On Dirac delta sequences and their generating functions

Quang A. Dang a, Matthias Ehrhardt b,∗

a Institute of Information Technology, Vietnamese Academy of Science and Technology (VAST), 18 Hoang Quoc Viet Road, Cau Giay, Hanoi, Viet Nam
b Lehrstuhl für Angewandte Mathematik und Numerische Analysis, Bergische Universität Wuppertal, Gaußstrasse 20, 42119 Wuppertal, Germany

1. Introduction

Recently, drawing from numerical simulation in quantum theory, Galapon [1] has posed the question of the existence of delta-convergent sequences that vanish at the support of the limit Dirac delta function (DDF) and constructed an example of sequences of this type. It is a sequence of even functions that do not have a compact support. Motivated by the question, in this note we develop some results concerning delta sequences and show more examples of delta sequences of the type with or without compact support and that are even or not even.

2. Generating functions of Dirac delta sequences

In order to design GDS, first we cite from [2, Section 3.3] a fundamental theorem, which will serve as a tool for constructing convergent DDS.

Theorem 1. Let α(x) be a nonnegative, locally integrable function in the n-dimensional space \( \mathbb{R}^n \) and

\[ \int_{\mathbb{R}^n} \alpha(x) \, dx = 1. \]  

Then the sequence of functions

\[ f_\epsilon(x) = \epsilon^{-n} \alpha(x/\epsilon) \]  

converges to the DDF \( \delta(x) \), \( \epsilon \to 0 \) (in the sense of generalized functions).
3. Delta sequences that vanish at the support of the DDF

Theorem 1 was originally stated as an exercise in [3] and later proved in [2]. Apparently, Galapon [1] was not aware of this theorem, and so he used another theorem from [4] to prove that his proposed sequence is indeed a DDS.

We shall call the function $\alpha(x)$ in Theorem 1 the generating function of the Dirac delta sequence $f_n(x)$. Sometimes instead of (2) it is convenient to use the sequence of functions $f_n(x) = e^{-\alpha(x)}$.

Below we list some generating functions and corresponding DDS functions. Alternatively, it can be convenient to use $F_m(x) = m \alpha(mx)$ for $m \to \infty$ instead.

### 2.1. Some generating functions of DDS without compact supports

The DDS $F_m(x)$ from A1 to A4 are given in [2] as examples and exercises, while A1 and A2 are given in [3] (see Table 1). The graphs of the above generating functions A1–A5 are depicted in Fig. 1.

### 2.2. Some generating functions of DDS with compact supports

The generating functions B0, B1, B5 are very popular (see e.g. [3, 2]); the functions B2–B4 are B-splines in [5] and B6 can be found in [6]. Notice that the function B4 is infinitely differentiable. The graphs of the above generating functions B1–B5 are depicted in Fig. 2.

<table>
<thead>
<tr>
<th>No.</th>
<th>$\alpha(x)$</th>
<th>$F_m(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{\pi} e^{-\frac{x^2}{4}}$</td>
<td>$\frac{1}{\pi} e^{-\frac{x^2}{4n}}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{\pi} e^{-\frac{x^2}{4}}$</td>
<td>$\frac{1}{\pi} e^{-\frac{x^2}{4n}}$</td>
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<td>$\frac{1}{\pi} e^{-\frac{x^2}{4}}$</td>
<td>$\frac{1}{\pi} e^{-\frac{x^2}{4n}}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{\pi} e^{-\frac{x^2}{4}}$</td>
<td>$\frac{1}{\pi} e^{-\frac{x^2}{4n}}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{\pi} e^{-\frac{x^2}{4}}$</td>
<td>$\frac{1}{\pi} e^{-\frac{x^2}{4n}}$</td>
</tr>
</tbody>
</table>

We remark that all the generating functions $\alpha(x)$ and $\beta(x)$ and the corresponding DDS have a maximum at $x = 0$ and are even functions, i.e., symmetric with respect to the $y$-axis. Below, in the next section, on the basis of Theorem 1 we construct DDS that vanish at $x = 0$, i.e., at the support of the DDF.

