Flight Dynamics and Control in Relation to Stall

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Abstract—Control of fixed-wing aircraft at high angles of attack is particularly challenging. In this case, aerodynamic forces can be subjected to strong variations, among which stall is certainly the most critical. This paper tackles flight dynamics and control for aerial vehicles subjected to the stall phenomenon. We propose a class of modeling functions for the lift coefficient, and we investigate the control problem. The equilibria analysis is addressed prior to the control design. We show that the stall phenomenon never forbids the existence of an equilibrium orientation for any reference trajectory, but the uniqueness of this orientation is not in general ensured. Consequently, the equilibrium orientation can be subjected to discontinuities leading to an ill-conditioned control problem. Feedback control laws are derived for reference velocities associated with continuous equilibrium orientations.

I. INTRODUCTION

Flight dynamics remains an active research domain after decades of studies in the subject. The complexity of aerodynamic effects and the diversity of wing profiles partly account for this continued interest. Lately, the emergence of small aerial vehicles for robotic applications (e.g. quad-rotors) has also renewed the interest of the control community in this subject. A major difficulty for the control of aerial vehicles is the non-linearity of the environmental reaction forces applied to the craft [1]. This paper aims at improving existing feedback control techniques by taking into account aerodynamic effects in the control design.

Fixed-wing aircraft are vehicles capable of flight using forward motion that generates lift-and-drag forces as the wing moves through the air. These aircraft can be roughly divided into two classes: airplanes and convertible vehicles. For the former, weight is compensated for by lift forces acting essentially on the wings, and propulsion, which cannot typically counteract the weight of the aircraft, is used to compensate for drag forces associated with large air velocities. By contrast, the latter can perform hovering flight at the expense of high energy consumption. During hovering, the convertible vehicle is typically controlled as a vertical take-off and landing (VTOL) vehicle since aerodynamic forces are of negligible intensity. On the other hand, classical airplane control techniques can be used in cruising flight.

Control design techniques for airplanes and VTOLs have been developed along different directions. Airplanes feedback control explicitly takes into account lift forces via linearized models at small angles of attack. Based on these models, stabilization is usually achieved through linear control techniques [1]. Consequently, the obtained stability is local and difficult to be quantified. In addition, linearized models do not take into account the aerodynamic stall phenomenon, which is an abrupt reduction of lift at large angles of attack. This phenomenon has a crucial role in practice since it is often indicated as primary cause for airplane crashes [2]. Linear techniques are also used for VTOLs, but several nonlinear feedback methods have been proposed in the last decade to obtain (semi) global stability [3]–[6]. These methods, however, are based on simplified models that neglect aerodynamic effects. Even drag forces, in fact, are but seldom taken into account [7]. The literature on the control of convertible vehicles, however, is scarce. This can be explained by the difficulty to operate transitions between hovering and cruising modes, in relation to strong variations of drag and lift forces at high angles of attack. Among these variations, stall is certainly the most critical.

This paper is dedicated to the flight dynamics analysis and control in the presence of the stall phenomenon. We propose a class of modeling functions for the lift coefficient associated with a generic bisymmetric vehicle, and we investigate the control problem. We show that the stall phenomenon never forbids the existence of the equilibrium orientation profile along any reference trajectory. However, the uniqueness of this orientation cannot be always ensured. Consequently, the equilibrium orientation can be subjected to discontinuities leading to an ill-conditioned asymptotic stabilization problem. Concerning the control design, we derive stabilizing feedback laws for reference trajectories associated with continuous equilibrium orientation profiles. For simplicity, the analysis is exposed for a vehicle moving in the vertical plane.

The paper is organized as follows. After specifying the notation used in the paper, general dynamic equations and modeling of aerodynamic forces are recalled in Section II. The modeling function for the lift coefficient is given in Section III. The equilibria analysis is presented in Section IV. Stabilizing feedback laws are given in Section V.

II. BACKGROUND

A. Notation

We assume that the controlled vehicle can be modeled by a single actuated body immersed in air.

The following notation is used:

$\bullet$ $G$ is the body’s center of mass and $m$ is the mass of the vehicle, assumed to be constant.
\[ \mathcal{I} = \{0; \bar{x}_0, \bar{y}_0\} \] is a fixed inertial frame with respect to (w.r.t.) which the vehicle’s absolute pose is measured. The vector of coordinates of \( G \) w.r.t. \( \mathcal{I} \) is denoted as \( x = (x_1, x_2)^T \). Therefore, \( \dot{\bar{O}} \bar{G} = x_1 \bar{x}_0 + x_2 \bar{y}_0 = (\bar{x}_0, \bar{y}_0)x. \)

\[ \mathcal{B} = \{G; \bar{i}, \bar{j}\} \] is a frame attached to the body. The vector \( \bar{v} \) is parallel to the thrust force axis, and this leaves two possible directions for this vector. The direction here chosen is consistent with the convention used for VTOL vehicles.

