Abstract—In this paper a new class of lattices called turbo lattices is introduced and established. We use the lattice Construction D to produce turbo lattices. This method needs a set of nested linear codes as its underlying structure. We benefit from turbo codes as our basis codes. Therefore, a set of nested turbo codes based on nested interleavers and nested convolutional codes is built. To this end, we employ both tail-biting and zero-tail convolutional codes. Using these codes, along with construction D, turbo lattices are created. Several properties of Construction D lattices and fundamental characteristics of turbo lattices including the minimum distance, coding gain, kissing number and an upper bound on the probability of error under a maximum likelihood decoder over AWGN channel are investigated. Furthermore, a multi-stage turbo lattice decoding algorithm based on iterative turbo decoding algorithm is given. Finally, simulation experiments provide strong agreement with our theoretical results.

Index Terms—Lattice, turbo codes, Construction D, interleaver, tail-biting, coding gain, iterative turbo decoder.

I. INTRODUCTION

There has been interest in constructing lattices with high coding gain, low kissing number and low decoding complexity [3, 22, 27]. It is known [15, 31] that we can send points of a lattice over an AWGN channel with noise variance $\sigma^2$ and then recover the received points using an appropriate decoding algorithm. This allows one to examine the strength of such lattices. The lattice version of the channel coding is to find an $n$-dimensional lattice $\Lambda$ which attains good error performance for a given value of volume-to-noise ratio (VNR) $\frac{V}{N} = 10^{131}$. Also an excellent performance of only $\frac{5}{25}$ dB away from capacity at SER of $10^{-5}$ is achieved for size $n = 10131$.

There are a wide range of applicable lattices in communications including the well-known root lattices [9], the recently introduced low-density parity-check lattices [22] (LDPC lattices) and the low-density lattice codes [27] (LDLC lattices). The former lattices have been extensively treated in the 1980’s and 1990’s [9]. After the year 2000, two classes of lattices based on the primary idea of LDPC codes have been established. These type of lattices have attracted a lot of attention in recent years [3, 6, 8, 10, 23, 28]. In the present work, we borrow the idea of turbo codes and construct a new class of lattices that we called turbo lattices.

Turbo codes were first introduced by Berrou et al. [5] in 1993 and have been largely treated since then. It has been shown [20] that these codes with an iterative turbo decoding algorithm can achieve a very good error performance close to Shannon capacity. Hence, constructing lattices based on these codes can be a promising tool.

The results by Forney et al. in [15] motivate us to apply Construction D lattices to design good turbo lattices. The authors of [15] proved the existence of capacity-achieving and sphere-bound-achieving lattices by means of Construction D lattices. This leads one to use Construction D method along with well-known turbo codes to produce turbo lattices. This is the first usage of turbo codes in constructing lattices. We benefit from structural properties of lattices and turbo codes to investigate and evaluate the basic parameters of turbo lattices such as minimum distance, volume, coding gain and kissing number. Then by using these parameters, one can find an appropriate upper bound on the performance of turbo lattices.

Various types of turbo codes have been constructed in terms of properties of their underlying constituent encoders and interleavers [20]. For example, encoders can be either block or convolutional codes and interleavers can be deterministic, pseudo-random or random [32]. Since Construction D deals with block codes, we treat turbo codes as block codes. Therefore, it seems more reasonable to use terminated convolutional codes. Since we use recursive and non-recursive convolutional codes, different types of termination methods can be applied to these component convolutional codes. Hence, we are interested in terminating trellises for both feed-back [26, 33] and feed-forward convolutional codes. To stay away from rate loss,
we employ tail-biting convolutional codes for short length turbo lattices. Also zero-tail convolutional codes are building blocks of turbo codes to use in construction of lattices with larger sizes [26, 33].

The closest vector problem (CVP) is one of the most important and hardest problems in lattice theory [9]. The main problem is to find the closest vector of an \( n \)-dimensional lattice \( \Lambda \) to a given point \( x \in \mathbb{R}^n \). This is called decoding of lattice \( \Lambda \). There are many well-known algorithms such as generalized min-sum algorithm [22], iterative decoding algorithms [8] and the algorithm in [27] for decoding newly introduced lattices. The basic idea behind these algorithms is to implement min-sum and sum-product algorithms and their generalizations.

Recently, a novel algorithm concerning decoding of lattices have been introduced [6]. This algorithm employs generator matrices of lattices for the decoding process and relies on a non-parametric belief propagation. Since we used turbo codes to construct turbo lattices, it is more reasonable to benefit from the underlying turbo structure of these lattices. In this case, we have to somehow relate the decoding of turbo lattices to the iterative turbo decoder [6] for turbo codes. This results in a multi-stage decoding algorithm based on iterative turbo decoders.

The present work is organized as follow. Two methods of constructing lattices are reviewed in Section II. We recall Construction \( A \) and its important parameters. Then, we state Construction \( D \) which can be viewed as a generalization of Construction \( A \). The crucial parameters of lattices which can be used to measure the efficiency of lattices are explained in that section. In Section III we recall the terminating methods for convolutional codes and introduce nested interleavers in a manner that can be used to build nested turbo codes. Section IV is devoted to the construction of nested turbo codes and consequently the construction of turbo lattices. In Section V a new multi-stage turbo lattice decoding algorithm is explained. This algorithm is based on the iterative turbo decoding algorithms of its underlying turbo codes. Section VI is dedicated to evaluate the critical parameters of turbo lattices based on the properties of their underlying turbo codes. We also demonstrate and discuss simulation results in this section. We conclude with final remarks on turbo lattices and further research topics in Section VII.

II. BACKGROUNDS ON LATTICES

In order to make this work self-contained, a background on lattices is essential. The general required information about critical parameters of Construction \( A \) and Construction \( D \) as well as parameters for measuring the efficiency of lattices are provided below.

A. General Notations for Lattices

A discrete additive subgroup \( \Lambda \) of \( \mathbb{R}^n \) is called lattice. Since \( \Lambda \) is discrete, it can be generated by \( m \leq n \) linearly independent vectors \( b_1, \ldots, b_m \) in \( \mathbb{R}^n \). The set \( \{b_1, \ldots, b_m\} \) is called a basis for \( \Lambda \). In the rest of this paper, we assume that \( \Lambda \) is an \( n \)-dimensional full rank (\( m = n \)) lattice over \( \mathbb{R}^n \). By using the Euclidean norm, \( \| \cdot \| \), we can define a metric on \( \Lambda \); that is, for every \( x, y \in \Lambda \) we have \( d(x, y) = \| x - y \|^2 \). The minimum distance of \( \Lambda \), \( d_{\min}(\Lambda) \), is

\[
d_{\min}(\Lambda) = \min_{x \neq y} \{ d(x, y) | x, y \in \Lambda \}.
\]

Let us put \( \{b_1, \ldots, b_m\} \) as the rows of a matrix \( B \), then we have \( \Lambda = \{x: x = zB, z \in \mathbb{Z}^n\} \). The matrix \( B \) is called a generator matrix for the lattice \( \Lambda \). The volume of a lattice \( \Lambda \) can be defined by \( \det(BB^T) \) where \( B^T \) is the transpose of \( B \). The volume of \( \Lambda \) is denoted by \( \det(\Lambda) \). For any point \( p \in \Lambda \) the Voronoi cell \( V(p) \) is

\[
\{ x = \sum_{i=1}^{n} \alpha_i b_i : \| x - p \| \leq \| x - q \|, \, \forall \, q \in \Lambda, \, \alpha_i \in \mathbb{R} \}.
\]

The Voronoi cell for the origin is denoted by \( V \). Two important parameters of every \( n \)-dimensional lattice \( \Lambda \) are the coding gain and the kissing number. The coding gain of a lattice \( \Lambda \) is defined by

\[
\gamma(\Lambda) = \frac{d_{\min}^2(\Lambda)}{\det(\Lambda)^{2/n}},
\]

where \( \det(\Lambda)^{2/n} \) is itself called the normalized volume of \( \Lambda \). This volume may be regarded as the volume of \( \Lambda \) per two dimensions. The coding gain used as a crude measure of the performance of a lattice. For any \( n \), \( \gamma(\mathbb{Z}^n) = 1 \). An uncoded system may be regarded as the one that uses a constellation based on \( \mathbb{Z}^n \). Thus the coding gain of an arbitrary lattice \( \Lambda \) may be considered to be the gain using a constellation based on \( \mathbb{Z}^n \) over an uncoded system using a constellation based on \( \mathbb{Z}^n \) [11, 12]. Therefore, coding gain is the saving in average of energy due to using \( \Lambda \) for the transmission instead of using the lattice \( \mathbb{Z}^n \) [16]. Geometrically, coding gain measures the increase in density of \( \Lambda \) over a baseline lattice \( \mathbb{Z}^n \).

If one put an \( n \)-dimensional sphere of radius \( d_{\min}(\Lambda)/2 \) centered at every lattice point of \( \Lambda \), then the kissing number of \( \Lambda \) is the maximum number of spheres that touch a fixed sphere. Hereafter we denote the kissing number of the lattice \( \Lambda \) by \( \tau(\Lambda) \).

There exist many ways to construct a lattice [9]. Lifting up discrete structures from groups and finite fields to the real case is a basic idea used in most of them. Some of these constructions rely on a set of underlying linear block codes. For example Constructions \( A, B, C, D \) and \( D' \) convert sets of linear block codes to lattices [9]. The crucial parameters of the derived lattices based on properties of the underlying codes can be computed.

In the following we give two algebraic constructions of lattices based on linear block codes [9]. The first one is Construction \( A \) which translates a block code to a lattice. Then a review of Construction \( D \) is given. These two constructions are the main structures of this work.

B. Lattice Constructions

In this subsection, a linear block code or a set of nested linear block codes are employed to build a lattice using Construction \( A \) or Construction \( D \). Then, one can derive parameters of these lattices by means of general properties of those underlying codes.
Let $C$ be a group code over $G = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, i.e. $C \subseteq G$, with minimum distance $d_{\text{min}}$. Define $\Lambda$ as a Construction $A$ lattice [9] derived from $C$ by:

$$\Lambda = \{2z_1+c_1, \ldots, 2z_n+c_n) : z_i \in \mathbb{Z}, c = (c_1, \ldots, c_n) \in C\}.$$  

(2)

The proof of the next theorem can be found in [9].

