

RATIONAL ELASTICITY OF FACTORIZATIONS IN KRULL DOMAINS

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ABSTRACT. For an atomic domain R , we define the elasticity of R as $\rho(R) = \sup\{m/n \mid x_1 \cdots x_m = y_1 \cdots y_n, \text{ for } x_i, y_j \in R \text{ irreducibles}\}$ and let $l_R(x)$ and $L_R(x)$ denote, respectively, the inf and sup of the lengths of factorizations of a nonzero nonunit $x \in R$ into the product of irreducible elements. We answer affirmatively two rationality conjectures about factorizations. First, we show that $\rho(R)$ is rational when R is a Krull domain with finite divisor class group. Secondly, we show that when R is a Krull domain, the two limits $l_R(x^n)/n$ and $L_R(x^n)/n$, as n goes to infinity, are positive rational numbers. These answer, respectively, conjectures of D. D. Anderson and D. F. Anderson, and D. F. Anderson and P. Puijs. (The second question has also been solved by A. Geroldinger and F. Halter-Koch.)

INTRODUCTION

If R is a UFD, then any two factorizations of a nonzero nonunit of R into the product of irreducible elements have the same length. Of course, this need not be true for an arbitrary atomic domain (an integral domain is *atomic* if each nonzero nonunit is a product of irreducible elements (atoms)). Following Zaks [15], we define an atomic domain R to be a *half-factorial domain* (HFD) if whenever $x_1 \cdots x_m = y_1 \cdots y_n$ with each $x_i, y_j \in R$ irreducible, then $m = n$. In order to measure how far an atomic domain R is from being a HFD, we define the *elasticity* of R as $\rho(R) = \sup\{m/n \mid x_1 \cdots x_m = y_1 \cdots y_n, \text{ for } x_i, y_j \in R \text{ irreducibles}\}$. Thus $1 \leq \rho(R) \leq \infty$ and $\rho(R) = 1$ if and only if R is a HFD. This concept was introduced by Valenza [14], who studied $\rho(R)$ for R the ring of integers in an algebraic number field. In an earlier paper, Steffan [13] studied $\rho(R)$ (without this notation) for a Dedekind domain R with finite divisor class group. Recently $\rho(R)$ has been studied in more detail in [1,

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5]. In [1, Theorem 3.2] it is shown that for any real number $r \geq 1$ or $r = \infty$, there is a Dedekind domain R with torsion divisor class group $\text{Cl}(R)$ such that $\rho(R) = r$. Moreover, if r is rational, then R may be chosen so that $\text{Cl}(R)$ is finite (this is also proved in [5]). This motivated the following question in [1]: if R is a Krull domain with $\text{Cl}(R)$ finite, is $\rho(R)$ rational, and moreover, does $\rho(R) = m/n$, where $x_1 \cdots x_m = y_1 \cdots y_n$ for some irreducibles $x_i, y_j \in R$?

In §1 we prove a general result (Theorem 4) about when certain quotients of real-valued functions on an additive submonoid of \mathbf{Z}_+^k obtain their maximum and minimum values. In the second section, Theorem 4 is then applied to answer two rationality conjectures about factorizations in Krull domains. Theorem 10 answers affirmatively the above question about the rationality of $\rho(R)$ in somewhat more generality. As an added bonus, Theorem 13 answers (part of) a conjecture of Anderson and Pruis [4] about the asymptotic behavior of length functions on integral domains; this conjecture has also recently been solved by Geroldinger and Halter-Koch [8]. The moral here is that certain finiteness conditions imply the rationality of factorization invariants.

For a nonzero nonunit x in an atomic domain R , let

$$l_R(x) = \inf\{n \mid x = x_1 \cdots x_n \text{ with each } x_i \in R \text{ irreducible}\},$$

$$L_R(x) = \sup\{n \mid x = x_1 \cdots x_n \text{ with each } x_i \in R \text{ irreducible}\},$$

and

$$\rho_R(x) = L_R(x)/l_R(x).$$

In [4] it is shown that the two limits $\bar{l}_R(x)$ and $\bar{L}_R(x)$ of $l_R(x^n)/n$ and $L_R(x^n)/n$, respectively, as n goes to infinity always exist ($\bar{L}_R(x)$ may be ∞). Moreover, for any $0 \leq \alpha \leq 1 \leq \beta \leq \infty$, there is an integral domain R and an irreducible $x \in R$ such that $\bar{l}_R(x) = \alpha$ and $\bar{L}_R(x) = \beta$. It was also conjectured that both limits are positive rational numbers when R is either a Krull domain or a Noetherian domain. In [8] this conjecture is shown to be true for Krull domains and several classes of Noetherian domains, but an example is given of a Noetherian domain R with $\bar{l}_R(x) = 0$ for an irreducible $x \in R$.