The vector of coordinates of the linear velocity of \( G \) w.r.t. \( \mathcal{I} \) is denoted as \( \dot{x} = (\dot{x}_1, \dot{x}_2)^T \), and as \( v = (v_1, v_2)^T \) when expressed in \( \mathcal{B} \), i.e. \( \bar{v} = \frac{\partial \bar{O} \bar{G}}{\partial \bar{x}} \dot{x} = (\bar{i}, \bar{j})v. \)

The wind velocity is denoted as \( \bar{v}_w = (\bar{x}_0, \bar{y}_0)\dot{x}_w = (\bar{i}, \bar{j})\bar{v}_w. \) The airspeed \( \bar{v}_a \) of the body is the difference between the velocity of \( G \) and \( \bar{v}_w. \)

The vehicle’s orientation is characterized by the angle \( \theta \) between \( \bar{x}_0 \) and \( \bar{i}. \) The rotation matrix of the angle \( \theta \) is \( \mathcal{R}(\theta). \)

\( \{e_1, e_2\} \) denotes the canonical basis in \( \mathbb{R}^2 \). \( S = R(\pi/2) \) is a skew-symmetric matrix. \( I = R(0) \) is the identity matrix.

The \( i_{th} \) component of a vector \( x \) is denoted as \( x_i \). Given a function \( f : \mathbb{R} \rightarrow \mathbb{R} \), its first and second derivative are denoted as \( f' \) and \( f'' \) respectively. Given a function \( f \) of several variables, the partial derivative of \( f \) w.r.t. one of them, say \( x \), is denoted as \( \partial_x f = \frac{df}{dx}. \) Given \( v, w \in \mathbb{R}^n \), the inner product between them is indicated as \( \langle v, w \rangle \equiv v^T w. \)

### B. System modeling

The equations of motion are derived by considering two control inputs. The first one is a thrust force \( \bar{T} \) along the body fixed direction \( \bar{i} (\bar{T} = -T \bar{i}) \) whose main role is to produce longitudinal motion. The second control input is a torque actuation, typically created via secondary propellers, rudders or flaps, control moment gyro's, etc. For the sake of simplification, we assume that any desired torque can be produced so that the vehicle’s angular velocity \( \omega \) is modified at will and used as a control variable (see [7] for complementary explanations concerning this assumption).

The external forces acting on the body are composed of the gravity \( m \bar{g} \) and the aerodynamic forces denoted by \( \bar{F}_a. \)

Applying the fundamental theorem of mechanics yields:

\[ m \ddot{x} = -TR(\theta)e_1 + mg e_1 + F_a(\bar{x}_a, \theta), \quad \dot{\theta} = \omega, \]  

with \( g \) the gravity constant and \( F_a \) the aerodynamic forces expressed in the inertial frame, i.e. \( F_a = (\bar{x}_0, \bar{y}_0)F_a. \)

### C. Aerodynamic forces

The Buckingham \( \pi \)-theorem [8, p. 34] points out that the static aerodynamic forces can be written as follows

\[ F_a = k_a[\bar{x}_a] \left[ c_L (R_e, M, \alpha)S - c_D (R_e, M, \alpha)I \right] \bar{x}_a, \]  

where \( k_a := \frac{\rho e^2}{2} \), \( \rho \) is the free stream air density, \( \Sigma \) is the characteristic surface of the vehicle’s body, \( c_L(\cdot) \) is the lift coefficient, \( c_D(\cdot) > 0 \) is the drag coefficient (\( c_L \) and \( c_D \) are called aerodynamic characteristics), \( R_e \) is the Reynolds number, \( M \) is the Mach number, and \( \alpha \) is the angle of attack.

The latter variable is defined as the angle between the body zero-lift line, along which the airspeed does not produce perpendicular forces, and the airspeed vector \( \bar{v}_a. \) By denoting the angle between the zero-lift line and the thrust direction as \( \mu \), and the airspeed angle w.r.t. \( \bar{x}_a \) as \( \gamma(\bar{x}_a) \), one has (Fig. 1):

\[ \alpha(\bar{x}_a, \theta) := \pi + \theta - \gamma(\bar{x}_a) - \mu. \]  

For low-subsonic regimes – Mach numbers smaller than 0.4– the dependence of the aerodynamic characteristics upon \( M \) can be neglected [8]. Fig. 2 depicts a typical behavior of the lift coefficient for the symmetric shape NACA 0021. The measurements (in blue) are obtained from [9] and [10].

At small angles of attack, the pressure decrease on the upper surface is much greater than the lower-surface pressure increase. This difference produces a linear relation between \( \alpha \) and the lift coefficient. As the angle of attack increases, it becomes difficult for the upper-surface flow to follow the surface, and a small vortex is created above the wing, causing flow separation; when the upper-surface flow separates, pressure differential and lift are lost, and the wing stalls [1]. The reduction of the lift coefficient for angles of attack exceeding the stall angle \( \alpha_s \) is called stall phenomenon.