**Theorem 1.** Let $\Lambda$ be a lattice constructed using Construction $A$. The minimum distance of $\Lambda$ is

$$d_{\text{min}}(\Lambda) = \min \left\{2, \sqrt{d_{\text{min}}} \right\}. \quad (3)$$

Its coding gain is

$$\gamma(\Lambda) = \begin{cases} 4^{\frac{k}{d_{\text{min}}}} & d_{\text{min}} \geq 4, \\ \frac{d_{\text{min}}}{2} & d_{\text{min}} < 4, \end{cases} \quad (4)$$

and its kissing number is

$$\tau(\Lambda) = \begin{cases} 2d_{\text{min}}A_{d_{\text{min}}} & d_{\text{min}} < 4, \\ 2n + 16A_4 & d_{\text{min}} = 4, \\ 2n & d_{\text{min}} > 4, \end{cases} \quad (5)$$

where $A_{d_{\text{min}}}$ denotes the number of codewords in $C$ with minimum weight $d_{\text{min}}$.

These definitions and theorems can be generalized to a more practical and nice lattice construction. The next section is devoted to this construction.

Now we use a set of nested linear block codes to give a more general lattice structure named Construction $D$. This construction plays a key role in this work. An exact formula for the determinant of every lattice constructed using Construction $D$ is given in [9]. Also, proper bounds for the other important parameters of these lattices including minimum distance and coding gain have been found with an extra condition on the minimum distance of the underlying nested codes [9]. We omit this restricting condition on the minimum distance of the underlying nested block codes and then generalize those bounds to a more useful form. The resulted expressions for minimum distance and coding gain are related to the underlying codes as we will see soon. In addition, an upper bound for the kissing number of every lattice generated using Construction $D$ is derived.

Let

$$C_0 \supseteq C_1 \supseteq \cdots \supseteq C_a$$

be a family of $a+1$ linear codes where $C_\ell[n, k_\ell, d_{\text{min}}^{(\ell)}]$ for $1 \leq \ell \leq a$ and $C_0$ is the $[n, n, 1]$ trivial code $\mathbb{F}_2^n$ such that

$$C_\ell = \langle c_1, \ldots, c_{k_\ell} \rangle.$$ 

For any element $x = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$ and for $1 \leq \ell \leq a$ consider the vector in $\mathbb{R}^n$ of the form:

$$\frac{1}{2^{\ell-1}}x = \left(\frac{x_1}{2^{\ell-1}}, \ldots, \frac{x_n}{2^{\ell-1}}\right).$$

Define $\Lambda \subseteq \mathbb{R}^n$ as all vectors of the form

$$z + \sum_{\ell=1}^{a} \sum_{j=1}^{k_\ell} \beta_j^{(\ell)} \frac{1}{2^{\ell-1}}c_j \quad \text{where } z \in 2(\mathbb{Z})^n \text{ and } \beta_j^{(\ell)} = 0 \text{ or } 1.$$ 

An integral basis for $\Lambda$ is given by the vectors

$$\frac{1}{2^{\ell-1}}c_j \quad (7)$$

for $1 \leq \ell \leq a$ and $k_{\ell+1} + 1 \leq j \leq k_\ell$ plus $n - k_1$ vectors of the form $\{0, \ldots, 0, 2, 0, \ldots, 0\}$. Let us consider vectors $c_j$ as integral in $\mathbb{R}^n$, with components 0 or 1. To be specific, this lattice $\Lambda$ can be represented by the following code formula

$$\Lambda_{TC} = C_1 + \frac{1}{2} C_2 + \cdots + \frac{1}{2^{a-1}} C_a + 2(\mathbb{Z})^n. \quad (8)$$

Next we investigate the minimum distance of Construction $D$ lattices based on the minimum distance of its underlying codes.

**Theorem 2.** Let $\Lambda$ be a lattice constructed based on Construction $D$. Then we have

$$d_{\text{min}}(\Lambda) = \min_{1 \leq \ell \leq a} \left\{2, \sqrt{d_{\text{min}}^{(\ell)}} \right\}, \quad (9)$$

where $d_{\text{min}}^{(\ell)}$ is the minimum distance of $C_\ell$ for $1 \leq \ell \leq a$.

The proof is given in the Appendix.

This theorem provides a relationship between the performance of the lattice $\Lambda$ and the performance of its underlying codes. It is also useful to bound the coding gain of $\Lambda$. The next theorem is cited form [4].

**Theorem 3.** Let $\Lambda$ be a lattice constructed using Construction $D$, then the volume of $\Lambda$ is

$$\det(\Lambda) = 2^n - \sum_{\ell=1}^{a} k_\ell. \quad \text{Furthermore, if } d_{\text{min}}^{(\ell)} \geq \frac{4^\ell}{2\ell}, \text{ for } 1 \leq \ell \leq a \text{ and } \beta = 1 \text{ or } 2, \text{ then the squared minimum distance of } \Lambda \text{ is at least } 4^\beta, \text{ and its coding gain satisfies}$$

$$\gamma(\Lambda) \geq \beta^{-1} 4^\beta - 1 \sum_{\ell=1}^{a} \frac{k_\ell}{4^\beta}. \quad \text{The kissing number of a Construction } D \text{ lattice can be bounded above based on the minimum distance and the number of minimum weight codewords of each underlying nested code.}$$

**Theorem 4.** If $\Lambda$ is an $n$-dimensional lattice constructed using Construction $D$, then the kissing number of $\Lambda$ has the following property

$$\tau(\Lambda) \leq 2n + \sum_{\ell=1}^{a} \sum_{\substack{1 \leq \ell \leq a \\text{ and } d_{\text{min}}^{(\ell)} \leq 4^\ell \\text{ and } d_{\text{min}}^{(\ell)} \leq 4^\ell}} A_{d_{\text{min}}^{(\ell)}} \quad (10)$$

where $A_{d_{\text{min}}^{(\ell)}}$ denotes the number of codewords in $C_\ell$ with minimum weight $d_{\text{min}}^{(\ell)}$. Furthermore, if $d_{\text{min}}^{(\ell)} > 4^\ell$ for every $1 \leq \ell \leq a$, then $\tau(\Lambda) \leq 2n$.

The proof is given in the Appendix.
C. Performance Measures of Lattices

Sending points of a specific lattice in the absence of power constraints has been studied. This is called coding without restriction [21]. For lattices, instead of coding rate and capacity, using of normalized logarithmic density and generalized capacity is suggested. In this subsection, these parameters are explained from [13, 21, 27].

For an n-dimensional lattice \( \Lambda \) over the AWGN channel with noise variance \( \sigma^2 \) the normalized logarithmic density (NLD) [21] is defined as

\[
NLD = \frac{1}{n} \ln \left( \frac{1}{\det(\Lambda)} \right)
\]

and the generalized capacity is defined as

\[
C_\infty = \frac{1}{2} \ln \left( \frac{1}{2\pi e\sigma^2} \right).
\]

The volume-to-noise ratio (VNR) of an n-dimensional lattice \( \Lambda \) is defined as

\[
\alpha^2 = \frac{\det(\Lambda)^{\frac{2}{n}}}{2\pi e\sigma^2}.
\]  

(11)

For large n, the VNR is the ratio of the normalized volume of \( \Lambda \) to the normalized volume of a noise sphere of squared radius \( n\sigma^2 \) which is defined as SNR in [22] and \( \alpha^2 \) in [15].

Since lattices have a uniform structure, we can assume 0 is transmitted and \( r \) is the received vector. Then \( r \) is a vector whose components are distributed based on a Gaussian distribution with zero mean and variance \( \sigma^2 \). The probability of error of a maximum likelihood decoder for a lattice \( \Lambda \) is the probability that a white Gaussian \( n \)-tuple with variance \( \sigma^2 \) falls outside the Voronoi cell \( \mathcal{V} \) of \( \Lambda \):

\[
P_e(\Lambda, \sigma^2) = 1 - \frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{\mathcal{V}} e^{-\frac{d^2}{2\sigma^2}} dx.
\]  

(12)

The following definitions are cited from [14].

**Definition 5.** A lattice \( \Lambda \) is sphere-bound-achieving if

\[P_e(\Lambda, \sigma^2) \approx 0\]

whenever \( \alpha^2 > 1 \). Furthermore, a class of packings of the Euclidean \( n \)-space is capacity-achieving for the AWGN channel with noise variance \( \sigma^2 \) per dimension if there exists a lattice \( \Lambda \) in the class with \( \alpha^2 = 1 \) and \( P_e(\Lambda, \sigma^2) \approx 0 \).

We note that \( \alpha^2 = 1 \) is equivalent to \( \text{NLD} = C_\infty \). Using the union bound [19], an appropriate estimation of the \( P_e(\Lambda, \sigma^2) \) is given for a minimum distance decoder:

\[P_e(\Lambda, \sigma^2) \leq \frac{\tau}{2} \text{erfc} \left( \frac{\rho}{\sqrt{2\sigma}} \right)\]

(13)

where \( \rho = \frac{d_{\min}}{2} \) is the packing radius of \( \Lambda \), \( \tau \) is the average number of code points at distance \( 2\rho \) from a code point, and

\[\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt,\]

\[1P_e(\Lambda, \sigma^2) \approx 0 \text{ means that } P_e(\Lambda, \sigma^2) \leq 10^{-5}.\]

is called complementary error function. Using the formula of coding gain and \( \alpha^2 \), the following bound on the probability of error [13] for maximum likelihood decoder is obtained:

\[P_e(\Lambda, \sigma^2) \leq \frac{\tau(\Lambda)}{2} \text{erfc}\left( \sqrt{\frac{\pi e}{4}} \gamma(\Lambda)\alpha^2 \right)\]

(14)

where \( \tau(\Lambda) \) is the kissing number of the lattice \( \Lambda \). It is known [22] that the normalized probability of error for \( \Lambda_{FC} \) with a minimum distance decoder over an AWGN channel with noise variance \( \sigma^2 \) is

\[P_e^*(\Lambda, \sigma^2) = \frac{2P_e(\Lambda, \sigma^2)}{n}.
\]

(15)

Thus, we get

\[P_e^*(\Lambda, \sigma^2) = \frac{2}{\tau(\Lambda) - 2} \text{erfc}\left( \sqrt{\frac{\pi e}{4}} \gamma(\Lambda)\alpha^2 \right).
\]

(16)

The normalized kissing number of an n-dimensional lattice \( \Lambda \) is defined as

\[\tau^*(\Lambda) = \frac{\tau(\Lambda)}{n}.
\]

(17)

