3-Consecutive Edge Coloring of a Graph

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Abstract

Three edges \(e_1, e_2\) and \(e_3\) in a graph \(G\) are consecutive if they form a path (in this order) or a cycle of length 3. The \(3\)-consecutive edge coloring number \(\psi'_{3c}(G)\) of \(G\) is the maximum number of colors permitted in a coloring of the edges of \(G\) such that if \(e_1, e_2\) and \(e_3\) are consecutive edges in \(G\), then \(e_1\) or \(e_3\) receives the color of \(e_2\). Here we initiate the study of \(\psi'_{3c}(G)\).

A close relation between \(3\)-consecutive edge colorings and a certain kind of vertex cuts is pointed out, and general bounds on \(\psi'_{3c}\) are given in terms of other graph invariants. Algorithmically, the distinction between \(\psi'_{3c} = 1\) and \(\psi'_{3c} = 2\) is proved to be intractable, while efficient algorithms are designed for some particular graph classes.

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1 Introduction

The classical theory of graph coloring began with the goal of minimizing the number of colors. On the other hand, there is now a fast-growing part of the literature where the number of colors is to be maximized under some conditions. The present work is a contribution in this direction; a new graph invariant (edge version of a previously studied kind of vertex coloring) is introduced and its connection with other important notions is shown.

1.1 3-consecutive edge coloring

Given a graph $G = (V, E)$, its 3-consecutive vertex coloring number $\psi_{3c}(G)$ is the maximum number of colors permitted in a coloring of the vertices of $G$ such that if $uv$ and $vw$ are different edges of $G$ then $u$ or $w$ receives the color of $v$. This parameter was introduced by E. Sampathkumar in [12], then studied in some detail in [13] and in a more general setting in [4]. Colorings under a slightly weaker condition were investigated in [5]. In this paper, we study the edge analogue of 3-consecutive vertex coloring of a graph.

In a graph $G$, three edges $e_1, e_2, e_3$ (in this fixed order) are called consecutive if $e_1 = xy, e_2 = yz$ and $e_3 = zu$ for some vertices $x, y, z, u$ (where $x = u$ is allowed). In other words, three edges are consecutive if they form a path or a cycle of length 3. An edge coloring $\varphi : E \rightarrow \mathbb{N}$ of $G = (V, E)$ is termed 3-consecutive if, for any three consecutive edges $e_1, e_2, e_3$, the edge $e_2$ receives the color of $e_1$ or $e_3$. The 3-consecutive edge coloring number $\psi'_{3c}(G)$ of $G$ is the maximum number of colors permitted in such a coloring. This coloring number was introduced by E. Sampathkumar in [12].

There is an alternative equivalent approach to this kind of coloring. Let us say that an edge coloring $\varphi$ of a graph $G$ is a strongly independent edge coloring, if it satisfies the requirement that $\{x, y\} \not\subseteq \bigcup_{c \neq \varphi(xy)} V(c)$ for all edges $xy$ of $G$, where $V(c)$ denotes the vertex set of the edges assigned to color $c$.

Our Proposition 1 will show that an edge coloring is 3-consecutive if and only if at least one end of each edge belongs to just one set $V(c)$. In this way, the definition of strongly independent edge coloring corresponds exactly to 3-consecutive edge coloring, therefore they can be considered equivalent alternatives to each other.

In this paper, we initiate a study of the 3-consecutive edge coloring number, $\psi'_{3c}$. Its exact values can be found for some known graphs; but, as we prove in Section 6, it is an algorithmically hard problem to determine $\psi'_{3c}$ in general, and even to decide whether $\psi'_{3c}(G) > 1$ for a generic input graph $G$. As a consequence, one cannot hope for a concise characterization of graphs having $\psi'_{3c}(G) = 1$. On the other hand, efficient algorithms can be designed for some classes of graphs, as we do in Section 7.

General estimates on $\psi'_{3c}(G)$ in terms of other graph invariants are presented
in Section 3, and the graphs $G$ attaining one of the extremal values $\psi_{3c}'(G) = 1$ or $\psi_{3c}'(G) = |E(G)|$ are studied in Section 5.

A substantial tool for the study of $\psi_{3c}'$ is Theorem 2 of Section 2, which transforms the lower bound $\psi_{3c}' \geq k$ concerning 3-consecutive edge colorings to an equivalent condition in terms of vertex cutsets with certain properties. It will be applied in several proofs later on.

One further remark should be added here. In nearly the entire literature of graph coloring theory, the edge colorings of a graph correspond to the vertex colorings of its line graph. This is not at all the case, however, with 3-consecutive colorings. Just on the opposite: it will be proved in [3] that for every graph $G$ of minimum degree at least 2 the equality $\psi_{3c}(G) = \psi_{3c}'(L(G))$ is valid.

1.2 Definitions and notation

- For a graph $G = (V,E)$ and a vertex set $X \subseteq V$, the subgraph of $G$ induced by $V \setminus X$ will be denoted by $G - X$.

- A vertex set $S \subseteq V$ is called stable or, with an equivalent term, independent if it contains no two adjacent vertices. The possible largest cardinality of an independent set in $G$ is the independence number of the graph and is denoted by $\beta_0(G)$. A vertex cover is a vertex set $T \subseteq V$ such that $V \setminus T$ is a stable set. The minimum cardinality of a vertex cover $T$ is the vertex covering number $\alpha_0(G)$ of $G$. By these definitions, $\alpha_0(G) + \beta_0(G) = |V|$ for every graph $G$.

- A cutset (or separator) of a connected graph $G$ is a vertex set $X \subseteq V$ for which $G - X$ has at least two components. A $k$-separator of a (not necessarily connected) graph $G$ is defined as a vertex set $X$ for which $G - X$ has at least $k$ components. If the graph is connected, a 2-separator means a cutset. A stable cutset or a stable $k$-separator means a cutset or a $k$-separator, respectively, which is also independent.

