Logarithmic correction to resistance

Antal A. Járai[1] Dante Mata López[2]

June 9, 2020

Abstract

We study the trace of the incipient infinite oriented branching random walk in $\mathbb{Z}^d \times \mathbb{Z}_+$ when the dimension is $d = 6$. Under suitable moment assumptions, we show that the electrical resistance between the root and level n is $O(n \log^{-\xi} n)$ for a $\xi > 0$ that does not depend on details of the model.

1 Introduction

Consider a critical branching random walk in $\mathbb{Z}^d$ conditioned to survive forever, starting with a single individual at the origin o. The space-time points visited by this process can be turned into a multi-graph, by placing an edge between (x, n) and $(y, n+1)$, whenever a particle located at x at time n produces an offspring located at y at time $n+1$. We call this multi-graph the *trace*. We regard the trace as an electrical network, where each edge has unit conductance. Let $R(n)$ denote the expected resistance between $(o, 0)$ and level n of the trace.

Barlow et al. [4, Example 1.8(iii)] showed that when $d > 6$, one has $R(n) \asymp n$ Answering a question of [4], Járai and Nachmias [14] showed that for $d \leq 5$ one has $R(n) = O(n^{1-\alpha})$ for a universal constant $\alpha > 0$, under suitable moment assumptions. In the present paper we show, under the same moment assumptions as in [14], that in $d = 6$ dimensions $R(n)$ is sub-linear by at least a logarithmic factor.

Theorem 1.1. *Consider the trace of a branching random walk in dimension $d = 6$ with progeny distribution that is critical, has positive variance and finite third moment, conditioned to survive forever. Assume that the random walk steps are symmetric, non-degenerate and have exponential tails. Then there exists a universal $\xi > 0$ such that*

$$R(n) = O\left(n \log^{-\xi} n\right).$$

[1]Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, United Kingdom. E-mail: A.Jarai@bath.ac.uk

[2]Department of Probability and Statistics, Centro de Investigación en Matemáticas A.C. Calle Jalisco S/N, C.P. 36240, Guanajuato, Mexico. E-mail: dante.mata@cimat.mx

Let us explain some background and our motivation. Given an infinite graph G, suitable bounds on the volume growth of G and electrical resistances in G provide quantitative information on the behaviour of the simple random walk on G, such as bounds on exit times from balls, and the heat kernel; see e.g. [4], [17]. One can distinguish two regimes.

On the one hand, several examples are known where the behaviour of the walk is characterised by the so-called Alexander-Orbach (AO) exponents [1]. See [15] for the case of a critical branching tree conditioned to survive, and [5] for more detailed estimates in the case of percolation on regular trees. The same exponents were also shown for: the incipient infinite cluster of oriented (spread-out) percolation in dimensions $d + 1 > 6 + 1$ [4]; for the oriented critical branching random in dimension $d + 1 > 6 + 1$ [4, Example 1.8(iii)] ; for unoriented percolation under the triangle condition [16], [12]; and for the uniform spanning forest in $d > 4$ and on non-amenable graphs [13]. In all of these examples, the underlying graph is similar, in a quantifiable sense, to a critical branching process, and the scaling limit of the walk is conjectured, and in some cases rigorously known, to be Brownian motion on the continuum random tree, see e.g. [7], [6].

On the other hand, there are 'low-dimensional' examples where the random walk exponents are different (either conjecturally, or rigorously). See [15], [9] for the incipient infinite percolation cluster in 2D; see [3] for the uniform spanning tree in 2D and [2] for the uniform spanning tree in 3D. Despite this progress, it is a major challenge, for example, to establish the scaling of resistances for 2D critical percolation. This is due to lack of a clear connection between resistance and conformally invariant quantities.

A consequence of [14] is that for the trace of oriented critical branching random walk, the AO exponents cannot hold in any dimension $d + 1 \leq 5 + 1$. This results from the fact that the resistance does not scale linearly: $R(n) = O(n^{1-\alpha})$ for some $\alpha > 0$. Although branching random walk is one of the simplest statistical physics models, determining the exact behaviour of the resistance for $d \leq 6$ is already very challenging in this case. To the best of our knowledge, no polynomial lower bound on $R(n)$ is known in $3 \leq d \leq 5$, where such is expected. In this paper we consider the trace of oriented branching random walk when $d = 6$, which is the conjectured critical dimension. By analogy with other statistical physics models, one expects a logarithmic correction: $R(n) \asymp n \log^{-\xi'} n$ for some $\xi' > 0$, and Theorem 1.1 provides an upper bound of this form. We expect that with additional work one can prove a lower bound of the same form (with an exponent $\xi'' < \infty$), and we outline a possible strategy for this at the end of Section 1.3. We also explain there why this is easier than proving meaningful lower bounds in $3 \leq d \leq 5$.

We follow a very similar setup to that of [14], in that we establish our upper bound by showing that sufficiently many intersections are present in the trace to reduce the resistance from $O(n)$ to $O(n \log^{-\xi} n)$. The main difference is that in $d = 6$ the intersections are more sparse than in $d \leq 5$. In particular, on each scale, there is only a logarithmically small probability to find intersecting paths on that scale. Establishing this intersection estimate is more delicate compared to its analogue in [14], and some of the other estimates also need improvement.

In order to facilitate the import of the setup from [14], we use the following convention: all notation that has the same meaning as in [14] is identical in the present paper, and notation that has closely related meaning is denoted by a prime. For clarity of the proofs, we found it necessary to spell out even smaller changes compared to [14]. However, we do take some arguments without change from [14], and hence familiarity with that paper is essential to understand our arguments.

1.1 Assumptions

Let $p(k)$, $k \geq 0$ be a progeny distribution that satisfies:
(i) $\sum_k kp(k) = 1$;
(ii) $\sum_k k(k-1)p(k) = \sigma^2 \in (0, \infty)$;
(iii) $\sum_k k^3 p(k) \leq C_3 < \infty$.
The *incipient infinite branching process* is obtained by conditioning on survival up to time n and taking the weak limit as $n \to \infty$. The limiting object admits the following alternative construction [15], [19]. Consider an infinite path $(V_0, V_1, \dots)$, and attach to each V_i, independently, a branching tree that in its first generation follows the size-biased distribution:

$$\widetilde{p}(k) = (k+1)p(k+1), \quad k \geq 0,$$

and follows p afterwards.

Let $\mathbf{p}^1(x, y)$ be a one-step random walk transition probability in $\mathbb{Z}^d$ that satisfies:
(i) $\sum_{x \in \mathbb{Z}^d} e^{b|x|} \mathbf{p}^1(o, x) < \infty$ for some $b > 0$;
(ii) $\{x \in \mathbb{Z}^d : \mathbf{p}^1(o, x) > 0\}$ generates $\mathbb{Z}^d$ as a group;
(iii) $\mathbf{p}^1(x, y) = \mathbf{p}^1(y, x)$.
We will denote by $\mathbf{p}^n(x, y)$ the n-step transition probabilities. The *incipient infinite branching random walk* is obtained by first drawing a sample $\mathcal{T}$ of the incipient infinite branching process, and then applying a random walk map $\Phi : \mathcal{T} \to \mathbb{Z}^d \times \mathbb{Z}_+$ defined as follows. We initialize Φ by requiring that the root ρ of $\mathcal{T}$ is mapped to $(o, 0)$. Then, recursively, if $\{U, V\}$ is an edge of $\mathcal{T}$ between generations n and $n+1$, such that $\Phi(U) = (x, n)$ has already been defined, we set $\Phi(V) = (y, n+1)$ with the displacement $y - x$ chosen according to $\mathbf{p}^1(x, y)$, independently between different edges. By the *trace* of the branching random walk we mean the multi-graph with vertex set $\Phi(\mathcal{T})$, and edge set consisting of $\{\Phi(U), \Phi(V)\}$ for every edge $\{U, V\}$ of $\mathcal{T}$.

1.2 Electrical resistance

For background on electrical resistance, see [20]. We denote by $R_{\text{eff}}(x \leftrightarrow y)$ the effective resistance between vertices x and y. We will frequently use the triangle inequality:

$$R_{\text{eff}}(x \leftrightarrow z) \leq R_{\text{eff}}(x \leftrightarrow y) + R_{\text{eff}}(y \leftrightarrow z). \tag{1.1}$$

We will also use the following *parallel law*: if $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ are graphs on the same vertex set but with disjoint edge sets, and $G = (V, E_1 \cup E_2)$, and if $R_1 = R_{\text{eff}}(x \overset{G_1}{\leftrightarrow} y)$,

$R_2 = R_{\text{eff}}(x \overset{G_2}{\leftrightarrow} y)$, then

$$R_{\text{eff}}(x \overset{G}{\leftrightarrow} y) \le \left(\frac{1}{R_1} + \frac{1}{R_2} \right)^{-1} \le \frac{1}{4}(R_1 + R_2), \tag{1.2}$$

where the second inequality uses that the harmonic mean is at most the arithmetic mean.

1.3 A finite approximation

Given $n \ge 1$ and $m \ge 2n$, we define the random tree $\mathcal{T}_{n,m}$ as in [14]:
(i) consider a backbone $V_0, \dots, V_n$ with marked root $\rho = V_0$;
(ii) attach to each V_i a critical tree that has distribution $\widetilde{p}$ in the first step, and p afterwards, and is conditioned to die out by time $m - i$.

We define $\gamma(n, x)$ as in [14]:

$$\gamma(n, x) = \sup_{m \ge 2n} \mathbf{E}_{\mathcal{T}_{n,m}} \Big[R_{\text{eff}}\left((o, 0) \leftrightarrow \Phi(V_n)\right) \Big| \Phi(V_n) = (x, n) \Big], \quad x \in \mathbb{Z}^d.$$

The following theorem is our main technical result, and is an analogue of [14, Theorem 1.2]. It is expressed in terms of the norm:

$$\|x\| := \sqrt{\frac{1}{d} \sum_{i,j=1}^{d} x_i Q_{ij}^{-1} x_j},$$

where $Q_{ij} = \sum_{x \in \mathbb{Z}^d} x_i x_j \mathbf{p}^1(o, x)$ is the covariance matrix of the random walk step distribution.

Theorem 1.2. *Assume $d = 6$. There exists a universal constant $\xi \in (0, 1/2)$ and $A = A(\sigma^2, C_3, \mathbf{p}^1) < \infty$ such that for all $n \ge 2$ we have*

$$\gamma(n, x) \le \begin{cases} A n (\log n)^{-\xi} & \text{when } \|x\| \le \sqrt{n}; \\ A n (\log n)^{-\xi} \left(1 - \frac{\log\left(\|x\|^2/n\right)}{\log n} \right)^{-\xi} & \text{when } \sqrt{n} < \|x\| \le n/2; \\ A n & \text{when } \|x\| > n/2. \end{cases}$$

Proof of Theorem 1.1 assuming Theorem 1.2. As in [14], we have

$$\begin{aligned} R(n) &\le \lim_{m \to \infty} \mathbf{E}_{\mathcal{T}_{n,m}} \Big[R_{\text{eff}}\left((o, 0) \leftrightarrow \Phi(V_n)\right) \Big] \\ &\le \sup_{m \ge 2n} \mathbf{E}_{\mathcal{T}_{n,m}} \Big[R_{\text{eff}}\left((o, 0) \leftrightarrow \Phi(V_n)\right) \Big] \\ &= \sum_{x \in \mathbb{Z}^d} \mathbf{p}^n(o, x)\, \gamma(n, x). \end{aligned}$$

From the bound in Theorem 1.2 we have

$$R(n) \le An(\log n)^{-\xi}\Bigg[\sum_{x:\|x\|\le\sqrt{n}} \mathbf{p}^n(o,x) + \sum_{x:\sqrt{n}<\|x\|\le n^{3/4}} \mathbf{p}^n(o,x)\,C$$

$$+ \sum_{x:n^{3/4}<\|x\|\le n/2} \mathbf{p}^n(o,x)\,C\,(\log n)^{\xi} + \sum_{x:n/2<\|x\|} \mathbf{p}^n(o,x)\,(\log n)^{\xi}\Bigg].$$

The first two sums are bounded by 1 and C, respectively. In the third and fourth sums we use that

$$(\log n)^{\xi} \le C\, n^{1/2} \le C\,\frac{\|x\|^2}{n}, \quad \text{when } \|x\| > n^{3/4}.$$

This gives the upper bound

$$C\sum_x \mathbf{p}^n(o,x)\,\frac{\|x\|^2}{n} = C.$$

□

1.4 Strategy for a lower bound

Let us now present a possible strategy for a lower bound on $R(n)$ in $d = 6$. Consider independent copies $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathcal{T}_{n,2n}$, and random walk mappings Φ_1 and Φ_2 initialized by $\Phi_1(\rho_1) = (o,0)$ and $\Phi_2(\rho_2) = (x,0)$, where $\|x\| \asymp \sqrt{n}$. The first and second moments of the number of intersections between $\Phi_1(\mathcal{T}_1)$ and $\Phi_2(\mathcal{T}_2)$ are $\asymp 1$ and $\asymp \log n$, respectively, in $d = 6$, as we will see. A key estimate we prove in this paper is the following (presented somewhat informally at this stage):

$$\mathbf{P}(\#\text{intersections of } \Phi_1(\mathcal{T}_1) \text{ and } \Phi_2(\mathcal{T}_2) \ge c\log n) \ge \frac{c}{\log n}. \tag{1.3}$$

Suppose for what follows that one could complement this with the bound:

$$\mathbf{P}(\Phi_1(\mathcal{T}_1) \cap \Phi_2(\mathcal{T}_2) = \emptyset) \ge 1 - \frac{C}{\log n}. \tag{1.4}$$

Consider the event that in $\Phi(\mathcal{T}_{2n,m})$, the edge $\{\Phi(V_n), \Phi(V_{n+1})\}$ is pivotal for connecting $(o,0)$ to level $2n$ of the trace. Let Tr_k denote the Φ-image of all critical trees attached to the path $\{V_{n-2^k}, \dots, V_{n-2^{k-1}}\}$, $k = 1, 2, \dots, \log_2 n$, and let $\widetilde{\mathrm{Tr}}$ denote the Φ-image of all critical trees attached to the path $\{V_{n+1}, \dots, V_{2n}\}$. By the FKG inequality, and a heuristic based on (1.4), we get

$$\mathbf{P}(\{\Phi(V_n), \Phi(V_{n+1})\} \text{ is pivotal}) \ge \prod_{k=1}^{\log_2 n} \mathbf{P}(\mathrm{Tr}_k \cap \widetilde{\mathrm{Tr}} = \emptyset) \ge \prod_{k=1}^{\log_2 n}\left(1 - \frac{C}{\log 2^k}\right) \ge c\,(\log n)^{-O(1)}.$$

This would imply that there are at least $n\log^{-\xi''} n$ pivotals along the backbone of $\mathcal{T}_{2n,m}$, and hence a lower bound on the resistance. For the same ideas to be fruitful in $3 \le d \le 5$, one would also need the exponent to be < 1, not only $O(1)$.

1.5 Random walk estimates

The following proposition collects some random walk estimates we take without change from [14]. Let $S(n)$, $n \geq 0$ denote a random walk with $S(0) = o$, and step distribution $\mathbf{p}^1$.

Proposition 1.3 ([14, Proposition 1.3])**.** *There exists $k_1 = k_1(\mathbf{p}^1), C > 0$ and $\delta_1 = \delta_1(d) > 0$ such that the following hold.*

(i) Whenever $k_1 \leq k \leq n$, $\|x\| \leq 4n/\sqrt{k}$, we have

$$\mathbf{E}\Big[\|S(k)\|^2 \,\Big|\, S(n) = x\Big] \leq Ck.$$

(ii) Whenever $k_1 \leq k \leq \delta_1 n$, $\|x\| \leq 4n/\sqrt{k}$, we have

$$\mathbf{E}\Big[\|S(k)\|^2 \,\Big|\, S(n) = x,\ \|S(k)\| > \sqrt{k}\Big] \leq Ck.$$

(iii) Whenever $k_1 \leq k \leq \delta_1 n$, $k_1 \leq k' \leq n - k$ and $\|x\| \leq \min\{4n/\sqrt{k}, 4n/\sqrt{k'}\}$, we have

$$\mathbf{E}\Big[\|S(k+k') - S(k)\|^2 \,\Big|\, S(n) = x,\ \|S(k)\| > \sqrt{k}\Big] \leq Ck'.$$

Let us write

$$D := \det(Q)^{1/2d}.$$

The following lemma is a special case of [14, Lemma 1.4] (take $\beta = 0$ there).

Lemma 1.4. *There exists $C = C(d)$ such that the following hold.*
(i) There exists $n_1 = n_1(\mathbf{p}^1)$ such that for all $y \in \mathbb{Z}^d$ we have

$$\mathbf{P}(S(n) = y) \leq \frac{2C}{D^d n^{d/2}}\,, \tag{1.5}$$

when $n \geq n_1$.
(ii) For any $0 < \varepsilon < 1$ and $0 < L < \infty$ there exists $n_2 = n_2(\mathbf{p}^1, \varepsilon, L)$ such that for all $y \in \mathbb{Z}^d$ such that $\|y\| \leq L\sqrt{n}$ we have

$$\begin{aligned} \mathbf{P}(S(n) = y) &\leq \frac{C(1+\varepsilon)}{D^d n^{d/2}} e^{-d\|y\|^2/(2n)}\,, \\ \mathbf{P}(S(n) = y) &\geq \frac{C(1-\varepsilon)}{D^d n^{d/2}} e^{-d\|y\|^2/(2n)}\,. \end{aligned} \tag{1.6}$$

when $n \geq n_2$.

Above we assumed that the random walk has period 1. Trivial modifications can be made when the period is 2, and we will not make this explicit in our arguments.