A first approximation describing the low frequency variations of the experimental characteristics, and for which important control results can be obtained [11], is given by

\[ c_L(\alpha) = c_{L0} + 2c_1 \sin^2(\alpha), \]  

\[ c_D(\alpha) = c_{D0} + 2c_1 \sin^2(\alpha), \]  

with \( c_{D0} \) being the drag coefficient at zero-angle of attack, and \( c_1 \in \mathbb{R} \). Observe that (5) and (6) satisfy (4). Fig. 2 shows a typical approximation result for the lift coefficient. The approximation result, in red, is good except for small angles of attack (modulo \( \pi \)) before the stall zone around \( \pm 10^\circ \). In particular, the model (5) does not characterize the initial high slope of the experimental lift coefficient, and the stall phenomenon. We will see that these discrepancies greatly affect the control problem associated with the dynamics (1).
III. A MODEL FOR THE LIFT COEFFICIENT

An extension of the modeling function (5) that provides a better approximation of the experimental lift coefficient when $\alpha \in (-45^\circ, 45^\circ)$ is obtained by setting

$$c_L(\alpha) = c_{L_1}(\alpha) \sin(2\alpha), \quad (7a)$$

$$c_{L_1}(\alpha) = c_1 + c_2 \cos^{2\alpha_3}(\alpha) - c_4 \cos^{2\alpha_5}(\alpha) \sin^{2\alpha_6}(\alpha), \quad (7b)$$

with $c_3, c_5, c_6 \in \mathbb{N}$, so that the bisymmetric constraints (4) are satisfied, and $c_1, c_2, c_4 \in \mathbb{R}$. Observe that for large angles of attack, the coefficient $c_{L_1}(\alpha)$ is roughly given by the constant $c_1$ and, consequently, the lift models (5) and (7a) basically coincide for these angles. Therefore, it is tempting to choose $c_1$ as specified in [11] by considering the experimental data for aerodynamic characteristics only at large angles of attack, i.e.

$$c_1 = \sum_{i=1}^{N} c_{L_M}(\alpha_i) \sin(2\alpha_i) + 2(c_{D_M}(\alpha_i) - c_{D_0}) \sin^2(\alpha_i) \quad (8)$$

with $\alpha_1, \ldots, \alpha_N$ the values of the angle of attack for which measurements $c_{L_M}(\alpha_i)$ and $c_{D_M}(\alpha_i)$ are available, and $\alpha_1 = 45^\circ$ and $\alpha_N = 135^\circ$. The coefficients $c_2, \ldots, c_6$ must be chosen so that the model (7) provides an approximation of the initial high slope of the experimental lift coefficient, and of the stall phenomenon. Besides numerical techniques that can be used to determine these coefficients, we observe that the experimental lift achieves two stationary points when $\alpha \in [0, 45^\circ]$: one at the stall angle $\alpha_s$ (local maximum) and one at the critical positive angle $\alpha_c$ (local minimum). Then, it is desirable that the model (7) satisfies

$$c_{L_s}(\alpha_s) = c_{L_M}(\alpha_s), \quad c_{L_c}(\alpha_c) = 0, \quad c_{L_c}(\alpha_c) = c_{L_M}(\alpha_c). \quad (9b)$$

To obtain an approximation of the initial slope of the experimental lift coefficient, it suffices to impose

$$c_{L}(0) = C_{L_a}, \quad (10)$$

where $C_{L_a}$ is supposed to be known from experimental data. In general, the coefficients $c_3, c_5$ and $c_6$ that make the above constraints satisfied are not integer numbers. A first attempt to satisfy the constraints (9)–(10) with $c_3, c_5, c_6 \in \mathbb{N}$ is given by the following choices:

$$c_2 = \frac{C_{L_a}}{2} - c_1, \quad (11a)$$

$$c_3 = \text{round}\left(\frac{C_{L_a} - 1 - 2\alpha_3^2}{6c_2^2\alpha_3^2}\right), \quad (11b)$$

$$q_1(\alpha) := c_1 + c_2 \cos^{2\alpha_3}(\alpha) - c_{L_M}(\alpha) \cot(\alpha_c) \quad (11c)$$

$$q_2 := c_2 c_3 \cos^{2\alpha_3}(\alpha_c) - c_{L_M}(\alpha_c) \cot(\alpha_c), \quad (11d)$$

$$c_5 = \text{round}\left(\frac{q_2}{q_1(\alpha_c)} + c_6 \cot^2(\alpha_c)\right), \quad (11e)$$

$$c_6 = \text{round}\left(\frac{0.5 \log\left(\frac{q_1(\alpha_c)}{q_1(\alpha_c)}\right) - 2}{\log\left(\frac{\sin^2(\alpha_c)}{\sin^2(\alpha_s)}\right) + \cot^2(\alpha_c) \log\left(\frac{\cos(\alpha_c)}{\cos(\alpha_s)}\right)}\right) \quad (11f)$$

$$c_4 = \frac{q_1(\alpha_s)}{\cos^{2\alpha_5}(\alpha_s) \sin^{2\alpha_6}(\alpha_s)}. \quad (11g)$$

The coefficient $c_2$ follows directly from (10). The coefficient $c_3$ has been obtained by imposing (9a) upon the third order Taylor expansion (at $\alpha = 0$) of model (7a). Finally, the coefficients $c_4, c_5$ and $c_6$ have been obtained by imposing (9b) upon the model (7a). Then, we have rounded off to the nearest integer numbers the obtained $c_3, c_5, c_6$. Fig. 3, which depicts a typical approximation result for the lift coefficient, shows that the approximation result is “good” everywhere. This figure is depicted by using the model (7) along with

$$c = (0.943, 1.789, 11, 8225.4, 27, 3, 0.0139), \quad (12)$$

where $c := (c_1, c_2, c_3, c_4, c_5, c_6, c_{D_0})$, and $c_1 \ldots c_6$ given by relations (8) and (11) in turn evaluated according to the experimental data shown in Fig. 2 (in blue).
IV. STALL RELATED CONTROL ISSUES

When the aerodynamic characteristics are given by a model as simple as (5)–(6), it is possible to recast the asymptotic stabilization problem into the one of controlling a spherical body [11], thus allowing for the application of previous control design methods. However, an important weakness of the aforementioned model is that it does not account for the stall phenomenon. Taking it into account adds considerable complexity to the vehicle’s dynamics, especially in the case of a vehicle moving within a fluid endowed with a large Reynolds number for which the stall phenomenon can no longer be neglected [12]. We show below that this phenomenon perturb the structural properties of the system dynamics (1) such as existence, multiplicity and continuity of equilibrium points. We assume low-subsonic flow and fixed Reynolds number, thus considering only the α dependence of $c_L$ and $c_D$. In view of (3), this in turn implies that $F_a$ only depends on the airspeed $\bar{v}$ and on the orientation $\theta$.