- The connectivity number of a connected incomplete graph $G$ is the minimum cardinality of a cutset of $G$. The connectivity number of the complete graph $K_n$ is defined to be $n - 1$. A biconnected graph is assumed to be connected and it remains connected after removing any vertex of it. Note that the property of having connectivity number at least 2 is equivalent to biconnectivity, with the only difference that the complete graph $K_2$ is regarded as biconnected but its connectivity number is only 1. A block is a maximal biconnected subgraph of a given graph. Every edge belongs to exactly one block, and two blocks can share at most one vertex.

- The line graph $L(G)$ of a graph $G$ has the edges of $G$ as its vertices, and two distinct edges of $G$ are adjacent in $L(G)$ if and only if they are incident
The neighborhood of vertex $v$ in graph $G$, that is the set of vertices adjacent to $v$, is denoted by $N_G(v)$, or simply by $N(v)$ if $G$ is understood.

For any further definitions on graphs we refer the book [7].

### 1.3 Some simple facts about $\psi'_{3c}$

It is worth stating two simple properties of $\psi'_{3c}(G)$ already at this early point. They will be applied later without referring to them.

- If $G$ is disconnected and has components $G_1, \ldots, G_c$ then its 3-consecutive edge coloring number can be calculated as $\psi'_{3c}(G) = \sum_{i=1}^{c} \psi'_{3c}(G_i)$. In particular, deletion/insertion of isolated vertices keeps the value of $\psi'_{3c}$ unchanged. We shall not refer to this fact in proofs where its application is obvious; but nevertheless, we allow isolated vertices to be present.

- If $G$ has a 3-consecutive edge coloring with exactly $k$ colors then for every integer $1 \leq k' \leq k$ there exists a 3-consecutive edge coloring of $G$ which uses precisely $k'$ colors.

The 3-consecutive edge coloring number can be easily determined for the following graphs:

- For $P_n$, a path on $n \geq 2$ vertices, $\psi'_{3c}(P_n) = \lceil n/2 \rceil$.
- For $C_n$, the cycle on $n \geq 3$ vertices, $\psi'_{3c}(C_n) = \lfloor n/2 \rfloor$.
- If $G$ is the complete graph on $n$ vertices or the wheel on $n+1$ vertices where $n \geq 3$, then $\psi'_{3c}(G) = 1$. 

![Figure 1: 3-consecutive edge coloring of Petersen graph with $\psi'_{3c} = 3$ colors.](image)
For the complete bipartite graph $K_{m,n}$, $\psi'_3(K_{m,n}) = \max\{m, n\}$.

If $G$ is the Petersen graph, then $\psi'_3(G) = 3$. (A possible coloring with 3 colors is shown in Figure 1.)

## 2 Stable separators

Stable cutsets are studied in the literature from several points of view, in relation to theoretical and practical problems as well. In this section we point out a close connection between 3-consecutive edge colorings and stable cutsets, which yields a necessary and sufficient condition for the existence of 3-consecutive edge colorings with exactly $k$ colors. This characterization will be a crucial tool in several later proofs.

Given a graph $G = (V, E)$ and a fixed edge coloring $\varphi : E \to \mathbb{N}$, a vertex $x \in V$ is said to be monochromatic if all edges incident to it have the same color. The equivalence of the 3-consecutive and the strongly independent edge colorings is shown by the following assertion.

**Proposition 1.** For a given graph $G = (V, E)$, a coloring $\varphi : E \to \mathbb{N}$ is a 3-consecutive edge coloring of $G$ if and only if at least one end of each edge is monochromatic.

**Proof.** We prove the equivalence of the negations. If an edge $e = xy$ has no monochromatic endpoint, there exist edges $f = xu$ and $g = yv$ (where $u = v$ is allowed) such that $\varphi(f) \neq \varphi(e) \neq \varphi(g)$. This implies that $\varphi$ is not a 3-consecutive edge coloring.

On the other hand, assume that $\varphi$ is not a 3-consecutive edge coloring of $G$. Hence, there exist three consecutive edges $f, e, g$, such that $e = xy$ has a color different from both $\varphi(f)$ and $\varphi(g)$. Consequently, the edge $e$ has no monochromatic end. \qed

**Theorem 2.** Let $G$ be a graph without isolated vertices. Then, for every integer $k \geq 2$, the 3-consecutive edge coloring number of $G$ is at least $k$ if and only if $G$ has a stable $k$-separator.

**Proof.** Assume that $\varphi$ is a 3-consecutive edge coloring of $G = (V, E)$ and $\varphi$ uses at least $k$ colors. Consider the set $M$ of monochromatic vertices of $G$. Due to Proposition 1, $M$ contains at least one vertex from every edge. Therefore, $M$ is a vertex cover and $V \setminus M$ is a stable set of $G$. The vertex set $M$ contains monochromatic vertices for each color used in $\varphi$. Moreover, there occur no edge connecting two vertices of $M$ being monochromatic with different colors. Therefore, $M$ induces a subgraph of $G$ having at least $|\varphi(V)| \geq k$ components, and hence, $V \setminus M$ is a stable $k$-separator of $G$. 

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On the other hand, suppose that $S$ is a stable $k$-separator of $G$. Let the vertex sets of the components of $G - S$ be denoted by $V_1, \ldots, V_\ell$, where $\ell \geq k$. Define an edge coloring such that, for each edge $e = xy$,

$$
\varphi(e) = i \iff x \in V_i \text{ or } y \in V_i.
$$

Since no edge can have its both ends in $S$, coloring $\varphi$ assigns at least one color to each edge of $G$; in fact exactly one, since $V_1, \ldots, V_\ell$ induce separated components in $G - S$ and no edge can have its ends in two different sets $V_i$ and $V_j$. Furthermore, every vertex of $V \setminus S$ is monochromatic in $\varphi$, and every edge has at least one of its ends in $V \setminus S$. Since we consider graphs without isolated vertices, each color between 1 and $\ell$ occurs at some vertex. Due to Proposition 1, this proves that $\varphi$ is a $3$-consecutive edge coloring of $G$ with $\ell \geq k$ colors. □

3 Bounds on $\psi'_{3c}$

This multi-part section presents various estimates on $\psi'_{3c}$. We shall mostly deal with general upper bounds.

