(i) \quad x^{0T} Bx^0 > 0, \quad x^{0T} Dx^0 = 0
\]

\[
\implies z^T \left[ \sum_{i=1}^{s} t_i^* \left( \nabla_x f(x^0, \bar{y}_i) + \frac{Bx^0}{(x^{0T} Bx^0)^{1/2}} - k_0 \nabla_x h(x^0, \bar{y}_i) \right) \right] + (z^T (k_0^2) Dz)^{1/2} < 0
\]
(ii) $x^0^TBx^0 = 0, x^0^TDx^0 > 0$

$$\Rightarrow z^T \left[ \sum_{i=1}^s t_i^s \left( \nabla_x f(x^0, \bar{y}_i) - k_0 \left( \nabla_x h(x^0, \bar{y}_i) - \frac{Bx^0}{(x^0^TBx^0)^{1/2}} \right) \right) \right] + (z^TDz)^{1/2} < 0$$

(iii) $x^0^TBx^0 = 0, x^0^TDx^0 = 0$

$$\Rightarrow z^T \left[ \sum_{i=1}^s t_i^s \left( \nabla_x f(x^0, \bar{y}_i) - k_0 \nabla_x h(x^0, \bar{y}_i) \right) \right] + (z^T(k_0^zDz)^{1/2} + (z^TBz)^{1/2} < 0$$

If we insert the condition $Z_{\bar{y}}(x^0) = \emptyset$ in Theorem A; then the result of Theorem A still holds.

3 First duality model

Using Theorem A, we formulate the wolf dual model to the problem (P) as follows:

$$\text{(D) Maximize} \quad \sup_{s, t, \bar{y}} \quad F(u)$$

$$\text{s.t.} \quad (s, t, \bar{y}) \in K, \quad (z, \mu, u, v) \in H_1(s, t, \bar{y})$$

where

$$F(z) = \sup_{y \in Y} \frac{f(z, y) + (z^TBz)^{1/2}}{h(z, y) - (z^TDz)^{1/2}} \equiv \sup_{y \in Y} \phi(z, y).$$

and

$$H_1(s, t, \bar{y}) = \{(z, \mu, u, v) \in \mathbb{R}^n \times \mathbb{R}^n_+ \times \mathbb{R}^n \times \mathbb{R}^n|(i)-(iii) \text{ hold}\}$$

Here, $t = (t_1, \ldots, t_s) \in \mathbb{R}^s_+$ and $\sum_{i=1}^s t_i = 1$ and conditions (i)-(iii) are given as follows:

(i) $\sum_{j=1}^r t_i \left( (h(z, \bar{y}_i) - (z^TDz)^{1/2})(\nabla_x f(z, \bar{y}_i) + Bu) - (f(z, \bar{y}_i) + (z^TBz)^{1/2})(\nabla_x h(z, \bar{y}_i) - Dv) \right.$

$$\left. + \sum_{j=1}^r \mu_j \nabla_x g_j(z) = 0, \right)$$

(ii) $\sum_{j=1}^r \mu_j g_j(z) \geq 0$ \hfill (10)

(iii) $u^TBu \leq 1, v^TDv \leq 1,$

$$(z^TBz)^{1/2} = z^TBu,$$

$$(z^TDz)^{1/2} = z^TDv$$

(11)

If the set $H_1(s, t, \bar{y}) = \emptyset$, we define the supremum of $F(z)$ over $H_1(s, t, \bar{y})$ equal $-\infty$. For convenience, we let

$$\phi(.) = \sum_{j=1}^s t_i \left\{ (h(z, \bar{y}_i) - z^TDv)(f(., \bar{y}_i) + (.)^TBu) - (f(z, \bar{y}_i) + z^TBu)(h(., \bar{y}_i) - (.)^TDv) \right\}$$
\begin{align*}
\gamma_i(.) &= f(\cdot, \bar{y}_i) + (\cdot)^T Bu \\
\beta_i(.) &= h(\cdot, \bar{y}_i) - (\cdot)^T Dv
\end{align*}

and

for each \(i, i = 1, \ldots, s\), therefore,

\[
\phi(.) = \sum_{j=1}^{s} t_i \{ \beta_i(z) \gamma_i(.) - \gamma_i(z) \beta_i(.) \}. \tag{12}
\]

Then we can establish the following duality theorems.