By dividing both sides of (10) by \( n \) we have:

\[\tau^*(\Lambda) = \frac{\tau(\Lambda)}{n} \leq 2 + \sum_{1 \leq \ell \leq \alpha} \frac{\gamma(d_{\min})^\ell}{d_{\min}^\ell} A_d^\ell.
\]

(18)

Hence,

\[P_e^*(\Lambda, \sigma^2) \leq \tau^*(\Lambda) \text{erfc}\left( \sqrt{\frac{\pi e}{4}} \gamma(\Lambda)\alpha^2 \right).
\]

(19)

This is a key observation of our paper. Fig. 1 plots the normalized error probability \( P_e^*(\mathbb{Z}^n, \sigma^2) \) for \( \mathbb{Z}^n \) for any \( n \). The curve \( P_e^*(\Lambda, \sigma^2) \) for a capacity-achieving lattice which is a vertical line at \( \alpha^2 = 1 \) (0 dB) is also provided. Since \( \gamma(\mathbb{Z}^n) = 1 \) and \( \tau(\mathbb{Z}^n) = 2n \), the union bound estimate for \( P_e^*(\mathbb{Z}^n, \sigma^2) \) is

\[2\text{erfc}\left( \sqrt{\frac{\pi e}{4}} \alpha^2 \right).
\]

Because \( \text{erfc}(.) \) is a decreasing function, the curve for \( P_e^*(\Lambda, \sigma^2) \) for a lattice \( \Lambda \) with coding gain \( \gamma(\Lambda) \) and normalized kissing number \( \tau^*(\Lambda) \) can be obtained from the curve for \( P_e^*(\mathbb{Z}^n, \sigma^2) \) by moving it to the left by \( \gamma(\Lambda) \) (in decibels) and up by a factor of \( \tau^*(\Lambda) \) (on a log_{10} scale). This can be considered as an applicable tool to investigate and analyze the performance of a lattice \( \Lambda \) with known coding gain and kissing number. This method is used to find, theoretically, an appropriate estimate upper bound on the error performance of turbo lattices.

III. CONVOLUTIONAL AND TURBO CODES

In this section, two approaches to convert a convolutional code to a block code are explained. More specifically, parallel concatenated terminated and tail-biting convolutional codes can be used to construct lattices based on Construction A. However, for Construction D we are allowed to use only parallel concatenated tail-biting convolutional codes. The reason for this situation is discussed below. In the following, we review
these two methods in details. Since recursive convolutional codes produce better turbo codes, we focus on tail-biting of feed-back convolutional codes.

A. Terminated Convolutional Codes

Let $C(N, K, \nu)$ be a systematic convolutional code of rate $\frac{K}{N}$ with constraint length $\nu$ and memory order $m$. In particular, it is usual to begin the encoding of a convolutional code in the all-zero state. Therefore, it is necessary to return to the all-zero state for performing the decoding process. It means that, after feeding the last information block $LK$, we have to input $m$ additional blocks of zero bits to force the encoder to get back to the all-zero state. Hence, as an output we obtain a codeword of length $(L + m)N$ including $mN$ output bits, resulting from termination. This is called a terminated convolutional code and it can be considered as an $[L + m]N, NK]$, block code. The rate of this code is $R = \frac{LK}{(L+m)N}$. It is obvious that, in this deformation from the convolutional code $C$ to the mentioned block code there exists a rate loss and a change in the size of the codewords while in Construction $D$ all the code lengths of the set of nested linear codes have to be equal. However, this termination method modifies the sizes of the underlying codes in each level. This code length modification results in a restriction which prevents the use of terminated convolutional codes in our derivation of lattices based on Construction $D$. In order to avoid this situation, an alternative method which is referred as tail-biting [26] can be used. Thus, terminated convolutional codes can only be employed to construct turbo codes which are appropriate for using along with Construction $A$.

B. Tail-Biting Convolutional Codes

Assume that $C$ is a feed-forward convolutional code of rate $\frac{K}{N}$ and constraint length $\nu$. Consider only input sequences of the form $(u_1, \ldots, u_K)$ where

$$u_i = (u_i, \ldots, u_i, 0, 1, \ldots, u_i, L)$$

for $1 \leq i \leq K$ and a positive integer $L$. This $L$ is called tail-biting length. We have $u_i, \ldots, u_i, 0, 1, \ldots, u_i, L$ for $1 \leq j \leq m - 1$, i.e. sequences of length $L + m - 1$, in which the first $m - 1$ symbols are repeated at the end. Assume that the $i$th shift register has memory $m_i$. Initially insert the last $m_i$ data bits in the $i$th input sequence along with $m - m_i$ zeros to the $i$th shift register, for $1 \leq i \leq K$ and $1 \leq m_i \leq m$. In the next step, discard the outputs of the first input sequence and then input the entire $LK$ information bits. In the usual manner an output of length $LN$ is expected. This results in an $[LN, NK]$ linear block code. The above procedure enforces the encoder to terminate in a state that it started at first. This is called tail-biting for a feed-forward convolutional encoder [26].

The algorithm for tail-biting a feed-back convolutional encoder is also introduced in [17-33]. However, tail-biting is not possible for all sizes. In other words, tail-biting of a feedback convolutional encoder is only possible for some special
Let $G(x)$ be a generator matrix of a systematic feed-back convolutional code $C(N, K, \nu)$ defined as follows
\[
\begin{bmatrix}
1 & \cdots & 0 & g_{1,K+1}(x) & \cdots & g_{1,N}(x) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & g_{K,K+1}(x) & \cdots & g_{K,N}(x)
\end{bmatrix},
\]
(20)

where $g_{i,j}(x) = \frac{q_{i,j}(x)}{p_{i,j}(x)}$ for coprime polynomials $q_{i,j}(x)$ and $p_{i,j}(x)$ for $1 \leq i \leq K$ and $K + 1 \leq j \leq N$. It is known that $G(x)$ is equivalent to $G'(x)$ defined by
\[
\begin{bmatrix}
p_{1}(x) & \cdots & 0 & q_{1,K+1}(x) & \cdots & q_{1,N}(x) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & p_{K}(x) & q_{K,K+1}(x) & \cdots & q_{K,N}(x)
\end{bmatrix},
\]
(21)

The set of encoder output codewords, generated by $G'(x)$ is exactly the same as the set of codewords generated by $G(x)$. However, there exist a different mapping between information bits and codewords in those two cases. Hence, with tail-biting we can make a one-to-one correspondence between a rate $K/N$ systematic feed-back convolutional encoder with constraint $\nu$ and a linear code $[LN, LK]$ with generator matrix
\[
G' = \begin{bmatrix}
P_1 & \cdots & 0 & Q_{1,K+1} & \cdots & Q_{1,N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & P_K & Q_{K,K+1} & \cdots & Q_{K,N}
\end{bmatrix},
\]
(22)

where $P_i$ and $Q_{i,j}$ are $L \times L$ circulant matrices with top row of length $L$ made from the $(i, j)$ entry of $G'(x)$ for $1 \leq i \leq K$ and $K + 1 \leq j \leq N$.

**Theorem 6.** Let $p_{i}(x)$ and $q_{i,j}(x)$ be polynomials of degree at most $L - 1$ and let $P_i$ and $Q_{i,j}$ be the associated circulant matrices for $1 \leq i \leq K$ and $K + 1 \leq j \leq N$. Then the block code $C[LN, LK]$ generated by $G'$ in (22) can also be generated by $G = [LN, LK][F]$, where $F$ is a circulant matrix if and only if $(p_{i}(x), x^{L} - 1) = 1$ for all $1 \leq i \leq K$. In this case, we get
\[
q_{i,j}(x) \equiv f_{i,j}(x)p_{i}(x) \pmod{x^{L} - 1}.
\]

The proof is given in the Appendix.

We observe that $F$ is an $LK \times L(N - K)$ circulant matrix consisting of $K \times (N - K)$ blocks of circulant $L \times L$ submatrices. The $L \times L$ circulant matrix, which must be placed in the $(i, j)$th block of $F$, is obtained using $f_{i,j}(x)$ as its top row, $1 \leq i \leq K$ and $K + 1 \leq j \leq N$. Also the identity matrix $I_{NK}$ can be written as an $K \times K$ identity block matrix with each of its nonzero entries replaced by an identity matrix $I_{L}$.

We close this subsection giving a proposition that relates our result in the above theorem and well-known results [29, 33] for eligible lengths of $L$ that can be applied to construct tail-biting feed-back convolutional codes. For the sake of brevity, we consider only feed-back convolutional codes of rate $N/N$. Let $G(x)$ be a generator matrix of a systematic feed-back convolutional code $C(N, N - 1, \nu)$ defined as follows
\[
\begin{bmatrix}
1 & \cdots & 0 & g_{1,N}(x) \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & g_{N-1,N}(x)
\end{bmatrix},
\]
(23)

where $g_{i,N}(x) = \frac{q_{i,N}(x)}{p_{i}(x)}$ for coprime polynomials $q_{i,N}(x)$ and $p(x)$ for $1 \leq i \leq N - 1$. Without loss of generality, we assume that $p(x) = p_0 + p_1 x + \cdots + p_m x^m$. If we realize this code in observer canonical form [33], then the state matrix is
\[
A = \begin{bmatrix}
0 & \cdots & 0 & p_m \\
1 & \cdots & 0 & p_{m-1} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & p_1
\end{bmatrix}.
\]
(24)

We see that in order to encode an $[LN, LK]$ tail-biting code with the method described in [33], the matrix $(A^L + I_m)$ has to be invertible. It should be noted that [33] realizing the encoder in controller canonical form and observer canonical form leads to the same set of possible sizes $L$.

**Proposition 7.** Let $A$, as in (24), be the state matrix of a convolutional code $C(N, N - 1, \nu)$ with generator matrix (23). Then det $(A^L + I_m) \neq 0$ if and only if $(p(x), x^{L} + 1) = 1$.

The proof is given in the Appendix.