3.1 Some simple graph operations

We observe a property of monotonicity first, namely that edge insertion in graphs without isolated vertices cannot increase $\psi'_{3c}$.

**Proposition 3.** If $H$ is a spanning subgraph of $G$, and each connected component of $H$ has order at least two, then $\psi'_{3c}(G) \leq \psi'_{3c}(H)$.

**Proof.** According to Theorem 2, $\psi'_{3c}(G) = k$ implies that $G$ has a stable $k$-separator $S$. Since $H$ is a spanning subgraph, $S$ is also a stable vertex set of $H$. Moreover, if two vertices belong to different components of $G - S$ then they cannot be contained in the same component of $H - S$ either. Consequently, $S$ is a stable $k$-separator in $H$. Furthermore, $H$ does not contain isolated vertices. Hence, by Theorem 2, $k \leq \psi'_{3c}(H)$ follows. □

Certainly, equality does not always hold in the above proposition, not even when $H$ is assumed to be connected. A small example is shown in Figure 2; one may also compare the path $P_n$ and the complete graph $K_n$.

If $G$ is a Hamiltonian graph on $p$ vertices, we can choose the spanning subgraph $H$ to be the Hamiltonian cycle. Hence, as an immediate consequence, we get:

**Corollary 4.** Let $G$ be a graph on $p$ vertices. If $G$ has a Hamiltonian path, then $\psi'_{3c}(G) \leq \lceil p/2 \rceil$; and if $G$ has a Hamiltonian cycle, then $\psi'_{3c}(G) \leq \lfloor p/2 \rfloor$.

Choosing the subgraph $H$ maximal cycle-free, we obtain:
Corollary 5. If $G$ is of order $p$ and has $c$ connected components, then $\psi'_{3c}(G) \leq p - c$.

While edge insertion cannot increase the value of $\psi'_{3c}$ (unless isolated vertices are also involved), it can dramatically drop the 3-consecutive edge coloring number as shown by the following result.

Theorem 6. If $u, v$ are two non-adjacent vertices in a graph $G$ of order $p \geq 4$, then $\psi'_{3c}(G) - \psi'_{3c}(G + uv) \leq p - 3$, and the bound is tight for all $p$.

Proof. We first observe that the decrease $p - 3$ is achievable. This is shown by the complete bipartite graph $K_{2, p - 2}$, which has $\psi'_{3c} = p - 2$, because we can assign a private color to each pair of edges incident to the vertices of the vertex class of size $p - 2$. Inserting an edge into the smaller vertex class, the graph becomes the union of $p - 2$ triangles, which all must be monochromatic.

By Corollary 5, $\psi'_{3c} \leq p - c \leq p - 1$ always holds. Hence, a decrease of $p - 2$ would be possible only if $\psi'_{3c}(G) = p - 1$ and $\psi'_{3c}(G + uv) = 1$. In particular, $G$ should be connected. Observe, however, that if $G$ contains two disjoint edges, say $e_1$ and $e_2$, then it is always possible to remove some edges from $G$ to obtain a graph $H$ without isolated vertices, in which $e_1$ and $e_2$ are in different components. In this case $\psi'_{3c}(G) \leq \psi'_{3c}(H) \leq p - 2$ (the first inequality is implied by Proposition 3), contrary to our assumption. Thus $G$ is a star and if we draw a new edge in a star then $\psi'_{3c}$ decreases to $p - 2$ rather than to 1. This completes the proof. $\square$

We next consider the block decomposition.

Proposition 7. If the blocks of graph $G$ are $B_1, \ldots, B_k$, then

$$\psi'_{3c}(G) \leq \sum_{i=1}^{k} \psi'_{3c}(B_i)$$

holds. In particular, if every block $B_i$ is complete, then $\psi'_{3c}(G) \leq k$. 

Figure 2: $\psi'_{3c}(G) = 3$, $\psi'_{3c}(H) = 4$. 
Proof. Consider a 3-consecutive edge-coloring $\varphi$ of $G$ which uses precisely $3c'(G)$ colors. Every edge of $G$ belongs to exactly one block. Hence,

$$
\psi_{3c}(G) = |\varphi(E(G))| \leq \sum_{i=1}^{k} |\varphi(E(B_i))| \leq \sum_{i=1}^{k} \psi_{3c}(B_i)
$$

follows. Moreover, if every block is a complete graph then $\psi_{3c}(B_i)$ equals 1 for all $1 \leq i \leq k$, and $\psi_{3c}(G) \leq k$ can be concluded. \qed

The following assertion may be viewed as the vertex analogue of Proposition 3, in the sense that under some well-defined simple conditions neither vertex- nor edge-insertion can increase $3c'$. We note that it is the occurrence of isolates that has to be excluded in both versions.

**Proposition 8.** Let $G = (V, E)$ be a graph, and $v \in V$ a vertex of degree at least 2.

(i) If the subgraph induced by $N(v)$ has no isolated vertices, then $\psi_{3c}(G) \leq \psi_{3c}(G - v)$ holds.

(ii) If the vertices of $N(v)$ are mutually adjacent, then $\psi_{3c}(G) = \psi_{3c}(G - v)$ holds.

Proof. If $x, y \in N(v)$ and $xy \in E$, then $\varphi(vx) = \varphi(vy) = \varphi(xy)$ must be valid in every 3-consecutive edge coloring $\varphi$ of $G$. Hence, $G - v$ contains all colors occurring in $G$. Moreover, the restriction of $\varphi$ to $E(G - v)$ remains a 3-consecutive edge coloring of $G - v$. This proves (i).