### 3. Delta sequences that vanish at the support of the DDF

In order to construct DDS that vanish at the support of the DDF we shall construct generating functions $\alpha(x)$ that have the above property, which we call the Galapon property. Using Theorem 1 it is easy to verify the following:
Theorem 2. If \( \alpha(x) \) is a generating function with the support \([-1, 1]\) that is continuous and even, then the function

\[
\beta(x) = \sum_{i=1}^{k} c_i a_i(x),
\]

\[
a_i(x) = \frac{(\alpha(x + 2i - 1) + \alpha(x - 2i + 1))}{2}, \quad c_i = \frac{2^{k-i}}{(2^k - 1)}
\]

is a generating function that has the Galapon property with a compact support. Furthermore, the function

\[
\beta(x) = \sum_{i=1}^{\infty} \frac{a_i(x)}{2^i}
\]

is a generating function with the Galapon property, without compact support.

Figs. 3–4 depict the DDS generated by the functions formed by (3) with \( k = 1 \) and \( k = 2 \) from the compactly supported functions B2 and B5. Notice that a DDS generated from the functions B2 is not smooth at \( x = 0 \) while a DDS generated from the functions B5 is infinitely differentiable at this point. From Theorem 2 we see that in principle it is possible to construct generating functions that are two-sided symmetric, have the Galapon property and do not have a compact support. But due to the construction they do not tend monotonically to zero.

Next we are interested in the design of the unsupported generating functions that tend monotonically to zero, which is not an easy task. Concerning this problem, we first remark that the DDS introduced by Galapon in [1]

\[
h_v(n, x) = \frac{1}{2^{2n+1} \Gamma(n + 1/2)} x^{n+1/2} e^{-vx^2/4}
\]
tending to DDF as $\nu \to \infty$ for any fixed $n$ has the generating function $G1$:

$$\alpha_n(x) = \frac{1}{2^{2n+1} \Gamma(n + 1/2)} x^{2n} e^{-x^2/4}.$$  

(5)

Besides this generating function of DDS due to the formula

$$\int_0^\infty x^n e^{-x} dx = n!$$  

(6)

[2, formula (14), p. 383] we conclude that

$$\alpha_n(x) = \frac{x^{2n} e^{-|x|}}{2(2n)!}$$

(7)

is also a generating function of a DDS with the Galapon property, which we denote by $G2$. By making the change of variables $x = y^2$ in the integral (6) we obtain

$$\int_0^\infty y^{2n+1} e^{-y^2} dy = \frac{n!}{2}.$$
Therefore, the function \( \alpha_n(x) = 2x^{2n+1}e^{-x^2/n} / n! \) is a generating function of a DDS with the Galapon property. The DDS generated by the functions G1 and G2 for \( n = 1 \) are given in Fig. 5.

Further, from (6), by making the change of variables \( x = 1/y \) it is easy to obtain

\[
\int_{0}^{\infty} \frac{y^{-2n}e^{-1/y}}{(2(n-1))!} \, dx = 1.
\]

Hence, the function

\[
\alpha(x) = \begin{cases} 
    x^{-2n}e^{-1/|x|}/2(2(n-1))!, & x \neq 0 \\
    0, & x = 0
\end{cases}
\]

(8)
is a generating function of a DDS with the Galapon property.

We point out that all the above DDS are even functions. However, it is possible to construct DDS that are not even functions. To this end, we observe that if \( \alpha(x) \) is an even generating function then the function

\[
\gamma(x) = \begin{cases} 
    2\alpha(x), & x > 0 \\
    0, & x \leq 0
\end{cases}
\]

(9)
also generates a DDS that is not even and vanishes at \( x = 0 \).

An example of such a generating function is

\[
\gamma(x) = \begin{cases} 
    x^{-2}e^{-1/x}, & x > 0 \\
    0, & x \leq 0
\end{cases}
\]

which is obtained from (8) for \( n = 1 \). This function can also be found in [7, p. 14] and we refer to this generating function as G3. The DDS generated by the function (9) for B2 and by G3 are depicted in Fig. 6.

4. Conclusion

We introduced the concept of the generating function of DDS and used it to systematize some known DDS with or without compact support. On the basis of this concept we propose a method for designing generating functions of DDS that vanish at the support of the DDF and are symmetric or not symmetric.

The solution of differential equations containing singular source terms attracts the attention of many researchers (see e.g. [8] and the bibliography therein). The central problem is the appropriate approximation of the delta function, which is usually called its regularization. The question of how to choose a method of regularization of the delta function is essential for reaching a high accuracy.

In the future we will devote our attention to the regularization of the delta function describing the location and the time of an accident of oil pollution in order to justify and improve on our recent results given in [9].
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References