Below we collect a few more frequently used facts. These are all standard (see [14, Section 1.5] for more information). There exists a constant $c > 0$ such that we have

$$\sum_{x:\|x\|\le L} 1 \ge cD^d L^d, \quad L \ge 1. \tag{1.7}$$

When $d \ge 3$, the Green function $G(x) := \sum_{n=0}^{\infty} \mathbf{p}^n(o,x)$ satisfies

$$G(x) \le \frac{C(d)}{D^d} \|x\|^{2-d}, \quad \text{when } \|x\| \ge L_1 = L_1(\mathbf{p}^1). \tag{1.8}$$

2 Induction Scheme

We set up the induction scheme as in [14], apart from the definition of the event $\mathcal{B}(i, c_0)$. For convenience of the reader, we provide Definitions 2.1–2.6 below, that are from [14]. Given an instance of $\mathcal{T}_{n,m}$, consider a small $\delta > 0$, such that δn is an integer. We write

$$X_i = V_{i\delta n}, \quad i = 0, 1, \dots, \lfloor \delta^{-1} \rfloor,$$

and write $x_i \in \mathbb{Z}^d$, $i = 0, 1, \dots, \lfloor \delta^{-1} \rfloor$ for the spatial location of X_i, so that $\Phi(X_i) = (x_i, i\delta n)$. Write $\mathcal{T}_{n,m}(\ell)$ for the subtree of $\mathcal{T}_{n,m}$ emanating from V_ℓ off the backbone (including V_ℓ).

Fix an integer K and write $n = NK\delta n + K'\delta n + \tilde{n}$, with $0 \le K' < K$ an integer and $\delta n \le \tilde{n} < 2\delta n$. The definitions to follow are illustrated in Figure 1.

Definition 2.1. For ℓ satisfying $i\delta n \le \ell < (i+1)\delta n$ we say that a backbone vertex V_ℓ has the *unique descendant property* (UDP) if among its descendants at level $(i+1)\delta n$ in $\mathcal{T}_{n,m}(\ell)$ there is a unique one that reaches level $(i+2)\delta n$. For any other vertex V of $\mathcal{T}_{n,m}$ at level $i\delta n$ we say that V has UDP if among its descendants at level $(i+1)\delta n$ there is a unique one that reaches level $(i+2)\delta n$.

Definition 2.2. Given an integer $K \ge 1$, a number $\delta > 0$ such that $K\delta \le (1/2)$ and an instance of $\mathcal{T}_{n,m}$ we say that a sequence $(i, i+1, \dots, i+K)$ of length $K+1$ is *K-tree-good* if the following holds:

(1) There exists a unique $i\delta n \le \ell_1 < (i+1)\delta n$ such that $\mathcal{T}_{n,m}(\ell_1)$ reaches height $(i+2)\delta n$. Moreover, this unique ℓ_1 satisfies $(i+1/4)\delta n \le \ell_1 \le (i+3/4)\delta n$.

(2) V_{ℓ_1} has UDP. We call the unique descendant $\mathcal{Y}_{i+1}$. For all i' satisfying $i+2 \le i' \le i+K$ we inductively define the vertices $\mathcal{Y}_{i'}$ of $\mathcal{T}_{n,m}(\ell_1)$ as follows. We require that $\mathcal{Y}_{i'-1}$ has UDP and call the unique descendant $\mathcal{Y}_{i'}$.

(3) There exists a unique $(i+K-1)\delta n \le \ell_2 < (i+K)\delta n$ such that $\mathcal{T}_{n,m}(\ell_2)$ reaches height $(i+K+1)\delta n$. Moreover, this unique ℓ_2 satisfies $(i+K-3/4)\delta n \le \ell_2 \le (i+K-1/4)\delta n$.

(4) V_{ℓ_2} has UDP, and we call the unique descendant $\mathcal{X}'_{i+K}$. The vertex $\mathcal{X}'_{i+K}$ has UDP, and we call the unique descendant $\mathcal{X}'_{i+K+1}$. Similarly, $\mathcal{Y}_{i+K}$ has UDP, and we call the unique descendant $\mathcal{Y}_{i+K+1}$.

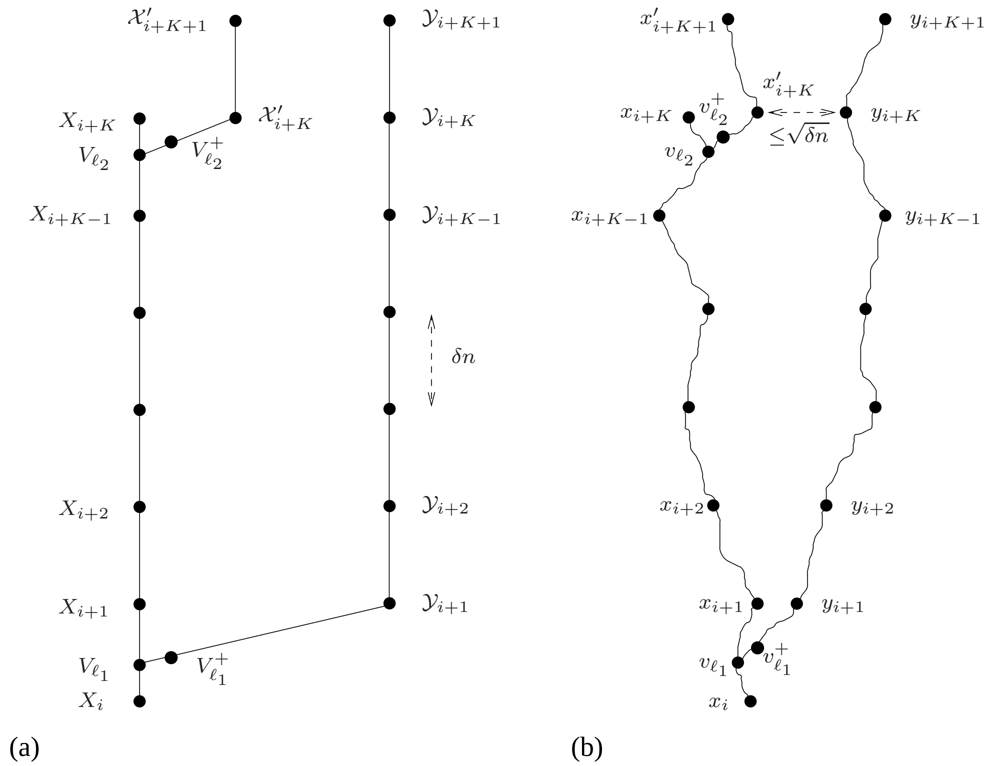

Figure 1: (a) Illustration of K-tree-good. (b) Illustration of K-spatially-good. All spatial distances between consecutive vertices are at most $\sqrt{\text{time difference}}$ and the spatial distance between x'_{i+K} and y_{i+K} is at most $\sqrt{\delta n}$. [Figure reused by permission of Springer, from *Commun. Math. Phys.* **331** (2014), 67–109 (Electrical resistance of the low-dimensional critical branching random walk. A.A. Járai and A. Nachmias) © (2014).]

Given a K-tree-good sequence $(i,\dots,i+K)$ we denote by $V^+_{\ell_1}$ (respectively $V^+_{\ell_2}$) the child of V_{ℓ_1} (respectively V_{ℓ_2}) leading to $\mathcal{Y}_{i+1}$ (respectively $\mathcal{X}'_{i+K}$). We further define the spatial locations $y_{i'}$ by $\Phi(\mathcal{Y}_{i'}) = (y_{i'}, i'\delta n)$ for $i+1 \le i' \le i+K+1$, and we similarly define x'_{i+K}, x'_{i+K+1}, v_{ℓ_1}, $v^+_{\ell_1}$, v_{ℓ_2}, $v^+_{\ell_2}$.

We will write $U \prec W$ to denote that W is a descendant of U, and write $h(U), h(W)$ for their respective heights in the tree (in particular, $h(W) > h(U)$).

Definition 2.3. Let $U \prec W$ be two tree vertices and let $u, w \in \mathbb{Z}^d$ be defined by $\Phi(U) = (u, h(U))$ and $\Phi(W) = (w, h(W))$. We say that U and W are *typically-spaced* if $\|w - u\| \le \sqrt{h(W) - h(U)}$. Denote this event by $\mathcal{TS}(U, W)$.

Definition 2.4. We say that a K-tree-good sequence $(i, \ldots, i + K)$ is *K-spatially-good* if the following holds.

(5)
- $\mathcal{TS}(X_i, V_{\ell_1})$,
- $\mathcal{TS}(V_{\ell_1+1}, X_{i+1})$,
- For each $i + 1 \le j \le i + K - 2$ we have $\mathcal{TS}(X_j, X_{j+1})$,
- $\mathcal{TS}(X_{i+K-1}, V_{\ell_2})$,
- $\mathcal{TS}(V_{\ell_2+1}, X_{i+K})$,

(6)
- $\mathcal{TS}(V^+_{\ell_1}, \mathcal{Y}_{i+1})$,
- For each $i + 1 \le j \le i + K - 1$ we have $\mathcal{TS}(\mathcal{Y}_j, \mathcal{Y}_{j+1})$,
- $\mathcal{TS}(V^+_{\ell_2}, \mathcal{X}'_{i+K})$,
- $\|x'_{i+K} - y_{i+K}\| \le \sqrt{\delta n}$.

Definition 2.5. When a sequence $(i, \ldots, i + K)$ is both K-tree-good and K-spatially-good we say that it is *K-good*. Let $\mathcal{A}(i)$ be the event that $(i, \ldots, i + K)$ is K-good.

Next, let $(i, \ldots, i + K)$ be a K-good sequence and let U_1, U_2 be two vertices at the same height such that $U_1 \succ \mathcal{X}'_{i+K}$ and $U_2 \succ \mathcal{Y}_{i+K}$. Given these, we write Z_1 for the highest common ancestor of U_1 and $\mathcal{X}'_{i+K+1}$ and Z_2 for the highest common ancestor of U_2 and $\mathcal{Y}_{i+K+1}$ (see Figure 2). Further, we denote by Z_1^+ (respectively Z_2^+) the child of Z_1 (respectively Z_2) leading to U_1 (respectively U_2).

Definition 2.6. We say that U_1, U_2 *intersect-well* if the following conditions hold:

1. $U_1 \succ \mathcal{X}'_{i+K}$, $U_2 \succ \mathcal{Y}_{i+K}$,
2. $(i + K + (5/6))\delta n \le h(U_1) = h(U_2) \le (i + K + 1)\delta n$;
3. $(i + K + (1/2))\delta n \le h(Z_1), h(Z_2) \le (i + K + (4/6))\delta n$;
4. $\mathcal{TS}(\mathcal{X}'_{i+K}, Z_1)$, $\mathcal{TS}(Z_1^+, U_1)$, $\mathcal{TS}(\mathcal{Y}_{i+K}, Z_2)$, $\mathcal{TS}(Z_2^+, U_2)$;
5. $\Phi(U_1) = \Phi(U_2)$.

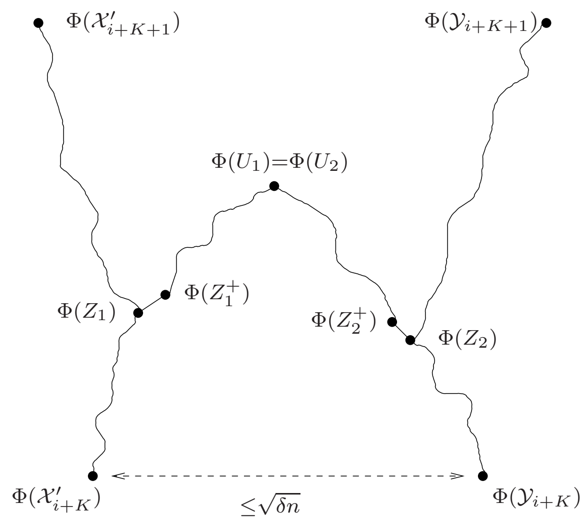

Figure 2: The labelling of vertices in the two (potentially) intersecting trees emanating from $\mathcal{X}'_{i+K}$ and $\mathcal{Y}_{i+K}$. [Figure reused by permission of Springer, from *Commun. Math. Phys.* **331** (2014), 67–109 (Electrical resistance of the low-dimensional critical branching random walk. A.A. Járai and A. Nachmias) © (2014).]

And define the random set $\mathcal{I}$ by

$$\mathcal{I} = \{(U_1, U_2) : U_1 \text{ and } U_2 \text{ intersect-well}\}\ , \tag{2.1}$$

Now we define the event $\mathcal{B}'(i, c_0')$ where $c_0' > 0$ is a constant to be chosen later:

$$\mathcal{B}'(i, c_0') = \mathcal{A}(i) \cap \left\{ |\mathcal{I}| \geq \frac{c_0' \sigma^4}{D^d} \log(\delta n) \right\}\ .$$

The following theorem replaces [14, Theorem 2.1], and will be proved in Section 3.

Theorem 2.7 (Intersections exist). *Assume $d = 6$. There exist constants $c_0', c_1' > 0$ and for any $K \geq 2$ there exists $c_2 = c_2(K) > 0$, and $n_3' = n_3'(\sigma^2, C_3, \mathbf{p}^1, K)$ such that for any $0 < \delta < (K+4)^{-1}$, whenever $\delta n \geq n_3'$ and x satisfies $\|x\| \leq \sqrt{2n/\delta}$, we have*

$$\mathbf{P}(\mathcal{A}(i) \,|\, \Phi(V_n) = (x, n)) \geq c_2,$$

and

$$\mathbf{P}(\mathcal{B}'(i, c_0') \,|\, \mathcal{A}(i),\ \Phi(V_n) = (x, n)) \geq \frac{c_1'}{\log(\delta n)},$$

for $i = 0, K, 2K, \ldots, (N-1)K$.

As in [14], we define

$$\gamma(n) = \sup_{x:\|x\|\le\sqrt{n}} \gamma(n,x).$$

The proof of the following theorem will be completed in Section 6.

Theorem 2.8 (Analysis of good blocks)**.** *There exists $K_0' < \infty$ and $n_4' = n_4'(\sigma^2, C_3, \mathbf{p}^1)$ such that if $K \ge K_0'$ and $\delta n \ge n_4'$, we have*

$$\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\Big|\mathcal{A}(i),\, \mathcal{B}'(i,c_0'),\, \Phi(V_n)=(x,n)\Big] \le \frac{3K}{4} \max_{1\le k\le \delta n} \gamma(k),$$

for $i = 0, K, 2K, \ldots, (N-1)K$.

3 Existence of Intersections

In this section we prove Theorem 2.7. The statement about the probability of $\mathcal{A}(i)$ is unchanged compared to [14, Theorem 2.1], and hence requires no proof. On the other hand, there are substantial changes to the proof of the estimate on the probability of $\mathcal{B}'(i,c_0')$, that we now detail. Some of the required estimates appeared in the MSc thesis [18], in a slightly different form (with less technical restrictions on the intersections). Here we adapt and complete the analysis of [18] in a form that suits the requirements of the present paper.

3.1 Sufficient intersections

We now proceed to prove the second statement of Theorem 2.7. We will need to assume that the progeny distribution is bounded by M, and approximate the original progeny distribution with bounded distributions $p_M(k)$, in such a way that

$$\begin{aligned} 1 &= \sum_{0\le k\le M} p_M(k) = \sum_{0\le k\le M} k p_M(k) \\ \sigma^2 &= \lim_{M\to\infty} \sum_{0\le k\le M} k(k-1)p_M(k) \\ C_3 &\ge \sum_{0\le k\le M} k^3 p_M(k). \end{aligned}$$

Given any n and m such that $m \ge 2n$ we regard the random tree $\mathcal{T}_{n,m}$ as a subtree of an infinite M-ary tree T_M with root ρ as follows: the root of $\mathcal{T}_{n,m}$ is mapped to ρ and if W is a vertex of $\mathcal{T}_{n,m}$ with k children we map the k edges randomly amongst the $\binom{M}{k}$ possible choices in T_M. Denote by $\mathcal{V}_n \in T_M$ the random vertex where the last backbone vertex of $\mathcal{T}_{n,m}$ was mapped to. The triple $(\mathcal{T}_{n,m}, \rho, \mathcal{V}_n)$ is a doubly rooted tree.

Let $(\mathcal{T}_1, \rho_1, \mathcal{V}_1)$ and $(\mathcal{T}_2, \rho_2, \mathcal{V}_2)$ be two independent copies of $(\mathcal{T}_{\delta n, 2\delta n}, \rho, \mathcal{V}_{\delta n})$, randomly imbedded into T_M. Let Φ_1 and Φ_2 be two independent random walk mappings of T_M such that $\Phi_1(\rho_1) = \Phi(\mathcal{X}'_{i+K})$ and $\Phi_2(\rho_2) = \Phi(\mathcal{Y}_{i+K})$. However, for notational convenience, and without loss of generality, we assume that $\Phi_1(\rho_1) = (o, 0)$ and $\Phi_2(\rho_2) = (x, 0)$, with $\|x\| \le \sqrt{\delta n}$. Recall that the random variable $|\mathcal{I}|$ introduced in (2.1) has the same distribution as the random variable (also denote $|\mathcal{I}|$ here):

$$|\mathcal{I}| = \sum_{U_1 \in \mathcal{T}_1,\, U_2 \in \mathcal{T}_2} \mathbf{1}_{(U_1, U_2) \text{ intersect well}}.$$

Our goal in this section is to show that when $d = 6$, we have

$$\mathbf{P}\left(|\mathcal{I}| \ge c'_0 \sigma^4 D^{-d} \log(\delta n)\right) \ge \frac{c'_1}{\log(\delta n)} \tag{3.1}$$

for suitable $c'_0, c'_1 > 0$. Note that once we prove this estimate for offspring distribution $p_M(k)$ (uniformly in M), we can let $M \to \infty$, and obtain the second statement of Theorem 2.7 in full generality.

For technical reasons, we will also need a slight modification of $\mathcal{I}$. The difference is that we make a stronger restriction on the spatial displacement between $\Phi(Z_1^+)$ and $\Phi(U_1)$ and between $\Phi(Z_2^+)$ and $\Phi(U_2)$, as well as a stronger restriction on the height of the intersection:

$$\mathcal{I}' = \left\{ (U_1, U_2) \in \mathcal{I} : \begin{array}{c} h(U_1) = h(U_2) \le (11/12)\delta n, \\ \|z_1^+ - u_1\| \le \frac{1}{2}\sqrt{h(U_1) - h(Z_1^+)}, \\ \|z_2^+ - u_2\| \le \frac{1}{2}\sqrt{h(U_2) - h(Z_2^+)} \end{array} \right\}.$$

The starting point for proving (3.1) is to look at the moments of $|\mathcal{I}|$. The following theorem is an analogue of [14, Theorem 3.8].