Suppose now that $N(v)$ induces a complete subgraph in $G$, and let $\varphi$ be a 3-consecutive edge coloring of $G - v$ with $\psi_{3c}(G - v)$ colors. By definition, all edges in $N(v)$ have the same color. Assign this color to all edges incident to $v$, too. We claim that this is a 3-consecutive edge coloring of $G$. Indeed, the condition cannot be violated inside $G - v$ and neither by triangles incident to $v$, nor by a $P_4$ having $v$ as an internal vertex. The only possibility would be some $vxyz \cong P_4$, where $x \in N(v)$ and $\varphi(vx) \neq \varphi(xy) \neq \varphi(yz)$. Let us choose a vertex $w \in N(v) \setminus \{x\}$. Then $wxz$ and $vxyz$ have the same color sequence in $\varphi$, hence we indeed have a 3-consecutive edge coloring of $G$. This proves (ii). \qed

We consider some further operations, which increase the value of $3c$ in a controlled way and will be applied in later proofs.

- **Attaching a path** of length $\ell$ means that we specify one vertex $x$ and supplement the graph with a path of length $\ell$ whose one end is $x$ and all of whose other vertices do not occur in the original graph.

- **Attaching an ear** of length $\ell$ means that we specify two distinct vertices $x, z$ and supplement the graph with a path of length $\ell$ whose ends are $x$ and $z$, and whose internal vertices do not occur in the original graph.
Proposition 9. Attaching a path of even length $\ell = 2t$ to a vertex, $\psi_{3c}$, increases by precisely $t$.

Proof. Let $G = (V, E)$ and $x \in V$. After the attachment a path of length $2t$ at $x$, denote the obtained graph by $G' = (V', E')$, where

\[ V' = V \cup Y \quad (Y = \{y_1, \ldots, y_{2t}\}, \ Y \cap V = \emptyset), \]

\[ E' = E \cup \{xy_1\} \cup \{y_iy_{i+1} \mid 1 \leq i \leq 2t - 1\}. \]

Due to Theorem 2, it is enough to prove that $G$ has a stable $s$-separator if and only if $G'$ has a stable $(s + t)$-separator.

First, suppose that $G$ has a stable $s$-separator $S$. To define a stable $(s + t)$-separator of $G'$, we distinguish between two cases:

- If $x \notin S$, consider the set $S' = S \cup \{y_1, y_3, \ldots, y_{2t-1}\}$. Obviously, $S'$ is a stable set of $G'$. Moreover, $G' - S'$ consists of the at least $s$ components of $G - S$ and further $t$ isolated vertices: $y_2, y_4, \ldots, y_{2t}$. Thus, $S'$ is a stable $s + t$-separator of $G'$.

- If $x \in S$, let the stable set $S' = S \cup \{y_3, \ldots, y_{2t-1}\}$ be considered. The components of $G' - S'$ are precisely $\{y_1, y_2\}, \{y_4\}, \ldots, \{y_{2t}\}$ and the at least $s$ components of $G - S$. This proves that $G'$ has a stable $(s + t)$-separator.

To prove the other direction, assume a stable $(s + t)$-separator $S''$ of $G'$. The graph $G' - S''$ has at least $s + t$ components and at most $t$ of them can meet the set $Y$. Consequently, at least $s$ components are entirely contained in $G$. This implies that $S'' \cap V$ is a stable $s$-separator in $G$. \qed

A similar increase occurs for attached ears:

Proposition 10. Attaching an ear of even length $\ell = 2t$ to two adjacent vertices of a graph, $\psi_{3c}$, increases by precisely $t - 1$.

Proof. Let $xy$ be an edge in graph $G = (V, E)$. We assume that the ear attached has $2t - 1$ new vertices $Y = \{y_1, y_2, \ldots, y_{2t-1}\}$ and $2t$ new edges $xy_1, y_1y_2, \ldots, y_{2t-2}y_{2t-1}, y_{2t-1}z$ (also $xz$ remains an edge). Denote by $G'$ the graph obtained. We will prove that $G'$ has a stable $(s + t - 1)$-separator if and only if $G$ has a stable $s$-separator. Due to Theorem 2, this will imply our proposition.

If $S$ is a stable $s$-separator in $G$, we have three cases:

- If $x \notin S$ and $z \notin S$ then take $S' = S \cup \{y_1, y_3, \ldots, y_{2t-1}\}$. This is a stable $(s + t - 1)$-separator in $G'$.

- If $x \in S$ then $z \notin S$. We can consider the set $S' = S \cup \{y_2, y_4, \ldots, y_{2t-2}\}$ which is a stable $(s + t - 1)$-separator in $G'$. (This yields $t - 1$ new singleton components, whilst $y_{2t-1}$ is linked to the component containing $z$.)
• If \( z \in S \) then \( x \notin S \) and the set \( S' = S \cup \{y_2, y_4, \ldots, y_{2t-2}\} \) is a stable \((s + t - 1)\)-separator in \( G' \), analogously to the previous case.

To prove the opposite direction, consider a stable \((s + t - 1)\)-separator \( S'' \) in \( G' \). The only possible case when \( Y \) meets at least \( t \) components of \( G' - S'' \) is when \( y_3, y_5, \ldots, y_{2t-3} \) are singleton components, moreover \( y_1 \) and \( y_{2t-1} \) belong to two further components. But, since \( x \in S'' \) and \( z \in S'' \) cannot hold simultaneously, at least one of the components containing \( y_1 \) or \( y_{2t-1} \) also meets the vertex set of \( G \). Hence, there exists at most \( t - 1 \) components in \( G' - S'' \) which are entirely contained in \( Y \). This implies that \( S'' \setminus Y \) is a stable \( s \)-separator in \( G \). \( \square \)

Viewing cycles as ‘closed ears’, the following analogue of Proposition 10 can be proved. Since the argument is quite similar to the previous ones, the proof is omitted.

**Proposition 11.** Let \( x \) be a vertex occurring in at least one stable \( \psi'_{3c} \)-separator of graph \( G \). Attaching a cycle of even length \( \ell = 2t \) to \( x \), \( \psi'_{3c} \) increases by precisely \( t \).