A. Existence of equilibrium orientations

Let $x_r(t)$ denote a reference trajectory. Then, in view of (1), the tracking error dynamics are governed by:

$$\dot{\bar{x}} = R(\theta)\bar{\nu};$$
$$\dot{\bar{v}} = -\omega R(\theta)^T f(\bar{x}, \theta, t), \quad \dot{\theta} = \omega,$$

where $u := T/m$, $f_a := F_a/m$,

$$f(\bar{x}, \theta, t) := g_e + f_a(\bar{x}_a, \theta) - \bar{x}_r(t),$$

$$\bar{x} := x - x_r$$ is the position tracking error, and $\bar{v} := R^T(\bar{x} - \bar{x}_r)$ is the velocity error in the body-fixed frame.

Asymptotic stabilization of tracking errors to zero requires the existence of equilibrium orientations, i.e. values $\theta_e$ that make $(\bar{x}, \bar{v}, \theta) = (0, 0, \theta_e)$ an equilibrium point of System (13)-(14). In particular, Eq (14) indicates that $\bar{v} \equiv 0$ implies:

$$-u_e + R(\theta)^T f(\bar{x}, \theta, t) = 0.$$ 

Thus, in order to guarantee the existence of the equilibrium state $\bar{v} = 0$, the following conditions must be satisfied:

$$\langle e_1, R^T f \rangle = u_e,$$
$$\langle e_2, R^T f \rangle = f_2(\bar{x}_r, \theta, t) \cos(\theta) - f_1(\bar{x}_r, \theta, t) \sin(\theta) = 0,$$

where $u_e$ is the value of the control input $u$ at $\bar{v} = 0$. General, the existence of an equilibrium orientation $\theta = \theta_e(t)$ that satisfies Eq. (17b) is not ensured a priori. It depends on the nature of $f$, and more precisely on the $\theta$-dependence of the aerodynamic acceleration $f_a$. For instance, when $f_a$ does not depend on $\theta$, or is given by (2) with the aerodynamic characteristics as (5)-(6), there always exists equilibrium angle $\theta_e$. The next lemma guarantees the existence of the equilibrium $\theta_e(t)$ when the bisymmetric lift-and-drag properties (4) are satisfied.

Lemma 1 If the aerodynamic characteristics $c_L(\alpha)$ and $c_D(\alpha)$ are continuous $\pi$-periodic functions (as in the case of bisymmetric body shapes) and $\bar{x}_r$ is differentiable, then there exists at least one equilibrium orientation curve $\theta_e(t)$, i.e. $\exists \theta_e(t)$ such that Eq. (17b) with $\theta = \theta_e(t)$ is satisfied $\forall t$.

The proof of this Lemma is given in Appendix A.

The problem of seeking the equilibrium orientations is equivalent to the problem of finding the values $\theta_e(t)$ such that $w|_{\theta=\theta_e(t)} = 0$ with

$$w := \frac{1}{g} \langle e_2, R(\theta)^T f(\bar{x}_r, \theta, t) \rangle.$$ 

In view of Eq. (3), the orientation $\theta$ in the above expression can be replaced by

$$\theta = \alpha_r - \pi + \gamma(\dot{x}_r w(t)) + \mu,$$

where $\dot{x}_r w(t) = \dot{x}_r(t) - \dot{x}_w(t)$. Then, the problem is to find the the equilibrium angles of attack $\alpha_r(t)$ such that $\alpha_r = \alpha_e(t)$ yields $w = 0$.

B. Ill-conditioning of the asymptotic stabilization problem resulting from multiple equilibrium orientations

This section shows how the existence of multiple equilibrium orientations can lead to an ill-conditioned asymptotic stabilization problem. For the sake of simplicity, we assume that no wind is blowing, i.e. $\dot{x}_w \equiv 0$, and that $\mu = 0$, i.e. $\bar{T}$ is aligned to the zero-lift direction.

1) Horizontal flight: desired horizontal flight implies that

$$\dot{x}_r = \nu(0, 1)^T,$$

where $\nu$ denotes a constant, here assumed to be positive. Under the assumptions of no wind and $\mu = 0$, Eq. (18), combined with (19) and (20), becomes

$$w(\alpha_r, a_\nu) = [1-a_L c_L(\alpha_r)] \cos(\alpha_r) - a_D c_D(\alpha_r) \sin(\alpha_r),$$

where $a_\nu$ is a dimensionless number defined as

$$a_\nu := \frac{ka^2}{mg},$$

with $k_a = \frac{1}{\rho \Sigma}$. The dimensionless property of $a_\nu$ can be used to determine dynamic similitude between different cases. For instance, two different aircraft admit the same equilibrium orientations at given $\nu$ if: i) the airplanes’ geometries and weights provide the same values for $a_\nu$; ii) the wings’ profiles together with flight conditions, which determine $R_e$, provide the same aerodynamic characteristics.