**Theorem 1** (Weak duality). Let \(x^0\) be a feasible solution for (P) and \((z, \mu, u, v, s, t, \bar{y})\) be a feasible solution of the dual problem (D). Suppose that any one of the following conditions holds:

(a) functions \(f(\cdot, \bar{y}_i) + (\cdot)^T Bu\) and \(-h(\cdot, \bar{y}_i) + (\cdot)^T Dv, i = 1, \ldots, s\) are \((F, \alpha, \rho, d)\)-convex and \(-\sum_{j=1}^{r} h_j g_j(.)\) is \((F, \alpha, \rho, d)\)-quasiconcave, in which \(\rho = \rho \sum_{i=1}^{s} t_i (\beta_i(z) + \gamma_i(z))\),

(b) \(\phi(.)\) is \((F, \alpha, \rho, d)\)-convex and \(-\sum_{j=1}^{r} \mu_j g_j(.)\) is \((F, \alpha, \rho, d)\)-concave,

(c) \(\phi(.)\) is \((F, \alpha, \rho, d)\)-pseudoconvex and \(-\sum_{j=1}^{r} \mu_j g_j(.)\) is \((F, \alpha, \rho, d)\)-quasiconcave,

(d) \(\phi(.)\) is \((F, \alpha, \rho, d)\)-quasiconvex and \(-\sum_{j=1}^{r} \mu_j g_j(.)\) is \((F, \alpha, \rho, d)\)-pseudoconcave.

Then

\[
\sup_{y \in Y} \left( \frac{f(x^0, y) + (x^0 T B x^0)^{1/2}}{h(x^0, y) - (x^0 T D x^0)^{1/2}} \right) \geq F(z).
\]

**Proof.** By contradiction suppose

\[
\sup_{y \in Y} \left( \frac{f(x^0, y) + (x^0 T B x^0)^{1/2}}{h(x^0, y) - (x^0 T D x^0)^{1/2}} \right) < F(z). \tag{13}
\]

Since \(\bar{y}_i \in Y(z), i = 1, \ldots, s\) we have

\[
F(z) = \left( \frac{f(z, \bar{y}_i) + (z T B z)^{1/2}}{h(z, \bar{y}_i) - (z T D z)^{1/2}} \right), \quad i = 1, \ldots, s \tag{14}
\]

By (13) and (14); we have

\[
(h(z, \bar{y}_i) - (z^T D z)^{1/2})(f(x^0, y) + (x^0 T B x^0)^{1/2}) - (f(z, \bar{y}_i) + (z^T B z)^{1/2})(h(x^0, y) - (x^0 T D x^0)^{1/2}) < 0
\]

for all \(i = 1, \ldots, s\) and \(y \in Y\).

From \(\bar{y}_i \in Y(z) \subset Y\) and \(t_i \in \mathbb{R}_+^s\) with \(\sum_{i=1}^{s} t_i = 1\), we have

\[
\sum_{i=1}^{s} t_i \left\{ (h(z, \bar{y}_i) - (z^T D z)^{1/2})(f(x^0, y) + (x^0 T B x^0)^{1/2}) - (f(z, \bar{y}_i) + (z^T B z)^{1/2})(h(x^0, y) - (x^0 T D x^0)^{1/2}) \right\} < 0. \tag{15}
\]
From (11), (15) and (2), we have
\[ \phi(x^0) = \sum_{j=1}^{s} t_i \left\{ (h(z, \bar{y}_i) - z^T Dv)(f(x^0, \bar{y}_i) + x^{0T} Bu) - (f(z, \bar{y}_i) + z^T Bu)(h(x^0, \bar{y}_i) - x^{0T} Dv) \right\} \]
\[ \leq \sum_{i=1}^{s} t_i \left\{ (h(z, \bar{y}_i) - (z^T Dz)^{1/2})(f(x^0, \bar{y}_i) + (x^{0T} Bx^0)^{1/2}) \right\} 
- (f(z, \bar{y}_i) + (z^T Bz)^{1/2})(h(x^0, \bar{y}_i) - (x^{0T} Dx^0)^{1/2}) \right\} < 0 = \phi(z) \]