Note that the side condition on \( x \) involving optimal stable separators cannot be omitted. For instance, we have \( \psi'_{3c}(K_{2,3}) = 3 \), and attaching \( C_4 \) to a degree-2 vertex, \( \psi'_{3c} \) increases to 4 instead of 5.

### 3.2 Independence and domination

Here we obtain some upper bounds on \( \psi'_{3c}(G) \) in terms of independence number, domination number and independent domination number of \( G \).

**Theorem 12.** For every graph \( G \), the inequality \( \psi'_{3c}(G) \leq \beta_0(G) \) holds, where \( \beta_0(G) \) is the independence number of \( G \).

**Proof.** Note first that we may remove all isolated vertices because the left-hand side of the inequality remains unchanged while the right-hand side decreases. After this operation, due to Theorem 2, \( \psi'_{3c}(G) = k \) implies the existence of a stable \( k \)-separator vertex set \( S \). The subgraph \( G - S \) consists of \( k \) components. Choosing one vertex from each, a \( k \)-element stable set of \( G \) is obtained. Thus, \( k = \psi'_{3c}(G) \leq \beta_0(G) \) follows. \( \square \)

An independent dominating set is a dominating set which is independent. (A set \( D \subseteq V(G) \) dominates \( G \) if every vertex \( v \notin D \) of \( G \) has at least one neighbor in \( D \).) Given a graph \( G \), the minimum cardinality of an independent dominating set is denoted by \( i(G) \) and called the independent domination number of \( G \).

**Theorem 13.** If \( G \) is a graph of order \( p \), then \( \psi'_{3c}(G) \leq p - i(G) \) holds, where \( i(G) \) is the independent domination number of \( G \).
Proof. Similarly to the previous proof, we may remove all isolated vertices from $G$ (if there are any) since this operation keeps the values on the two sides of the inequality unchanged. Assume now that $\psi^{'}_{sc}(G) = k$ and consider a stable $k$-separator $S$ of $G = (V, E)$. Such a vertex set $S$ exists, due to Theorem 2. Moreover, $S$ can be extended to an inclusion-wise maximal stable set $S'$. Every maximal stable set is an independent dominating set as well. Therefore, $|S'| \geq i(G)$ and $|V \setminus S'| \leq p - i(G)$ are valid.

On the other hand, $S'$ is also a $k$-separator. Indeed, consider a component $G_i$ of $G - S$. Since $G$ is assumed not to contain isolated vertices, $V(G_i) \cup S$ cannot be stable. Thus, every component $G_i$ contains some vertex $x_i \notin S'$, and the number of components does not decrease when $S$ is extended to $S'$. Since $V \setminus S'$ is divided into $k$ components,

$$\psi^{'}_{sc}(G) = k \leq |V \setminus S'| \leq p - i(G)$$

can be concluded. $\square$

The following inequality can be obtained as a corollary of Theorem 12 and Theorem 13, as well. Let $p$, $\alpha_0(G)$ and $\gamma(G)$ respectively denote the order, the vertex covering number and the domination number of $G$. Theorem 12 and the well known relations $\alpha_0(G) + \beta_0(G) = p$ and $\gamma(G) \leq \alpha_0(G)$ (if isolated vertices are excluded) together imply the upper bound $p - \gamma(G)$. On the other hand, starting with Theorem 13, one can apply the inequality $i(G) \geq \gamma(G)$ and the same result is obtained.

**Corollary 14.** If $G$ is a graph of order $p$ then $\psi^{'}_{sc}(G) \leq p - \gamma(G)$ holds.

To show that the upper bounds in Theorems 12, 13 and Corollary 14 are sharp, consider the star $S_n = K_{1,n}$ on $n + 1$ vertices ($n \geq 1$). Clearly, $\beta_0(S_n) = n$, $\gamma(S_n) = i(S_n) = 1$, whilst a 3-consecutive edge coloring can be obtained if the edges get pairwise different colors. Consequently, $\psi^{'}_{sc}(S_n) = n = \beta_0(S_n) = (n + 1) - \gamma(S_n) = (n + 1) - i(S_n)$.

### 3.3 Vertex degrees

The bounds in the previous subsection imply that the inequality

$$\psi^{'}_{sc}(G) \leq p - \frac{p}{\Delta(G) + 1}$$

is valid for all graphs of order $p$, and actually this bound is tight for graphs all of whose components are stars of the same size. For connected graphs, however, an improvement proportional to $p/\Delta^2$ (which is constant times $p$ whenever the maximum degree is fixed) can be achieved, as follows.
Theorem 15. If $G$ is a connected graph of order $p$ and maximum degree $\Delta$, then
\[ \psi'_{3c}(G) \leq p - \frac{p - 1}{\Delta} \]
holds. Moreover, the bound is tight for infinitely many $p$, for every fixed $\Delta \geq 2$.

Proof. We first prove the upper bound. Let $G = (V, E)$ be a connected graph of order $p$, and suppose that $\psi'_{3c}(G) = k$. Applying the characterization Theorem 2, we select a stable $k$-separator and supplement it to an inclusionwise maximal stable set $S \subseteq V$. This set $S = \{v_1, \ldots, v_m\}$ is also a $k$-separator in $G$. Next, we consider the ‘neighborhood hypergraph’ associated to the set $S$, that is $H = (X, E)$ with vertex set $X = V \setminus S$ and edge set
\[ E = \{N(v_i) \mid 1 \leq i \leq m\}. \]
Note that $|E_i| \leq \Delta$ holds for all edges $E_i \in E$, by the degree condition in $G$. Moreover, by the maximality of $S$, the equality $\bigcup_{1 \leq i \leq m} N(v_i) = X$ is also satisfied.