**C. Parallel Concatenated Codes; Structure of Turbo Codes**

Turbo codes can be assumed as block codes by fixing their interleaver lengths; but they have not been analyzed from this point of view except in [32]. Next we briefly introduce turbo codes from [20] and then we use them to produce a new type of lattices called turbo lattices. The basic encoder for a classical turbo code consists of an input information sequence, two systematic feed-back convolutional encoders, and an interleaver, denoted by $\Pi$. The information sequence is considered to be a block of length $K$ and is represented by the vector $(u_1, \ldots, u_K)$. The information sequence is the first transmitted sequence. The first encoder $C_1$ generates the parity sequence $v^{(1)} = (v^{(1)}_1, \ldots, v^{(1)}_K)$. The interleaver $\Pi$ permutes the $K$ bits in the information block so that the second encoder $C_2$ receives a permuted sequence different from the first one. If $v^{(2)} = (v^{(2)}_1, \ldots, v^{(2)}_K)$ represents the second set of parity sequences generated by $C_2$, then the final transmitted sequence (codeword) is given by the vector
\[
v = (u_1v^{(1)}_1, \ldots, u_Kv^{(1)}_K, v^{(2)}_1, \ldots, v^{(2)}_K).
\]

We assume that an interleaver II and a recursive convolutional encoder $E$ with parameters $(N, K, \nu)$ are used for constructing a turbo code of size $k = KL$.

The information block (interleaver size) $k$ has to be selected large enough to achieve performance close to Shannon limit. Improving minimum free distance of turbo codes is possible by designing good interleavers. In other words, interleavers make a shift from lower-weight codewords to higher-weight codewords. This shifting has been called spectral thinning [20]. Such interleaving matches the codewords with lower weight of the first encoder to the high-weight parity sequences of the second encoder. More precisely, for large values of interleaver size $k$ the multiplicities of the low-weight codewords in the turbo code weight spectrum are reduced by a factor of $k$. This reduction by a factor of $k$ is called interleaver gain. Hence, it is apparent that interleavers have a key role in the heart of turbo codes and it is important to have random-like interleavers [20, 32].
IV. NESTED TURBO CODES AND TURBO LATTICES

We exploit a set of nested tail-biting convolutional codes and a nested interleaver along with Construction D to form turbo lattices. Also terminated convolutional codes and Construction A are employed for the same purpose. An explicit explanation of these two approaches is given next.

A. Constructing Nested Turbo Codes

Consider a turbo code TC with two component codes generated by a generator matrix $G(x)$ of size $k \times n$ of a convolutional code and a random interleaver $I$, of size $k = LK$. Assume that both encoders are systematic feed-back convolutional encoders. Every interleaver $I$ can be represented by a matrix $P_{k \times k}$ which has only one 1 in each column and row. It is easy to see that the generator matrix of TC can be written as follows

$$G_{TC} = [I_k | P | PF]_{k \times n} \quad (25)$$

where $P$ is a $L \times (L(N-K))$ submatrix of $G$, the tail-bited generator matrix of $G(x)$, including only parity columns of $G$. The matrix $I_k$ is the identity matrix of size $k$. Therefore, we can assume that $G_{TC}$ is a $k \times n$ matrix with $k = LK$ rows and $n = 2LN - LK$ columns.

The above representation (25) can be extended to construct a generator matrix for a parallel concatenated code with $b$ branches. Each branch has its own interleaver $I_i$ with matrix representation $P_i$ and a recursive encoder $E_i$ for $1 \leq i \leq b$. Assume that all the encoders are the same $(N, K, \nu)$ convolutional encoder and the block of information bits has length $k = KL$. Thus, the corresponding generator matrix of this turbo code is

$$G_{TC} = [I_k | P_1F | P_2F | \cdots | P_bF]_{k \times n_e} \quad (26)$$

where $F$ is a $L \times L(N-K)$ as above and $n_e = KL + bL(N-K)$.

In order to design a nested set of turbo codes, the presence of a nested interleaver is essential. Hence, a new concept of nested interleavers has to be given.

**Definition 8.** The interleaver $I$ of size $k$ is a $(k_1, \ldots, k_1)$-nested interleaver if the following conditions hold

1) $0 < k_a < k_{a-1} < \cdots < k_1 = k$,
2) for every $1 \leq \ell \leq a$ we have

$$x \in \{1, \ldots, k_\ell\} \implies I(x) \in \{1, \ldots, k_\ell\}. \quad (27)$$

A $(k_2, k_1)$-nested interleaver is called a $k_2$-nested interleaver.

**Example 9.** Let $K = 1$. The permutation

$$I_1 = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
3 & 4 & 2 & 1 & 5 & 7 & 8 & 6
\end{pmatrix}$$

is a 4-nested interleaver because $k_1 = k = L = 8$ and $k_2 = 4$.

In order to have a set of nested block turbo codes, a set of nested matrices has to be defined. These nested turbo codes are appropriate to use in both Construction A and Construction D for producing turbo lattices.

**Definition 10.** Let $TC$ be a parallel concatenated convolutional code with two equivalent systematic convolutional codes generated by $G(x)$. Let $G$ be the generator matrix of tail-biting of $G(x)$, and $I$ be the interleaver of size $k = LK$ with the $(k_a, \ldots, k_1)$-nested property that is used to construct a turbo code TC. Then $G_{TC}$ is as of (25). Define a set of turbo codes

$$TC = TC_1 \supseteq TC_2 \supseteq \cdots \supseteq TC_a. \quad (28)$$

In fact, a generator matrix $G_\ell$ of size $k_\ell \times n$ is a submatrix of $G_{TC}$ consisting of the first $k_\ell$ rows of $G_{TC}$ for every $1 \leq \ell \leq a$.

**Example 11.** Consider a $(4, 3, 3)$ systematic convolutional code with the following generator matrix

$$G(x) = \begin{pmatrix}
1 & 0 & 0 & 1 + x + x^2 + x^4 \\
0 & 1 & 0 & 1 + x + x^2 + x^4 \\
0 & 0 & 1 & 1 + x + x^2 + x^4
\end{pmatrix}.$$

The matrix $G(x)$ is equivalent to $G'(x)$ given by

$$\begin{pmatrix}
p_1(x) & 0 & 0 & q_{1,4}(x) \\
p_2(x) & 0 & q_{2,4}(x) \\
p_3(x) & 0 & q_{3,4}(x)
\end{pmatrix},$$

where $p_1(x) = p_2(x) = p_3(x) = 1 + x^2 + x^4$ and also $q_{1,4}(x) = 1 + x + x^3 + x^4$, $q_{2,4}(x) = 1 + x^3 + x^4$ and $q_{3,4}(x) = 1 + x + x^2 + x^4$. Let $L = 8$, then $(p_1(x), x^8 - 1) = 1$ and $q_{i,4}(x) = p_1(x) f_1(x) (\mod x^8 - 1)$, for $1 \leq i \leq 3$. One can use the Euclidean algorithm to find $f_1(x)$. Therefore, we get

$$\begin{align*}
f_{1,4}(x) &= x^2 + x^3 + x^5 + x^6, \\
f_{2,4}(x) &= x + x^2 + x^3 + x^6 + x^7, \\
f_{3,4}(x) &= 1 + x + x^3 + x^7.
\end{align*}$$

Hence,

$$G = \begin{pmatrix}
I_8 & 0 & 0 & F_{1,4} \\
0 & I_8 & 0 & F_{2,4} \\
0 & 0 & I_8 & F_{3,4}
\end{pmatrix}_{24 \times 32},$$

where $F_{i,4}$ is a circulant matrix of size 8 defined by top row $f_{i,4}(x)$, $1 \leq i \leq 3$. For instance

$$F_{1,4} = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}.$$
where $P_{j,j}$ is another permutation matrix of size $8 \times 8$. Then $G_{TC}$, a generator matrix for our nested turbo code is

$$G_{TC} = \begin{bmatrix} I_{24} & F & PF \end{bmatrix}.$$ 

Now we have $TC = TC_1 \supseteq TC_2 \supseteq TC_3$ such that a generator matrix for $TC_2$ is consisting of the first 8 rows and a generator matrix for $TC_3$ is consisting of the first 16 rows of $G_{TC}$.

Now, we are prepared to formulate the basic characteristics of nested turbo codes. The next theorem studies the structural properties of a set of nested turbo codes in terms of properties of its subcodes. Let $\Pi$ be an $(k_a, k_{a-1}, \ldots, k_1)$-nested interleaver and

$$TC = TC_1 \supseteq TC_2 \supseteq \cdots \supseteq TC_a$$

be a set of nested turbo codes constructed as above. Then, we have $d_{\min} = d_{\min}^{(1)} \leq d_{\min}^{(2)} \leq \cdots \leq d_{\min}^{(a)}$ where $d_{\min}^{(a)}$ denotes the minimum distance of $TC_a$. Also the rate of $TC_\ell$ is equal to $R_\ell = \frac{k_\ell}{n-k_\ell}$ for $1 \leq \ell \leq a$. Furthermore, we have $R_{TC_\ell} = R_1 \geq R_2 \geq \cdots \geq R_a$. The rate of each $TC_\ell$ can be increased to $\frac{k_\ell}{n-k_\ell}$ because we have $k - k_\ell$ all-zero columns in $G_\ell$. In fact, these columns can be punctured out to avoid from generating zero bits, but we can still keep them. Since producing turbo lattices and measuring the performance of them are in mind, $R_\ell = \frac{k_\ell}{n-k_\ell}$ is more useful than the actual rate in the turbo lattices.

**Theorem 12.** Let $\Pi$ be an $(k_a, k_{a-1}, \ldots, k_1)$-nested interleaver and

$$TC = TC_1 \supseteq TC_2 \supseteq \cdots \supseteq TC_a$$

be a set of nested turbo codes constructed as above. If we increase by scaling the tail-biting length $L$ and parameters $k_\ell$’s in the construction of the generator matrix of the turbo codes and induced set of nested turbo codes by a scale factor of $t$, then the rates of the resulting nested turbo codes remain intact.

The proof is given in the Appendix.

The above theorem reveals the fact that the rates of nested turbo codes stay unchanged when the interleaver sizes are increased. The only impact of this is on the increasing of the minimum distance (via interleaver gain and spectral thinning), on the coding gain (via change in the numerator not in denominator of the formula) and on the kissing number of turbo lattices. These results are shown more explicitly in Section \[7\]

**B. Interleaver Design**

We observe that to produce nested turbo codes, an interleaver which satisfies the $(k_a, \ldots, k_1)$-nested property is necessary. In other words, we put two conditions in Definition\[8\] in a manner that, along with Definition\[10\] each $TC_\ell$ determines a turbo code.

It is well-known \[32\] that three major types of interleavers are random, pseudorandom and deterministic interleavers. The pseudorandom interleavers are usually defined by the $S$-random property. An interleaver $\Pi$ is called $S$-random if for every $1 \leq i, j \leq k$, the condition $|j - i| < S$ implies that $|\Pi(j) - \Pi(i)| > S$. A deterministic interleaver can be induced from a deterministic function with random properties \[25\] \[30\].