Theorem 3.1. *Assume that $d = 6$ and $\|x\| \le \sqrt{\delta n}$. There exist constants $C' < \infty$ and $c' > 0$ and $n'_9 = n'_9(\sigma^2, C_3, \mathbf{p}^1) < \infty$ such that for $\delta n \ge n_9$ we have*

$$\mathbf{E}|\mathcal{I}| \ge \mathbf{E}|\mathcal{I}'| \ge \frac{c'\sigma^4}{D^6}, \tag{3.2}$$

and

$$\mathbf{E}|\mathcal{I}|^2 \le \frac{C'\sigma^8}{D^{12}} \log(\delta n). \tag{3.3}$$

Since the proof of this theorem requires only minor adaptations compared to the proof of [14, Theorem 3.8], we omit it.

Applying the Paley-Zygmund inequality to $|\mathcal{I}|$ shows that $\mathbf{P}(|\mathcal{I}| > 0) \ge c'/\log(\delta n)$, but comes short of proving (3.1). In order to prove this, we will consider the number of intersections conditional on $(U_1, U_2) \in \mathcal{I}'$ for fixed $U_1, U_1 \in T_M$, and show that this is at least of order $\log(\delta n)$ with conditional probability bounded away from 0.

For the statement of the next lemma we fix

$$0 \leq k_1 \leq \delta n - 1 \qquad\qquad k_1 + 1 \leq h_u \leq \delta n.$$

Given $V \in T_M$ at level δn and $U \in T_M$ at level h_u let $Z \in T_M$ be the highest common ancestor of V and U and let Z^+ be the unique child of Z leading towards U. We assume that Z is at height k_1. Given a tree $t \subset T_M$ such that $V, U \in t$ and V does not have any children in t, we have a unique decomposition of t into edge disjoint trees $(t^A, \rho, Z), (t^B, Z^+, U), t^C$ and t^D, see Figure 3. The doubly rooted tree (t^A, ρ, Z) contains all the descendants of ρ that are not descendants of Z. The doubly rooted tree (t^B, Z^+, U) contains all the descendants of Z^+ that are not descendants of U. The tree t^C contains all the descendants of U and finally the tree t^D contains all other edges, namely, all the descendants of Z that are not descendants of Z^+ (in particular, the edge Z, Z^+ is in t^D).

For $W \in T_M$ let Θ_W denote the tree isomorphism that takes W to ρ and the descendants subtree of W onto T_M.

The following lemma is taken without change from [14].

Lemma 3.2 ([14, Lemma 6.4])**.** *Let $V, U \in T_M$ be at heights δn and h_u, respectively, and let $(\mathcal{T}, \rho, \mathcal{V})$ be distributed as $(\mathcal{T}_{\delta n, 2\delta n}, \rho, \mathcal{V}_{\delta n})$. Conditionally on the event $\{\mathcal{V} = V,\, U \in \mathcal{T}\}$ we have that*

$$(\mathcal{T}^A,\, \rho,\, Z) \stackrel{\mathrm{d}}{=} (\mathcal{T}_{k_1, 2\delta n},\, \rho,\, \mathcal{V}_{k_1}) \,|\, \mathcal{V}_{k_1} = Z,$$

and

$$\Theta_{Z^+}((\mathcal{T}^B,\, Z^+,\, U)) \stackrel{\mathrm{d}}{=} (\mathcal{T}_{h_u - k_1 - 1, 2\delta n - k_1 - 1},\, \rho,\, \mathcal{V}_{h_u - k_1 - 1}) \,|\, \mathcal{V}_{h_u - k_1 - 1} = \Theta_{Z^+}(U).$$

Let us fix vertices $V_1, V_2 \in T_M$ at height δn, and vertices $U_1, U_2 \in T_M$, both at height h_u, where $(5/6)\delta n \leq h_u \leq (11/12)\delta n$, such that $(1/2)\delta n \leq h(Z_1), h(Z_2) \leq (4/6)\delta n$. Let us denote the vertices on the path in T_M between Z_1^+ and U_1 as

$$Z_1^+ = W_1^{(k_1+1)}, W_1^{(k_1+2)}, \ldots, W_1^{(h_u-1)}, W_1^{(h_u)} = U_1,$$

and vertices on the path in T_M between Z_2^+ and U_2 as

$$Z_2^+ = W_2^{(k_2+1)}, W_2^{(k_2+2)}, \ldots, W_2^{(h_u-1)}, W_2^{(h_u)} = U_2.$$

For any choice of (V_1, V_2, U_1, U_2) as above, let us define the event

$$\mathcal{IW}' = \mathcal{IW}'(V_1, V_2, U_1, U_2) = \{\mathcal{V}_1 = V_1,\, \mathcal{V}_2 = V_2\} \cap \{(U_1, U_2) \in \mathcal{I}'\}.$$

We will also use the shorthand

$$\mathbf{1}_{\mathcal{IW}'} = \mathbf{1}_{\mathcal{IW}'(V_1, V_2, U_1, U_2)}.$$

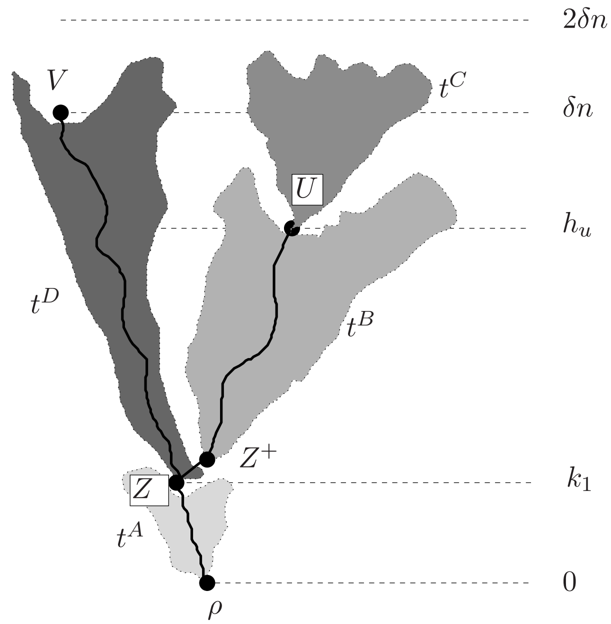

Figure 3: Illustration of the decomposition into edge-disjoint trees t^A, t^B, t^C, t^D appearing in Lemma 3.2 ($2\delta n$ and δn are not to scale). [Figure reused by permission of Springer, from *Commun. Math. Phys.* **331** (2014), 67–109 (Electrical resistance of the low-dimensional critical branching random walk. A.A. Járai and A. Nachmias) © (2014).]

For the next definition, recall the constants n_1 and n_2 of Lemma 1.4, and put

$$n^* = \max\{n_1, n_2(\mathbf{p}^1, \varepsilon = 1/2, L = 1)\}.$$

When the event $\mathcal{IW}'$ occurs, we say that a pair of vertices (Y_1, Y_2) is a *good extra intersection*, if the following events all occur:

- $Y_1 \in \mathcal{T}_1^B$ and $Y_2 \in \mathcal{T}_2^B$;
- Y_1 is a descendant of $W_1^{(r_1)}$ for some $(9/12)\delta n \le r_1 \le h_u - n^*$;
- Y_2 is a descendant of $W_2^{(r_2)}$ for some $(9/12)\delta n \le r_2 \le h_u - n^*$;
- (Y_1, Y_2) intersect-well;
- $h_u \le h(Y_1) = h(Y_2)$.

Observe that the last two conditions require in particular that $(5/6)\delta n \le h_u \le h(Y_1) = h(Y_2) \le \delta n$, and $\Phi_1(Y_1) = \Phi_2(Y_2)$. We define the random set:

$$\widehat{I}_{V,U} = \begin{cases} \{(Y_1, Y_2) : (Y_1, Y_2) \text{ is a good extra intersection}\} & \text{when } \mathcal{IW}' \text{ occurs;} \\ \emptyset & \text{when } \mathcal{IW}' \text{ does not occur,} \end{cases}$$

and the random variable:

$$J = J(c_0') = \sum_{\substack{V_1, V_2 \in T_M \\ U_1, U_2 \in T_M}} \mathbf{1}_{\left|\widehat{I}_{V,U}\right| \ge c_0' \frac{\sigma^4}{D^d} \log(\delta n)}.$$

A crucial property is that we have the following inclusion of events:

$$\{J > 0\} \subset \left\{ \exists V_1, V_2, U_1, U_2 : \left|\widehat{I}_{V,U}\right| \ge c_0' \frac{\sigma^4}{D^d} \log(\delta n) \right\}$$
$$\subset \left\{ |\mathcal{I}| \ge c_0' \frac{\sigma^4}{D^d} \log(\delta n) \right\} = \mathcal{B}'(c_0').$$

Hence in order to conclude, we need to show that $\mathbf{P}(J > 0) \ge c_1' / \log(\delta n)$ for suitable $c_0', c_1' > 0$ This follows immediately from the theorem and proposition stated below, that are the main results of this section.

Theorem 3.3. *Assume $d = 6$ and $\|x\| \le \sqrt{\delta n}$. There exists $n_9' = n_9'(\sigma^2, C_3, \mathbf{p}^1) < \infty$ such that for $\delta n \ge n_9'$ and all $V_1, V_2, U_1, U_2 \in T_M$ we have*

$$\mathbf{E}\Big[\left|\widehat{I}_{V,U}\right| \,\Big|\, \mathcal{IW}'(V_1, V_2, U_1, U_2) \Big] \ge \frac{c' \sigma^4}{D^6} \log(\delta n),$$

and

$$\mathbf{E}\Big[\left|\widehat{I}_{V,U}\right|^2 \,\Big|\, \mathcal{IW}'(V_1, V_2, U_1, U_2) \Big] \le \frac{C' \sigma^8}{D^{12}} \log^2(\delta n).$$

In particular, taking $c_0' = \frac{1}{2} c'$, we have

$$\mathbf{P}\Big(\left|\widehat{I}_{V,U}\right| \ge c_0' \frac{\sigma^4}{D^6} \log(\delta n) \,\Big|\, \mathcal{IW}'(V_1, V_2, U_1, U_2) \Big) \ge 4 \frac{(c')^2}{C'} > 0. \tag{3.4}$$

Proposition 3.4. *Assume $d = 6$ and $\|x\| \le \sqrt{\delta n}$. For $\delta n \ge n_9'$ and with the choice of c_0' in the conclusion* (3.4) *of Theorem 3.3, we have*

$$\mathbf{E} J(c_0') \ge \frac{c'\sigma^4}{D^6},$$

and

$$\mathbf{E} J(c_0')^2 \le \frac{C'\sigma^8}{D^{12}} \log(\delta n).$$

In particular, we have

$$\mathbf{P}(\mathcal{B}'(c_0')) \ge \mathbf{P}(J(c_0') > 0) \ge \frac{(c')^2}{C'} \frac{1}{\log(\delta n)}.$$

Proof of Proposition 3.4 assuming Theorem 3.3. Due to (3.4) of Theorem 3.3, we have

$$\begin{aligned} \mathbf{E} J(c_0') &= \sum_{\substack{V_1,V_2\\U_1,U_2}} \mathbf{P}(\mathcal{IW}'(V_1,V_2,U_1,U_2))\, \mathbf{P}\Big(\Big|\widehat{I}_{V,U}\Big| \ge c_0' \frac{\sigma^4}{D^6} \log(\delta n) \,\Big|\, \mathcal{IW}'(V_1,V_2,U_1,U_2) \Big) \\ &\ge c' \sum_{\substack{V_1,V_2\\U_1,U_2}} \mathbf{P}(\mathcal{IW}'(V_1,V_2,U_1,U_2)) = c'\, \mathbf{E}|\mathcal{I}'|. \end{aligned}$$

Due to Theorem 3.1, this is at least $c'\sigma^4 D^{-6}$, proving the first statement.

For the second statement we use that $J(c_0') \le |\mathcal{I}|$, and hence the upper bound on $\mathbf{E} J(c_0')^2$ follows immediately from Theorem 3.1.

The last statement follows from the Paley-Zygmund inequality:

$$\mathbf{P}(J(c_0') > 0) \ge \frac{[\mathbf{E} J(c_0')]^2}{\mathbf{E} J(c_0')^2}.$$

□

Hence it remains to prove Theorem 3.3, that we do in the next two sections.

3.2 Lower bound on the first moment

We start with the arguments for the lower bound on the first moment. For the next lemma, assume that the event $\mathcal{IW}'(V_1,V_2,U_1,U_2)$ occurs. Recall that this defines vertices $Z_1, Z_1^+ \in \mathcal{T}_1$ and $Z_2, Z_2^+ \in \mathcal{T}_2$ at levels k_1, k_1+1 and k_2, k_2+1, respectively, as well as the vertices $W_1^{(r_1)}$ at levels $k_1+1 \le r_1 \le h_u$ and $W_2^{(r_2)}$ at levels $k_2+1 \le r_2 \le h_u$. We will write $\Phi(W_1^{(r_1)}) = (w_1^{(r_1)}, r_1)$ and $\Phi(W_r^{(r_r)}) = (w_2^{(r_2)}, r_2)$.

Lemma 3.5. *There exists an absolute constant $c' > 0$ with the following property. Assume that the event $\mathcal{IW}'(V_1, V_2, U_1, U_2)$ occurs. Then for $\delta n \geq 12 n_2(\mathbf{p}_1, \varepsilon = 1/2, L = 1)$ and any $(9/12)\delta n \leq r_1, r_2 \leq h_u - n_2$ we have*

$$\mathbf{P}\Big(\|w_j^{(r_j)} - z_j^+\| \leq \frac{1}{\sqrt{2}} \sqrt{r_j - k_j - 1},\, \|u_j - w_j^{(r_j)}\| \leq \frac{1}{8} \sqrt{h_u - r_j},\, j = 1, 2 \,\Big|\, \mathcal{IW}' \Big) \geq c'. \tag{3.5}$$

Proof. Condition on the event $\mathcal{IW}'$, and let us further condition on the spatial locations z_1^+, z_2^+ and $u_1 = u_2 = u$. Due to Lemma 3.2, the conditional distribution of the path between $(z_j^+, k_j + 1)$ and (u, h_u), is a random walk started at z_j^+ and conditioned to arrive at u at time $h_u - k_j - 1$, $j = 1, 2$. Let us write

$$\Omega_j = \left\{ w \in \mathbb{Z}^d : \|w - z_j^+\| \leq \frac{1}{\sqrt{2}} \sqrt{r_j - k_j - 1},\, \|u - w\| \leq \frac{1}{8} \sqrt{h_u - r_j} \right\}.$$

Therefore, the probability in the statement of the lemma equals

$$\prod_{j \in \{1,2\}} \frac{\sum_{w \in \Omega_j} \mathbf{p}^{r_j - k_j - 1}(z_j^+, w)\, \mathbf{p}^{h_u - r_j}(w, u)}{\mathbf{p}^{h_u - k_j - 1}(z_j^+, u)}.$$

Since $h_u - k_j - 1 \geq (5/6)\delta n - (4/6)\delta n - 1 \geq (1/12)\delta n \geq n_2$, and $h_u - r_j \geq n_2$, the local limit theorem (Lemma 1.4) implies that there exists $C = C(d)$ and $c = c(d) > 0$, such that for all $w \in \Omega_j$ we have

$$\begin{aligned} \mathbf{p}^{h_u - k_j - 1}(z_j^+, u) &\leq \frac{C}{D^d\, (h_u - k_j - 1)^{d/2}} \leq \frac{C}{D^d\, (\delta n)^{d/2}} \\ \mathbf{p}^{r_j - k_j - 1}(z_j^+, w) &\geq \frac{c}{D^d\, (r_j - k_j - 1)^{d/2}} \geq \frac{c}{D^d\, (\delta n)^{d/2}} \\ \mathbf{p}^{h_u - r_j}(w, u) &\geq \frac{c}{D^d\, (h_u - r_j)^{d/2}}. \end{aligned} \tag{3.6}$$

Since we are conditioning on $\mathcal{IW}'$, the restriction $\|u - z_j^+\| \leq \frac{1}{2}\sqrt{h_u - k_j - 1}$ holds. We claim that this implies that

$$|\Omega_j| \geq c' D^d (h_u - r_j)^{d/2}. \tag{3.7}$$

Indeed, writing $t = h_u - k_j - 1$, $\alpha t = h_u - r_j$, $(1 - \alpha)t = r_j - k_j - 1$, we have $\alpha \leq 1/2$, and this implies the inequalities $\frac{1}{8}\sqrt{\alpha t} \leq \frac{1}{\sqrt{2}}\sqrt{(1 - \alpha)t}$ and $\frac{1}{2}\sqrt{t} \leq \frac{1}{\sqrt{2}}\sqrt{(1 - \alpha)t}$. These in turn imply that at least half of the ball of radius αt centred at u (namely the half lying in the direction of z_j^+) is included in Ω_j. Putting together the bounds (3.6) and (3.7) gives the statement of the lemma. □

For the next lemma, assume the event $\mathcal{IW}'(V_1, V_2, U_1, U_2)$, and consider vertices $Y_1 \in \mathcal{T}_1^B$ and $Y_2 \in \mathcal{T}_2^B$ at common height $h_u \leq h(Y_1) = h_y = h(Y_2) \leq \delta n$. Let Y_1 be a descendant of $W_1^{(r_1)}$, and Y_2 be a descendant of $W_2^{(r_2)}$.

Lemma 3.6. *Fix V_1, V_2, U_1, U_2, and assume the event $\mathcal{IW}'(V_1, V_2, U_1, U_2)$. Let $Y_1 \in \mathcal{T}_1^B$ and $Y_2 \in \mathcal{T}_2^B$ be vertices at height $h_u \le h(Y_1) = h_y = h(Y_2) \le \delta n$, and assume they are descendants of $W_1^{(r_1)}$ and $W_2^{(r_2)}$, respectively. Assume also that $h_y - r_1, h_y - r_2 \ge n_2(\mathbf{p}^1, \varepsilon = 1/2, L = 1)$. We have*

$$\mathbf{P}\big((Y_1, Y_2) \in \widehat{I}_{V,U} \,\big|\, \mathcal{T}^1, \mathcal{T}^2, \mathcal{IW}'(V_1, V_2, U_1, U_2)\big) \ge \frac{c'}{D^d} \frac{1}{(2h_y - r_1 - r_2)^{d/2}}. \tag{3.8}$$

Proof. Without loss of generality, we assume that $h_y - r_1 \le h_y - r_2$ (the opposite case is handled analogously). Condition on the event in the statement, and let us further condition on the spatial locations of $W_1^{(r_1)}$ and $W_2^{(r_2)}$, that we denote by w_1 and w_2, for short. Due to Lemma 3.5, we may assume, at the cost of a constant factor, that the event in (3.5) hold. Assuming that this is the case, let

$$\Omega = \{y \in \mathbb{Z}^d : \|w_1 - y\| \le \frac{1}{\sqrt{2}}\sqrt{h_y - r_1},\ \|w_2 - y\| \le \frac{1}{\sqrt{2}}\sqrt{h_y - r_2}\}.$$

We show that $|\Omega| \ge cD^d(h_y - r_1)^{d/2}$. For this, it is enough to show that all points y satisfying the condition on $\|w_1 - y\| \le \frac{1}{8}\sqrt{h_y - r_1}$ automatically satisfy the condition on $\|w_2 - y\|$ in the definition of Ω.