Hence
\[ \phi(x^0) < \phi(z). \quad (16) \]

Now if condition (a) holds, for each \( i, i = 1, \ldots, s \),
\[ \gamma_i(x^0) - \gamma_i(z) \geq F(x^0, z, \alpha(x^0, z)\nabla_x \gamma_i(z)) + \rho d^2(x^0, z) \quad (17) \]
\[ -\beta_i(x^0) + \beta_i(z) \geq F(x^0, z, -\alpha(x^0, z)\nabla_x \beta_i(z)) + \rho d^2(x^0, z) \quad (18) \]

We sum up the inequalities (17) and (18) after multiplying by \( t_i \beta_i(z) \) and \( t_i \gamma_i(z) \), respectively, then we have
\[ \sum_{i=1}^{s} t_i \{ \beta_i(z)\gamma_i(x^0) - \gamma_i(z)\beta_i(x^0) \} \geq F(x^0, z, \alpha(x^0, z) \sum_{j=1}^{s} t_i \nabla_x \gamma_i(z)\beta_i(z)) \]
\[ + F(x^0, z, -\alpha(x^0, z) \sum_{j=1}^{s} t_i \gamma_i(z)\nabla_x \beta_i(z)) + \rho d^2(x^0, z) \sum_{j=1}^{s} t_i (\beta_i(z) + \gamma_i(z)) \].

By sublinearity of \( F \) and definition of \( \bar{\rho} \) and (12) we have
\[ \phi(x^0) - \phi(z) \geq F(x^0, z, \alpha(x^0, z)\nabla_x \phi(z)) + \bar{\rho} d^2(x^0, z). \]

In view of (9) and \( (F, \alpha, \bar{\rho}, d) \)-quasiconcavity of \( -\sum_{j=1}^{r} \mu_j g_j(\cdot) \) we have
\[ \phi(x^0) - \phi(z) \geq F(x^0, z, -\alpha(x^0, z) \sum_{j=1}^{r} \mu_j \nabla_x g_j(z)) + \bar{\rho} d^2(x^0, z) \geq - \sum_{j=1}^{r} \mu_j g_j(x^0) + \sum_{j=1}^{r} \mu_j g_j(z). \]

Since \( x^0 \in X, \mu \in R_+^r \) and by (10) we have
\[ \phi(x^0) - \phi(z) \geq 0. \]
So
\[ \phi(x^0) \geq \phi(z) \]
which contradicts the fact of (16) and so (13) is false.

If condition (b) holds, and under the contrary inequality (13) then by the \( (F, \alpha, \rho, d) \)-convexity of \( \phi(\cdot) \), we get from (16) that
\[ F(x^0, z, \alpha(x^0, z)\nabla_x \phi(z)) < -\rho d^2(x^0, z) \]
From (9) we get
\[ F(x^0, z, -\alpha(x, z) \sum_{j=1}^{r} \mu_j \nabla_x g_j(z)) + \rho d^2(x^0, z) < 0. \]

The \((F, \alpha, \rho, d)\)-concavity of \(- \sum_{j=1}^{r} \mu_j g_j(.)\) implies that
\[ - \sum_{j=1}^{r} \mu_j g_j(x^0) + \sum_{j=1}^{r} \mu_j g_j(z) < 0 \]

From \(x^0 \in X\) and \(\mu \in \mathbb{R}^r_+\) we have
\[ \sum_{j=1}^{r} \mu_j g_j(z) < 0 \]
This contradicts (10).

