GEOMETRICAL PLENOPTIC SAMPLING

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ABSTRACT

In this paper, we present a general framework for analysis of plenoptic sampling by investigating the spectral analysis of plenoptic imaging. The proposed framework provides a unified representation that generalizes several existing methods for plenoptic sampling. The plenoptic sampling theory presented extends the study of plenoptic imaging to parallel cameras that are not restricted to lie on a plane corresponding to a camera array. Furthermore, we conduct an analysis of plenoptic sampling for unstructured camera systems, in which the cameras can reside at arbitrary locations and orientations. Finally, we establish necessary and sufficient conditions for unoccluded imaging, which is a critical precondition for plenoptic sampling. These conditions are shown to introduce constraints on the position and curvature of the surface of the scene as well as the position and orientation of the cameras.

Index Terms— Sampling, Plenoptics, Occlusion.

1. INTRODUCTION

Plenoptic function [1], \( p(\theta, \phi, \lambda, t, V_x, V_y, V_z) \), is an idealized 7D function used to represent the image of a scene from anywhere at any time. With the plenoptics, new images of scenes can be generated independent of the scene geometry, and it is one form of image-based rendering (IBR).

Various models have been proposed for projecting the plenoptic samples, such as the unit sphere, the cube, and the cylinder. The concepts of light field [2] and lumigraph [3] parameterize the plenoptic function into 4D by placing the object into the double bounding boxes. Then Chai et al. [4] conduct a spectral analysis of light field signals, and conclude that the spectral support of a light field signal is bounded by the minimum and maximum depths only. This study is based on the assumption that all cameras can see all points on the object. Zhang et al. [5] release this assumption by applying the surface plenoptic function and mapping images to the object, and achieve the same conclusion. They also show that the constant depth in concentric mosaic is very similar to the constant depth in light field by using polar coordinate and linear approximation. But the mapping is usually very complex, and even the scene on a tilted line has not be solved analytical. Lin et al. [6] show that the minimum sampling rate is also related with the depth variation of the scene. This is a geometric approach and based on bilinear interpolation.

Although both spectral and geometric analysis have been addressed, the spectral properties of plenoptic data from cameras with arbitrary position and orientation remain unclear. In this paper, we employ the method in [4] and extend their study to unstructured cameras. We also specify the unoccluded image conditions of the surface given the camera array. The remainder of this paper is organized as follows: In Section 2, we firstly focus on spectral analysis of arbitrary camera positions, and then extend to arbitrary orientations. In Section 3, we specify the curvature constraints because of the unoccluded image condition. Finally a conclusion is given in Section 4.

2. SPECTRAL ANALYSIS

In Fig. 1, \((t_o, z_o)\) is an object point, \(f\) is the focal length of the camera, and \(v\) indicates the projected point on images. By using pinhole camera model and local coordinate system, we can see that the coordinate of the projection point is always related to the angle between the camera’s main axis and the connection line of the object point and the camera, i.e., \((v, f) = (f \tan \theta, f)\). This conclusion can be directly extended to 3D case, thus only a 2D case is considered in the following for simplicity.

![Fig. 1. Projection with pinhole camera model.](image)

2.1. Parallel Cameras

A camera array is considered in [4], where cameras with same orientations are arranged on a single plane. We discuss the case that cameras have various depth to the object in this subsection. As can be seen from Fig. 2, we will always set Camera 1 as the reference, and the position of Cameras 2 can be computed as

\[
\begin{align*}
    t_2 & = z_o \tan \theta_1 - (z_o - z_2) \tan \theta_2 \\
    & = \frac{d_2}{d_1} \tan \theta_1 - \frac{f}{d_1} t_2 ,
\end{align*}
\]

where \(d\) represents depth. Accordingly we have,

\[
f \tan \theta_1 = \frac{d_2}{d_1} f \tan \theta_2 + \frac{f}{d_1} t_2 .
\]

Also if Lambertian surfaces are assumed, the light ray \(l(v, t)\) from an object point will be the same to an observer regardless of
Fig. 2. Cameras with arbitrary position.

the observer’s angle of view. Consequently,  
\[ l(v_2, t_2) = l(f \tan \theta_2, t_2) \]
\[ = l(v_1, 0) = l(f \tan \theta_1, 0) \]
\[ = l(d_2 \frac{f \tan \theta_2 + f}{d_1}, 0) \]
\[ = l(d_2 \frac{f \tan \theta_1 + f}{d_1}, 0) . \]  
(3)