Write $a = h_u - r_1$, $b = h_u - r_2$, $c = h_y - h_u$, so that $0 \le a \le b$ and $c \ge 0$. Then we have

$$\|w_2 - w_1\| \le \frac{1}{8}\sqrt{a} + \frac{1}{8}\sqrt{b} \le \frac{1}{4}\sqrt{a+b}.$$

Hence we are left to show that

$$\frac{1}{8}\sqrt{a+c} + \frac{1}{4}\sqrt{a+b} \le \frac{1}{\sqrt{2}}\sqrt{b+c}.$$

It is easy to see that this follows from $a + c \le b + c$ and $a + b \le 2(b + c)$.

With the estimate on the size of Ω at hand, and using that $h_y - r_1, h_y - r_2 \ge n_2$, we can apply the local CLT (Lemma 1.4) to get that the conditional probability in (3.8) is at least:

$$\begin{aligned}\sum_{y \in \Omega} \mathbf{p}^{h_y - r_1}(w_1, y)\, \mathbf{p}^{h_y - r_2}(w_2, y) &\ge |\Omega| \, \frac{c}{D^d}(h_y - r_1)^{-d/2} \, \frac{c}{D^d}(h_y - r_2)^{-d/2} \\ &\ge \frac{c}{D^d}(h_y - r_2)^{-d/2} \ge \frac{c}{D^d}(h_y - r_2 + h_y - r_1)^{-d/2},\end{aligned}$$

as claimed. □

Lemma 3.7. *Assume $d = 6$. Then for $\delta n \ge (24n^*)^2$ (where $n^* = \max\{n_1, n_2(\mathbf{p}^1, \varepsilon = 1/2, L = 1)\}$), we have*

$$\mathbf{E}\big(|\widehat{I}_{V,U}| \,\big|\, \mathcal{IW}'\big) \ge \frac{c'\,\sigma^4}{D^d}\log(\delta n).$$

Proof. Due to Lemma 3.6, we have

$$\mathbf{E}\big(|\widehat{I}_{V,U}| \,\big|\, \mathcal{IW}'\big) \ge \frac{c'}{D^d} \sum_{r_1=(9/12)\delta n}^{h_u-n_2} \sum_{r_2=(9/12)\delta n}^{h_u-n^*} \sum_{h_y=h_u}^{\delta n} \frac{1}{(2h_y-r_1-r_2)^3} \mathbf{E}\mathcal{L}(h_y,r_1)\mathbf{E}\mathcal{L}(h_y,r_2), \quad (3.9)$$

where, analogously to [14, Lemma 3.10], $\mathcal{L}(h_y,r_1)$ denotes the number of vertices Y_1 of $\mathcal{T}_1^B$ at level h_y that are descendants of $W_1^{(r_1)}$. As in [14, Lemma 3.10], we have $\mathbf{E}\mathcal{L}(h_y,r_1) \ge c\sigma^2$ and $\mathbf{E}\mathcal{L}(h_y,r_2) \ge c\sigma^2$. This gives that the right hand side of (3.9) is at least

$$\frac{c'\,\sigma^4}{D^d} \sum_{r_1=(9/12)\delta n}^{h_u-n^*} \sum_{r_2=(9/12)\delta n}^{h_u-n^*} \sum_{h_y=h_u}^{\delta n} \frac{1}{(2h_y-r_1-r_2)^3}.$$

Let us write $s_1=h_u-r_1$, $s_2=h_u-r_2$ and $h^*=h_y-h_u$, so that the last expression satisfies

$$\begin{aligned}
&\ge \frac{c'\,\sigma^4}{D^d} \sum_{s_1=n^*}^{(1/24)\delta n} \sum_{s_2=n^*}^{(1/24)\delta n} \sum_{h^*=0}^{(1/12)\delta n} \frac{1}{(2h^*+s_1+s_2)^3} \\
&\ge \frac{c'\,\sigma^4}{D^d} \sum_{s_1=n^*}^{(1/24)\delta n} \sum_{s_2=n_2}^{(1/24)\delta n} \frac{1}{(s_1+s_2)^2} \\
&\ge \frac{c'\,\sigma^4}{D^d} \sum_{s_1=n^*}^{(1/24)\delta n} \frac{1}{s_1} \\
&\ge \frac{c'\,\sigma^4}{D^d} \log(\delta n),
\end{aligned}$$

using in the last step that $\log(n^*)+\log(24) \le \frac{1}{2}\log(\delta n)$. □

3.3 Upper bound on the second moment

We fix V_1, V_2, U_1, U_2 such that $(5/6)\delta n \le h(U_1) = h_u = h(U_2) \le (11/12)\delta n$, and recall that we condition on the event $\mathcal{IW}'$. Fix heights $h_u \le h_y, h_{\widetilde{y}} \le \delta n$, and consider a pair of vertices $(Y_1, \widetilde{Y}_1)$ that are both in $\mathcal{T}_1^B$ such that $h(Y_1) = h_y$ and $h(\widetilde{Y}_1) = h_{\widetilde{y}}$. There then exist unique heights $(9/12)\delta n \le r_1, \widetilde{r}_1 \le h_u - n_1$ such that $Y_1 \succ W_1^{(r_1)}$ and $\widetilde{Y}_1 \succ W_1^{(\widetilde{r}_1)}$. Let $\widetilde{Z}_1$ denote the highest common ancestor of Y_1 and $\widetilde{Y}_1$, and let $k_1' = h(\widetilde{Z}_1)$. Note that we can have $r_1 = \widetilde{r}_1$, in which case $k_1' \ge r_1 = \widetilde{r}_1$, while if $r_1 \ne \widetilde{r}_1$, we have $k_1' = r_1 \wedge \widetilde{r}_1$. Let us write $\mathcal{L}'(h_y, h_{\widetilde{y}}, k_1', r_1, \widetilde{r}_1)$ for the number of pairs $(Y_1, \widetilde{Y}_1)$ that satisfy the above height restrictions with given $h_y, h_{\widetilde{y}}, k_1', r_1, \widetilde{r}_1$. The following is an analogue of [14, Lemma 3.11].

Lemma 3.8. *We have*

$$\mathbf{E}\Big[\mathcal{L}'(h_y, h_{\widetilde{y}}, k_1', r_1, \widetilde{r}_1) \,\Big|\, \mathcal{IW}'\Big] \le \begin{cases} \sigma^4 & \text{when } r_1 = \widetilde{r}_1 < k_1' < h_y, h_{\widetilde{y}}; \\ \sigma^2 & \text{when } r_1 = \widetilde{r}_1 < k_1' = h_y \wedge h_{\widetilde{y}}; \\ C_3 & \text{when } r_1 = \widetilde{r}_1 = k_1' < h_y, h_{\widetilde{y}}; \\ \sigma^4 & \text{when } r_1 \neq \widetilde{r}_1,\ k_1' = r_1 \wedge \widetilde{r}_1 < h_y, h_{\widetilde{y}}; \end{cases}$$

(Observe that there is no case $r_1 = \widetilde{r}_1 = k_1' = h_y \wedge h_{\widetilde{y}}$, since $r_1 = \widetilde{r}_1 < h_u \le h_y, h_{\widetilde{y}}$.)

Proof. Conditional on $\mathcal{IW}'$ the distribution of $\mathcal{T}_1^B$ is the same as that of $\mathcal{T}_{h_u-k_1-1,2\delta n-k_1-1}$ (cf. Lemma 3.2). Hence the proof boils down to the same (straightforward) branching process calculations as the proof of [14, Lemma 3.11]. □

The following 'diagrammatic estimate' is taken without change from [14]. Recall the constant $n_1 = n_1(\mathbf{p}_1)$ from Lemma 1.4(i), and the constant $L_1 = L_1(\mathbf{p}_1)$ from (1.8). See Figure 4(a).

Lemma 3.9 ([14, Lemma 3.12])**.** *Suppose $d \ge 3$. There are constants $C = C(d) > 0$ and $C_2 = C_2(\mathbf{p}^1)$ such that for all $\widetilde{z}_1, \widetilde{z}_2 \in \mathbb{Z}^d$ we have*

$$\sum_{h: k_1' \vee k_2' \le h \le \delta n} \mathbf{p}^{2h-k_1'-k_2'}(\widetilde{z}_1, \widetilde{z}_2) \le \frac{C}{D^d} f(k_1', k_2', \widetilde{z}_1, \widetilde{z}_2)\,,$$

where

$$f(k_1', k_2', \widetilde{z}_1, \widetilde{z}_2) := \begin{cases} |k_1' - k_2'|^{(2-d)/2} & \text{if } \|\widetilde{z}_1 - \widetilde{z}_2\| \le |k_1' - k_2'|^{1/2} \\ & \text{and } |k_1' - k_2'| \ge n_1; \\ C_2 & \text{if } \|\widetilde{z}_1 - \widetilde{z}_2\| \le |k_1' - k_2'|^{1/2} < \sqrt{n_1}; \\ \|\widetilde{z}_1 - \widetilde{z}_2\|^{2-d} & \text{if } \|\widetilde{z}_1 - \widetilde{z}_2\| > |k_1' - k_2'|^{1/2} \\ & \text{and } \|\widetilde{z}_1 - \widetilde{z}_2\| \ge L_1; \\ C_2 & \text{if } |k_1' - k_2'|^{1/2} < \|\widetilde{z}_1 - \widetilde{z}_2\| < L_1. \end{cases}$$

Proposition 3.10. *Assume $d = 6$. We have*

$$\mathbf{E}\big(|\widehat{I}_{V,U}|^2 \,\big|\, \mathcal{IW}'\big) \le \frac{C'\,\sigma^8}{D^{2d}} \log^2(\delta n).$$

Proof. The proof broadly follows the outline of the proof of [14, Theorem 3.8]. In addition to the event $\mathcal{IW}'$, let us further condition on the spatial location u of the intersection $(U_1, U_2) \in \mathcal{I}'$, as well as the spatial locations z_1^+ and z_2^+. Let $Y_1, \widetilde{Y}_1 \in \mathcal{T}_1^B$ and $Y_2, \widetilde{Y}_2 \in \mathcal{T}_2^B$ be pairs of tree

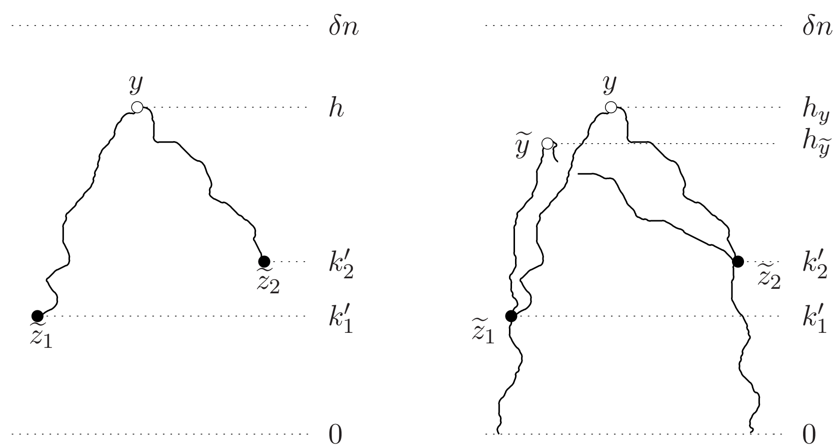

Figure 4: (a) Illustration of the quantity bounded in Lemma 3.9. The curves represent random walk transition probabilities between the indicated space-time points. The expression is summed over y to obtain $\mathbf{p}^{2h-k_1'-k_2'}(\widetilde{z}_1, \widetilde{z}_2)$ and then summed over h. When $\|\widetilde{z}_1 - \widetilde{z}_2\| > |k_1' - k_2'|^{1/2}$, we get the Green function decay from the spatial separation $\|\widetilde{z}_1 - \widetilde{z}_2\|$. When $\|\widetilde{z}_1 - \widetilde{z}_2\| \leq |k_1' - k_2'|^{1/2}$ we get decay from the time separation $|k_1' - k_2'|$. (b) Illustration of the quantity appearing in $q_{a,1}$ that contains "two copies" of f. [Figure adapted by permission of Springer, from *Commun. Math. Phys.* **331** (2014), 67–109 (Electrical resistance of the low-dimensional critical branching random walk. A.A. Járai and A. Nachmias) © (2014).]

vertices, such that $h(Y_1) = h_y = h(Y_2)$ and $h(\widetilde{Y}_1) = h_{\widetilde{y}} = h(\widetilde{Y}_2)$. Recall the notation introduced at the beginning of this section, and extend it to the tree $\mathcal{T}_2^B$; e.g. $\widetilde{Z}_2$ is the highest common ancestor of $Y_2, \widetilde{Y}_2$ at height k_2', etc.

We give separate bounds in the following four cases.
Case (a). $r_1 = \widetilde{r}_1 < k_1'$ and $r_2 = \widetilde{r}_2 < k_2'$;
Case (b1). $r_1 = \widetilde{r}_1 < k_1'$ and $r_2 \wedge \widetilde{r}_2 = k_2'$;
Case (b2). $r_1 \wedge \widetilde{r}_1 = k_1'$ and $r_2 = \widetilde{r}_2 < k_2'$;
Case (c). $r_1 \wedge \widetilde{r}_1 = k_1'$ and $r_2 \wedge \widetilde{r}_2 = k_2'$.
We then have

$$\begin{aligned} &\mathbf{E}\big(|\widehat{I}_{V,U}|^2 \,\big|\, \mathcal{IW}', \Phi(U_1) = (u, h_u) = \Phi(U_2), \Phi(Z_1^+) = (z_1^+, k_1+1), \Phi(Z_2^+) = (z_2^+, k_2+1)\big) \\ &\quad = S_a + S_{b1} + S_{b2} + S_c, \end{aligned} \tag{3.10}$$

where the four terms represent contributions from intersecting pairs satisfying the criteria of the respective cases. Then the proposition follows from the three lemmas below. □

Lemma 3.11. *We have*

$$S_a \le \frac{C'\,\sigma^8}{D^{2d}} \log^2(\delta n).$$

Lemma 3.12. *We have*

$$S_{b1} + S_{b2} \le \frac{C'\,\sigma^8}{D^{2d}} \log^2(\delta n).$$

Lemma 3.13. *We have*

$$S_c \le \frac{C'\,\sigma^8}{D^{2d}} \log^2(\delta n).$$

Proof of Lemma 3.11. By symmetry, we can restrict to $r_1 \ge r_2$. We then have

$$\begin{aligned} S_a \le 2 &\sum_{(9/12)\delta n \le r_2 \le r_1 \le h_u - n_1} \sum_{\substack{r_1 < k_1' \le \delta n \\ r_2 < k_2' \le \delta n}} \sum_{h_y, h_{\widetilde{y}} = k_1' \vee k_2' \vee h_u}^{\delta n} \mathbf{E}\mathcal{L}'(h_y, h_{\widetilde{y}}, r_1, r_1, k_1')\, \mathbf{E}\mathcal{L}'(h_y, h_{\widetilde{y}}, r_2, r_2, k_2') \\ &\times p_a(h_y, h_{\widetilde{y}}, k_1', k_2', r_1, r_2), \end{aligned}$$

where p_a is the probability, given the conditioning in (3.10) that $\Phi(Y_1) = \Phi(Y_2)$ and $\Phi(\widetilde{Y}_1) = \Phi(\widetilde{Y}_2)$. This indeed only depends on the heights in the argument of p_a, and can be written as follows. Write $\widetilde{\mathbf{p}}(w_1)$, $\widetilde{\mathbf{p}}(w_2)$ for the conditional distributions of the spatial locations w_1, w_2, respectively, we have:

$$\begin{aligned} \widetilde{\mathbf{p}}(w_1) &= \frac{\mathbf{p}^{r_1 - k_1 - 1}(z_1^+, w_1)\, \mathbf{p}^{h_u - r_1}(w_1, u)}{\mathbf{p}^{h_u - k_1 - 1}(z_1^+, u)} \quad w_1 \in \mathbb{Z}^d; \\ \widetilde{\mathbf{p}}(w_2) &= \frac{\mathbf{p}^{r_2 - k_2 - 1}(z_2^+, w_2)\, \mathbf{p}^{h_u - r_2}(w_2, u)}{\mathbf{p}^{h_u - k_2 - 1}(z_2^+, u)} \quad w_2 \in \mathbb{Z}^d. \end{aligned}$$

Then we have

$$p_a(h_y, h_{\widetilde{y}}, k'_1, k'_2, r_1, r_2) = \sum_{w_1,w_2\in\mathbb{Z}^d} \sum_{\widetilde{z}_1,\widetilde{z}_2\in\mathbb{Z}^d} \sum_{y,\widetilde{y}\in\mathbb{Z}^d} \widetilde{\mathbf{p}}(w_1)\,\widetilde{\mathbf{p}}(w_2)\,\mathbf{p}^{k'_1-r_1}(w_1,\widetilde{z}_1)\,\mathbf{p}^{k'_2-r_2}(w_2,\widetilde{z}_2)$$
$$\times\,\mathbf{p}^{h_y-k'_1}(\widetilde{z}_1,y)\,\mathbf{p}^{h_y-k'_2}(\widetilde{z}_2,y)\,\mathbf{p}^{h_{\widetilde{y}}-k'_1}(\widetilde{z}_1,\widetilde{y})\,\mathbf{p}^{h_{\widetilde{y}}-k'_2}(\widetilde{z}_2,\widetilde{y})$$
$$= \sum_{w_1,w_2\in\mathbb{Z}^d} \sum_{\widetilde{z}_1,\widetilde{z}_2\in\mathbb{Z}^d} \widetilde{\mathbf{p}}(w_1)\,\widetilde{\mathbf{p}}(w_2)\,\mathbf{p}^{k'_1-r_1}(w_1,\widetilde{z}_1)\,\mathbf{p}^{k'_2-r_2}(w_2,\widetilde{z}_2)$$
$$\times\,\mathbf{p}^{2h_y-k'_1-k'_2}(\widetilde{z}_1,\widetilde{z}_2)\,\mathbf{p}^{2h_{\widetilde{y}}-k'_1-k'_2}(\widetilde{z}_1,\widetilde{z}_2).$$