Because of the non-linearities of the stall phenomenon, the problem of finding the explicit expression of the equilibrium angles $\alpha_e = \phi(a_\nu)$ from $w(\alpha_e, a_\nu) = 0$ does not have a straightforward solution. A pattern of equilibrium angles $\alpha_e$ can be found by drawing the function $w(\alpha_r, a_\nu)$ versus $\alpha_r$ for different values of the constant $a_\nu$. Fig. 4(a) depicts the function (21) evaluated with the aerodynamic characteristics given by (6) and (7) which are in turn evaluated with the coefficients $c_{L\alpha}, c_{D\alpha}$, and $\bar{c}_\alpha$ given by (12).

By inspection, we see that the equilibrium angle $\alpha_e$ is not unique when the reference velocity $\nu$ is such that $a_\nu$ belongs to a neighborhood of the value 1.36. The loss of the $\theta_e$-uniqueness for these velocities is due to the change of slope of the function $w(\alpha_r, a_\nu)$. With an eye to Eq. (21), we deduce that this change of slope is essentially caused by the
When considering \( \omega \) control input, it cannot be well posed to be continuous. The fact that the continuity of the reference velocity \( \nu \) with \( \nu \) were admissible, and this is clearly not feasible in practice.

Horizontal reference velocity is of small intensity almost configuration. At time \( t_3 \), the intensity of horizontal velocity whose norm increases monotonically, i.e., \( \alpha_e \) does not in general imply the continuity of the equilibrium orientation \( \theta_e(t) \) is a necessary condition for the asymptotic stabilization problem to be well posed. This problem requires, in fact, that the control input \( \omega \) at the equilibrium configuration, namely \( \omega = \dot{\theta}_e(t) \), is defined for any time \( t \), and this is not the case if \( \theta_e(t) \) is discontinuous. In general, the延续 of the continuity of the reference velocity \( \dot{x}_r(t) \) does not in general imply the continuity of the equilibrium orientation \( \theta_e(t) \) is visually clear from Fig. 4(b) when considering transition maneuvers between hovering and high-velocity cruising. For example, consider a reference horizontal velocity whose norm increases monotonically, i.e.

\[
\dot{x}_r = \nu(0, t)^T, \tag{23}
\]

with \( \nu \) a positive number. On the time interval \( t \in (0, t_1) \) (see Fig. 4(b)), the equilibrium angle \( \alpha_e \) is large because the horizontal reference velocity is of small intensity (almost vertical configuration). As time goes by, the intensity of the reference velocity increases, and this in turn implies smaller values of the angle of attack at the equilibrium configuration. At time \( t_3 \), the equilibrium attitude \( \alpha_e(t) \) instantaneously jumps from \( 19^\circ \) to \( 8^\circ \). Such a discontinuity renders the asymptotic stabilization problem ill-conditioned and, consequently, it does not exist a local stabilizer that makes the reference velocity (23) asymptotically stable.

2) Other velocities directions:

Assume now that the reference velocity \( \dot{x}_r \) is of the form

\[
\dot{x}_r = \nu \begin{pmatrix} \cos(\gamma_r) \\ \sin(\gamma_r) \end{pmatrix}^T, \tag{24}
\]

with \( \nu \in \mathbb{R}^+ \) and \( \gamma_r \in S^1 \) two constant values. Under the assumptions of no wind and \( \mu = 0 \), Eq. (18), combined with (19) and (24), becomes

\[
w = \nu \left[ c \sin(\gamma_r) - c L \cos(\gamma_r) \right] \cos(\alpha_r + \gamma_r) \tag{25}
\]

where the dimensionless positive constant \( \alpha_r \) is still given by Eq. (22). Because of the aforementioned difficulties in finding the expression of the equilibrium angles \( \alpha_e \), the couples \((\alpha_e, \gamma_r)\) such that \( w(\alpha_e, \gamma_r, \alpha_r) = 0 \) can be identified by drawing the level curves \( \{ (\alpha_e, \gamma_r) : w(\alpha_e, \gamma_r) = 0 \} \) for different values of the constant \( \alpha_r \). Figs. 5 depicts these curves by using the aerodynamic characteristics given by (6) and (7) in turn evaluated with \( c_1 \ldots c_6, c_{D_0} \) given by (12).

To illustrate how these curves can be used, focus on the level curve associated with \( \alpha_r = 1.36 \), and fix the reference velocity direction \( \gamma_r \) at 80°. Imagine now a horizontal line drawn at this angle. Then, the equilibrium angles \( \alpha_e \) are given by the intersection of this virtual line and the level curve, namely \( \alpha_{e1} \approx 11^\circ, \alpha_{e2} \approx 14^\circ \) and \( \alpha_{e3} \approx 23^\circ \). Repeating this simple procedure, we find out that the system admit three equilibrium orientations as long as \( \gamma \in [42^\circ, 100^\circ] \). Consequently, the stall phenomenon induces several equilibrium orientations not only in cruising flight, but also for other reference velocity directions. Hence, there exist discontinuous equilibrium orientations associated with generic continuous reference velocity \( \dot{x}_r(t) \).