The Fourier transform of the plenoptic signal is
\[ L(\Omega_v, \Omega_t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l(v, t) e^{-j(\Omega_v v + \Omega_t t)} dv dt \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l(v, 0) e^{-j(\Omega_v v + \Omega_t t)} dv dt \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l\left(\frac{d}{d_1} v, t\right) e^{-j(\Omega_v v + \Omega_t t)} dv dt \]
\[ = \int_{-\infty}^{\infty} l\left(\frac{d}{d_1} v, 0\right) e^{-j\Omega_v v} dv \int_{-\infty}^{\infty} e^{-j(\Omega_t + \frac{f}{d_1}) t} dt \]
\[ = 2\pi \left(\frac{d_1}{d}\right) L'(\frac{d_1}{d} \Omega_v) \delta(\Omega_t - \frac{f}{d_1} \Omega_v) , \]  
(4)

where
\[ L'(\Omega_v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} l(v, 0) e^{-j\Omega_v v} dv . \]  
(5)

Fig. 3. Frequency support for plenoptic data.

In this paper we assume that the spectrum is mainly bounded by the maximum and minimum depth. [4] also suggest to use \( d_{opt} = 2/(1/d_{min} + 1/d_{max}) \) as the best depth to render the scene.

If we set \( d = d_1 \), i.e. all cameras are in the same depth, Eq. (4) becomes
\[ L(\Omega_v, \Omega_t) = 2\pi L'(\Omega_v) \delta(\Omega_t - \frac{f}{d_1} \Omega_v) . \]  
(6)

Note in [4], it is \( L(\Omega_v, \Omega_t) = 2\pi L'(\Omega_v) \delta(\Omega_t + \frac{f}{d_1} \Omega_v) \), because they use the absolute values. Eq. (6) is the same as [5]. If we only take geometry information into consideration, we will have the following theorem.

Theorem 1. Plenoptic signals taken by parallel cameras will be bandlimited.

2.2. Unstructured Cameras

2.2.1. General Results

As shown in Fig. 2 and Fig. 4, we add an arbitrary rotation to the arbitrarily located camera to get
\[ l(v_3, t_3) = l(f \tan \theta_3, t_3) = l(f \tan(\theta_2 - \alpha), t_3) . \]  
(7)

According to
\[ \tan \theta_2 = \frac{\tan(\theta_2 - \alpha) + \tan \alpha}{1 - \tan \alpha \tan(\theta_2 - \alpha)} = \frac{\tan \theta_3 + \tan \alpha}{1 - \tan \alpha \tan \theta_3} , \]  
(8)

and Eq. (2), we have
\[ f \tan \theta_1 = \frac{d_3}{d_1} f \tan \theta_2 + \frac{f}{d_1} t_3 \]
\[ = \left(\frac{d_3}{d_1} f \tan \theta_3 + f \tan \alpha\right) + \frac{f}{d_1} t_3 . \]  
(9)
Similarly as before, we obtain
\[ l(v_3, t_3) = l(f \tan \theta_3, t_3) = \frac{d_3}{d_1} v_3 + f \tan \alpha + f(t_3, 0) \]
\[ = l\left(\frac{d_3}{d_1} v_3 + f \tan \alpha, t_3\right) + f(t_3, 0) \]
\[ = l\left(\frac{d_3}{d_1} v_3 + \frac{f^2 a}{d_1} + \frac{1}{a} \frac{1}{v_3} + f(t_3, 0) \right) \]
\[ = -\frac{d_3}{d_1} a + \frac{d_3}{d_1} a \cdot \frac{1}{a} \frac{1}{v_3} + f(t_3, 0) \]  \hspace{0.5cm} (10)