We perform the summation over $h_y, h_{\widetilde{y}}$ using Lemmas 3.8 and 3.9. Restricting the sum to $h_y, h_{\widetilde{y}} > k'_1 \vee k'_2$ we get an upper bound of the form:

$$S_{a,1} = \frac{C\,\sigma^8}{D^{2d}} \sum_{(9/12)\delta n\le r_2\le r_1\le h_u-n_1} \sum_{\substack{r_1<k'_1\le\delta n\\ r_2<k'_2\le\delta n}} q_{a,1}(k'_1,k'_2,r_1,r_2),$$

where

$$q_{a,1}(k'_1,k'_2,r_1,r_2) = \sum_{w_1,w_2\in\mathbb{Z}^d} \sum_{\widetilde{z}_1,\widetilde{z}_2\in\mathbb{Z}^d} \widetilde{\mathbf{p}}(w_1)\,\widetilde{\mathbf{p}}(w_2)\,\mathbf{p}^{k'_1-r_1}(w_1,\widetilde{z}_1)\,\mathbf{p}^{k'_2-r_2}(w_2,\widetilde{z}_2)\,f(k'_1,k'_2,\widetilde{z}_1,\widetilde{z}_2)^2.$$

Similarly, summing over $h_y > k'_1 \vee k'_2$ and $h_{\widetilde{y}} = k'_1 \vee k'_2$, and when the roles of y and $\widetilde{y}$ are interchanged, yields the upper bound:

$$S_{a,2} = \frac{C\,\sigma^6}{D^{d}} \sum_{(9/12)\delta n\le r_2\le r_1\le h_u-n_1} \sum_{\substack{r_1<k'_1\le\delta n\\ r_2<k'_2\le\delta n}} q_{a,2}(k'_1,k'_2,r_1,r_2),$$

where

$$q_{a,2}(k'_1,k'_2,r_1,r_2) = \sum_{w_1,w_2\in\mathbb{Z}^d} \sum_{\widetilde{z}_1,\widetilde{z}_2\in\mathbb{Z}^d} \widetilde{\mathbf{p}}(w_1)\,\widetilde{\mathbf{p}}(w_2)\,\mathbf{p}^{k'_1-r_1}(w_1,\widetilde{z}_1)\,\mathbf{p}^{k'_2-r_2}(w_2,\widetilde{z}_2)$$
$$\times\,\mathbf{p}^{|k'_1-k'_2|}(\widetilde{z}_1,\widetilde{z}_2)\,f(k'_1,k'_2,\widetilde{z}_1,\widetilde{z}_2).$$

And finally, our bound when $h_y = h_{\widetilde{y}} = k'_1 \vee k'_2$ is

$$S_{a,3} = C\,\sigma^4 \sum_{(9/12)\delta n\le r_2\le r_1\le h_u-n_1} \sum_{\substack{r_1<k'_1\le\delta n\\ r_2<k'_2\le\delta n}} q_{a,3}(k'_1,k'_2,r_1,r_2),$$

where

$$q_{a,3}(k'_1,k'_2,r_1,r_2) = \sum_{w_1,w_2\in\mathbb{Z}^d} \sum_{\widetilde{z}_1,\widetilde{z}_2\in\mathbb{Z}^d} \mathbf{p}^{k'_1-r_1}(w_1,\widetilde{z}_1)\,\mathbf{p}^{k'_2-r_2}(w_2,\widetilde{z}_2)\,\mathbf{p}^{|k'_1-k'_2|}(\widetilde{z}_1,\widetilde{z}_2)\,\mathbf{p}^{|k'_1-k'_2|}(\widetilde{z}_1,\widetilde{z}_2).$$

This altogether gives an upper bound of the form:

$$S_a \le S_{a,1} + S_{a,2} + S_{a,3},$$

and hence we bound each of the three terms separately.

We start with bounding $S_{a,1}$, and we split the summation over $\widetilde{z}_1, \widetilde{z}_2$ into:
(I)+ $\|\widetilde{z}_2 - \widetilde{z}_1\| \le |k_1' - k_2'|^{1/2}$ and $k_2' \ge h_u$;
(I)– $\|\widetilde{z}_2 - \widetilde{z}_1\| \le |k_1' - k_2'|^{1/2}$ and $r_2 < k_2' < h_u$;
(II)+ $\|\widetilde{z}_2 - \widetilde{z}_1\| > |k_1' - k_2'|^{1/2}$ and $k_2' \ge h_u$;
(II)– $\|\widetilde{z}_2 - \widetilde{z}_1\| > |k_1' - k_2'|^{1/2}$ and $r_2 < k_2' < h_u$.
We start with (I)+, and initially restrict to $|k_1' - k_2'| \ge n_1$. Then using Lemma 3.9 and the local limit theorem (Lemma 1.4) to bound $\mathbf{p}^{k_2'-r_2}(w_2, \widetilde{z}_2)$, we have

$$\begin{aligned}
q_{a,1,(I)+}(k_1', k_2', r_1, r_2) &\le \frac{C}{D^d} |k_1' - k_2'|^{2-d} (k_2' - r_2)^{-d/2} \sum_{w_1, w_2 \in \mathbb{Z}^d} \sum_{\widetilde{z}_1 \in \mathbb{Z}^d} \sum_{\widetilde{z}_2 : \|\widetilde{z}_2 - \widetilde{z}_1\| \le |k_1' - k_2'|^{1/2}} \\
&\qquad \widetilde{\mathbf{p}}(w_1)\, \widetilde{\mathbf{p}}(w_2)\, \mathbf{p}^{k_1' - r_1}(w_1, \widetilde{z}_1) \\
&\le C\, |k_1' - k_2'|^{2-d/2} (k_2' - r_2)^{-d/2} \sum_{w_1 \in \mathbb{Z}^d} \sum_{\widetilde{z}_1 \in \mathbb{Z}^d} \widetilde{\mathbf{p}}(w_1)\, \mathbf{p}^{k_1' - r_1}(w_1, \widetilde{z}_1) \\
&\le C\, |k_1' - k_2'|^{2-d/2} (k_2' - r_2)^{-d/2} \\
&= C |k_1' - k_2'|^{-1} (k_2' - r_2)^{-3}.
\end{aligned}$$

Performing the sum over k_1' yields a factor $\log(\delta n)$, while performing the sums over k_2', r_2 and r_1 yield:

$$\sum_{(9/12)\delta n \le r_1 \le h_u - n_1} \sum_{r_2 \le r_1} \sum_{k_2' \ge h_u} (k_2' - r_2)^{-3} \le C \log(\delta n).$$

The contribution to (I)+ from $|k_1' - k_2'| < n_1$ is of lower order, since:

$$\begin{aligned}
q_{a,1,(I)+} &\le \frac{C\, C_2^2}{D^d} (k_2' - r_2)^{-d/2} \sum_{w_1, w_2 \in \mathbb{Z}^d} \sum_{\widetilde{z}_1 \in \mathbb{Z}^d} \sum_{\widetilde{z}_2 : \|\widetilde{z}_2 - \widetilde{z}_1\| \le |k_1' - k_2'|^{1/2}} \widetilde{\mathbf{p}}(w_1)\, \widetilde{\mathbf{p}}(w_2)\, \mathbf{p}^{k_1' - r_1}(w_1, \widetilde{z}_1) \\
&\le C\, C_2^2\, |k_1' - k_2'|^3 (k_2' - r_2)^{-3}.
\end{aligned}$$

Summing over k_1' yields a factor n_1^4, while the rest of the sums contribute a factor $\log(\delta n)$.

In case (I)– we again initially restrict to $|k_1' - k_2'| \ge n_1$. We use Lemma 3.9 and we bound

$\widetilde{\mathbf{p}}(w_2)$ using the local limit theorem to get:

$$q_{a,1,(I)-}(k_1',k_2',r_1,r_2) \le \frac{C}{D^d}\,|k_1'-k_2'|^{2-d}\,(h_u-r_2)^{-d/2} \sum_{w_1,w_2\in\mathbb{Z}^d}\sum_{\widetilde{z}_1\in\mathbb{Z}^d}\sum_{\widetilde{z}_2:\|\widetilde{z}_2-\widetilde{z}_1\|\le|k_1'-k_2'|^{1/2}} \widetilde{\mathbf{p}}(w_1)\,\mathbf{p}^{k_1'-r_1}(w_1,\widetilde{z}_1)\mathbf{p}^{k_2'-r_2}(w_2,\widetilde{z}_2)$$
$$\le C\,|k_1'-k_2'|^{2-d/2}\,(h_u-r_2)^{-d/2}\sum_{w_1\in\mathbb{Z}^d}\sum_{\widetilde{z}_1\in\mathbb{Z}^d}\widetilde{\mathbf{p}}(w_1)\,\mathbf{p}^{k_1'-r_1}(w_1,\widetilde{z}_1)$$
$$\le C\,|k_1'-k_2'|^{2-d/2}\,(h_u-r_2)^{-d/2}$$
$$= C|k_1'-k_2'|^{-1}\,(h_u-r_2)^{-3}.$$

Summing over k_1' yields a factor of $\log(\delta n)$, while the sums over k_2', r_2, r_1 yield:

$$\sum_{(9/12)\delta n\le r_1\le h_u-n_1}\ \sum_{r_2\le r_1}\ \sum_{r_2<k_2'<h_u}(h_u-r_2)^{-3}\le C\,\log(\delta n).$$

The contribution to (I)– from $|k_1'-k_2'|<n_1$ is again of order $n_1^4\,\log(\delta n)$.

We turn to the bound for (II)+, where we initially restrict to $\|z_1-z_2\|\ge L_1$. We apply the local limit theorem to $\mathbf{p}^{k_2'-r_2}(w_2,\widetilde{z}_2)$, which gives

$$q_{a,1,(II)+}(k_1',k_2',r_1,r_2) \le \frac{C}{D^d}\,(k_2'-r_2)^{-d/2}\sum_{w_1,w_2\in\mathbb{Z}^d}\sum_{\widetilde{z}_1\in\mathbb{Z}^d}\sum_{\widetilde{z}_2:\|\widetilde{z}_2-\widetilde{z}_1\|>|k_1'-k_2'|^{1/2}} \|\widetilde{z}_2-\widetilde{z}_1\|^{4-2d}\,\widetilde{\mathbf{p}}(w_1)\,\widetilde{\mathbf{p}}(w_2)\,\mathbf{p}^{k_1'-r_1}(w_1,\widetilde{z}_1)$$
$$\le C\,|k_1'-k_2'|^{2-d/2}\,(k_2'-r_2)^{-d/2}\sum_{w_1\in\mathbb{Z}^d}\sum_{\widetilde{z}_1\in\mathbb{Z}^d}\widetilde{\mathbf{p}}(w_1)\,\mathbf{p}^{k_1'-r_1}(w_1,\widetilde{z}_1)$$
$$\le C\,|k_1'-k_2'|^{-1}\,(k_2'-r_2)^{-3}.$$

This is the same expression as in case (I)+, and hence yields $C\log^2(\delta n)$ upon summation. The contribution from $\|z_1-z_2\|<L_1$ is bounded by $C\,L_1^7\,\log(\delta n)$.

In the case (II)–, we again restrict to $\|z_1-z_2\|\ge L_1$, and apply the local limit theorem to $\widetilde{\mathbf{p}}(w_2)$. This yields:

$$q_{a,1,(II)-}(k_1',k_2',r_1,r_2) \le \frac{C}{D^d}\,(h_u-r_2)^{-d/2}\sum_{w_1,w_2\in\mathbb{Z}^d}\sum_{\widetilde{z}_1\in\mathbb{Z}^d}\sum_{\widetilde{z}_2:\|\widetilde{z}_2-\widetilde{z}_1\|>|k_1'-k_2'|^{1/2}} \|\widetilde{z}_2-\widetilde{z}_1\|^{4-2d}\,\widetilde{\mathbf{p}}(w_1)\,\mathbf{p}^{k_1'-r_1}(w_1,\widetilde{z}_1)\,\mathbf{p}^{k_2'-r_2}(w_2,\widetilde{z}_2)$$
$$\le C\,|k_1'-k_2'|^{2-d/2}\,(h_u-r_2)^{-d/2}\sum_{w_1\in\mathbb{Z}^d}\sum_{\widetilde{z}_1\in\mathbb{Z}^d}\widetilde{\mathbf{p}}(w_1)\,\mathbf{p}^{k_1'-r_1}(w_1,\widetilde{z}_1)$$
$$\le C\,|k_1'-k_2'|^{-1}\,(h_u-r_2)^{-3}.$$

This is the same expression as in case (I)–, and hence we conclude as in that case. The contribution from $\|z_1 - z_2\| < L_1$ is again bounded by $C\, L_1^7 \log(\delta n)$.

The above altogether shows that if $\delta n \geq \max\{e^{n_1^4}, e^{L_1^7}\}$, then $S_{a,1} \leq C\, \sigma^8\, D^{-2d} \log^2(\delta n)$.

Similar arguments for $S_{a,2}$ yield the lower order bound:

$$S_{a,2} \leq \frac{C\, \sigma^6}{D^{2d}} \log(\delta n),$$

and for $S_{a,3}$ the lower order bound:

$$S_{a,3} \leq \frac{C\, \sigma^4}{D^{2d}} \log(\delta n).$$

□

Proof of Lemma 3.13. We split the summations according to whether:
Subcase (c,1): $r_1 \neq \widetilde{r}_1$ and $r_2 \neq \widetilde{r}_2$;
Subcase (c,2): either $r_1 = \widetilde{r}_1$ and $r_2 \neq \widetilde{r}_2$, or $r_1 \neq \widetilde{r}_1$ and $r_2 = \widetilde{r}_2$;
Subcase (c,3): $r_1 = \widetilde{r}_1$ and $r_2 = \widetilde{r}_2$.
This gives $S_c = S_{c,1} + S_{c,2} + S_{c,3}$, where we bound each term separately. Using the last case in Lemma 3.8 and using Lemma 3.9 to sum over $h_y, h_{\widetilde{y}}$ we have

$$S_{c,1} \leq \sigma^8 \sum_{(9/12)\delta n \leq r_1 \neq \widetilde{r}_1 \leq h_u - n_1} \; \sum_{(9/12)\delta n \leq r_2 \neq \widetilde{r}_2 \leq h_u - n_1} \; \sum_{h_y, h_{\widetilde{y}} = h_u}^{\delta n} q_{c,1}(r_1, \widetilde{r}_1, r_2, \widetilde{r}_2, h_y, h_{\widetilde{y}}),$$

where

$$q_{c,1} = \sum_{w_1, \widetilde{w}_1 \in \mathbb{Z}^d} \; \sum_{w_2, \widetilde{w}_2 \in \mathbb{Z}^d} \widetilde{\mathbf{p}}(w_1, \widetilde{w}_1)\, \widetilde{\mathbf{p}}(w_2, \widetilde{w}_2)\, \mathbf{p}^{2h_y - r_1 - r_2}(w_1, w_2)\, \mathbf{p}^{2h_{\widetilde{y}} - \widetilde{r}_1 - \widetilde{r}_2}(\widetilde{w}_1, \widetilde{w}_2),$$

with $\widetilde{\mathbf{p}}(w_1, \widetilde{w}_1)$ and $\widetilde{\mathbf{p}}(w_2, \widetilde{w}_2)$, respectively, denoting the joint distributions of the spatial locations $w_1, \widetilde{w}_1$ and $w_2, \widetilde{w}_2$, respectively.

We explain the bound in the case when $r_2 \leq r_1$ and $\widetilde{r}_2 \leq \widetilde{r}_1$. There are three very similar other cases, that can be handled analogously. We use the local limit theorem to bound

$$\mathbf{p}^{2h_y - r_1 - r_2}(w_1, w_2) \leq \frac{C}{D^d}(h_y - r_2)^{-d/2}$$
$$\mathbf{p}^{2h_{\widetilde{y}} - \widetilde{r}_1 - \widetilde{r}_2}(\widetilde{w}_1, \widetilde{w}_2) \leq \frac{C}{D^d}(h_{\widetilde{y}} - \widetilde{r}_2)^{-d/2},$$

and then sum over $w_1, w_2, \widetilde{w}_1, \widetilde{w}_2$ to get the upper bound:

$$\begin{aligned}
&\frac{C\,\sigma^8}{D^{2d}} \sum_{r_2 \le r_1 \le h_u - n_1} \sum_{h_y \ge h_u} (h_y - r_2)^{-3} \sum_{\widetilde{r}_2 \le \widetilde{r}_1 \le h_u - n_1} \sum_{h_{\widetilde{y}} \ge h_u} (h_{\widetilde{y}} - \widetilde{r}_2)^{-3} \\
&\quad \le \frac{C\,\sigma^8}{D^{2d}} \sum_{r_2 \le r_1 \le h_u - n_1} (h_u - r_2)^{-2} \sum_{\widetilde{r}_2 \le \widetilde{r}_1 \le h_u - n_1} (h_u - \widetilde{r}_2)^{-2} \\
&\quad \le \frac{C\,\sigma^8}{D^{2d}} \sum_{r_2 \le h_u - n_1} (h_u - r_2)^{-1} \sum_{\widetilde{r}_2 \le h_u - n_1} (h_u - \widetilde{r}_2)^{-1} \\
&\quad \le \frac{C\,\sigma^8}{D^{2d}} \log^2(\delta n).
\end{aligned}$$

The subcase $S_{c,2}$ leads to the lower order bound $S_{c,2} \le C\,C_3\,\sigma^4/D^{2d}$, and $S_{c,3}$ leads to the lower order bound $S_{c,3} \le C\,C_3^2/D^{2d}$. □

Proof of Lemma 3.12. Due to symmetry, it is enough to bound S_{b1}. We show that the bound follows from the bounds on S_a and S_c already established. We can write S_{b1} in the form

$$S_{b1} = \sum_{y, \widetilde{y} \in \mathbb{Z}^d} \sum_{h_y, h_{\widetilde{y}} = h_u}^{\delta n} g_a(y, \widetilde{y}, h_y, h_{\widetilde{y}})\, g_c(y, \widetilde{y}, h_y, h_{\widetilde{y}}),$$

where g_a involves summation over the variables $r_1, k_1', w_1, \widetilde{z}_1$, and g_c involves summation over the variables $r_2, \widetilde{r}_2, w_2, \widetilde{w}_2$. Applying the Cauchy-Schwarz inequality we get

$$\begin{aligned}
S_{b1} &\le \left[\sum_{y, \widetilde{y} \in \mathbb{Z}^d} \sum_{h_y, h_{\widetilde{y}} = h_u}^{\delta n} g_a(y, \widetilde{y}, h_y, h_{\widetilde{y}})^2 \right]^{1/2} \left[\sum_{y, \widetilde{y} \in \mathbb{Z}^d} \sum_{h_y, h_{\widetilde{y}} = h_u}^{\delta n} g_c(y, \widetilde{y}, h_y, h_{\widetilde{y}})^2 \right]^{1/2} \\
&= S_a^{1/2}\, S_c^{1/2} \le \frac{C\sigma^8}{D^{2d}} \log^2(\delta n),
\end{aligned}$$

where we use Lemmas 3.11 and 3.13 in the last step. □

4 Analysis of Tree Bad Blocks

Most of the analysis of tree bad blocks does not require any change compared to [14, Section 4]. However, we need to improve the bounds of [14, Lemma 4.1] and [14, Lemma 4.3], as the error terms of the form $(1 + C_4\delta)$ present in those lemmas are not sufficient for the induction argument in $d = 6$. In fact, it turns out that the $C_4\delta$ term can be completely removed, and we prove this improvement below.