The problem of characterizing the reference trajectories associated with continuous orientation profiles is beyond the scope of the present paper, and it will be addressed in future studies. In this respect, we believe that the trajectories representing the transition maneuvers between hovering and cruising flight for convertible vehicles must guarantee the existence of a continuous equilibrium orientation profile.

Simulations show that the equilibria analysis performed with experimental data for the aerodynamic characteristics (lift shown in Fig. 2, for drag see [10] or [11]) is basically equivalent to the one performed with (6)–(7) and (12).
V. BASICS OF THE CONTROL DESIGN: VELOCITY CONTROL

Given a reference velocity, assume the continuity of the equilibrium orientation \( \theta_e(t) \), which in turn implies the asymptotic stabilization problem be well posed. Under this condition, we propose feedback controllers ensuring the asymptotic stabilization of this reference velocity. Define \( \kappa_a := k_a/m \) and

\[
\begin{align*}
\Lambda &:= R(\theta) \begin{bmatrix} \sin(\theta) - \cos(\theta) \\ 0 & 0 \end{bmatrix}, \\
f_p &:= f_p + \kappa_a [\lambda(\theta) |c'_L S - c'_D|] \dot{x}_a, \\
f_v &:= ge_1 + f_p(\tilde{x}_a, \theta) - \tilde{x}_r(t).
\end{align*}
\]

(26a) (26b) (26c)

We make the following assumption.

**Assumption 1** There exists an equilibrium orientation \( \theta_e(t) \) \( \in C^0 \) and \( \delta \in \mathbb{R}_+ \), such that \( |f_v(\tilde{x}_r(t), \theta_e(t), t)| > \delta, \forall t \in \mathbb{R}_+ \).

As a consequence, one obtains the following result.

**Proposition 1** If Assumption 1 is satisfied, then the equilibrium orientation \( \theta_e(t) \) is locally unique, i.e. for any \( t \in \mathbb{R}_+ \) there exists a neighborhood \( U_t \) of \( \theta_e(t) \) such that

\[
\langle \epsilon_2, R(\theta)^T f(\tilde{x}_r(t), \theta, t) \rangle \neq 0, \forall \theta \in U_t \setminus \{ \theta_e(t) \}
\]

The proof of this Proposition is given in Appendix B. Based on Assumption 1, a solution to the local asymptotic stabilization problem of \( \tilde{v} \) to zero can be proposed.

**Proposition 2** Assume the following regularity conditions:

(i) \( f_p \) is continuously differentiable and its partial derivatives are bounded uniformly w.r.t. \( \tilde{x}_a \) in compact sets,

(ii) the vectors \( \tilde{x}_w, \tilde{x}_v, \tilde{x}_w, \tilde{x}_r, \tilde{x}_r \) and \( \tilde{x}_r \) are bounded in norm on \( \mathbb{R}_+ \).

Let \( k_i > 0, i = 1, 2, 3 \) and apply the control

\[
\begin{align*}
u &= k_1 f_v \tilde{v} + \int_0^t \omega, \\
\omega &= k_v \left[ k_2 |f_v| \tilde{v}_2 + \frac{k_3 |f_v| |\tilde{v}_2|}{|f_v| + |\tilde{v}_2|} - \frac{\tau S f_v}{|f_v|} \right],
\end{align*}
\]

(27a) (27b)

to System (14) with \( \bar{f} = R^T f, f \) given by (15), \( \bar{f} = R^T f_v, f_v \) by (26c),

\[
f_S := \partial_{\tilde{x}_a} f_p[\tilde{x} - \tilde{x}_w] - \tilde{x}_r(t),
\]

\( \tilde{x} \) given by (1) and \( k_v \) by:

\[
k_v := k_v(\tilde{x}_a, \theta) := \left[ 1 - \bar{f}_2 \frac{\tau S f_v}{|f_v|} \right]^{-1},
\]

(28)

with

\[
df_v := \partial^2 f_v \tilde{x}_a - 2S \partial f_v f_a.
\]

Suppose that:

(i) The aerodynamic characteristics \( c_{L}(\alpha) \) and \( c_{D}(\alpha) \) are continuous \( \pi \)-periodic functions, so that Lemma 1 holds,

(ii) Assumption 1 is satisfied with \( f_v \) given by (26c).

Then,

(i) the control laws (27) are well defined in the neighborhood of the reference velocity,

(ii) \( \tilde{(v, \theta)} = (0, \theta_e(t)) \) is a locally asymptotically stable equilibrium point of System (14).

The proof of this result is given in Appendix C. It follows from (28) that \( k_v \) is equal to one when the aerodynamic forces \( f_v \) do not depend on the attitude \( \theta \), i.e. \( \partial \theta f_v \equiv 0 \Rightarrow df_v \equiv 0 \). In this case, the control law (27) coincides with the velocity control presented in [7, III B] (when adapted to the 2D case) which is derived by using the spherical body assumption, i.e. \( \partial \theta f_v \equiv 0 \). Then, we view (27) as an extension of the velocity control presented in [7, III B]. In addition, observe that \( df_v \equiv 0 \) also when the aerodynamic characteristics \( c_L \) and \( c_D \) are given by (5)-(6). In fact, in view of (3) one has \( \partial \theta f_v = \partial \theta f_a \), which in turn implies

\[
df_v = \kappa_a [\bar{x} \bar{a}] \left((c''_L + 2c'_D)S - (c''_L - 2c'_L)I\right) \tilde{x}_a.
\]

This relation is a \((2 \times 1)\) null-vector when evaluated with the model (5)-(6) since it satisfies \( c''_L + 2c'_D \equiv 0 \) and \( c''_L - 2c'_L \equiv 0 \). Then, it is possible to verify that the law (27) also includes the control result presented in [11] when the velocity control [7, III B] is applied to the transformed system.