For condition (c), it follows by the same lines as the proof given in (b).
If the condition (d) is holds and under the contrary inequality (13) then by the \((F, \alpha, \rho, d)\)-quasiconvexity of \(\phi(.\)) we get from (16) that
\[ F(x^0, z, \alpha(x^0, z) \nabla_x \phi(z)) \leq -\rho d^2(x^0, z) \]
From (9) we get
\[ F(x^0, z, -\alpha(x^0, z) \sum_{j=1}^{r} \mu_j \nabla_x g_j(z)) \leq -\rho d^2(x^0, z) \]
and by \((F, \alpha, \rho, d)\)-pseudoconcavity of \(- \sum_{j=1}^{r} \mu_j g_j(.)\) we have
\[ - \sum_{j=1}^{r} \mu_j g_j(x^0) + \sum_{j=1}^{r} \mu_j g_j(z) < 0. \]
Therefore
\[ \sum_{j=1}^{r} \mu_j g_j(z) < 0 \]
This contradicts the fact of (10). Hence the proof is complete.

**Theorem 2** (Strong duality). Let \(x^*\) be an optimal solution of problem (P). Suppose that \(x^*\) satisfies the set \(Z_{y^*}(x^*)\) defined in the remark 5 so that \(Z_{y^*}(x^*) = \emptyset\) for (P) and \(\nabla_x g_j(x^0), j \in J(x^0)\) being linear independent. Then there exist \((s^*, t^*, y^*) \in K(x^*)\) and \((x^*, \mu^*, u^*, v^* ) \in H(s^*, t^*, y^*)\) such that \((x^*, \mu^*, u^*, v^*, s^*, t^*, y^*)\) is a feasible solution of (D). Furthermore; if for each feasible solution of (D), the hypothesis of Theorem 1 are also fulfilled; then the point \((x^*, \mu^*, u^*, v^*, s^*, t^*, y^*)\) is an optimal solution for (D); and the problem (P) and its dual problem (D) have the same optimal values.

**Proof.** See [23].
Theorem 3 (Strictly converse duality). Let $x^*$ and $(z, v, w, t, \tilde{y})$ be optimal solution of $(P)$ and $(D)$, respectively. Suppose that the assumptions of Theorem 2 are fulfilled; and that any one of the following conditions holds:

(a) one of the functions $f(., \tilde{y}_i) + (.)^T B u$ and $-h(., \bar{y}_i) + (.)^T D v$, $i = 1, \ldots, s$ are strictly $(F, \alpha, \rho, d)$-convex,

(b) either $\phi(.)$ is $(F, \alpha, \rho, d)$-convex and $-\sum_{j=1}^r \mu_j g_j(.)$ is strictly $(F, \alpha, \rho, d)$-concave or $\phi(.)$ is strictly $(F, \alpha, \rho, d)$-convex and $-\sum_{j=1}^r \mu_j g_j(.)$ $(F, \alpha, \rho, d)$-concave,

(c) $\phi(.)$ is strictly $(F, \alpha, \rho, d)$-pseudoconcave and $-\sum_{j=1}^r \mu_j g_j(.)$ is $(F, \alpha, \rho, d)$-quasiconcave.

Then $z = x^*$; that is, $z$ is also an optimal solution of $(P)$.

Proof. We mention here only on the case that condition (c) holds. While the other cases (a) and (b); one can follow along with the same arguments as given the case of condition (c). We assume that $z \neq x^*$ and exhibit a contradiction. Since $x^*$ is an optimal solution of $(P)$ and the corresponding set $Z_\tilde{g}(x^*)$ is empty, then there exist $(s^*, t^*, \tilde{y}^*) \in K(x^*)$ and $(x^*, \mu^*, u^*, v^*) \in H(s^*, t^*, \tilde{y}^*)$ such that $(x^*, \mu^*, u^*, v^*, t^*, \tilde{y}^*)$ is a optimal solution of $(D)$. By the assumptions of Theorem, $(z, \mu, v, w, t, \tilde{y})$ is optimal solution of $(D); therefore, F(x^*) = F(z)$ i.e.

$$\sup_{y \in Y} \left( \frac{f(x^*, y) + (x^* T B x^*)^{1/2}}{h(x^*, y) - (x^* T D x^*)^{1/2}} \right) = \sup_{y \in Y} \left( \frac{f(z, y) + (z^T B z)^{1/2}}{h(z, y) - (z^T D z)^{1/2}} \right).$$