Factorization theory can be studied at several levels of generality; much of it can be abstracted to semigroups, as in [8, 9, 11, 12]. As our main interest is for integral domains, we will take the middle of the road and work with semigroups only when the added generality substantially simplifies the problem. Throughout, R will always be an integral domain with group of units $U(R)$ and nonzero elements R^* . Our notation and definitions will follow [7] for Krull domains and [10] for semigroups. For R a Krull domain, $X^{(1)}(R)$ is its set of height-one prime ideals and $\text{Div}(R)$ is its group of divisorial ideals under the v -operation. $\text{Div}(R)$ is free abelian on $X^{(1)}(R)$. For an additive submonoid Γ of a free abelian monoid $\bigoplus \mathbf{Z}_+$, \leq denotes the usual product order. As usual, $\infty/a = a/0 = \infty/0 = \infty$ for any real number $a > 0$. For a study of several factorization properties weaker than unique factorization in an arbitrary integral domain, see [2].

1. MAX AND MINS

In this section, Γ will be a commutative, cancellative monoid, written additively, with Γ^* its subsemigroup of noninvertible elements (thus $\Gamma^* = \Gamma \setminus \{0\}$ when Γ is an additive submonoid of a free abelian monoid). A function

$f : \Gamma \rightarrow \mathbf{R}_+$ is a *semilength function* on Γ if (1) $f(x + y) = f(x) + f(y)$ for all $x, y \in \Gamma$ and (2) $f(x) > 0$ for all $x \in \Gamma^*$. Given two semilength functions f and g on Γ , define $\phi(f, g) = \phi : \Gamma^* \rightarrow \mathbf{R}_+$ by $\phi(x) = f(x)/g(x)$ for all $x \in \Gamma^*$. Note that $\phi(kx) = \phi(x)$ for all integers $k \geq 1$. We define $L(\Gamma, \phi) = \sup\{\phi(x) \mid x \in \Gamma^*\}$ and $l(\Gamma, \phi) = \inf\{\phi(x) \mid x \in \Gamma^*\}$. Then $0 < L(\Gamma, \phi) \leq \infty$ and $0 \leq l(\Gamma, \phi) < \infty$. Note that if $\bar{\phi} = \phi(g, f)$, then $\bar{\phi} = 1/\phi$ and $l(\Gamma, \phi) = 1/L(\Gamma, \bar{\phi})$. We next give three elementary lemmas.

Lemma 1. *Let $a, b, c,$ and d be positive real numbers. Then $a/c < (a + b)/(c + d)$ if and only if $(a + b)/(c + d) < b/d$.*

Proof. $a/c < (a + b)/(c + d) \Leftrightarrow ac + ad < ac + bc \Leftrightarrow ad < bc \Leftrightarrow ad + bd < bc + bd \Leftrightarrow (a + b)/(c + d) < b/d$. \square

Lemma 2. *Let f and g be semilength functions on Γ and $\phi = \phi(f, g)$. Let $x, y \in \Gamma^*$ such that $y - x \in \Gamma^*$. If $\phi(x) < \phi(y)$ then $\phi(y) < \phi(y - x)$.*

Proof. Because $f(x)/g(x) = \phi(x) < \phi(y) = f(y)/g(y) = [f(x) + f(y - x)]/[g(x) + g(y - x)]$, we have $\phi(y) = f(y)/g(y) < f(y - x)/g(y - x) = \phi(y - x)$ by Lemma 1. \square

Lemma 3. *Let Γ be an additive submonoid of a free abelian monoid and $\phi : \Gamma^* \rightarrow G$ be any function into a totally ordered abelian group G . Then there is a function $\psi : \Gamma^* \rightarrow \Gamma^*$ such that (1) $\psi(x) \leq x$ for all $x \in \Gamma^*$ and (2) if $x, y \in \Gamma^*$ with $x \leq y$ then $\phi(x) \leq \phi(\psi(y))$. (Note that ψ need not be uniquely determined.)*

Proof. For $x \in \Gamma^*$, $\hat{x} = \{z \in \Gamma^* \mid z \leq x\}$ is finite. Thus we may choose $\psi(x) = w \in \hat{x}$ with $\phi(w)$ maximum, i.e., $\phi(z) \leq \phi(w)$ for all $z \in \hat{x}$. Clearly (1) and (2) hold. \square

For our main theorem, we specialize to Γ an additive submonoid of \mathbf{Z}_+^k for some integer $k \geq 1$ and $\phi = \phi(f, g)$.