For $k \leq n$ we define

$$\bar{\gamma}(k;(x,n)) = \sum_{y \in \mathbb{Z}^d} \frac{\mathbf{p}^k(o,y)\mathbf{p}^{n-k}(y,x)}{\mathbf{p}^n(o,x)} \gamma(k,y). \tag{4.1}$$

For $a = 1, \dots, 6$ we define $\mathcal{E}_{(a)}$ to be the event that conditions (1) to $(a-1)$ in Definitions 2.2 and 2.4 are satisfied, but condition (a) is not.

Lemma 4.1 (Strengthening of [14, Lemma 4.1])**.** *We have*

$$\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(1)}, \Phi(V_n) = (x,n)\Big] \leq K\bar{\gamma}(\delta n;(x,n)).$$

For the proof we will need the following stochastic monotonicity result.

Lemma 4.2. *Let $i\delta n \leq j \leq (i+1)\delta n - 1$, and consider the event $\mathcal{F}(j)$ that $\mathcal{T}_{n,m}(V_j)$ reaches level $(i+2)\delta n$. Then conditioned on $\mathcal{F}(j)$, the distribution of $\mathcal{T}_{n,m}(V_j)$ is stochastically larger than unconditionally, which is stochastically lager than conditionally on $\mathcal{F}(j)^c$.*

Proof. We show that the distribution of $\mathcal{T}_{n,m}(V_j)$ enjoys the FKG property. This is not immediate, since in the definition of $\mathcal{T}_{n,m}$ we are conditioning on the decreasing event that level m is not reached. We realize $\mathcal{T}_{n,m}(V_j)$ on the following probability space. Let $\mathbb{T}_{j,m}$ denote an infinitary tree with root V_j and with finite height $m-1-j$ (that is, the vertices at distance $m-1-j$ from V_j are the leaves of $\mathbb{T}_{j,m}$). If $V, W \in \mathbb{T}_{j,m}$, and W is a child of V, we write $n(W;V) \geq 1$ for the index of W among all children of V. Let

$$\Omega_{j,m} = \left\{ (\eta_U) \in \mathbb{N}^{\mathbb{T}_{j,m}} : \begin{array}{l} \eta_W = 0 \text{ when } W \text{ is a child of } V \text{ and } n(W;V) > \eta_V; \text{ and} \\ \eta_W = 0 \text{ for all } W \text{ at height } m-1-j \end{array} \right\}.$$

Elements of $\Omega_{j,m}$ are in bijection with finite trees of height at most $m-1-j$, where V_j has η_{V_j} children, each child W of V_j has, respectively, η_W children, etc. Let us write $T = T(\eta)$ for the tree represented by $\eta \in \Omega_{j,m}$. Then we have

$$\mathbf{P}[\mathcal{T}_{n,m}(V_j) = T(\eta)] = \frac{1}{Z_{m,j}} \tilde{p}(\eta_{V_j}) \times \prod_{U \in T : U \neq V_j} p(\eta_U),$$

where $Z_{m,j}$ is a normalizing constant.

We use the criterion of [10, Theorem 4.11]. That theorem requires that η_U be bounded. Therefore, in applying the theorem, we first take a large M, replace $p(M)$ by $\sum_{k \geq M} p(k)$, and apply the theorem to this progeny distribution bounded by M. Then we let $M \to \infty$. The theorem also requires that the probability of the maximal element be positive. This is the case, if $p(M) > 0$, which we may assume without loss of generality. Another requirement is that configurations can be transformed into one another by changing one coordinate at a time. This is easily verified: any tree can be transformed into the tree containing only the root, by removing leaves. It is left to check the monotonicity property of one site conditional distributions: given any configurations ζ and ρ on the vertices $\{W : W \neq V\}$, such that $\zeta \geq \rho$ we need to check that

$$\begin{aligned}\mathbf{P}[\eta_V \leq k \,|\, \eta_W = \zeta_W,\, W \neq V]\\ \leq \mathbf{P}[\eta_V \leq k \,|\, \eta_W = \rho_W,\, W \neq V], \quad k = 0, \dots, M.\end{aligned} \tag{4.2}$$

We do this by checking some cases separately. First note that we may assume without loss of generality, that either V is the root, or, if V' is the parent of V, that $n(V; V') \leq \rho_{V'} \leq \zeta_{V'}$. Indeed, if $\rho_{V'} < n(V; V')$, then the right hand side of (4.2) equals 1 already for $k = 0$ (and hence for all k). We may also assume without loss of generality that the height of V is less than $m - 1 - j$, otherwise both sides in (4.2) are 1 for $k = 0$ (and hence for all k). Under these assumptions, let

$$\begin{aligned}k_\zeta &= \max\{j \geq 0 : \zeta_W > 0 \text{ for the child } W \text{ of } V \text{ of index } n(W; V) = j\}\\ k_\rho &= \max\{j \geq 0 : \rho_W > 0 \text{ for the child } W \text{ of } V \text{ of index } n(W; V) = j\}.\end{aligned}$$

Since $\zeta \geq \rho$, we have $k_\zeta \geq k_\rho$. We distinguish the following two cases.

(i) $k < k_\zeta$: In this case, the left hand side of (4.2) is 0, and hence (4.2) holds.

(ii) $k \geq k_\zeta \geq k_\rho$: In this case, the left hand side in (4.2) is

$$\frac{1}{Z_\zeta} \sum_{\ell = k_\zeta}^{k} p(\ell) p(0)^{\ell - k_\zeta}, \qquad \text{with} \qquad Z_\zeta = \sum_{\ell = k_\zeta}^{M} p(\ell) p(0)^{\ell - k_\zeta},$$

and the right hand side is

$$\frac{1}{Z_\rho} \sum_{\ell = k_\rho}^{k} p(\ell) p(0)^{\ell - k_\rho} \qquad \text{with} \qquad Z_\rho = \sum_{\ell = k_\rho}^{M} p(\ell) p(0)^{\ell - k_\rho}.$$

Thus the required inequality (4.2) boils down to showing that

$$\begin{aligned}\sum_{\ell_1 = k_\rho}^{M} p(\ell_1) p(0)^{\ell_1 - k_\rho} \sum_{\ell_2 = k_\zeta}^{k} p(\ell_2) p(0)^{\ell_2 - k_\zeta}\\ \leq \sum_{\ell_1 = k_\zeta}^{M} p(\ell_1) p(0)^{\ell_1 - k_\zeta} \sum_{\ell_2 = k_\rho}^{k} p(\ell_2) p(0)^{\ell_2 - k_\rho},\end{aligned}$$

which reduces to

$$\sum_{\ell_1 = k_\rho}^{M} p(\ell_1) p(0)^{\ell_1} \sum_{\ell_2 = k_\zeta}^{k} p(\ell_2) p(0)^{\ell_2} \leq \sum_{\ell_1 = k_\zeta}^{M} p(\ell_1) p(0)^{\ell_1} \sum_{\ell_2 = k_\rho}^{k} p(\ell_2) p(0)^{\ell_2}.$$

This is easily verified, by checking that the terms appearing in the left hand side form a subset of the terms appearing in the right hand side. Hence (4.2) also holds in this case.

Letting $M \to \infty$ we obtain that the distribution of $\mathcal{T}_{n,m}(V_j)$ has the FKG property, and this implies the statement of the lemma. □

Proof of Lemma 4.1. Using the triangle inequality for effective resistance, we have

$$\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \,\Big|\, \mathcal{E}_{(1)},\, \Phi(V_n) = (x,n)\Big] \\
&\quad \le \sum_{i'=i}^{i+K-1} \mathbf{E}\Big[R_{\text{eff}}(\Phi(X_{i'}) \leftrightarrow \Phi(X_{i'+1})) \,\Big|\, \mathcal{E}_{(1)},\, \Phi(V_n) = (x,n)\Big].
\end{aligned}$$

For $i' = i+1, \dots, i+K-1$, the conditioning on $\mathcal{E}_{(1)}$ does not affect the distribution of the tree between $X_{i'}$ and $X_{i'+1}$, and hence the sum of the terms over those i' are bounded by $(K-1)\bar{\gamma}(\delta n; (x,n))$.

It is left to bound the $i' = i$ term by $\bar{\gamma}(\delta n; (x,n))$.

When $\mathcal{E}_{(1)}$ occurs, we have exactly one of the following three cases:

(i) There are no levels in $[i\delta n, (i+1)\delta n)$ that reach height $2\delta n$,

(ii) There is more than one such level,

(iii) There is a unique such level ℓ_1, but $\ell_1 \notin [(i+1/4)\delta n, (i+3/4)\delta n)]$.

Let us handle these cases separately. If (i) occurs, each tree $\mathcal{T}_{n,m}(V_j)$ for $j = i\delta n, \dots, (i+1)\delta n - 1$ is conditioned to not reach level $(i+2)\delta n$, and hence the the part of $\mathcal{T}_{n,m}$ between X_i and X_{i+1} is distributed according to the law of $\mathcal{T}_{\delta n, 2\delta n}$. This gives

$$\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1}))\mathbf{1}_{(i)} \,\Big|\, \Phi(V_n) = (x,n)\Big] \\
&\quad \le \bar{\gamma}(\delta n; (x,n))\, \mathbf{P}((i) \,|\, \Phi(V_n) = (x,n)),
\end{aligned}$$

since in the definition of $\bar{\gamma}(\delta n; (x,n))$ we take a supremum over $m \ge 2\delta n$.

If (ii) occurs, let $j_1, \dots, j_k$ be the levels whose tree reaches height $2\delta n$ ($k \ge 2$), and denote by $\mathcal{F}(j_1, \dots, j_k)$ the event that (ii) occurs with exactly these levels. Due to Lemma 4.2, on the event $\mathcal{F}(j_1, \dots, j_k)$, the trees $\mathcal{T}_{n,m}(V_{j_s})$, $s = 1, \dots, k$ are stochastically larger than unconditionally, and hence stochastically larger than on the event (i). Since R_{eff} is a decreasing random variable, we have

$$\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1})) \,\Big|\, \mathcal{F}(j_1, \dots, j_k),\, \Phi(V_n) = (x,n)\Big] \\
&\quad \le \mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1})) \,\Big|\, (i),\, \Phi(V_n) = (x,n)\Big] \\
&\quad \le \bar{\gamma}(\delta n; (x,n)).
\end{aligned}$$

Finally, if (iii) occurs, an argument similar to the case (ii) (with $k = 1$) shows that

$$\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1}))\,\Big|\,(iii),\, \Phi(V_n) = (x,n)\Big]$$
$$\leq \bar{\gamma}(\delta n; (x,n)).$$

This completes the proof of the lemma. □

Lemma 4.3 ([14, Lemma 4.2])**.**

$$\mathbf{E}\big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\big|\,\mathcal{E}_{(2)},\, \ell_1,\, \Phi(V_n) = (x,n)\big]$$
$$\bar{\gamma}(\ell_1 - i\delta n; (x,n)) + 1 + \bar{\gamma}((i+1)\delta n - \ell_1 - 1; (x,n))$$
$$+ (K-1)\bar{\gamma}(\delta n; (x,n)).$$

Lemma 4.4 (Strengthening of [14, Lemma 4.3])**.** *We have*

$$\mathbf{E}\big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\big|\,\mathcal{E}_{(3)},\, \ell_1,\, \Phi(V_n) = (x,n)\big]$$
$$\leq \bar{\gamma}(\ell_1 - i\delta n; (x,n)) + 1 + \bar{\gamma}((i+1)\delta n - \ell_1 - 1; (x,n))$$
$$+ (K-1)\bar{\gamma}(\delta n; (x,n)).$$

Proof. The modifications required compared to [14, Lemma 4.3] are exactly the same as in Lemma 4.1. □

Lemma 4.5 ([14, Lemma 4.4])**.**

$$\mathbf{E}\big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\big|\,\mathcal{E}_{(4)},\, \ell_1,\, \ell_2,\, \Phi(V_n) = (x,n)\big]$$
$$\leq \bar{\gamma}(\ell_1 - i\delta n; (x,n)) + 1 + \bar{\gamma}((i+1)\delta n - \ell_1 - 1; (x,n))$$
$$+ (K-2)\bar{\gamma}(\delta n; (x,n)) + \bar{\gamma}(\ell_2 - (i+K-1)\delta n; (x,n)) + 1$$
$$+ \bar{\gamma}((i+K)\delta n - \ell_2 - 1; (x,n)).$$

5 Analysis of Spatially Bad Blocks

The analysis of spatially bad blocks does not require any change compared to [14]. For the convenience of the reader, we repeat the required definitions, and state the results.

We analyze what happens when condition (5) or (6) in Definition 2.4 fails, that is, some spatial displacement is "not typical", and also what happens when $\mathcal{B}'(i, c_0')$ fails. We write $\mathcal{G}_{\text{tree}}$ for the event

$$\mathcal{G}_{\text{tree}} = \{(1)\text{–}(4), \ell_1, \ell_2, \Phi(V_n) = (x,n)\}.$$

We define a set of times $i\delta n = T_0 < T_1 < \cdots < T_{K+4} = (i+K)\delta n$, time differences $t_1, t_2, \ldots, t_{K+4}$ and spatial locations $z_0, \ldots, z_{K+4} \in \mathbb{Z}^d$ by

$$\begin{array}{lll}
z_0 = x_i & & T_0 = i\delta n \\
z_1 = v_{\ell_1} & t_1 = \ell_1 - i\delta n & T_1 = \ell_1 \\
z_2 = v_{\ell_1+1} & t_2 = 1 & T_2 = \ell_1 + 1 \\
z_3 = x_{i+1} & t_3 = (i+1)\delta n - \ell_1 - 1 & T_3 = (i+1)\delta n \\
z_4 = x_{i+2} & t_4 = \delta n & T_4 = (i+2)\delta n \\
z_5 = x_{i+3} & t_5 = \delta n & T_5 = (i+3)\delta n \\
\vdots & \vdots & \vdots \\
z_{K+1} = x_{i+K-1} & t_{K+1} = \delta n & T_{K+1} = (i+K-1)\delta n \\
z_{K+2} = v_{\ell_2} & t_{K+2} = \ell_2 - (i+K-1)\delta n & T_{K+2} = \ell_2 \\
z_{K+3} = v_{\ell_2+1} & t_{K+3} = 1 & T_{K+3} = \ell_2 + 1 \\
z_{K+4} = x_{i+K} & t_{K+4} = K\delta n - \ell_2 - 1 & T_{K+4} = (i+K)\delta n
\end{array}$$

Observe that conditional on $\mathcal{G}_{\text{tree}}$, the times T_s and time differences t_s are non-random but the spatial locations z_s are random. We define for any $s = 1, \ldots, K+4$

$$\mathbf{q}_s(z) = \sum_{\substack{\|y_r\| \le \sqrt{t_r} \\ r=1,\ldots,s-1 \\ y_1+\cdots+y_{s-1}=z}} \prod_{r=1}^{s-1} \mathbf{p}^{t_r}(0, y_r).$$

For any $s = 1, \ldots, K+4$ we define the event

$$\mathcal{E}^s_{(5)} = \bigcap_{r=1}^{s-1} \Big\{ \|z_r - z_{r-1}\| \le \sqrt{t_r} \Big\} \bigcap \Big\{ \|z_s - z_{s-1}\| > \sqrt{t_s} \Big\}.$$

Note that

$$\mathbf{P}(\mathcal{E}^s_{(5)} \,|\, \mathcal{G}_{\text{tree}}) = \sum_{z,y: \|y\| > \sqrt{t_s}} \mathbf{q}_s(z) \frac{\mathbf{p}^{t_s}(z, z+y)\mathbf{p}^{n-T_s+T_0}(z+y, x)}{\mathbf{p}^n(o, x)}. \tag{5.1}$$

Lemma 5.1 ([14, Lemma 5.1])**.** *For any $s = 1, \ldots, K+4$ and $s' = 1, \ldots, K+4$, the quantity*

$$\mathcal{R}_{s',s} = \mathbf{E}\Big[R_{\text{eff}}((z_{s'-1}, T_{s'-1}) \leftrightarrow (z_{s'}, T_{s'}) \mathbf{1}_{\mathcal{E}^s_{(5)}} \,\Big|\, \mathcal{G}_{\text{tree}} \Big],$$

satisfies:

$$\mathcal{R}_{s',s} \le \mathbf{P}(\mathcal{E}^s_{(5)} \,|\, \mathcal{G}_{\text{tree}}), \quad \textit{when } s' = 2, K+3,$$
$$\mathcal{R}_{s',s} \le \mathbf{P}(\mathcal{E}^s_{(5)} \,|\, \mathcal{G}_{\text{tree}})\gamma(t_{s'}), \quad \textit{when } s' < s,\ s' \ne 2, K+3,$$
$$\mathcal{R}_{s',s} \le \sum_z \mathbf{q}_s(z) \sum_{y:\|y\|>\sqrt{t_s}} \frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y,x-z)}{\mathbf{p}^n(o,x)}\gamma(t_s,y),$$

when $s' = s,\ s' \ne 2, K+3$,

$$\mathcal{R}_{s',s} \le \sum_{\substack{z,y_s,y \\ \|y_s\|>\sqrt{t_s}}} \mathbf{q}_s(z)\frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{t_{s'}}(o,y)\mathbf{p}^{n-t_{s'}-T_s}(y,x-z-y_s)}{\mathbf{p}^n(o,x)}\gamma(t_{s'},y),$$

when $s' > s,\ s' \ne 2, K+3$.