Assume that $\Pi_1, \Pi_2, \ldots, \Pi_4$ is a sequence of interleavers of sizes $k_a, k_{a-1} - k_a, \ldots, k_1 - k_2$ where $\Pi_i(i) = j_i$ for $1 \leq \ell \leq a, 1 \leq i, j_i \leq k_{\ell+1} - k_\ell$ and $k_{a+1} = 0$. We can simply append these interleavers to produce a $(k_a, \ldots, k_1)$-nested interleaver. More precisely, for every $1 \leq i \leq k$ let $1 \leq \ell_0 \leq a$ be such that $k_{\ell_0+1} + 1 \leq i \leq k_{\ell_0}$, then define

$$\Pi(i) = \Pi_{\ell_0}(i - k_{\ell_0+1}) + k_{\ell_0+1}.$$  \(29\)

By this definition, the interleaver $\Pi$ is of length $k = k_1$.

**Example 13.** The interleaver $\Pi_3$ in the previous example can be derived by appending two interleavers $\Pi_8$ and $\Pi_{10}$ of sizes 4 and $8 - 4 = 4$. It is easy to check that

$$\Pi_8 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}, \quad \Pi_{10} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}.$$  

In the next theorem the relationship between the interleavers $\Pi_\ell, \ldots, \Pi_{\ell}$ and the interleavers $\Pi_\ell$ for every $TC_\ell$ is disclosed, $1 \leq \ell \leq a$.

**Theorem 14.** Assume that $\Pi$, an interleaver of size $k$ for $TC_\ell$, is built by appending a set of interleavers $\Pi_\ell, \ldots, \Pi_{\ell}$ of sizes $k_a, k_{a-1} - k_a, \ldots, k_1 - k_2$ respectively as in (29). Then $\Pi_i$, the interleaver for the $i$th turbo code $TC_\ell$, is an $(k_a, \ldots, k_\ell)$-nested interleaver and it can be obtained by appending $\Pi_{\ell}, \ldots, \Pi_{\ell}$, $1 \leq \ell \leq a$.

The proof is given in the Appendix.

The append operation preserves the deterministic and pseudorandom properties. Indeed, it is clear that if we append a deterministic interleaver, then a deterministic interleaver can be defined by a function including at most $a$ cases. Therefore, the following proposition is clear and we do not give its proof.

**Proposition 15.** Let $\Pi_\ell, \ldots, \Pi_{\ell}$ be a set of deterministic interleavers. Then the appended interleaver $\Pi_\ell$ is a deterministic interleaver.

The following theorem states that we can produce an $S$-random $(k_a, \ldots, k_1)$-nested interleaver using a set of pseudorandom interleavers.

**Theorem 16.** Let $\Pi_\ell, \ldots, \Pi_{\ell}$ be a set of pseudorandom interleavers such that $\Pi_\ell$ is an $S_\ell$-random interleaver and $\Pi_{\ell}(1) = 1, 1 \leq \ell \leq a$. Then the appended interleaver $\Pi$ is an $S$-random $(k_a, \ldots, k_1)$-nested interleaver where $S = \min_{1 \leq \ell \leq a}(S_\ell)$.

The proof is given in the Appendix.

Overall, a set of nested turbo codes can be induced from a set of nested interleaver and a set of tail-bited turbo code. Now, we are in a position that we can introduce and construct turbo lattices.

**C. Turbo Lattices**

Next, turbo codes and their nested versions are used to derive lattices using Construction $D$ or Construction $A$. In
order to do so, we need a set of nested linear block codes
\[ P_2^n = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_a. \]

We use (28) and their corresponding generator matrices as nested codes which we need for producing a lattice based on Construction D. Now a generator matrix for a lattice constructed using Construction D can be derived from a set of generator vectors for the largest underlying code as in (7). Hence, for finding a generator matrix for \( \Lambda \), we have to multiply the rows of \( G_{TC} \) with index numbers between \((k\ell+1) + 1 \) and \( k\ell \) by \( \frac{1}{2\ell+1} \), \( 0 \leq \ell \leq a \). The resulting matrix along \( n-k \) vectors of the form \((0, \ldots, 0, 2, 0, \ldots, 0) \) of length \( n \) form an integral basis for a lattice \( \Lambda_{TC} \).

**Definition 17.** A lattice \( \Lambda_{TC} \) constructed using Construction D is called a turbo lattice if its largest underlying code is a turbo code TC.

It is easy to verify that we can form a turbo lattice \( \Lambda_{TC} \) using a turbo code \( TC \) with generator matrix \( G_{TC} \) as in (25). If the level of our construction, \( a \), is larger than 1, then we have to use turbo codes which come from tail-bited convolutional codes. However, if \( a = 1 \) we have a degree of freedom in using a turbo code built from terminated or tail-bited convolutional codes.

**Example 18.** Let \( G_{TC} \) be as in the previous example. In order to obtain a generator matrix of \( \Lambda_{TC} \), a lattice constructed using Construction D and \( G_{TC} \) as its largest underlying code, we have to multiply the rows with indices \( \{1, \ldots, 8\} \) by \( \frac{1}{4} \) and the rows with indices \( \{9, \ldots, 16\} \) by \( \frac{1}{2} \). The resulting matrix along with 16 additional rows of the form \((0, \ldots, 0, 2, 0, \ldots, 0) \) produce a generator matrix for \( \Lambda_{TC} \). Hence, a generator matrix \( G_{TL} \) for the produced turbo lattice is

\[
G_{TL} = \begin{bmatrix}
\frac{1}{4} I_8 & 0 & 0 & \frac{1}{4} P_{1,1} F_{1,4} & \frac{1}{4} P_{1,1} F_{1,4} \\
0 & \frac{1}{2} I_8 & 0 & \frac{1}{2} F_{1,4} & \frac{1}{2} F_{1,4} \\
0 & 0 & I_8 & F_{3,4} & P_{3,3} F_{3,4} \\
0 & 0 & 0 & 2 I_8 & 2 I_8 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}_{40 \times 40},
\]

where

\[
P = \begin{bmatrix}
P_{1,1} & 0 & 0 \\
0 & P_{2,2} & 0 \\
0 & 0 & P_{3,3}
\end{bmatrix}_{24 \times 24}
\]

is the matrix of an \((8,16)\)-nested interleaver. Each \( P_{j,j} \) is another permutation, \( 1 \leq j \leq 3 \), coming from an interleaver of size 8. In other words, the interleaver corresponding to \( P \) can be constructed by the appending method with underlying interleavers \( P_{j,j} \), \( 1 \leq j \leq 3 \).

The above example benefited from a set of nested turbo codes. These turbo codes have tail-bited recursive convolutional codes as their component codes. Also they used nested interleavers. We can also simply provide a turbo code based on an interleaver and two terminated convolutional codes. In this case, Construction A may be used to obtain a turbo lattice. An example of a lattice constructed using construction A and turbo code TC which uses terminated convolutional codes as its constituent codes is given next.

**Example 19.** Let

\[
G_A(x) = \begin{bmatrix} 1 \\ \frac{1+x^2}{1+x+x^2} \end{bmatrix}
\]

be the generator matrix of a recursive convolutional code. In this case, we have \( K = 1, N = 2 \) and \( \nu = 2 \). Let \( L = 8 \), then we get a turbo code TC of rate

\[
\frac{L_K}{(2K-1)(LK+\nu)} = \frac{8}{30}.
\]

If we use terminated version of these recursive convolutional codes along with an interleaver of size 8, a linear block code \( TC[30,8] \) can be obtained. Now consider this turbo code as a base code of Construction A to induce a turbo lattice. The minimum distance, coding gain and kissing number of this turbo lattice is closely related to the minimum distance of its underlying turbo code. Since the minimum distance of this turbo code can be increased or decreased by a selection of interleaver, the performance analysis of this turbo lattice relies on the choice of its interleaver.

V. DECODING ALGORITHM

In the previous sections we used a set of nested turbo codes to produce turbo lattices \( \Lambda_{TC} \). Now our aim is to solve a closest lattice point problem for \( \Lambda_{TC} \). There exist many decoding algorithms for finding the closest point in a lattice \([6, 6, 24, 25]\). Let us suppose that a vector \( x \in \Lambda \) is sent over an unconstrained AWGN channel with noise variance \( \sigma^2 \) and a vector \( r \in \mathbb{R}^n \) is received. The closest point search algorithms attempt to compute the lattice vector \( \hat{x} \in \Lambda \) such that \( \|x - r\| \) is minimized. In the next subsection we review the structure of an iterative turbo decoder. By applying this soft decision decoding algorithm, we derive a soft decoding algorithm for decoding every lattice constructed based on Construction A and using a turbo code as its underlying code.

Then we generalize and investigate a multi-stage soft decision decoding algorithm for decoding turbo lattices constructed based on Construction D. Finally, in the last subsection we analyze the decoding complexity of the proposed algorithm.

A. Iterative Turbo Decoding Algorithm

The great performance of turbo codes is due to an efficient decoding algorithm called **iterative turbo decoding algorithm** \([5]\). This algorithm is a soft-input soft-output (SISO) one that uses well-known maximum a posteriori (MAP) algorithms like BCJR, SOVA as its constituent decoders for decoding a turbo code \([20]\).