Suppose that $H$ has precisely $c$ connected components. Then, as is well known, the inequality
\[ |V| - |S| - c = |X| - c \leq \sum_{i=1}^{m} (|E_i| - 1) \leq (\Delta - 1)|S|, \]
is valid, from which we obtain
\[ |S| \geq \frac{p - c}{\Delta}. \]
Moreover, $G$ is supposed to be connected, hence there exist paths between any two distinct $v_i, v_j \in S$, and therefore there must occur at least $c - 1$ connections between the distinct components of $H$ in $G$ (recall that $S$ is independent in $G$). Thus, $X$ cannot induce a subgraph with more than $|V| - |S| - (c - 1)$ connected components in $G$. Consequently,
\[ \psi'_{3c}(G) \leq p - (|S| + c - 1) \leq p + 1 - c - \frac{p - c}{\Delta} \leq p - \frac{p - 1}{\Delta} \]
as $c \geq 1$.

Tightness for $\Delta = 2$ is shown by paths of odd order $p$, they have $\psi'_{3c} = (p + 1)/2$. For $\Delta \geq 3$ we consider trees where all internal vertices have degree $\Delta$ and all leaves are at the same distance, say $r$, from a central vertex $x$ which will be viewed as the root of the tree.

We select the stable separator $S$ as the vertices at distance $r - 1, r - 3, \ldots$ from $x$; that is, $V \setminus S$ consists of the vertices $v$ such that
\[ d(v, x) \equiv r \pmod{2}. \]
We are going to prove the equality

$$|V \setminus S| = (\Delta - 1)|S| + 1$$

from which tightness of the upper bound immediately follows because the entire set $V \setminus S$ is independent and hence has $p - |S|$ components. Although the equality above can be verified numerically, the next argument fortunately avoids tedious calculation.

We observe that every $v \in S \setminus \{x\}$ has precisely $\Delta - 1$ children in $V \setminus S$. Now, if $x \in S$ (i.e., if $r$ is odd), then $x$ has $\Delta$ children instead of $\Delta - 1$, yielding the claimed ‘$+1$’ in the formula. And if $x \notin S$ (i.e., if $r$ is even), then $x$ itself yields the term ‘$+1$’. □

**Corollary 16.** Every graph $G$ with $p$ vertices, $c$ connected components and maximum degree $\Delta$ has

$$\psi_{3c}'(G) \leq p - \frac{p - c}{\Delta}$$

and the bound is tight for infinitely many values of $p$ for every fixed $c \geq 1$ and $\Delta \geq 2$.

We can also obtain a general bound depending on minimum degree.

**Proposition 17.** (i) If $G$ is a graph of order $p$ and minimum degree $\delta$, then $\psi_{3c}'(G) \leq p - \delta$, and the bound is tight for all $\delta \leq p/2$.

(ii) For $\delta > p/2$ there exists a graph with order $p$, minimum degree $\delta$, and $\psi_{3c}'(G) \geq k$, if and only if

$$p + \lfloor \delta/k \rfloor \geq 2\delta + 1.$$  

**Proof.** (i) Let $S$ be a stable separator in $G = (V, E)$. Any $v \notin S$ has degree at least $\delta$, therefore $V \setminus S$ cannot induce more than $p - \max\{|S|, \delta\}$ connected components. This implies the upper bound on $\psi_{3c}'(G)$. Tightness is shown by the complete bipartite graphs $K_{\delta, p-\delta}$.

(ii) Let $k = \psi_{3c}'(G)$, and consider a stable $k$-separator $S$ according to Theorem 2. Then $V \setminus S$ induces precisely $k$ connected components, we denote their vertex sets by $V_1, \ldots, V_k$. Let us write $s = |S|$ and $p_i = |V_i|$ for $i = 1, \ldots, k$. We necessarily have

$$p - s \geq \delta$$

because $S$ is independent and a vertex $v \in S$ has all its neighbors in $V \setminus S$. This means $|S| \leq p - \delta$. Moreover, the smallest $p_i$ is at most $\left\lfloor \frac{p - s}{k} \right\rfloor$. Since a vertex $v \in V_i$ has all its neighbors in $V_i \cup S$,

$$s + \left\lfloor \frac{p - s}{k} \right\rfloor - 1 \geq \delta$$
also has to hold. The weakest condition from the two inequalities above is obtained when $s = p - \delta$, and this implies the ‘only if’ part of (ii).

To see the other direction, let $|V| = p$. Select a subset $S \subseteq V$ with $|S| = s = p - \delta$, and partition $V \setminus S$ into $k$ parts $V_i$ of cardinality $\left\lceil \frac{p-s}{k} \right\rceil$ or $\left\lfloor \frac{p-s}{k} \right\rfloor$ each. Draw all edges between $S$ and $V \setminus S$, and make each $V_i$ a complete graph. This graph satisfies the conditions with the given parameters. □

3.4 Bipartite graphs

We close this section with some observations with regard to the 3-consecutive edge coloring number of bipartite graphs.

**Proposition 18.** If $G$ is a bipartite graph with bipartition $V = V_1 \cup V_2$, and $G$ has no isolated vertices, then $\max\{|V_1|,|V_2|\} \leq \psi_{3c}'(G) \leq \beta_0(G)$.

**Proof.** In view of Theorem 12, we need to establish only the lower bound. Let us assume that $|V_1| \geq |V_2|$ and $V_1 = \{v_1, v_2, ..., v_m\}$. Color the edges of $G$ such that all edges incident to $v_i$ are assigned with color $i$ for all $1 \leq i \leq m$. Clearly, this yields a 3-consecutive edge coloring of $G$ with $m$ colors. This implies that $\max\{|V_1|,|V_2|\} \leq \psi_{3c}'(G)$. □

The above coloring method does not yield a maximum 3-consecutive edge coloring for every bipartite graph. As a counterexample, we can consider the tree in Figure 3, for which $\max\{|V_1|,|V_2|\} = 4$ whereas $\psi_{3c}'(G) = 5$.

If $G$ is a bipartite graph with a perfect matching, then $\beta_0(G) = |V_1| = |V_2|$ holds, and hence, $\psi_{3c}'(G) = \beta_0(G)$. This conclusion remains valid also under a weaker condition.

**Proposition 19.** If $G$ is a bipartite graph without isolated vertices, and there exists a matching which covers one of its partition classes, then $\psi_{3c}'(G) = \beta_0(G)$ holds.