Theorem 4. *Let Γ be an additive submonoid of \mathbf{Z}_+^k , f and g be semilength functions on Γ , and $\phi = \phi(f, g)$. If $y - x \in \Gamma$ for all $x, y \in \Gamma$ with $x \leq y$, then $L(\Gamma, \phi) = \phi(x_0)$ and $l(\Gamma, \phi) = \phi(y_0)$ for some $x_0, y_0 \in \Gamma^*$. Thus $L(\Gamma, \phi)$ and $l(\Gamma, \phi)$ are each positive real numbers. If f and g are each rational-valued, then $L(\Gamma, \phi)$ and $l(\Gamma, \phi)$ are each positive rational numbers.*

Proof. We prove the theorem for $L(\Gamma, \phi)$. Since $l(\Gamma, \phi) = 1/L(\Gamma, 1/\phi)$, the result follows for $l(\Gamma, \phi)$ as well. Let ψ be the function from Lemma 3 applied to ϕ . Suppose that $\phi(x) < L(\Gamma, \phi)$ for all $x \in \Gamma^*$. Choose $y_1 \in \Gamma^*$, and let $x_1 = \psi(y_1)$. Suppose that $y_1, \dots, y_n \in \Gamma^*$ have been chosen so that the $x_i = \psi(y_i)$'s are incomparable and $\phi(x_1) < \dots < \phi(x_n)$. Pick $y_{n+1} \in \Gamma^*$ with $\phi(y_{n+1}) > \phi(x_n)$, and let $x_{n+1} = \psi(y_{n+1})$. We claim that x_{n+1} is not comparable to any x_1, \dots, x_n . If $x_{n+1} < x_i$ then $x_{n+1} < x_i = \psi(y_i) \leq y_i$. Hence $\phi(x_{n+1}) \leq \phi(\psi(y_i)) = \phi(x_i)$ by Lemma 3, a contradiction. On the other hand, if $x_i < x_{n+1}$ then $x_{n+1} - x_i \in \Gamma^*$ and $\phi(x_i) < \phi(x_{n+1})$. Thus $\phi(x_{n+1}) < \phi(x_{n+1} - x_i)$ by Lemma 2. However, $0 < x_{n+1} - x_i < x_{n+1} = \psi(y_{n+1}) \leq y_{n+1}$; so $\phi(x_{n+1} - x_i) \leq \phi(\psi(y_{n+1})) = \phi(x_{n+1})$ by Lemma 3 again, a contradiction. Hence we can construct an infinite set $\{x_n\}$ of incomparable elements in $\Gamma \subset \mathbf{Z}_+^k$, a contradiction [6, Theorem 9.18]. \square

Remark 5. (a) We first show that Theorem 4 may fail if we do not assume that $y-x \in \Gamma$ when $x, y \in \Gamma$ with $x \leq y$. Let $a, b \in [0, \infty]$ with $0 \leq a < b \leq \infty$. Then $\Gamma = \{(m, n) \in \mathbf{Z}_+^2 \mid m, n \geq 1 \text{ and } a < m/n < b\} \cup \{(0, 0)\}$ is an additive submonoid of \mathbf{Z}_+^2 and $f, g : \Gamma \rightarrow \mathbf{R}_+$ given by $f(m, n) = m$ and $g(m, n) = n$ are semilength functions on Γ . For $\phi = \phi(f, g)$, we have $l(\Gamma, \phi) = a$ and $L(\Gamma, \phi) = b$; however, $a < \phi(m, n) = m/n < b$ for all $(m, n) \in \Gamma^*$. Note that there are integers $m, n \geq 1$ with $(m+1, n), (m, n) \in \Gamma$, but $(m+1, n) - (m, n) = (1, 0) \notin \Gamma$.