B. A Soft Decoder for Construction A Lattices

In order to take advantage from the excellent performance of iterative turbo decoder, we should modify this algorithm so it can be used for lattices constructed using Construction A. The task of this algorithm is to find \( \hat{x}_\ell \) in \( \Lambda_\ell = C_\ell + 2(\mathbb{Z})^n \) for the received vector \( r_\ell \), where \( C_\ell \) is a turbo code. In fact, we are looking for an efficient soft-input soft-output decoding algorithm associated with the lattice \( \Lambda_\ell \). We observe that the lattice \( \Lambda_\ell \) is a lattice constructed using the turbo code \( C_\ell \) as its underlying code following Construction A. We can simply use the iterative turbo decoder associated with \( C_\ell \) for decoding lattice \( \Lambda_\ell \). Given any \( r_\ell \), first reduce all the components \( r_j \)
of \( r_\ell \) to the interval \(-1 \leq r_\ell \leq 1\) by subtracting a vector \( w \in 2(\mathbb{Z}^n) \). To this end, find the closest even and odd integers \( e_j \) and \( o_j \) to each coordinate \( r_\ell j \) of \( r_\ell \), \( 1 \leq j \leq n \). Compute \( s_j = 2(\frac{e_j + o_j}{2} \mp r_\ell j) \) where the upper signs are taken if \( e_j < o_j \) and the lower ones if \( e_j > o_j \). Hence \( s_j \) is within the interval \(-1 \leq s_j \leq 1\). Now a soft decision decoding algorithm for \( C_t \) can be used for decoding \( s_\ell = (s_1, \ldots, s_n) \). In fact, since \( s_\ell \) is now in the unit cube \([-1, 1]^n\), an iterative turbo decoder for \( C_t \) can be applied to decode \( s_\ell \). Suppose that an output \( c_\ell \in C_t \) is obtained. Then \( v = c_\ell \mp w \) is the closest point of \( \Lambda_c \) to the original vector \( r_\ell \). It is obvious that no point of \( \Lambda_c \) is closer to \( r_\ell \) than the closest codeword of \( C_t \). Assume (by contradiction) that \( u \neq v \) be a closest lattice point to \( r_\ell \). Subtracting a suitable vector in \( 2(\mathbb{Z}^n) \) would yield all the components of \( r_\ell \) to be between \( \pm 1 \) and produce another point of \( \Lambda_c \), let us say \( u_1 \), which is in \( C_t \) as well. It means that \( u_1 \) is closer to \( r_\ell \) than \( u \), a contradiction. Thus every soft decision decoding algorithm for \( C_t \) may be used as a decoding algorithm for \( \Lambda_c \).

Now our next target is to use this algorithm to decode a lattice constructed using Construction \( D \) with a set of nested turbo codes.

### C. A Multi-stage Turbo Lattice Decoder

To have an efficient decoding algorithm, the main idea is to employ the iterative turbo decoding structure in the decoding process of a turbo lattice. As it is shown in Section III every lattice constructed using Construction \( D \) benefits from a nice layered code structure. This building block consists of a set of nested linear block codes which is a set of nested turbo codes in turbo lattices. The goal is to use \( a \), the number of levels of the construction, serially matching iterative turbo decoding algorithms. We start by a received vector \( r \) and a decoding algorithm for \( C_a \), the finest turbo code used to build \( \Lambda_{TC} \). Let us consider that the decoded vector obtained from this goes as the input data (received vector \( r_{a-1} \)) to the \((a - 1)\)-th iterative turbo decoder associated to the turbo code \( C_{a-1} \). We do this subsequently until reaching \( r_1 \). For the induction suppose that we used \( 2^{a-1}r = r_a \) as the initial input to the whole process of decoding and succeeded obtaining are \( r_{a-1}, \ldots, r_1+1 \) as the outputs. As the output of \((a - i - 1)\)-th stage we get an \( n\)-dimensional vector \( r_i+1 \) which has to feed the \((i + 1)\)-th decoder associated to \( C_{i+1} \) in the \((a - i)\)-th stage. Thus, we have \( r_1 \) as the output and use it as the input of the \( i \)-th decoder in the next stage.

More explicitly this algorithm can be expressed as follows. One can restate (8) as

\[
\Lambda_0 = 2^{a-1}\Lambda_{TC} = C_a + 2C_{a-1} + \cdots + 2^{2-1}C_1 + 2^2(\mathbb{Z})^n. \tag{30}
\]

The above representation of \( \Lambda_{TC} \) states that every \( x \in \Lambda_{TC} \) can be represented by

\[
\begin{align*}
x &= x_1 + \frac{1}{2}x_2 + \cdots + \frac{1}{2^{a-1}}x_a + w, \quad \text{or} \\
2^{a-1}x &= x_a + 2x_{a-1} + \cdots + 2^{a-1}x_1 + 2^{a-1}w,
\end{align*}
\tag{31}
\]

where \( x_\ell \in C_\ell \) and \( w \in 2(\mathbb{Z})^n \), \( 1 \leq \ell \leq a \). Use \( 2^{a-1}r = r_a \) as the input to a decoder associated with the lattice \( \Lambda_a = 2(\mathbb{Z})^n + C_a \) and obtain \( x_a \). Then use \( r_{a-1} = \frac{r_a - x_a}{2} \) as the input to the decoder associated with the lattice \( \Lambda_{a-1} = 2(\mathbb{Z})^n + C_{a-1} \) and get \( x_{a-1} \) and so on. Therefore, we get the sequence \( x_a, \ldots, x_1, x_0 \) of decoded lattice points in \( \Lambda_a, \Lambda_{a-1}, \ldots, \Lambda_1 \), respectively. Hence, \( y = x_a + 2x_{a-1} + \cdots + 2^{a-1}x_1 + 2^{a-1}w \) is the closest point in the lattice \( 2^{a-1}\Lambda_{TC} \) to the received vector \( 2^{a-1}r \). It means that \( \frac{y}{2} = \bar{y} \) is the closest lattice point of \( \Lambda_{TC} \) to the received vector \( r \).

#### Decoding Algorithm for Turbo Lattices

**Input:** \( r \) an \( n \)-dimensional vector in \( \mathbb{R}^n \).

**Output:** a closest vector \( \bar{x} \) to \( r \) in \( \Lambda_{TC} \).

- **Step 1** Put \( r_a = 2^{a-1}r \).
- **Step 2**
  - for \( \ell = a \) downto 1 do
    - Decode \( r_\ell \) to the closest point \( x_\ell \in \Lambda_\ell = C_\ell + 2(\mathbb{Z})^n \).
    - Compute \( r_{\ell-1} = \frac{r_\ell - x_\ell}{2} \).
- **Step 3** Evaluate \( \bar{x} = x_a + 2x_{a-1} + \cdots + 2^{a-1}x_1 + 2^{a-1}w \).

The next theorem shows that the above algorithm can find the closest lattice point of \( \Lambda_{TC} \) to the received vector \( r \) when the points of \( \Lambda_{TC} \) are sent over an unconstrained AWGN channel with noise variance \( \sigma^2 \).

**Theorem 20.** Given an \( n \)-tuple \( r \), if there is a point \( \bar{x} \) in \( \Lambda_{TC} \) such that \( ||r - \bar{x}||^2 < d_{\min}^2(\Lambda_{TC})/4 \), then the algorithm decodes \( r \) to \( \bar{x} \).

The proof is given in the Appendix.

Similar expressions, algorithms and theorems can be found in [13]. In fact Forney in [13] uses a code formula along with a multi-stage decoding algorithm to solve a CVP for a lattice based on Construction \( B \) [13][13]. We note that Construction \( C \) is equal to Construction \( D \) when the set of underlying codes used to construct a lattice have an extra nested property.

### D. Decoding Complexity

Since the operations for computing the nearest odd and even integer numbers close to the components of a received vector \( r \) are negligible, the decoding complexity of a lattice \( \Lambda_\ell = C_\ell + 2(\mathbb{Z})^n \) constructed using Construction \( A \) is equal to the complexity of decoding the turbo code \( C_\ell \) via an iterative turbo decoder. As shown a turbo lattice decoder uses exactly \( a \) subsequent and successive turbo decoder algorithms for \( \Lambda_\ell \), \( 1 \leq \ell \leq a \), thus the overall decoding complexity of the proposed turbo lattice decoding algorithm can not exceed \( a \) times the decoding complexity of an iterative turbo decoder.

### VI. Performance Analysis and Simulation Results

In this section some fundamental properties of turbo lattices such as minimum distance, coding gain and kissing number are studied. It gives us the possibilities to obtain information from the underlying turbo codes in order to theoretically check the efficiency of the constructed turbo lattices.

#### A. Analysis of Turbo Lattices

Now we look at the turbo lattice \( \Lambda_{TC} \) closer. The next theorem provides some formulas and an inequality about
performance measures of a turbo lattice $\Lambda_{TC}$ constructed following Construction D.

**Theorem 21.** Let $\Lambda_{TC}$ be a turbo lattice constructed following Construction D with nested turbo codes

$$TC = TC_1 \supseteq TC_2 \supseteq \cdots \supseteq TC_a$$

as its underlying linear block codes with parameters $[n, k, d_{\min}^{(l)}]$ and rate $R_i = \frac{k_r}{t_r}$ for $1 \leq \ell \leq a$. Then the minimum distance of $\Lambda_{TC}$ satisfies

$$d^2_{\min}(\Lambda_{TC}) = \min_{1 \leq \ell \leq a} \left\{ 4, \frac{d_{\min}^{(\ell)}}{4^{t-1}} \right\}. \quad (32)$$

The coding gain is

$$\gamma(\Lambda_{TC}) = 4^{(\sum_{i=1}^{a} R_i) - 1} \min_{1 \leq \ell \leq a} \left\{ 4, \frac{d_{\min}^{(\ell)}}{4^{t-1}} \right\}, \quad (33)$$

and for the kissing number of $\Lambda_{TC}$ we have

$$\tau^*(\Lambda_{TC}) \leq 2 + \sum_{1 \leq \ell \leq a} \frac{2d_{\min}^{(\ell)}}{\min \{ A^{(\ell)}_{d_{\min}^{(\ell)}}, n \}} \quad (34)$$

where $A^{(\ell)}_{d_{\min}^{(\ell)}}$ denotes the number of codewords in $C_{\ell}$ with minimum weight $d_{\min}^{(\ell)}$.

The proof is given in the Appendix.

Considering the above theorem, if one increases tail-biting length $L$ and factors $(k_a, \ldots, k_1)$ in the construction process of the underlying nested turbo codes by a factor of $t$, then $d_{\min}^{(\ell)}$'s are also increased due to interleaveer gain and spectral thinning (because recursive systematic convolutional codes are used as our constituent encoders). Therefore, there clearly exists an intensification on the minimum distance of $\Lambda_{TC}$; see Equation (32).

As it is shown in Theorem 12, the rates of the underlying nested turbo codes remain unchanged if we increase $L$ and the parameters $k_a$. However $d_{\min}^{(\ell)}(\Lambda_{TC})$ at the right hand side of (33) guarantees the growth in the coding gain of $\Lambda_{TC}$. Hence, by increasing the interleaveer size, larger minimum distance and higher coding gain are reached.

It has to be noted that Construction D has been chosen because it has a good potential to be sphere-bound-achieving and capacity-achieving due to their better coding gain and minimum distance compare with Construction A. Construction A suffers from a lack of nested structure, and this is why lattices constructed based on this construction have poor coding gain and minimum distance.