The domain of attraction of the equilibrium point \( \tilde{(v, \theta)} = (0, 0) \) is related to the set on which \( k_v^{-1} \neq 0 \), which is in turn related to the \( \theta \)-variations of the aerodynamic forces \( f_v \) (via \( df_v \)). This suggests a relation between the stability constraint \( k_v^{-1} \neq 0 \) and the stall phenomenon, typically associated with large variations of lift forces w.r.t. the angle of attack.

In practice, the control law (27) must be complemented with integral correction terms in order to compensate for almost constant unmodeled additive perturbations. Let

\[
I_v := \int_0^t \tilde{x}(s) ds,
\]

and denote a smooth bounded strictly positive function defined on \([0, +\infty)\) satisfying the properties [7, Sec. III.C]. It then suffices to replace the definitions of \( f \) and \( f_v \) by:

\[
\begin{align*}
f &= ge_1 + f_v - \tilde{x}_r + h(|I_v|^2)I_v \\
f_v &= ge_1 + f_p - \tilde{x}_r + h(|I_v|^2)I_v
\end{align*}
\]

(30a) (30b)

in (27) to obtain a control which incorporates an integral correction action and for which local asymptotic stability and convergence can still be proven.

The control law (27b) uses terms that may introduce singularities. To provide a control law that is well-defined everywhere, we modify the expression (27b) by setting \( k_v \equiv 1 \) and by multiplying the terms \( 1/(|f_v| + |\tilde{f}_2|)^2 \) and \( 1/|f_v|^2 \) by an adequate function. The choice of fixing \( k_v = 1 \) is justified by the fact that it is equal to one close to the reference trajectory \( \tilde{f}_2 \approx 0 \). Doing so we do not destroy the local stability property of Proposition 2. As for the aforementioned function, we can use the class \( \mathcal{C}^1 \) function \( \mu_r : [0, +\infty) \to [0, 1] \) defined by:

\[
\mu_r(s) = \begin{cases} 
\sin \left( \frac{s^2}{2} \right), & \text{if } s \leq \tau \\
1, & \text{otherwise}
\end{cases}
\]

(31)

with \( \tau > 0 \). This yields the well-defined control expression

\[
\begin{align*}
u &= k_1 |f_v| \tilde{v}_1 + \int_0^t \omega, \\
\omega &= k_2 |f_v| \tilde{v}_2 + \mu_r(|f_v|) \left( \frac{k_3 |f_v| |\tilde{v}_2|}{(|f_v| + |\tilde{v}_2|)^2} - \mu_r(|f_v|) \frac{\tau S f_v}{|f_v|} \right)
\end{align*}
\]
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REFERENCES

[10] CYBERIAD, http://www.cyberiad.net/foildata.htm, Cyberiad is a team of Australian-based consultants providing mathematical, engineering, educational, administrative, and statistical services.

APPENDIX

A. Proof of Lemma 1

The definition (3) implies that \( \alpha(\dot{x}_a, \theta + \pi) = \alpha(\dot{x}_a, \theta) + \pi \), together with the assumption of \( \pi \)-periodicity of the aerodynamic characteristics, in turn implies

\[
F_a(\dot{x}_a, \theta + \pi) = k_a \dot{x}_a \left[ c_l(\cdot) S - e_D(\cdot) I \right]_{\alpha+\pi} = F_a(\dot{x}_a, \theta).
\]

This fact in turn implies that \( f(\dot{x}, \theta + \pi, t) = f(\dot{x}, \theta, t) \), where \( f \) is given by (15). From there, one verifies that

\[
\langle e_2, R(\theta)T f(\dot{x}_r, \theta, t) \rangle = -\langle e_2, R(\theta+\pi) T f(\dot{x}_r, \theta+\pi, t) \rangle.
\]

Then, the proof of the existence of an equilibrium orientation \( \theta_c(t) \) such that \( \langle e_2, R(\theta_c(t)) T f(\dot{x}_r(t), \theta_c(t), t) \rangle = 0 \) \( \forall t \) is a direct application of the intermediate value theorem since, by assumption, \( f \) is defined \( \forall t \) (\( \dot{x}_r \) is differentiable), and also continuous versus \( \theta \) (\( c_l \) and \( e_D \) are continuous).