(b) In Theorem 4, it is also necessary to assume that Γ is an additive submonoid of a finitely generated free abelian monoid, rather than of just a free abelian monoid (cf. the proof of [1, Theorem 3.2]). \square

2. APPLICATIONS

In this section, we apply Theorem 4 to two rationality conjectures in factorization theory. Here, H will always be a commutative, cancellative monoid, written multiplicatively, with group of units $U(H)$. We say that H is *reduced* if $U(H) = \{1\}$. As in [12] a monoid H is *atomic* if each nonunit of H is a product of irreducible elements. For an atomic monoid H , we define the *elasticity* of H as $\rho(H) = \sup\{m/n \mid x_1 \cdots x_m = y_1 \cdots y_n, \text{ for } x_i, y_j \in H \text{ irreducibles}\}$. Then $1 \leq \rho(H) \leq \infty$. Many of the results of [1, 5] on $\rho(R)$, for R an atomic domain, also apply to $\rho(H)$, for H an atomic monoid. In particular, if $f : H \rightarrow \mathbf{R}_+$ is a semilength function on H , then $\rho(H) \leq M^*/m^*$, where $M^* = \sup\{f(x) \mid x \in H \text{ is irreducible, but not prime}\}$, $m^* = \inf\{f(x) \mid x \in H \text{ is irreducible, but not prime}\}$, and $M^* = m^* = 1$ if all irreducible elements of H are prime (cf. [1, Theorem 2.1]).

Let R be an integral domain. Then $H_1 = R^*$, $H_2 = R^*/U(R)$, and $H_3 = \text{Prin}(R)_+ = \{xR \mid x \in R^*\}$ are all multiplicative monoids and $H_2 \approx H_3$. Clearly R is atomic $\Leftrightarrow H_1$ is atomic $\Leftrightarrow H_2$ is atomic $\Leftrightarrow H_3$ is atomic. Moreover, if R is atomic, then $\rho(R) = \rho(H_1) = \rho(H_2) = \rho(H_3)$. More generally, if H is any monoid, then H is atomic if and only if the reduced monoid $\bar{H} = H/U(H)$ is atomic, and in this case, $\rho(\bar{H}) = \rho(H)$. One can sometimes calculate $\rho(H)$ by relating H to a simpler atomic monoid. Our next result, whose simple proof will be omitted, is sufficient for our purposes.

Lemma 6. *Let $\partial : H_1 \rightarrow H_2$ be a homomorphism of atomic monoids.*

- (a) *If $\partial(x)$ is irreducible whenever $x \in H_1$ is irreducible, then $\rho(H_1) \leq \rho(H_2)$.*
- (b) *If in addition $H_2 \subset H_1$ and $\partial(x) = x$ for each $x \in H_2$, then $\rho(H_1) = \rho(H_2)$. \square*

Our next theorem, along with Theorem 12, are the two applications of Theorem 4.

Theorem 7. *Let H be an atomic monoid with only a finite number of nonassociate irreducible elements. Then $\rho(H)$ is rational. Moreover, $\rho(H) = m/n$, where $x_1 \cdots x_m = y_1 \cdots y_n$ for some irreducibles $x_i, y_j \in H$.*

Proof. We may assume that H is reduced and z_1, \dots, z_k are the irreducible elements of H . Then $\Gamma = \{(a_1, \dots, a_k, b_1, \dots, b_k) \in \mathbf{Z}_+^{2k} \mid z_1^{a_1} \cdots z_k^{a_k} =$

$z_1^{b_1} \cdots z_k^{b_k}$ is an additive submonoid of \mathbf{Z}_+^{2k} , and $f, g : \Gamma \rightarrow \mathbf{R}_+$ given by $f(a_1, \dots, b_k) = a_1 + \cdots + a_k$ and $g(a_1, \dots, b_k) = b_1 + \cdots + b_k$ are semilength functions on Γ . Clearly Γ and $\phi = \phi(f, g)$ satisfy the hypotheses of Theorem 4 and $\rho(H) = L(\Gamma, \phi)$. Thus, $\rho(H) = \phi(x_0)$ for some $x_0 \in \Gamma^*$, and the theorem follows. \square

We next apply Theorem 7 to $\rho(R)$, for R an atomic domain. As an immediate corollary of Theorem 7, we have that $\rho(R)$ is rational when R has only a finite number of (nonassociate) irreducible elements. Such integral domains are called *Cohen-Kaplansky* (CK) domains and have recently been studied extensively in [3]. We state this as (cf. [1, Corollary 2.13])