**Proof.** Assume that $G = (V, E)$ has partition classes $V_1, V_2$ where $|V_1| \leq |V_2|$. If there exists a matching of size $|V_1|$, then we have $\beta_0(G) \leq |V| - |V_1| = |V_2| \leq \beta_0(G)$. □
\( \beta_0(G) \), hence \( \beta_0(G) = |V_2| \) follows. By Proposition 18, this implies \( \psi'_{3c}(G) = \beta_0(G) \).

Since the \( n \)-cube \( Q_n \) is a bipartite graph with a perfect matching, we have

**Corollary 20.** For every integer \( n \geq 1 \), the 3-consecutive edge coloring number of the \( n \)-dimensional hypercube graph \( Q_n \) is \( \psi'_{3c}(Q_n) = 2^{n-1} \).

The following simple assertion characterizes bipartite graphs having \( \psi'_{3c} = 2 \). Interestingly enough, later we shall see that a similar short description for graphs with \( \psi'_{3c} = 2 \) (and even \( \psi'_{3c} = 1 \)) in general does not exist.

**Proposition 21.** Let \( G \) be a connected bipartite graph. Then, \( \psi'_{3c}(G) = 2 \) if and only if \( G \cong P_3, P_4 \) or \( C_4 \).

**Proof.** Suppose that \( \psi'_{3c}(G) = 2 \). It is clear that \( \Delta(G) \leq \max\{|V_1|, |V_2|\} \leq \psi'_{3c}(G) = 2 \). Hence, \( G \) is either a path or an even cycle.

- If \( G \) is a path \( P_n \), then \( 2 = \psi'_{3c}(G) = \lceil n/2 \rceil \). This implies that \( n = 3 \) or \( 4 \); that is, \( G \cong P_3 \) or \( G \cong P_4 \).
- If \( G \) is an even cycle \( C_n \), then \( 2 = \psi'_{3c}(C_n) = \lfloor n/2 \rfloor \). This implies \( n = 4 \); that is, \( G \cong C_4 \). □

## 4 Extremal values of \( \psi'_{3c} \)

In this section we investigate graphs whose 3-consecutive edge coloring number is the possible largest or smallest one; that is, \( \psi'_{3c}(G) \) equals \( |E(G)| \) or 1, respectively. The former case admits a simple characterization, which also implies that the decision problem whether \( \psi'_{3c}(G) = |E(G)| \) holds, can be solved efficiently. This is far from being true for testing whether \( \psi'_{3c}(G) = 1 \).

**Proposition 22.** Let \( G \) be a graph with \( q \) edges. Then \( \psi'_{3c}(G) = q \) holds if and only if each component of \( G \) is a star or an isolated vertex.

**Proof.** We prove only the necessary part, as sufficiency is obvious. Assume that \( \psi'_{3c}(G) = q \). If there exists a component of \( G \) which is neither a star nor a single vertex, then \( G \) contains three consecutive edges. At least two of them must have a common color in every 3-consecutive edge coloring. Hence, \( \psi'_{3c}(G) < q \) would follow which is a contradiction. □

The above case of \( \psi'_{3c}(G) = q \) equivalently means that every assignment \( \varphi : E(G) \to \mathbb{N} \) is a 3-consecutive edge coloring of \( G \). The other extremal case is \( \psi'_{3c}(G) = 1 \) when the graph admits only the trivial monochromatic 3-consecutive edge coloring.
Proposition 23. For a graph $G$ without isolated vertices, $\psi'_{3c}(G) = 1$ holds if and only if $G$ is connected and contains no stable cutset.

Proof. If $G$ consists of more than one component, each of them contains some edges and hence, $\psi'_{3c}(G) > 1$. Otherwise, due to Theorem 2, $\psi'_{3c}(G) \geq 2$ if and only if the graph has a stable cutset. \hfill \Box

We will see in Section 5 that deciding whether there exists a 3-consecutive edge coloring different from the trivial one is an $\mathbf{NP}$-complete problem already on some restricted graph classes. Thus, unless $\mathbf{P} = \mathbf{NP}$, there is no hope of obtaining a characterization which can be checked in polynomial time. So, it is worth giving some sufficient and some necessary conditions which can be easily tested.

A graph $G$ is chordal if every cycle of length 4 or more in $G$ has a chord (an edge connecting two non-consecutive vertices of the cycle).

Proposition 24. If $G$ is a biconnected chordal graph, then $\psi'_{3c}(G) = 1$.

Proof. Since the assertion is obvious for $K_2$, we assume that $G$ has at least three vertices. Consider any 3-consecutive edge coloring of $G$. We will prove that every cycle $C$ of $G$ is monochromatic. This is clear by definition if $C$ is a triangle. With this anchor, we can apply induction on cycle length. If $C$ is a cycle longer than 3, it can be split into two shorter cycles, say $C'$ and $C''$ sharing an edge $e$ which is a chord of $C$. By the induction hypothesis, both $C'$ and $C''$ are monochromatic in the color of $e$, hence the entire $C$ has the same color.

Since $G$ is 2-connected, any two of its edges are contained in some cycle, which then must be monochromatic. Thus, just one color can occur. \hfill \Box

The converse implication is not true as it is shown by the graph in Figure 4.

Conditions necessary for $\psi'_{3c} = 1$ can be obtained as the negations of conditions sufficient for $\psi'_{3c} \geq 2$.

Proposition 25. Let $G$ be a connected graph of order at least three.
(i) If $G$ has a cut vertex then $\psi'_{3c}(G) \geq 2$.

(ii) If $G$ has a vertex $v$ not belonging to any triangle, then $\psi'_{3c}(G) \geq 2$.

(iii) If $G$ has $p$ vertices and at most $2p - 4$ edges, then $\psi'_{3c}(G) \geq 2$.