where \( a = \tan \alpha / f \). Also in the frequency domain, we get
\[
\begin{align*}
L(\Omega_0, \Omega_0) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l(v, t) e^{-j(\Omega_0 v + \Omega_0 t)} dv \, dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( -\frac{d_3}{d_1} a + \frac{d_3}{d_1} a \cdot \frac{1}{a} \frac{1}{v_3} + f(t_3, 0) \right) e^{-j(\Omega_0 v + \Omega_0 t)} dv \, dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( -\frac{d_3}{d_1} a + \frac{d_3}{d_1} a \cdot \frac{1}{a} \frac{1}{v_3} + f(t_3, 0) \right) e^{-j(\Omega_0 v + \Omega_0 t)} dv \, dt
\end{align*}
\]
\[
\begin{align*}
&= 2\pi \left( \frac{d_1}{f} \int \int \frac{d_1}{d_1} \Omega_0 e^{-j \frac{d_1}{d_1} \Omega_0} F[e^{-j \frac{d_1}{d_1} \Omega_0}] \right) \\
&= 2\pi \left( \frac{d_1}{f} \int \int \frac{d_1}{d_1} \Omega_0 e^{-j \frac{d_1}{d_1} \Omega_0} |r| e^{-j \frac{d_1}{d_1} \Omega_0} G(r \Omega_0) \right) \\
&= 2\pi \left( \frac{d_1}{f} \int \int \frac{d_1}{d_1} \Omega_0 |r| e^{-j \frac{d_1}{d_1} \Omega_0} G(r \Omega_0) \right),
\end{align*}
\]  \hspace{0.5cm} (11)

where \( F[\cdot] \) represents the Fourier transform function, and
\[
r = -\frac{d_1}{f} a^2 + 1 \Omega_0,
\]  \hspace{0.5cm} (12)

\[
G(\Omega_0) = \frac{\Omega_0}{2\sqrt{\Omega_0} + |\Omega_0|}
\]
\[
= \sqrt{2\pi d_1} \Omega_0 - j \frac{\Omega_0}{2\sqrt{\Omega_0} + |\Omega_0|}
\]
\[
\{ 4(\Omega_0 - |\Omega_0|)B_{2K}[1, 2 \sqrt{\Omega_0}] \\
- \pi \Omega_0 (H_{2K}[1, 2 \sqrt{\Omega_0}] + H_{2K}[1, 2 \sqrt{\Omega_0}]) \}
\]
\[
= \left\{ \begin{array}{ll}
-\sqrt{\frac{2\pi d_1}{\Omega_0}} B[1, 2 \sqrt{\Omega_0}] & \Omega_0 > 0 \\
0 & \Omega_0 < 0
\end{array} \right.,
\]  \hspace{0.5cm} (13)

\[
B[1, 2 \sqrt{\Omega_0}] = \frac{1}{2} H_{2K}[1, 2 \sqrt{\Omega_0}] + H_{2K}[1, 2 \sqrt{\Omega_0}].
\]  \hspace{0.5cm} (14)

We denote \( B[n, x] \) the Bessel function, \( B_{2K}[n, x] \) is the Modified Bessel function, and \( H_{2K}[1, 2 \sqrt{\Omega_0}] \) and \( H_{2K}[1, 2 \sqrt{\Omega_0}] \) are Hankel functions [7]. Eq. (13) with \( \Omega_0 > 0 \) is plotted in Fig. 5. Obviously, Eq. (11) is not bandlimited. Hence we have the following theorem.

**Theorem 2.** Plenoptic signals taken by unparallel cameras will not be bandlimited.

### 2.2.2. Limited Rotation and FOV

Eq. (8) can be linearized as
\[
\tan \theta_2 \approx \tan(\theta_2 - \alpha) + \tan \alpha = \tan \theta_3 + \tan \alpha,
\]  \hspace{0.5cm} (15)

which requires both FOV (field of view) of cameras and rotation \( \alpha \) to be small. If FOV is \((-20^\circ, 20^\circ)\), there is 4.1% maximum error. Accordingly, we have
\[
\begin{align*}
& f \tan \theta_1 = \frac{d_3}{d_1} f \tan \theta_2 + f \frac{d_3}{d_1} t_3 \\\n& = \frac{d_3}{d_1} (f \tan \theta_3 + f \tan \alpha) + f \frac{d_3}{d_1} t_3,
\end{align*}
\]  \hspace{0.5cm} (16)

\[
\begin{align*}
l(v_3, t_3) &= l(f \tan \theta_3, t_3) = l(v_1, 0) = l(f \tan \theta_1, 0) \\
&= l\left(\frac{d_3}{d_1} (f \tan \theta_3 + f \tan \alpha) + f \frac{d_3}{d_1} t_3, 0\right) \cdot (17)
\end{align*}
\]