Since $\tilde{y}_i \in \bar{Y}(z)$, for all $i = 1, \ldots, s$,

$$\sup_{y \in Y} \left( \frac{f(x^*, y) + (x^* T B x^*)^{1/2}}{h(x^*, y) - (x^* T D x^*)^{1/2}} \right) = \left( \frac{f(z, \tilde{y}_i) + (z^T B z)^{1/2}}{h(z, \tilde{y}_i) - (z^T D z)^{1/2}} \right).$$

Therefore

$$\frac{f(x^*, y) + (x^* T B x^*)^{1/2}}{h(x^*, y) - (x^* T D x^*)^{1/2}} \leq \frac{f(z, \tilde{y}_i) + (z^T B z)^{1/2}}{h(z, \tilde{y}_i) - (z^T D z)^{1/2}}.$$

for each $y \in Y$. Also since $\bar{Y}(z) \subset Y$, we have

$$\frac{f(x^*, \tilde{y}_i) + (x^* T B x^*)^{1/2}}{h(x^*, \tilde{y}_i) - (x^* T D x^*)^{1/2}} \leq \frac{f(z, \tilde{y}_i) + (z^T B z)^{1/2}}{h(z, \tilde{y}_i) - (z^T D z)^{1/2}}. \quad (19)$$

for all $i = 1, \ldots, s$.

By the generalized Schwartz inequality (1) and (11) we have

$$x^* D v \leq (x^* T D x^*)^{1/2}$$

$$x^* B u \leq (x^* T B x^*)^{1/2} \quad (20)$$

From (11), (19) and (20) we have

$$\frac{f(x^*, \tilde{y}_i) + (x^* T B u)}{h(x^*, \tilde{y}_i) - (x^* T D v)} \leq \frac{f(z, \tilde{y}_i) + (z^T B u)}{h(z, \tilde{y}_i) - (z^T D v)}.$$

for all $i = 1, \ldots, s$.  

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The relation (21) can be written as
\[(h(z, \bar{y}_i) - (z^T Dv))(f(x^*, \bar{y}_i) + (x^T Bu)) - (f(z, \bar{y}_i) + (z^T Bu))(h(x^*, \bar{y}_i) - (x^T Dv)) \leq 0\]
for all \(i = 1, \ldots, s\).

We sum up the above inequality after multiplying by \(t_i \geq 0, i = 1, \ldots, s\). then by definition of \(\phi(.)\) we have
\[
\phi(x^*) = \sum_{i=1}^{s} t_i \left\{ (h(z, \bar{y}_i) - (z^T Dv))(f(x^*, \bar{y}_i) + (x^T Bu)) - (f(z, \bar{y}_i) + (z^T Bu))(h(x^*, \bar{y}_i) - (x^T Dv)) \right\} \leq 0.
\]
Therefore
\[
\phi(x^*) \leq 0 = \phi(z). \tag{22}
\]

On the other hand, since \(x^*\) is feasible of (P) and \(\mu \geq 0\) then,
\[
-\sum_{j=1}^{r} \mu_j g_j(x^*) \geq 0. \tag{23}
\]

Also since \((z, \mu, v, w, s, t, \bar{y})\) is feasible solution of (D); we get
\[
-\sum_{j=1}^{r} \mu_j g_j(z) \leq 0. \tag{24}
\]

From (23) and (24) we have
\[
-\sum_{j=1}^{r} \mu_j g_j(x^*) \geq -\sum_{j=1}^{r} \mu_j g_j(z)
\]
which, because of \((F, \alpha, \rho, d)\)-quasiconcavity of \(-\sum_{j=1}^{r} \mu_j g_j(.)\), implies
\[
F(x^*, z, -\alpha(x^*, z) \sum_{j=1}^{r} \mu_j \nabla x g_j(z)) \geq -\rho d^2(x^*, z).
\]

Using this in (i), we have
\[
F(x^*, z, -\alpha(x^*, z) \sum_{j=1}^{r} \mu_j \nabla x \phi(z)) \geq -\rho d^2(x^*, z).
\]

By strictly \((F, \alpha, \rho, d)\)-pseudoconvexity of \(\phi(.)\), this implies
\[
\phi(x^*) > \phi(z).
\]

This contradicts (22). Hence \(x^* = z\) i.e., \(z\) is an optical solution of (P). This completes the proof.
References