B. Proof of Proposition 1

First, it follows from (15) and (26) that

\[
\mathcal{F}_2 = \langle e_2, R^T f \rangle = \langle e_2, R^T f \rangle = \mathcal{F}_2,
\]

since \( \langle e_2, R^T \Lambda \rangle = 0 \). Equality (32) implies that

\[
\partial \langle e_2, R^T f \rangle = \langle e_2, R^T \partial f \rangle = \langle e_2, R^T \partial f \rangle = \langle e_2, S R^T f \rangle.
\]

Given \( \Lambda \) and \( f \) as in (26a) and (26c), one can verify that

\[
\partial \langle e_2, R^T f \rangle = \partial \langle e_2, R^T f \rangle = \Lambda \left[ \partial^2 f_a - 2 \delta \partial f_a \right],
\]

which implies

\[
\langle e_2, R^T \partial f \rangle = \langle e_2, R^T \partial f \rangle = 0.
\]

Substituting (34) in (33) yields

\[
\mathcal{F}_2 = \langle e_2, R^T f \rangle = -\partial \langle e_2, R^T f \rangle.
\]

Now, a sufficient condition for the local uniqueness of the equilibrium orientation \( \theta_c(t) \) is

\[
\partial \langle e_2, R(\theta)T f(\dot{x}_r(t), \theta, t) \rangle \bigg|_{\theta=\theta_c(t)} = 0, \quad \forall t \in \mathbb{R}^+.
\]

From Assumption 1, one has that \( |f_a|^2 \) never vanishes at the equilibrium configuration. Let us recall that at this configuration one has \( \mathcal{F}_2 \equiv 0 \). Now observe that

\[
|f_a|^2 = \mathcal{F}_2^2 = \mathcal{F}_2^2 + \mathcal{F}_2^2 = \mathcal{F}_2^2 + \mathcal{F}_2^2.
\]

In view of (35) and Assumption 1, at the equilibrium one has

\[
|f_a|^2 = \mathcal{F}_2^2 = \langle e_2, R(\theta)T f \rangle^2 > 0.
\]

This suffices to prove (36).

C. Proof of Proposition 2

Given (32), System (14) can be rewritten as

\[
\ddot{\tilde{v}} = -\omega \tilde{v} - u_a e_1 + R^T f, \quad \tilde{\theta} = \omega,
\]

where \( u_a := u + \mathcal{F}_2 \dot{x}_r - \mathcal{F}_2 \) and \( \mathcal{F}_2 = R^T f, \quad \mathcal{F}_2 = R^T f \). Let \( \tilde{\theta} \in (-\pi, \pi) \) denote the angle between the two unit vectors \( e_1 \) and \( \mathcal{F}_2 / |f_a| \), so that \( |f_a|^2 \cos (\tilde{\theta}) = \mathcal{F}_2 |\mathcal{F}_2| \). Then, the control objective is the asymptotic stabilization of \( \tilde{\theta} \) to zero. Consider the candidate Lyapunov function \( V \) defined by:

\[
V = \frac{1}{2} |\tilde{v}|^2 + \frac{1}{\omega^2} \left( 1 - \frac{\mathcal{F}_2 |\mathcal{F}_2|}{|f_a|^2} \right).
\]

Via direct calculations one can verify that

\[
\frac{d}{dt} \left( 1 - \frac{\mathcal{F}_2 |\mathcal{F}_2|}{|f_a|^2} \right) = -\frac{\mathcal{F}_2 |\mathcal{F}_2|}{|f_a|^2} \left( \omega + \frac{\omega^T S f_a}{|f_a|^2} \right).
\]

In view of \( \dot{\tilde{v}} = \partial \langle e_2, f \rangle \), \( \mathcal{F}_2 = \mathcal{F}_2 \dot{x}_r - \mathcal{F}_2 \dot{x}_r \), and the definition of \( u_a, \mathcal{V} \), \( \tilde{V} \) along the solutions of System (37) is

\[
\tilde{V} = \tilde{v} \left( \mathcal{F}_2 u_a - \mathcal{F}_2 \mathcal{F}_2 \right) \left( 1 + \frac{\omega^T S f_a}{|f_a|^2} \right) \left( \omega + \frac{\omega^T S f_a}{|f_a|^2} - k_2 |f_a| / |\mathcal{F}_2| \right).
\]

Now, in view of Eq. (34), the term multiplying \( \omega \) is

\[
1 + \frac{\omega^T S f_a}{|f_a|^2} = 1 + \frac{\mathcal{F}_2 S R^T \delta f_a}{|f_a|^2} = 1 + \mathcal{F}_2 \frac{R^T \delta f_a}{|f_a|^2},
\]

which is clearly different from zero close to the equilibrium point (i.e. \( \tilde{v} \approx 0 \)). In view of (27), (28), (32), (40) and the fact that \( e_1 R^T \delta f_a \equiv -e_1 R^T f \partial f_a \), one verifies that the obtained expression for \( \tilde{V} \) is

\[
\tilde{V} = -k_1 \mathcal{F}_2 |\mathcal{F}_2| - \frac{k_2 \mathcal{F}_2 |\mathcal{F}_2|}{k_2 (1 + |f_a|^2 |\mathcal{F}_2|)} = -k_1 |f_a| |\mathcal{F}_2| - k_2 \tan^2 (\tilde{\theta}/2),
\]

since \( \tan^2 (\tilde{\theta}/2) > 0 \).

Because \( \tilde{V} \) is negative semi-definite, the velocity error term \( \tilde{\tilde{v}} \) is bounded. The next step of the proof consists in showing the uniform continuity of \( \tilde{V} \) along every system’s solution and, using Barbalat’s lemma, one deduces the convergence of \( \tilde{\tilde{v}} \) and \( \tilde{\theta} \) to zero. This part of the proof is omitted since it is similar to the proof developed in [7, Appendix C].