Similar to Section 2.1, we have
\[
\begin{align*}
L(\Omega_0, \Omega_0) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l(v, t) e^{-j(\Omega_0 v + \Omega_0 t)} dv \, dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l(v, t) e^{-j(\Omega_0 v + \Omega_0 t)} dv \, dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l\left(\frac{d_1}{f} (v + f \tan \alpha), 0\right) e^{-j(\Omega_0 v + \Omega_0 t)} dv \, dt \\
&= \int_{-\infty}^{\infty} l\left(\frac{d_1}{f} (v + f \tan \alpha), 0\right) e^{-j(\Omega_0 v + \Omega_0 t)} dv \, dt \\
&= 2\pi \left( \frac{d_1}{f} \int \int \frac{d_1}{d_1} \Omega_0 \delta(\Omega_0 - f \frac{d_1}{d_1} \Omega_0) \right).
\end{align*}
\]  \hspace{0.5cm} (18)

If \( \alpha = 0 \), Eq. (18) becomes Eq. (4).

**Theorem 3.** Plenoptic data taken by unparallel cameras with limited FOV and rotations can be approximated to be bandlimited.

### 3. UNOCCULDED IMAGE CONSTRAINTS

The discuss in Section 2 is based on the assumption that all cameras can see all points. We will clarify the unoccluded image conditions in this section with an example of camera array, but the results can be easily extend to the case with parallel cameras. With an assumption of known object geometry, we have derived the curvature constraints in three situations, where the surface function is differentiable, continuous, or discontinuous, but only the example of differentiable surfaces is presented here.

![Fig. 5. Plot of G(\Omega_0)](image-url)
In Fig. 6, we assume that the length of the camera line is $2L$, and set the middle point in the camera line as the origin of the coordinate system. We suppose the surface curve is

$$\begin{align*}
  t &= f(x) \\
  z &= g(x) 
\end{align*}$$

(19)

Then the tangent passing through $(f(x_1), g(x_1))$ is

$$z = \frac{g'(x_1)}{f'(x_1)}(t - f(x_1)) + g(x_1),$$

(20)

which will intersect the $t$ axis at the point $(f(x_1) - g(x_1)/g'(x_1), 0)$. To avoid occlusion by other parts of the scene, the intersection should be out of the range of the camera array; that is

$$f(x_1) - g(x_1) \frac{f'(x_1)}{g'(x_1)} > L.$$  

(21)

Similarly, the tangent at $(f(x_2), g(x_2))$ will pass through $(f(x_2) - g(x_2)[f'(x_2)/g'(x_2)], 0)$, and we have

$$f(x_2) - g(x_2) \frac{f'(x_2)}{g'(x_2)} < -L.$$  

(22)

Consequently, we have the following theorem.

**Theorem 4.** If any point $(f(x), g(x))$ on the surface of the scene is differentiable, there is no occlusion with the surface if and only if

$$\|f(x) - g(x) \frac{f'(x)}{g'(x)}\| > L.$$  

(23)

In comparison, the unoccluded image condition in [5][8] is

$$\max_z \| \frac{g'(x)}{f'(x)} \| < \frac{1}{\tan(\theta_{max})},$$

(24)

which means that the slope of the surface should not be greater than that of any possible light ray captured. However, this condition is replaced by Eq. (23), because $L$ is limited. An example is exhibited in Fig. 7. The gradient at $(f(x_3), g(x_3))$ is larger than $1/\tan(\theta_{max})$, but each camera can still see all points on the surface of the scene.

### 4. CONCLUSION

According to the theory of lightfield sampling, the spectral support of a light field signal is bounded by the minimum and maximum depths of the scene [4]. In this paper, we extended this conclusion to parallel cameras that are not restricted to lie on a plane corresponding to a camera array. Furthermore, we conducted an analysis of plenoptic sampling for unstructured camera systems, in which the cameras can reside at arbitrary locations and orientations. We showed that the conclusion of plenoptic sampling can also be extended to unstructured cameras with limited rotation and FOV. This extension establishes a general framework for analysis of plenoptic sampling by investigating the spectral analysis of plenoptic signals.

The proposed framework provides a unified representation that generalizes several existing methods for plenoptic sampling. The derivation of the theory of lightfield sampling is based on the assumption that every camera can view every point on the object and thus imposes a constraint on the image to be unoccluded from all cameras. We derived necessary and sufficient conditions for unoccluded imaging which introduce constraints on the position and curvature of the surface of the scene as well as the position and orientation of the cameras.

### 5. REFERENCES