Lemma 5.2 ([14, Lemma 5.2])**.**

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(6)} \cup \mathcal{E}_{(7)},\, \mathcal{G}_{\text{tree}}\Big] \\ \le (K-2)\gamma(\delta n) + \gamma(\ell_1 - i\delta n) + 1 + \gamma((i+1)\delta n - \ell_1 - 1) \\ + \gamma(\ell_2 - (i+K-1)\delta n) + 1 + \gamma((i+K)\delta n - \ell_2 - 1).\end{aligned}$$

Let us write n in the form $n = NK\delta n + K'\delta n + \tilde{n}$, where $0 \le K' < K$ and $\delta n \le \tilde{n} < 2\delta n$. Write $i^{\text{last}} = KN$.

Lemma 5.3 ([14, Lemma 5.3])**.**

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_{i^{\text{last}}}) \leftrightarrow \Phi(V_n))\,\Big|\,\Phi(V_n) = (x,n)\Big] \\ \le K'\bar{\gamma}(\delta n; (x,n)) + \bar{\gamma}(\tilde{n}; (x,n)).\end{aligned}$$

6 Analysis of good blocks

In this section we estimate expectations of resistances given

$$\mathcal{G}'_{\text{good}} = \left\{\Phi(V_n) = (x,n), \mathcal{A}(i), \mathcal{B}'(i,c'_0), \ell_1, \ell_2\right\}.$$

The following lemma is essentially [14, Lemma 6.1], with $\mathcal{G}_{\text{good}}$ replaced by $\mathcal{G}'_{\text{good}}$, and requires no modification in its proof.

Lemma 6.1. *Given* $\mathcal{G}'_{\text{good}}$*, we have*

1. $\mathbf{E}\big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(V_{\ell_1})) \mid \mathcal{G}'_{\text{good}}\big] \le \gamma(\ell_1 - i\delta n)$.

2. $\mathbf{E}\big[R_{\mathrm{eff}}(\Phi(V_{\ell_1}) \leftrightarrow \Phi(X_{i+1})) \mid \mathcal{G}'_{\mathrm{good}}\big] \le \gamma((i+1)\delta n - \ell_1 - 1) + 1.$

3. For all $i+1 \le j \le i+K-2$ *we have*

$$\mathbf{E}\big[R_{\mathrm{eff}}(\Phi(X_j) \leftrightarrow \Phi(X_{j+1})) \mid \mathcal{G}'_{\mathrm{good}}\big] \le \gamma(\delta n)\,.$$

4. $\mathbf{E}\big[R_{\mathrm{eff}}(\Phi(X_{i+K-1}) \leftrightarrow \Phi(V_{\ell_2})) \mid \mathcal{G}'_{\mathrm{good}}\big] \le \gamma(\ell_2 - (i+K-1)\delta n).$

5. $\mathbf{E}\big[R_{\mathrm{eff}}(\Phi(V_{\ell_2}) \leftrightarrow \Phi(X_{i+K})) \mid \mathcal{G}'_{\mathrm{good}}\big] \le \gamma((i+K)\delta n - \ell_2 - 1) + 1.$

6. $\mathbf{E}\big[R_{\mathrm{eff}}(\Phi(V_{\ell_2}) \leftrightarrow \Phi(X'_{i+K})) \mid \mathcal{G}'_{\mathrm{good}}\big] \le \gamma((i+K)\delta n - \ell_2 - 1) + 1.$

7. $\mathbf{E}\big[R_{\mathrm{eff}}(\Phi(V_{\ell_1}) \leftrightarrow \Phi(Y_{i+1})) \mid \mathcal{G}'_{\mathrm{good}}\big] \le \gamma((i+1)\delta n - \ell_1 - 1) + 1.$

8. For all $i+1 \le j \le i+K-1$ *we have*

$$\mathbf{E}\big[R_{\mathrm{eff}}(\Phi(Y_j) \leftrightarrow \Phi(Y_{j+1})) \mid \mathcal{G}'_{\mathrm{good}}\big] \le \gamma(\delta n)\,.$$

The following lemma replaces [14, Lemma 6.2].

Lemma 6.2. *Assume* $d = 6$*. There exists* $C'_5 < \infty$ *such that we have*

$$\mathbf{E}\Big[R_{\mathrm{eff}}(\Phi(X'_{i+K}) \leftrightarrow \Phi(Y_{i+K}))\,\Big|\,\mathcal{G}'_{\mathrm{good}}\Big] \le C'_5 \max_{1 \le \ell \le \delta n} \gamma(\ell),$$

whenever $\delta n \ge \max\{n_1(\mathbf{p}^1),\, n'_9(\sigma^2, C_3, \mathbf{p}^1)\}$*.*

The proof is a straightforward adaptation of the proof of [14, Lemma 6.2]. We first assume that the progeny distribution is bounded by M, prove the statement with this restriction relying on Theorem 2.7, and then let $M \to \infty$. We omit the details.

Proof of Theorem 2.8. This can be completed exactly as in [14, Section 6]. □

7 Proof of Theorem 1.2

Let K_0 be the constant in Theorem 2.8. We fix $K = K_0$ for the remainder of the proof. Let

$$n_0 = \max\{n'_3(\sigma^2, C_3, \mathbf{p}^1, K),\, n'_4(\sigma^2, C_3, \mathbf{p}^1),\, 4k_1(\mathbf{p}^1)\},$$

where n'_3 and n'_4 are the constants from Theorems 2.7 and 2.8, and k_1 is the constant from Proposition 1.3. Let $\delta_0 > 0$, $\xi \in (0, 1/2)$ and $A > 0$ be constants. These will be chosen below in the order: δ_0, ξ, A, and among others we will require that

$$2\delta_0 \le (K+4)^{-1}, \qquad 2\delta_0 \le \delta_1, \tag{7.1}$$

where δ_1 is the constant from Proposition 1.3(ii). Once δ_0 and ξ will be chosen, we choose A to satisfy:

$$A \geq \max\left\{\frac{n_0}{\delta_0}, \frac{1}{\delta_0^2}\right\}, \qquad \exp\left(-\frac{1}{2}A^{1/\xi}\right) \leq \frac{1}{\sqrt{n_0}}, \qquad A^{-1} \leq \xi \frac{n_0}{(\log n_0)^{1+\xi}}. \tag{7.2}$$

We prove the theorem by induction. Due to the first condition on A, the theorem holds for $n < \max\{n_0/\delta_0, 1/\delta_0^2\}$, so we may assume $n \geq n_0/\delta_0$ and $n \geq 1/\delta_0^2$. Our induction hypothesis is that for all $n' < n$ and for all $x \in \mathbb{Z}^d$ we have

$$\gamma(n', x) \leq \begin{cases} An'(\log n')^{-\xi} & \text{when } \|x\| \leq \sqrt{n'}; \\ An'(\log n')^{-\xi} \left(1 - \frac{\log\left(\|x\|^2/n'\right)}{\log n'}\right)^{-\xi} & \text{when } \sqrt{n'} < \|x\| \leq n'/2; \\ An' & \text{when } \|x\| > n'/2, \end{cases} \tag{7.3}$$

and given the hypothesis, we prove it for n. Sometimes, instead of (7.3), it will be convenient to use the following consequence of (7.3) that involves simpler expressions: there is a universal constant $C > 0$ such that

$$\gamma(n', x) \leq \begin{cases} An'(\log n')^{-\xi} & \text{when } \|x\| \leq \sqrt{n'}; \\ An'(\log n')^{-\xi} \left(1 + \frac{C\xi}{\log n'} \frac{\|x\|^2}{n'}\right) & \text{when } \|x\| > \sqrt{n'}. \end{cases} \tag{7.4}$$

These bounds follow from the elementary inequalities:

$$\left(1 - \frac{\log(\|y\|^2/k)}{\log k}\right)^{-\xi} \leq 1 + \frac{C\xi}{\log k}\frac{\|y\|^2}{k}, \qquad \sqrt{k} < \|y\| \leq k/2,\ 0 \leq \xi \leq 1/2, \tag{7.5}$$

and

$$(\log k)^{\xi} \leq 1 + \frac{C\xi}{\log k}\frac{\|y\|^2}{k}, \qquad \|y\| > k/2,\ 0 \leq \xi \leq 1/2. \tag{7.6}$$

Since $\gamma(n, x) \leq n$, we claim that it suffices to prove the statement of the theorem when $\|x\| \leq n\exp(-\frac{1}{2}A^{1/\xi})$. Indeed, suppose that $\|x\| > n\exp(-\frac{1}{2}A^{1/\xi})$. Then if we have $\|x\| \leq \sqrt{n}$, then also $\exp(-\frac{1}{2}A^{1/\xi}) < n^{-1/2}$, and hence $(\log n)^{-\xi} > A^{-1}$, and hence

$$\gamma(n, x) \leq n = AnA^{-1} < An(\log n)^{-\xi}.$$

On the other hand, if we have $\sqrt{n} < \|x\| \leq n/2$, then $\|x\|^2 > n^2\exp(-A^{1/\xi})$ implies that $A^{-1} < (\log n^2/\|x\|^2)^{-\xi}$, and hence

$$\begin{aligned}\gamma(n, x) \leq n = AnA^{-1} &< An\left(\log n + \log\frac{n}{\|x\|^2}\right)^{-\xi} \\ &= An(\log n)^{-\xi}\left(1 - \frac{\log\|x\|^2/n}{\log n}\right)^{-\xi}.\end{aligned}$$

Hence from now on we assume the upper bound $\|x\| \le n\exp(-\frac{1}{2}A^{1/\xi})$. Note that due to the second inequality in (7.2), this implies

$$\|x\| \le n\exp\left(-\frac{1}{2}A^{1/\xi}\right) \le \frac{n}{\sqrt{n_0}}. \tag{7.7}$$

Given such x, fix

$$\delta = \min\Big\{\eta : \eta \ge \min\Big\{\delta_0,\ \frac{n}{\|x\|^2}\Big\},\ \eta n \text{ is an integer}\Big\}. \tag{7.8}$$

Observe that

$$\delta \le \delta_0 + \frac{1}{n} \le \delta_0 + \frac{\delta_0}{n_0} \le 2\delta_0, \tag{7.9}$$

and similarly, using (7.7), we have

$$\delta \le \frac{n}{\|x\|^2} + \frac{1}{n} \le \frac{n}{\|x\|^2} + \frac{1}{n_0}\frac{n}{\|x\|^2} \le \frac{2n}{\|x\|^2}. \tag{7.10}$$

We also have

$$\delta n \ge \min\left\{\delta_0 n,\ \frac{n^2}{\|x\|^2}\right\} \ge \min\{n_0,\ n_0\} = n_0. \tag{7.11}$$

Finally, note that

$$\|x\| \le \sqrt{2n/\delta},$$

which can be seen by considering separately the cases $\delta_0 \le n/\|x\|^2$ and $\delta_0 > n/\|x\|^2$. Therefore, Theorem 2.7 can be applied to (x, n).

Consider the sequences

$$(0,\dots,K),(K,\dots,2K),\dots,((N-1)K,\dots,NK),$$

where $n = NK\delta n + K'\delta n + \tilde{n}$, with $0 \le K' < K$, $\delta n \le \tilde{n} < 2\delta n$. Fix any integer $m \ge 2n$, and define

$$\gamma_m(n,x) = \mathbf{E}_{\mathcal{T}_{n,m}}\big[R_{\text{eff}}\big((o,0) \leftrightarrow \Phi(V_n)\big) \,\big|\, \Phi(V_n) = (x,n)\big],$$

where the resistance is considered in the graph $\Phi(\mathcal{T}_{n,m})$, so that $\gamma(n,x) = \sup_{m\ge 2n}\gamma_m(n,x)$. We bound $\gamma_m(n,x)$ by estimating

$$\mathbf{E}\big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}) \,\big|\, \Phi(V_n) = (x,n)\big] \tag{7.12}$$

for each $i = 0, K, 2K, \dots, (N-1)K$ and adding the estimates, using the triangle inequality for resistance (1.1), and also adding the estimates for the final stretch from $NK\delta n$ to n.

Fix $0 \le i \le N-1$. We split the expectation in 7.12 according to whether $\mathcal{A}(i) \cap \mathcal{B}'(i, c_0')$ occurred or not. By Theorem 2.8 we have that

$$\begin{aligned} &\mathbf{E}\big[R_{\mathrm{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\mathbf{1}_{\mathcal{A}(i)\cap\mathcal{B}'(i,c_0')} \,\big|\, \Phi(V_n) = (x,n)\big] \\ &\quad \le \frac{3K \max_{1\le k\le \delta n} \gamma(k)}{4} \mathbf{P}(\mathcal{A}(i) \cap \mathcal{B}'(i, c_0') \,|\, \Phi(V_n) = (x,n)) \\ &\quad \le \frac{3KA(\delta n)\,(\log(\delta n))^{-\xi}}{4} \mathbf{P}(\mathcal{A}(i) \cap \mathcal{B}'(i, c_0') \,|\, \Phi(V_n) = (x,n)), \end{aligned} \tag{7.13}$$

where in the last step we used the induction hypothesis.

We now estimate the expectation on the event when either $\mathcal{A}(i)$ or $\mathcal{B}'(i, c_0')$ fail. Recall that we may write:

$$\mathcal{A}(i)^c \cup \mathcal{B}'(i, c_0')^c = \bigcup_{a=1}^{6} \mathcal{E}_{(a)} \cup (\mathcal{A}(i) \cap \mathcal{B}'(i, c_0')^c),$$

where $\mathcal{E}_{(a)}$ was defined in Section 4. We need a few lemmas to estimate the contribution from these terms.

Lemma 7.1. *There exists $C_6' > 0$ such that, assuming the induction hypothesis, for any $\delta n/4 \le k \le 2\delta n$ we have*

$$\bar\gamma(k; (x,n)) \le \left(1 + \frac{C_6'\xi}{\log \delta n}\right) Ak(\log k)^{-\xi},$$

where $\bar\gamma$ is defined in (4.1).

Proof. By the induction hypothesis and its consequence (7.4) we have

$$\bar\gamma(k; (x,n)) \le Ak(\log k)^{-\xi} H_1(\xi),$$

where

$$\begin{aligned} H_1(\xi) = &\sum_{y:\|y\|\le\sqrt{k}} \frac{\mathbf{p}^k(o,y)\mathbf{p}^{n-k}(y,x)}{\mathbf{p}^n(o,x)} \\ &+ \sum_{y:\|y\|>\sqrt{k}} \frac{\mathbf{p}^k(o,y)\mathbf{p}^{n-k}(y,x)}{\mathbf{p}^n(o,x)} \left(1 + \frac{C\xi}{\log k}\frac{\|y\|^2}{k}\right). \end{aligned}$$

We have $H_1(0) = 1$ and

$$\begin{aligned} H_1'(\xi) &= \frac{C}{\log k} \sum_{y:\|y\|>\sqrt{k}} \frac{\mathbf{p}^k(o,y)\mathbf{p}^{n-k}(y,x)}{\mathbf{p}^n(o,x)} \frac{\|y\|^2}{k} \\ &\le \frac{C}{\log k}\frac{1}{k}\mathbf{E}\big[\|S(k)\|^2 \,\big|\, S(n) = x\big]. \end{aligned}$$

Since $k \geq \delta n/4 \geq n_0/4 \geq k_1$ and $\|x\| \leq \sqrt{2n/\delta} \leq \sqrt{2}n/\sqrt{\delta n} \leq 4n/\sqrt{k}$, we can apply Proposition 1.3(i) to the expectation on the right hand side to see that $H_1'(\xi) \leq C/\log k$, and the lemma follows. □

Lemma 7.2. *There exists $C_6' > 0$ such that, assuming the induction hypothesis, for all $s' \geq s$ with $s' \neq 2, K+3$ we have*

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}((z_{s'-1},T_{s'-1}) \leftrightarrow (z_{s'},T_{s'}))\mathbf{1}_{\mathcal{E}^s_{(5)}} \,\Big|\, \mathcal{G}_{\text{tree}}\Big] \\ \leq \left(1+\frac{C_6'\xi}{\log \delta n}\right) At_{s'}(\log t_{s'})^{-\xi}\mathbf{P}(\mathcal{E}^s_{(5)} \,|\, \mathcal{G}_{\text{tree}}).\end{aligned}$$

Proof. We first consider the case $s' = s$ and $s' \neq 2, K+3$. Appealing to Lemma 5.1 and using consequence (7.4) of the induction hypothesis, we get that the expectation in the claim of the lemma is at most $At_s(\log t_s)^{-\xi}H_2(\xi)$, where

$$H_2(\xi) = \sum_z \mathbf{q}_s(z) \sum_{y:\|y\|>\sqrt{t_s}} \frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y,x-z)}{\mathbf{p}^n(o,x)}\left(1+\frac{C\xi}{\log t_s}\frac{\|y\|^2}{t_s}\right).$$

By (5.1), we have

$$H_2(0) = \mathbf{P}(\mathcal{E}^s_{(5)} \,|\, \mathcal{G}_{\text{tree}}).$$

Similarly to the previous lemma, we have

$$H_2'(\xi) = \frac{C}{\log t_s}\sum_z \mathbf{q}_s(z) \sum_{y:\|y\|>\sqrt{t_s}} \frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(z+y,x)}{\mathbf{p}^n(o,x)}\frac{\|y\|^2}{t_s}. \tag{7.14}$$

Let us multiply and divide by $\mathbf{p}^{n-T_{s-1}+T_0}(o,x-z)$, which allows us to rewrite (7.14) as

$$\begin{aligned}&\frac{C}{\log t_s}\frac{1}{t_s}\sum_z \mathbf{q}_s(z)\frac{\mathbf{p}^{n-T_{s-1}+T_0}(o,x-z)}{\mathbf{p}^n(o,x)} \\ &\quad\times \sum_{y:\|y\|>\sqrt{t_s}} \|y\|^2\frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y,x-z)}{\mathbf{p}^{n-T_{s-1}+T_0}(o,x-z)}.\end{aligned} \tag{7.15}$$

Let us now fix z. We want to apply Proposition 1.3(ii) to the sum over y in (7.15). For this, observe that

$$\begin{aligned}\|x-z\| &\leq \|x\|+\|z\| \leq \sqrt{2n/\delta}+(K+4)\sqrt{\delta n}+n\left(\frac{\sqrt{2}}{\sqrt{\delta n}}+\frac{\delta(K+4)}{\sqrt{\delta n}}\right) \\ &\leq n\frac{3}{\sqrt{\delta n}},\end{aligned}$$

where in the last step we used (7.1). This implies that we have $\|x-z\| \le 3n/\sqrt{\delta n} \le 3n/\sqrt{t_s}$. We also have $t_s \ge \delta n/4 \ge n_0/4 \ge k_1$, where k_1 is the constant in Proposition 1.3. In addition, $t_s \le \delta n \le \delta_1 n$, due to (7.1). Hence we can apply Proposition 1.3(ii). This gives:

$$\begin{aligned}
&\sum_{y:\|y\|>\sqrt{t_s}} \|y\|^2 \frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y,x-z)}{\mathbf{p}^{n-T_{s-1}+T_0}(o,x-z)} \\
&\quad = \mathbf{E}\Big[\|S(t_s)\|^2\,\mathbf{1}_{\|S(t_s)\|>\sqrt{t_s}}\,\Big|\,S(n)=x-z\Big] \\
&\quad \le Ct_s\mathbf{P}\Big(\|S(t_s)\|>\sqrt{t_s}\,\Big|\,S(n)=x-z\Big) \\
&\quad = Ct_s \sum_{y:\|y\|>\sqrt{t_s}} \frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y,x-z)}{\mathbf{p}^{n-T_{s-1}+T_0}(o,x-z)}.
\end{aligned}$$

Substituting this bound back into (7.15) and cancelling the factors $\mathbf{p}^{n-T_{s-1}+T_0}(o,x-z)$, we get

$$\begin{aligned}
H_2'(\xi) &\le \frac{C}{\log t_s} \sum_{\substack{z,y:\\ \|y\|>\sqrt{t_s}}} \mathbf{q}_s(z)\frac{\mathbf{p}^{t_s}(o,y)\mathbf{p}^{n-T_s+T_0}(y,x-z)}{\mathbf{p}^n(o,x)} \\
&= \frac{C}{\log t_s}\mathbf{P}(\mathcal{E}^s_{(5)}\,|\,\mathcal{G}_{\text{tree}}).
\end{aligned}$$

This gives the statement of the lemma in the case $s'=s$.