If the interleaveer size $k$ and its relative parameters $(k_a, \ldots, k_1)$ are increased by a factor of $t$, then the dimension of the constructed lattice $\Lambda_{TC}$ increases by the same factor. As mentioned, by this modification and due to the interleaveer gain and spectral thinning, the minimum distance of nested turbo codes, $d_{\min}^{(\ell)}$, increase slightly or remain unchanged. This increase can not be faster than logarithmically with the code length $n$ [7]. Thus, in (34), $2d_{\min}^{(\ell)}$ decreases. Also the number of minimum weight codewords in these turbo codes decreases by a factor of $t$. Hence, our upper bound (18) for the normalized kissing number of $\Lambda_{TC}$ decreases.

Now, let us put all the above discussion together. We can control (increasing of) minimum distance, (increasing of) coding gain and (decreasing of) kissing number of the constructed turbo lattice $\Lambda_{TC}$ only by setting up a good interleaveer of size $k$ and adjusting its size. Furthermore, if one produces a set of nested turbo codes

$$TC = TC_1 \supseteq TC_2 \supseteq \cdots \supseteq TC_a$$

where $d_{\min}^{(\ell)} > \frac{4^t}{\beta}$ such that $\beta = 1$ or 2, then we get the following bounds

$$d_{\min}(\Lambda_{TC}) \geq \frac{4}{\beta}, \quad \gamma(\Lambda_{TC}) \geq \frac{4^{(\sum_{i=1}^{a} R_i)}}{\beta},$$

and

$$\tau(\Lambda_{TC}) \leq 2n \quad \text{or} \quad \tau^*(\Lambda_{TC}) \leq 2.$$

Therefore, in this case, we have

$$P_e(\lambda, \sigma^2) \leq 2\text{erfc} \left( \frac{\pi e \sum_{i=1}^{a} R_i}{4 \beta \alpha^2} \right). \quad (35)$$

This means that, if we use strong enough underlying turbo codes such that $d_{\min}^{(\ell)} > \frac{4^t}{\beta}$ for $1 \leq \ell \leq a$, then for $\alpha^2$ close to 1 we have

$$P_e(\lambda, \sigma^2) \leq 2\text{erfc} \left( \frac{\pi e \sum_{i=1}^{a} R_i}{4 \beta} \right) \leq 2e^{-\frac{\pi e \sum_{i=1}^{a} R_i}{4 \beta}},$$

where the last inequality uses the fact that $\text{erfc}(w) \leq e^{-w^2}$. Therefore, in order to have a reliable communication the following condition must hold

$$4e^{-\frac{\pi e \sum_{i=1}^{a} R_i}{4 \beta}} \leq 10^{-5},$$

which is equivalent to $\sum_{i=1}^{a} R_i \geq 1.29$. It can be seen that (35) is a generalization of an expression for normalized probability of error for a turbo lattice formed following construction A with underlying turbo code TC and with minimum distance greater than 4. This is true because selecting $a = 1$, Construction A can be obtained.

The above discussion about turbo lattices shows that if a set of nested turbo codes along with Construction D have been used to obtain a turbo lattice, then it has an underlying structure. This setting results in larger minimum distance, better coding gain and lower kissing number compare with the turbo lattices which come from parallel concatenated of terminated recursive convolutional codes and Construction A. It means that Construction D turbo lattices have a good chance to achieve channel capacity, as well as sphere-bound due to their efficient parameters such as minimum distance, coding gain and kissing number. However, this geometrical properties of an a level Construction D turbo lattices make their decoding algorithm more complex.
B. Simulation Results for Performance of Turbo Lattices

Guidelines to choose tail-biting convolutional codes that are especially suited for parallel concatenated schemes are given in [33]. The authors of [33] also tabulate tail-biting convolutional codes of different rate and length. The minimum distance of their associated turbo codes are also provided.

Assume that a tail-biting version of a systematic recursive convolutional code of rate $\frac{2}{3}$ with memory 3 and generator matrix

\[
G_1 = \begin{pmatrix}
1 & 0 & \frac{1 + x + x^2 + x^3}{1 + x + x^2 + x^3} \\
0 & 1 & \frac{1 + x + x^2 + x^3}{1 + x + x^2 + x^3}
\end{pmatrix}
\]

is used to form a nested turbo code. The resulting turbo code has rate $R_1 = \frac{1}{2}$ and based on [33], it has minimum distance $d_{\min}^{(1)} = 13$ for block information bits of length 400. Now let us define $TC_2$. To this end, consider only the first row of the generator matrix of $TC_1$. Therefore, the component encoders of $TC_2$ have generator matrices

\[
G_2 = \begin{pmatrix}
1 & 0 & \frac{1 + x + x^2 + x^3}{1 + x + x^2 + x^3} \\
0 & 1 & \frac{1 + x + x^2 + x^3}{1 + x + x^2 + x^3}
\end{pmatrix}
\]

The zero bits can be punctured out, so we get

\[
G_2 = \begin{pmatrix}
1 & 0 & \frac{1 + x + x^2 + x^3}{1 + x + x^2 + x^3} \\
0 & 1 & \frac{1 + x + x^2 + x^3}{1 + x + x^2 + x^3}
\end{pmatrix}
\]

A block turbo code which uses $G_2$ as its constituent codes has rate $R_2 = \frac{1}{3}$ and according to the information in [33], the minimum distance of this code is $d_{\min}^{(2)} = 28$ for information block length of 576. For instance suppose that a block of information bits of size 1000 is used. Since $TC_1$ is a rate-$\frac{1}{2}$ block turbo code, the lattice points are in $\mathbb{R}^{2000}$. Therefore, a square generator matrix of size 2000 for this turbo lattice $G_{TL}$ can be formed following the approach in Example 18. Hence, $G_{TL}$ is

\[
\begin{bmatrix}
\frac{1}{2}I_{576} & 0 & \frac{1}{2}P_{1,1}F_{1,3} \\
0 & I_{324} & \frac{1}{2}P_{2,1}F_{2,3} \\
0 & 0 & 2I_{500}
\end{bmatrix},
\]

where

\[
P = \begin{pmatrix}
P_{1,1} & 0 \\
0 & P_{2,2}
\end{pmatrix}
\]

of size 1000 is a 576-nested interleaver. In other words $P$ is an interleaver for $TC_1$ and $P_{1,1}$ is an interleaver of size 576 for $TC_2$. Now the fundamental parameters of this turbo lattice $\Lambda_{TC}$ constructed with 2 levels of Construction $D$ can be found. Since $d_{\min}^{(1)} = 13$ and $d_{\min}^{(2)} = 28$, Theorem [21] implies that

\[
d_{\min}^2(\Lambda_{TC}) = \min_{1 \leq \ell \leq 2} \left\{ \frac{d_{\min}^{(1)}}{4^{\ell - 1}}, \frac{d_{\min}^{(2)}}{4^{\ell - 1}} \right\}
\]

\[
= \min_{1 \leq \ell \leq 2} \left\{ \frac{13}{4^{\ell - 1}}, \frac{28}{4^{\ell - 1}} \right\} = 4.
\]
The symbol error rate (SER) is showed in Fig. 2. This curve is a shifted version of Fig. 1 to the left by $5\,\text{dB}$. Also the kissing number of $\Lambda_{TC}$ is a lower bound, hence we have at least $4\,\text{dB}$ coding gain versus uncoded system. It means that, interleavers of larger sizes or interleavers of better quality may result in higher coding gain and being closer to the sphere-bound than $1\,\text{dB}$.

Since $d_{\min}^{(1)} > 4$ and $d_{\min}^{(1)} > 4^2$, the summation in the above inequality disappears and we get $\tau(\Lambda_{TC}) \leq 4000$ or equivalently $\gamma(\Lambda_{TC}) \leq 2\,\text{dB}$. In the worst scenario, this turbo lattice achieves an SER of $10^{-5}$ at an $\alpha^2$ of $1.75\,\text{dB}$ for size 2000. The obtained coding gain is a lower bound, hence we have at least $6.75 - 1.75 = 5\,\text{dB}$ coding gain versus uncoded system. It means that, interleavers of larger sizes or interleavers of better quality may result in higher coding gain and being closer to the sphere-bound than $1.75\,\text{dB}$.

According to the discussion and example described above, we can take advantage from a wide range of aspects of these lattices. To be more specific, these turbo lattices are generated by Construction $D$ using two nested block turbo codes. Their underlying codes are two tail-biting recursive convolutional codes. Thus, this class provides an appropriate link between two approaches of block and convolutional codes. The tail-biting method gives us the opportunity to combine profits of recursive convolutional codes (such as memory) with the advantages of block codes. It is worth pointing this remark out that the nested property of turbo codes induces higher coding gain (see (33)). Also, excellent performance of parallel concatenating systematic feed-back convolutional codes imply efficient turbo lattices with great fundamental parameters.

All simulation results presented are achieved using an AWGN channel, systematic recursive convolutional codes in the parallel concatenated scheme, and iterative turbo lattice decoder all discussed earlier. Indeed, we investigate turbo lattice designed using two identical terminated convolutional codes with generator matrix $G_A(x)$. Turbo lattices of different lengths are examined. Furthermore, the performance of these turbo lattices are evaluated using BCJR algorithms [12, 20] as constituent decoders for the iterative turbo decoder. Moreover, $S$-random interleavers of sizes $2^5$, $7^3$ and $15^3$ such that $S$ equals to $3, 10$ and $30$ have been used respectively. These results in turbo lattices of dimensions $(2^5 + 2) \times 3 = 102, (7^3 + 2) \times 3 = 1035$ and $(15^3 + 2) \times 3 = 10131$. The number of iterations for the iterative turbo decoder is fixed. It is equal to ten in all cases. Fig. 3 shows a comparison between turbo lattices formed with turbo codes of different lengths. These turbo lattices achieve an SER of $10^{-5}$ at $\alpha^2 = 2.75\,\text{dB}$ for size 102, an $\alpha^2 = 1.25\,\text{dB}$ for frame length 1035. Also an
SER of $10^{-5}$ is attained at an $\alpha^2 = 0.5$ dB for size 10131.

In the following we compare these results for turbo lattices with other newly introduced lattices including LDPC lattices [22] and LDLC lattices [27]. The comparison is presented in Table I in Fig. 3 and for turbo lattices of sizes $n = 102, 1035, 10131$, at SER of $10^{-5}$, we achieve $\alpha^2 = 2.75, 1.25$ and .5 dB away from capacity while for $n = 100, 1000, 10000, 100000$, LDLC lattices [27] can work as close as 3.7, 1.5, 0.8 and 0.6 dB from capacity, respectively. Thus, the excellent performance of turbo lattices when compared with other lattices is clear.