Corollary 8. *Let R be a CK domain. Then $\rho(R)$ is rational. Moreover, $\rho(R) = m/n$, where $x_1 \cdots x_m = y_1 \cdots y_n$ for some irreducibles $x_i, y_j \in R$. \square*

In order to prove similar results for more general atomic domains, certain finiteness conditions are needed. For the Krull domain case, we will need

Lemma 9. *Let R be a Krull domain and $P_1, \dots, P_n \in X^{(1)}(R)$. Then $\{x \in R^* \mid xR = (P_1^{a_1} \cdots P_n^{a_n})_v \text{ for integers } a_i \geq 0\}$ has only a finite number of nonassociate irreducible elements.*

Proof. Let $\Gamma = \{(a_1, \dots, a_n) \in \mathbf{Z}_+^n \mid (P_1^{a_1} \cdots P_n^{a_n})_v \text{ is principal}\}$. If $xR = (P_1^{a_1} \cdots P_n^{a_n})_v$ for $(a_1, \dots, a_n) \in \Gamma$, then x is irreducible if (a_1, \dots, a_n) is minimal in Γ . Since Γ has only a finite number of minimal elements, the result follows. \square

We can now prove the conjecture on the rationality of $\rho(R)$ stated in the introduction.

Theorem 10. *Let R be a Krull domain such that only a finite number of divisor classes contain prime ideals. Then $\rho(R)$ is rational. Moreover, $\rho(R) = m/n$, where $x_1 \cdots x_m = y_1 \cdots y_n$ for some irreducibles $x_i, y_j \in R$. In particular, this is the case if $\text{Cl}(R)$ is finite.*

Proof. Suppose that $\{[P] \mid P \in X^{(1)}(R)\} \subset \text{Cl}(R)$ is finite. Pick $P_1, \dots, P_n \in X^{(1)}(R)$ such that for any $Q \in X^{(1)}(R)$, $[Q] = [P_i]$ for a unique integer $1 \leq i \leq n$. Let H be the subgroup of $\text{Div}(R)$ generated by P_1, \dots, P_n . Define $\alpha : \text{Div}(R) \rightarrow H$ by defining it on the basis elements by $\alpha(Q) = P_i$ if $[Q] = [P_i]$. Note that α restricts to a homomorphism from $\text{Prin}(R)_+$ to $H \cap \text{Prin}(R)_+$. Moreover, $\alpha(xR)$ is irreducible if and only if xR is irreducible. Thus $\rho(\text{Prin}(R)_+) = \rho(H \cap \text{Prin}(R)_+)$ by Lemma 6(b). By Lemma 9, $H \cap \text{Prin}(R)_+$ has only a finite number of irreducible elements. Hence by Theorem 7, $\rho(R) = \rho(\text{Prin}(R)_+) = \rho(H \cap \text{Prin}(R)_+)$ is rational and $\rho(R) = \rho_R(x)$ for some nonunit $x \in R^*$. \square

Remark 11. (a) Since $\{[P] \mid P \in X^{(1)}(R)\}$ generates $\text{Cl}(R)$, the hypotheses in Theorem 10 force $\text{Cl}(R)$ to be finitely generated, and thus to be finite if $\text{Cl}(R)$ is torsion.

(b) If $\{[P] \mid P \in X^{(1)}(R)\}$ is infinite, $\rho(R)$ need not be rational or even finite. Even if $\rho(R)$ is rational, we may have $\rho_R(x) < \rho(R)$ for each nonunit $x \in R^*$ (cf. the remarks after [1, Theorem 3.2]).

We next apply Theorem 4 to the asymptotic behavior of length functions. Let H be an atomic monoid and $x \in H$ a nonunit. As in [4, 8], we de-

fine $l_H(x) = \inf\{n \mid x = x_1 \cdots x_n \text{ with } x_i \in H \text{ irreducible}\}$ and $L_H(x) = \sup\{n \mid x = x_1 \cdots x_n \text{ with } x_i \in H \text{ irreducible}\}$. We then define

$$\bar{l}_H(x) = \lim_{n \rightarrow \infty} l_H(x^n)/n \quad \text{and} \quad \bar{L}_H(x) = \lim_{n \rightarrow \infty} L_H(x^n)/n.$$

Both limits always exist ($\bar{L}_H(x)$ may be ∞), and furthermore, $\bar{l}_H(x) = \inf\{m/n \mid x^n = x_1 \cdots x_m \text{ with } x_i \in H \text{ irreducible}\}$ and $\bar{L}_H(x) = \sup\{m/n \mid x^n = x_1 \cdots x_m \text{ with } x_i \in H \text{ irreducible}\}$ [4, Theorem 3; 8, Theorem 1]. We can now use Theorem 4 to give a very elementary proof of the key result of Geroldinger and Halter-Koch [8, Theorem 2].