The case $s'>s$ is similar. Appealing to the last statement of Lemma 5.1 yields that the required quantity is at most $At_{s'}(\log t_{s'})^{-\xi}H_3(\xi)$, where

$$\begin{aligned}
H_3(\xi) = &\sum_{\substack{z,y_s,y:\\ \|y_s\|>\sqrt{t_s}}} \mathbf{q}_s(z)\frac{\mathbf{p}^{t_s}(o,y_s)\mathbf{p}^{t_{s'}}(o,y)\mathbf{p}^{n-t_{s'}-T_s}(y,x-z-y_s)}{\mathbf{p}^n(o,x)} \\
&\times\left(1+\frac{C\xi}{\log t_{s'}}\frac{\|y\|^2}{t_{s'}}\mathbf{1}_{\|y\|>\sqrt{t_{s'}}}\right).
\end{aligned}$$

Setting $\xi=0$ and performing the sum over y, we see using (5.1) that $H_3(0)=\mathbf{P}(\mathcal{E}^s_{(5)}\,|\,\mathcal{G}_{\text{tree}})$. The derivative $H_3'(\xi)$ can be analyzed similarly to $H_2'(\xi)$, this time appealing to Proposition 1.3(iii). This gives $H_3'(\xi) \le (C/\log t_{s'})\mathbf{P}(\mathcal{E}^s_{(5)}\,|\,\mathcal{G}_{\text{tree}})$, and proves the statement of the lemma when $s'>s$. □

We now assemble our resistance bounds on the event $\mathcal{A}(i)^c \cup \mathcal{B}'(i,c_0')^c$. Lemmas 4.1 and 7.1 and the induction hypothesis yield

$$\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i)\leftrightarrow\Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(1)},\,\Phi(V_n)=(x,n)\Big] \\
&\quad \le AK\left(1+\frac{C_6'\xi}{\log\delta n}\right)\delta n(\log\delta n)^{-\xi}.
\end{aligned}$$

Lemmas 4.3 and 7.1 yield

$$
\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(2)},\,\ell_1,\,\Phi(V_n)=(x,n)\Big]\\
&\quad\le A\left(1+\frac{C_6'\xi}{\log\delta n}\right)\Big[(\ell_1-i\delta n)(\log(\ell_1-i\delta n))^{-\xi}+\\
&\qquad+((i+1)\delta n-\ell_1)(\log((i+1)\delta n-\ell_1))^{-\xi}\\
&\qquad+(K-1)\delta n(\log\delta n)^{-\xi}\Big]+1.
\end{aligned}
$$

Writing $\nu = \frac{\ell_1}{\delta n} - i \in [1/4, 3/4]$, the first two terms inside the square brackets can be written as

$$
\begin{aligned}
&\delta n(\log\delta n)^{-\xi}\left[\nu\left(\frac{\log(\nu\delta n)}{\log\delta n}\right)^{-\xi}+(1-\nu)\left(\frac{\log(1-\nu)\delta n}{\log\delta n}\right)^{-\xi}\right]\\
&\quad=\delta n(\log\delta n)^{-\xi}\left[\nu\left(1+\frac{\log\nu}{\log\delta n}\right)^{-\xi}+(1-\nu)\left(1+\frac{\log(1-\nu)}{\log\delta n}\right)^{-\xi}\right]\\
&\quad\le\delta n(\log\delta n)^{-\xi}\left[\nu\left(1-\frac{2\xi\log\nu}{\log\delta n}\right)+(1-\nu)\left(1-\frac{2\xi\log(1-\nu)}{\log\delta n}\right)\right]\\
&\quad=\delta n(\log\delta n)^{-\xi}\left[1+\frac{2\xi}{\log\delta n}\left(-\nu\log\nu-(1-\nu)\log(1-\nu)\right)\right]\\
&\quad\le\delta n(\log\delta n)^{-\xi}\left[1+\frac{2(\log 2)\xi}{\log\delta n}\right].
\end{aligned}
$$

Hence, writing $C_7' = 2\log 2$, we have

$$
\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(2)},\,\ell_1,\,\Phi(V_n)=(x,n)\Big]\\
&\quad\le A\left(1+\frac{(C_6'+C_7'+C_6'C_7')\xi}{\log\delta n}\right)K\delta n(\log\delta n)^{-\xi}+1.
\end{aligned}
$$

Lemmas 4.4 and 7.1 give

$$
\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{E}_{(3)},\,\ell_1,\,\Phi(V_n)=(x,n)\Big]\\
&\quad\le A\left(1+\frac{C_6'\xi}{\log\delta n}\right)\Big[(\ell_1-i\delta n)(\log(\ell_1-i\delta n))^{-\xi}+\\
&\qquad+((i+1)\delta n-\ell_1)(\log((i+1)\delta n-\ell_1))^{-\xi}\\
&\qquad+(K-1)\delta n(\log\delta n)^{-\xi}\Big]+1\\
&\quad\le A\left(K+\frac{C_7'\xi}{\log\delta n}\right)\left(1+\frac{C_6'\xi}{\log\delta n}\right)\delta n(\log\delta n)^{-\xi}+1.
\end{aligned}
$$

Lemmas 4.5 and 7.1 give

$$
\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \,\Big|\, \mathcal{E}_{(4)},\, \ell_1,\, \ell_2,\, \Phi(V_n) = (x,n)\Big] \\
&\quad \le A\left(1 + \frac{C_6'\xi}{\log \delta n}\right)\Big[(\ell_1 - i\delta n)(\log(\ell_1 - i\delta n))^{-\xi} + \\
&\qquad + ((i+1)\delta n - \ell_1)(\log((i+1)\delta n - \ell_1))^{-\xi} \\
&\qquad + (K-2)\delta n(\log \delta n)^{-\xi} \\
&\qquad + (\ell_2 - (i+K-1)\delta n)(\log(\ell_2 - (i+K-1)\delta n))^{-\xi} + \\
&\qquad + ((i+K)\delta n - \ell_2)(\log((i+K)\delta n - \ell_2))^{-\xi}\Big] + 2 \\
&\quad \le A\left(1 + \frac{C_6'\xi}{\log \delta n}\right)\left(K + \frac{2C_7'\xi}{\log \delta n}\right)\delta n(\log \delta n)^{-\xi} + 2.
\end{aligned}
$$

Lemmas 5.1 and 7.2 and the induction hypothesis give that for any $s = 1, \dots, K+4$ and any $s' = 1, \dots, K+4$ we have that

$$
\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}((z_{s'-1}, T_{s'-1}) \leftrightarrow (z_{s'}, T_{s'})) \,\Big|\, \mathcal{E}^s_{(5)}, \mathcal{G}_{\text{tree}}\Big] \\
&\quad \le \begin{cases} 1 & \text{if } s' = 2, K+3; \\ A t_{s'}(\log t_{s'})^{-\xi} & \text{if } s' < s,\ s' \ne 2, K+3; \\ A\left(1 + \frac{C_6'\xi}{\log \delta n}\right) t_{s'}(\log t_{s'})^{-\xi} & \text{if } s' \ge s,\ s' \ne 2, K+3. \end{cases}
\end{aligned}
$$

By the triangle inequality for resistance we get that for all $s = 1, \dots, K+4$ we have

$$
\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \,\Big|\, \mathcal{E}^s_{(5)},\, \mathcal{G}_{\text{tree}}\Big] \\
&\quad \le AK\left(1 + \frac{C_6'\xi}{\log \delta n}\right)\left(1 + \frac{2C_7'\xi}{\log \delta n}\right)\delta n(\log \delta n)^{-\xi} + 2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \,\Big|\, \mathcal{E}_{(5)},\, \ell_1,\, \ell_2,\, \Phi(V_n) = (x,n)\Big] \\
&\quad \le AK\left(1 + \frac{C_6'\xi}{\log \delta n}\right)\left(1 + \frac{2C_7'\xi}{\log \delta n}\right)\delta n(\log \delta n)^{-\xi} + 2.
\end{aligned}
$$

Lemma 5.2 and the induction hypothesis (recall that $\mathcal{E}_{(7)} = \mathcal{A}(i) \cap \mathcal{B}'(i, c_0')^c$) imply that

$$
\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \,\Big|\, \mathcal{E}_{(6)} \cup \mathcal{E}_{(7)},\, \mathcal{G}_{\text{tree}}\Big] \\
&\quad \le A\left(K + \frac{2C_7'\xi}{\log \delta n}\right)\delta n(\log \delta n)^{-\xi} + 2.
\end{aligned}
$$

Putting the above estimates together implies that there exists $C_8' = C_8'(K)$ such that

$$\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\mathcal{A}(i)^c \cup \mathcal{B}'(i,c_0)^c,\ \Phi(V_n) = (x,n)\Big]$$
$$\leq A\left(K + C_8' \frac{\xi}{\log \delta n}\right) \delta n (\log \delta n)^{-\xi} + 2.$$

With (7.13) this gives

$$\begin{aligned}
&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))\,\Big|\,\Phi(V_n) = (x,n)\Big] \\
&\leq \frac{3AK\delta n(\log \delta n)^{-\xi}}{4} \mathbf{P}(\mathcal{A}(i) \cap \mathcal{B}'(i,c_0')) \,|\, \Phi(V_n) = (x,n)) \\
&\quad + \left[A\left(K + C_8' \frac{\xi}{\log \delta n}\right) \delta n (\log \delta n)^{-\xi} + 2\right] \\
&\qquad \times \mathbf{P}(\mathcal{A}(i)^c \cup \mathcal{B}'(i,c_0') \,|\, \Phi(V_n) = (x,n)).
\end{aligned} \tag{7.16}$$

Due to Theorem 2.7, there exists a constant $c' = c'(K) \in (0,1)$ such that the expression in the right hand side of (7.16) is at most

$$A\delta n (\log \delta n)^{-\xi} \left[\frac{c' 3K}{4 \log \delta n} + \left(1 - \frac{c'}{\log \delta n}\right)\left(K + C_8' \frac{\xi}{\log \delta n} + \frac{2\xi}{\log \delta n}\right)\right],$$

where we bounded the remaining term $2A^{-1}(\delta n)^{-1}(\log \delta n)^{\xi}$ by

$$2A^{-1}(\delta n)^{-1}(\log \delta n)^{\xi} \leq \frac{2\xi}{\log \delta n} \frac{n_0}{(\log n_0)^{1+\xi}} \frac{(\log \delta n)^{1+\xi}}{\delta n} \leq \frac{2\xi}{\log \delta n}$$

using the third inequality in (7.2) and (7.11). We now choose δ_0 and ξ (depending only on $K = K_0$). In addition to (7.1) that we already required for δ_0, let us also have

$$\delta_0 \leq \frac{c'}{8(C_8' + 2)}, \qquad \big(K(2\delta_0) + 4\delta_0\big) C_6' \delta_0 \leq \frac{c'}{16} \tag{7.17}$$

Let $0 < \xi \leq 1/2$ satisfy:

$$\xi \leq \delta_0, \qquad \left(1 - \frac{c'}{16 \log n}\right)\left(1 - \frac{\log(1/\delta_0)}{\log n}\right)^{-\xi} \leq 1. \tag{7.18}$$

The second condition is satisfied for small ξ, since $n \geq 1/\delta_0^2$ implies that $(\log 1/\delta_0)/\log n \leq 1/2$, and hence

$$\left(1 - \frac{\log 1/\delta_0}{\log n}\right)^{-\xi} \leq 1 + 2^{1+\xi}\,\xi \frac{\log 1/\delta_0}{\log n} \leq 1 + 2^{3/2}\,\xi \frac{\log 1/\delta_0}{\log n}.$$

The first condition in (7.17) gives that

$$\begin{aligned}\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_i)\leftrightarrow\Phi(X_{i+K}))\,\Big|\,\Phi(V_n)=(x,n)\Big]&\\ \le AK\left(1-\frac{c'}{8\log\delta n}\right)\delta n(\log\delta n)^{-\xi}&\\ = An(\log\delta n)^{-\xi}K\delta\left(1-\frac{c'}{8\log\delta n}\right)&\end{aligned}\tag{7.19}$$

for $i=0,K,\ldots,(N-1)K$. For the final stretch, Lemmas 5.3 and 7.1 and the induction hypothesis give that

$$\begin{aligned}&\mathbf{E}\Big[R_{\text{eff}}(\Phi(X_{i^{\text{last}}})\leftrightarrow\Phi(V_n)\,\Big|\,\Phi(V_n)=(x,n)\Big]\\ &\quad\le K'A\delta n(\log\delta n)^{-\xi}\left(1+\frac{C_6'\xi}{\log\delta n}\right)+A\tilde n(\log\tilde n)^{-\xi}\left(1+\frac{C_6'\xi}{\log\delta n}\right).\end{aligned}\tag{7.20}$$

Since $K'<K$ and $\tilde n<2\delta n$, the right hand side of (7.20) is at most:

$$\begin{aligned}&An(\log\delta n)^{-\xi}\left[(K'\delta+\tilde n/n)\left(1+\frac{C_6'\xi}{\log\delta n}\right)\right]\\ &\quad\le An(\log\delta n)^{-\xi}\left[(K'\delta+\tilde n/n)+(K\delta+2\delta)\frac{C_6'\xi}{\log\delta n}\right].\end{aligned}\tag{7.21}$$

Using (7.9) and the second inequality in (7.17), the expression in (7.21) is at most

$$An(\log\delta n)^{-\xi}\left[K'\delta+\tilde n/n+\frac{c'}{16\log\delta n}\right].$$

Let us sum (7.19) over $i=0,K,\ldots,(N-1)K$ using the triangle inequality, and add (7.20). This gives

$$\begin{aligned}\gamma(n,x)&=\sup_{m\ge 2n}\gamma_m(n,x)\\ &\le An(\log\delta n)^{-\xi}\Big[NK\delta+K'\delta+\tilde n/n-\frac{c'}{8\log\delta n}+\frac{c'}{16\log\delta n}\Big]\\ &=An(\log\delta n)^{-\xi}\left(1-\frac{c'}{16\log\delta n}\right),\end{aligned}\tag{7.22}$$

where we used that $NK\delta+K'\delta+\tilde n/n=1$.

We can conclude the argument as follows. If in the definition (7.8) of δ we have $\delta_0 \le n/\|x\|^2$, then we have

$$
\begin{aligned}
(\log \delta n)^{-\xi}\left(1-\frac{c'}{16\,\log \delta n}\right) &= (\log n)^{-\xi}\left(1-\frac{\log 1/\delta}{\log n}\right)^{-\xi}\left(1-\frac{c'}{16\,\log \delta n}\right)\\
&\le (\log n)^{-\xi}\left(1-\frac{\log 1/\delta_0}{\log n}\right)^{-\xi}\left(1-\frac{c'}{16\,\log n}\right)\\
&\le (\log n)^{-\xi},
\end{aligned}
$$

due to the second requirement on ξ in (7.18). Thus the right hand side of (7.22) is at most $An(\log n)^{-\xi}$. When $\|x\| \le \sqrt{n}$, this is the claimed inequality, and when $\sqrt{n} < \|x\| \le \sqrt{n}/\delta_0$, it is stronger than than the claimed inequality.

On the other hand, if in (7.8) we have $\delta_0 > n/\|x\|^2$, then we have $\delta \ge n/\|x\|^2$ and hence $\log 1/\delta \le \log \|x\|^2/n$. This implies that the right hand side of (7.22) is

$$
\begin{aligned}
An(\log \delta n)^{-\xi}\left(1-\frac{c'}{16\,\log \delta n}\right) &= An(\log n)^{-\xi}\left(1-\frac{\log 1/\delta}{\log n}\right)^{-\xi}\left(1-\frac{c'}{16\,\log \delta n}\right)\\
&\le An(\log n)^{-\xi}\left(1-\frac{\log \|x\|^2/n}{\log n}\right)^{-\xi},
\end{aligned}
$$

as claimed. This completes the induction and the proof of Theorem 1.2.

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