### VII. Conclusion and Further Research Topics

Using Construction $D$ method for lattices along with a set of newly introduced nested turbo codes, the concept of turbo lattices is established. To this end, tail-biting and terminated convolutional codes are concatenated in parallel. This parallel concatenation to induce turbo codes was supported with nested $S$-random interleavers. This gives us the possibility to combine the characteristics of convolutional codes and block codes to produce good turbo lattices. The fundamental parameters of turbo lattices for investigating the error performance are provided. This includes minimum distance, coding gain, kissing number and an upper bound on the probability of error. Finally, our experimental results show excellent performances for turbo lattices as expected by the theoretical results. More precisely, for example at SER of $10^{-5}$ and for $n = 10131$ we can work as close as $\alpha^2 = 0.5$ dB from capacity.

Interleavers play an important role in turbo codes [32]. Consequently, a key point of turbo lattices are also interleavers. They should have random-like properties and avoid some specific patterns to induce a good minimum distance. For a turbo code ensemble using uniform interleaver technique, one can show turbo codes are good in the following sense [18]. That is, the minimum distance of parallel concatenated codes with $b$ parallel branches and recursive component codes grows as $n \sim b^{2} [19]$. Also the average maximum-likelihood decoder block error probability approaches zero, at least as fast as $n^{-b+2}$ [18]. Since increase in coding gain and decrease in normalized kissing number is completely and straightforwardly related to the increase of minimum distance, it is reasonable to use more than 2 branches. Therefore, analyzing other factors and parameters of $b$-branches turbo lattices such as sphere packing, covering and quantization problem is also of great interest. Another interesting research problem is to find the error performance of turbo lattices designed by other types of interleavers including deterministic interleavers [25] [32].

Since the performance of turbo lattices depends on the performance of their underlying codes, then search for other well-behaved turbo-like codes [1] would be interesting.

### Appendix

**Proofs**

Theorem 2 Let $c^i$ be a codeword with minimum weight in $C_L$ for $1 \leq \ell \leq a$. There exist $\beta_j^{(\ell)} \in \{0,1\}$ such that $c^i = \sum_{j=1}^{k_{\ell}} \beta_j^{(\ell)} c_j$. Since $\frac{1}{2^\ell} c^i$ is in the form of (6), it belongs to $\Lambda$. Thus, we have

$$d_{min}(\Lambda) \leq \left\| \frac{1}{2^\ell-1} c^i \right\|^{1/2} = \frac{1}{2^\ell-1} \sqrt{d_{min}^{(\ell)}}.$$ 

This means that $d_{min}(\Lambda) \leq \min_1^{\ell} \leq a \left\{ 2, \frac{1}{2^\ell-1} \sqrt{d_{min}^{(\ell)}} \right\}$. On the other hand the number 2 in this formula happens when $\beta_j^{(\ell)} = 0$ for all $j \in \ell$ and $z = (0,0,\ldots,0,2,0,\ldots,0)$. Now, we set $L_0 = (2Z)^n$ and

$$L_\ell = \left\{ x + \sum_{j=1}^{k_\ell} \beta_j^{(\ell)} \frac{c_j}{2^\ell-1} \right\}$$

where $\beta_j^{(\ell)} = 0$ or 1 and $x \in L_{\ell-1}$. Hence, $L_\ell = \Lambda$. It is easy to check that $L_\ell$ is a lattice. Let $0 \neq v$ be a vector of level $\ell$. If $\ell = 0$ then $\|v\| = 4$. If $\ell > 0$ then according to the definition of $L_\ell$ we can write $v = x + y$ where $x \in L_{\ell-1}$ and $y \in L_\ell$. The vector $y$ has at least $d_{min}^{(\ell)}$ components since it is in level $\ell$. It means that the norm of the vector $y$ and therefore the norm of the vector $v$ is at least $d_{min}^{(\ell)}$. Thus,

$$d_{min}(\Lambda) \geq \min_1^{\ell} \leq a \left\{ 2, \frac{1}{2^\ell-1} \sqrt{d_{min}^{(\ell)}} \right\}.$$ 

Theorem 4 The only points in $\Lambda$ that achieve $d_{min}(\Lambda)$ are the $2n$ points $\pm 2e_1$ for $1 \leq i \leq n$ where $e_i$ is the $i$th unit vector plus the points which are in $C_L$’s satisfying (9) for $1 \leq \ell \leq a$. In other words when we have $d_{min}^{(\ell)} = \frac{d^{(\ell)}}{2^\ell}$ for some $1 \leq \ell \leq a$, then $4 \geq d_{min}^{(\ell)}$ and the codewords with minimum weight in $C_L$ are the candidates to produce spheres which can touch the sphere with center 0 and radius $\frac{d_{min}^{(\ell)}}{2^\ell}$. Therefore, these points must be in $C_L$ with weight $d_{min}^{(\ell)}$ such that $d_{min}^{(\ell)} \leq 4^\ell$. It means that the kissing number of $\Lambda$ is upper bounded by

$$2n + \sum_{1 \leq \ell \leq a} 2 d_{min}^{(\ell)} A_d^{(\ell)}. \quad (36)$$

The coefficient $2 d_{min}^{(\ell)}$ appears since the nonzero entries of each lattice vector of $\Lambda$ can be positive or negative. We note that if $d_{min}^{(\ell)} > 4^\ell$, then the right hand side of (9) is equal to 2 and the summations in (10) and (36) disappear.

Theorem 5 The vector

$$(0,\ldots,0,1,0,\ldots,0,f_i,K+1(x),\ldots,f_i,N(x))$$

is in the code if and only if there exists a polynomial $t_i(x)$ such that

$$t_i(x) (0,\ldots,0,p_i(x),0,\ldots,0,q_i,K+1(x),\ldots,q_i,N(x)) \equiv$$
Let \( p_i(x) \) has an inverse \( p_i^{-1}(x) \) (mod \( x^L - 1 \)).

This happens if and only if every \( p_i(x) \) has an inverse \( p_i^{-1}(x) \) (mod \( x^L - 1 \)) for \( 1 \leq i \leq K \) and \( K + 1 \leq j \leq N \) because \( p_i^{-1}(x) = t_i(x) \) for \( 1 \leq i \leq K \).

**Proposition** Since the characteristic polynomial of matrix \( A \) is \( p(x) \), we conclude that \( p(A) = 0 \). We have that \( \text{det}(p(x), x^L + 1) = 1 \) if and only if there exist two polynomials \( r(x) \) and \( t(x) \) such that \( r(x)p(x) + t(x)(x^L + 1) = 1 \). Put \( x = A \), we get \( t(A)(x^L + I_m) = I_m \). Hence, we have \( \text{det}(A + I_m) \neq 0 \) if and only if \( (p(x), x^L + 1) = 1 \).

**Theorem 14** Let \( R_0^\ell = \frac{k_0^\ell}{n_\ell^L} \) denote the rate of the \( \ell \)th component of the nested turbo codes when we scale \( L \) and \( k_\ell \)'s by a factor of \( t \). Then, we have

\[
R_0^\ell = \frac{k_0^\ell}{n_\ell^L} = \frac{tk_\ell}{(2Lt)N - (Lt)K} = R_t.
\]

We observe that in this case the interleaver size is \( k_0^\ell = tk_\ell = (Lt)K \) and the interleaver \( \Pi \) is a \((k_0^\ell, \ldots, tk_\ell^\ell)\)-nested interleaver. Also this is true for the actual rate of our nested turbo codes. Suppose \( n_\ell^0, k_\ell^0 \) and \( R_0^\ell \) be as above, then

\[
R_0^\ell = \frac{k_\ell^0}{n_\ell^L - k_\ell^0 + k_\ell^0} = \frac{tk_\ell}{(2Lt)N - (Lt)K - tk + tk_\ell} = \frac{k_\ell^0}{n_\ell^L - k_\ell^0 + k_\ell^0} = R_t.
\]

**Theorem 14** First of all, for every \( \ell \), an interleaver \( \Pi_\ell \) constructed using \( k_\ell^0 \) along with \( \Pi_\ell, \ldots, \Pi_L \) has length \( k_\ell = k_\ell^0 + (k_\ell^0 - k_\ell^0) + \cdots + (k_\ell - k_\ell^0 + 1) \). In addition, if we append these \( a - \ell + 1 \) interleavers \( \Pi_\ell, \ldots, \Pi_L \) we get

\[
\Pi_\ell(i - k_\ell^0 + k_\ell^0) = k_\ell^0 + 1 \leq i \leq k_\ell^0, \\
\Pi_\ell(i - k_\ell^0 + k_\ell^0 - 1) = k_\ell^0 + 1 + i \leq k_\ell^0, \\
\vdots
\]

Then the interleaver \( \Pi_\ell \) acts as the following

\[
\begin{pmatrix}
1 & 2 & \cdots & k_\ell \\
\Pi_\ell(1) & \Pi_\ell(2) & \cdots & \Pi_\ell(k_\ell)
\end{pmatrix}
\]

where \( \Pi_\ell(i) = \Pi_\ell(i - k_\ell^0 + k_\ell^0) + k_\ell^0 \) is the minimum index that satisfies \( k_\ell^0 + 1 \leq i \leq k_\ell^0 \), it means that \( \Pi_\ell \) is a \((k_\ell^0, \ldots, k_\ell)\)-nested interleaver of size \( k_\ell \) for \( TC_\ell \) by Definition 10.

**Theorem 13** Based on the Theorem 14, it is clear that \( \Pi \) is a \((k_\ell^0, \ldots, k_\ell)\)-nested interleaver. Since \( \Pi_\ell(1) = 1 \), for every \( k_\ell^0 + 1 \leq r \leq k_\ell \) such that \( (r - k_\ell^0) - 1 < S_{\ell t} \) and based on the definition of an \( S_t \)-random interleaver, we get \( |\Pi_\ell(r - k_\ell^0 + 1)| = |\Pi_\ell(r - k_\ell^0 + 1) - 1| > S_{\ell t} \).

Hence, we have \( \Pi_\ell(r - k_\ell^0 + 1) > S_{\ell t} + 1 \), and

\[
\Pi_\ell(r) > k_\ell^0 + S_{\ell t} + 1.
\]
REFERENCES