Theorem 12. *Let H be an atomic monoid and $x \in H$ a nonunit such that $\{y \in H \mid y \mid x^n \text{ for some integer } n \geq 1\}$ has only a finite number of nonassociate irreducible elements. Then $\bar{l}_H(x)$ and $\bar{L}_H(x)$ are each positive rational numbers. Moreover, there are integers $m, n \geq 1$ so that $\bar{l}_H(x) = l_H(x^{km})/km$ and $\bar{L}_H(x) = L_H(x^{kn})/kn$ for all integers $k \geq 1$.*

Proof. We may assume that H is reduced and that z_1, \dots, z_r are the irreducible divisors of powers of x . Then $\Gamma = \{(n_1, \dots, n_r) \in \mathbf{Z}'_+ \mid z_1^{n_1} \cdots z_r^{n_r} = x^k \text{ for some integer } k \geq 0\}$ is an additive submonoid of \mathbf{Z}'_+ , and $f, g : \Gamma \rightarrow \mathbf{R}_+$ given by $f(n_1, \dots, n_r) = n_1 + \cdots + n_r$ and $g(n_1, \dots, n_r) = k$ if $z_1^{n_1} \cdots z_r^{n_r} = x^k$ are semilength functions on Γ . Clearly Γ and $\phi = \phi(f, g)$ satisfy the hypotheses of Theorem 4, and $\bar{L}_H(x) = L(\Gamma, \phi)$ and $\bar{l}_H(x) = l(\Gamma, \phi)$. Hence $L(\Gamma, \phi) = \phi(x_0)$ and $l(\Gamma, \phi) = \phi(y_0)$ for some $x_0, y_0 \in \Gamma^*$. Note that for any $y \in \Gamma^*$, $\phi(y) = \phi(ky)$ for all integers $k \geq 1$. The result is now clear. \square

By Lemma 9, the hypotheses of Theorem 12 hold for $H = R^*$ when R is a Krull domain, and in this case, $\bar{l}_R(x) = \bar{l}_H(x)$ and $\bar{L}_R(x) = \bar{L}_H(x)$ for each nonunit $x \in R^*$. Thus, as a corollary, we obtain the following result of Geroldinger and Halter-Koch [8, Theorem 3], which answers part of the conjecture of Anderson and Pruis [4]. (Note that in both Theorems 12 and 13 we may of course take $m = n$.)

Theorem 13. *Let R be a Krull domain and $x \in R^*$ a nonunit. Then $\bar{l}_R(x)$ and $\bar{L}_R(x)$ are each positive rational numbers. Moreover, there are integers $m, n \geq 1$ so that $\bar{l}_R(x) = l_R(x^{km})/km$ and $\bar{L}_R(x) = L_R(x^{kn})/kn$ for all integers $k \geq 1$. \square*

We end with an observation about the asymptotic behavior of the ρ_R function. For any atomic domain R and $x \in R^*$ a nonunit, we define $\bar{\rho}_R(x) = \lim_{n \rightarrow \infty} \rho_R(x^n)$ ($= \bar{L}_R(x)/\bar{l}_R(x) = \sup\{\rho_R(x^n) \mid n \geq 1 \text{ an integer}\}$). By [4], this limit always exists and $1 \leq \bar{\rho}_R(x) \leq \infty$. Moreover, for any $\gamma \in [1, \infty]$, there is an integral domain R and an irreducible $x \in R$ with $\bar{\rho}_R(x) = \gamma$. If R is a Krull domain, then by Theorem 13, $\bar{\rho}_R(x)$ is a positive rational number, and moreover, there is an integer $n \geq 1$ such that $\bar{\rho}_R(x) = L_R(x^{kn})/l_R(x^{kn})$ for all integers $k \geq 1$.

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