Proof. A cut vertex is a stable cutset. Moreover, if there is no triangle in $G$ incident to a fixed vertex $v$ and $G$ is not a star centered at $v$, then $N(v)$ is a stable cutset. Thus, Proposition 23 implies that $\psi'_{3c}(G) > 1$ necessarily holds under each of conditions (i) and (ii).

To prove (iii), we apply a result from [6] which states that if the number of edges is not greater than $2p - 4$, then $G$ necessarily contains a stable cutset. Due to Proposition 23, this conclusion is equivalent to $\psi'_{3c}(G) > 1$. □

Consider the complete bipartite graph $K_{2,4}$ and supplement it with two independent edges in the vertex class of size 4. The obtained graph $K^*_{2,4}$ has 3-consecutive edge coloring with two colors, but fulfills none of the conditions (i)–(iii).

5 Intractability

In this section, and also in the next one, we will study the algorithmic complexity of the following two problems:

3-CONSECUTIVE EDGE $k$-COLORABILITY

Instance: A graph $G = (V, E)$.

Question: Does $G$ admit a 3-consecutive edge coloring with precisely $k$ colors?

It is worth noting that the existence of 3-consecutive edge colorings with ‘precisely $k$ colors’ and with ‘at least $k$ colors’ are equivalent, therefore 3-CONSECUTIVE EDGE $k$-COLORABILITY exactly means asking whether $\psi'_{3c}(G) \geq k$ holds.

STABLE $k$-SEPARATOR

Instance: A graph $G = (V, E)$ without isolated vertices.

Question: Does $G$ have a stable set $S \subseteq V(G)$ such that $G - S$ has at least $k$ components?

According to Theorem 2 these two problems are equivalent for graphs which do not contain isolated vertices. Moreover, for connected graphs, STABLE 2-SEPARATOR means the problem STABLE CUTSET whose complexity status was discussed in several papers. It was proved in [8] that STABLE CUTSET is NP-complete in general. Then, similar hardness was pointed out on restricted graph classes. From these results we can obtain immediate consequences concerning 3-consecutive edge 2-colorings.
Theorem 26. The 3-Consecutive Edge 2-Colorability problem is NP-complete on each of the following classes restricted to connected graphs:

(a) perfect graphs;
(b) $K_4$-free planar graphs with $\Delta = 5$;
(c) graphs with connectivity number 2;
(d) line graphs with $\Delta = 5$;
(e) line graphs of bipartite graphs;
(f) for every fixed $\epsilon > 0$: graphs with $q \leq (2 + \epsilon)p$ (where $p$ and $q$ denote the number of vertices and edges, respectively);
(g) and also on the intersection of the graph classes given in (d)–(e)–(f).

Proof. In view of Proposition 23, it suffices only to refer to the NP-completeness results concerning the problem Stable Cutset on the classes listed above. This was proved for classes (b) and (c) in [9] and [2], respectively. Intractability of Stable Cutset on classes (a) and (e) was stated explicitly in [2] as immediate conclusion of a result from [11]. For the intersection of (d)–(e)–(f), NP-completeness was proved in [10]. □

Now, we consider the more general problem, whether $\psi_3^*(G) \geq k$ holds, where $k$ is an arbitrarily fixed integer greater than 1. The following result shows that these problems are NP-complete in general, and on some restricted graph classes, too.

Theorem 27. For every fixed integer $k \geq 2$, the problems

- 3-Consecutive Edge $k$-Colorability
- Stable $k$-Separator

are NP-complete on each of the following classes restricted to connected graphs:

(i) perfect graphs;
(ii) $K_4$-free planar graphs with $\Delta = 5$;
(iii) graphs with connectivity number 2;
(iv) line graphs with $\Delta = 5$;
(v) line graphs of bipartite graphs;
(vi) for every fixed $\epsilon > 0$: graphs with $q \leq (2 + \epsilon)p$ (where $p$ and $q$ denote the number of vertices and edges, respectively);

(vii) and also on the intersection of the graph classes given in (iv)–(v)–(vi).

Proof. Assume a fixed integer $k \geq 2$. Due to Theorem 26, if $k = 2$ then the problems are $\mathsf{NP}$-complete. Hence, it can be assumed that $k \geq 3$ and we apply reduction from the problem 3-CONSECUTIVE EDGE 2-COLORABILITY to 3-CONSECUTIVE EDGE $k$-COLORABILITY. It will be readily seen in each case that the reduction takes polynomial time (actually, constant or linear).

First, let us consider the cases (i)–(ii) together. For a connected graph $G$ of order at least two, we attach a path of length $2k - 4$ to a vertex $x$ (chosen arbitrarily for (i) and specified later for (ii)) and denote by $G'$ the graph obtained. Applying Proposition 9 with $t = k - 2$, we obtain that $\psi'_{3c}(G') = \psi'_{3c}(G) + k - 2$; and hence, $\psi'_{3c}(G) \geq 2$ if and only if $\psi'_{3c}(G') \geq k$. This means that the decision problem of 2-colorability is reduced to that of $k$-colorability. Moreover, if $G$ is perfect or belongs to (ii), then so does $G'$ as well; for the latter we need to note that the graphs obtained from the reduction in [9] have minimum degree at most 3 and this makes it possible to choose $x$ properly. Thus, it follows that 3-CONSECUTIVE EDGE $k$-COLORABILITY and equivalently STABLE $k$-SEPARATOR are $\mathsf{NP}$-complete problems on classes (i)–(ii).

Similarly, (iii) follows from part (c) of Theorem 26, by applying Proposition 10 with $t = k - 1$, because graphs obtained by attaching an ear to a 2-connected graph are again 2-connected.

The reduction for (vii) is quite similar to the previous ones, but we need a little modification.

Let us first concentrate just on line graphs, without any further restrictions. Hence, let $G = L(H)$ be the line graph of a generic graph $H$, and choose a vertex $x$ of any degree $d \geq 2$ in $H$. We attach a pendant edge $e = xx'$ to $x$. Then the neighborhood of $e$ in the line graph of this extended graph $H'$ is $K_d$, therefore Proposition 8 implies $\psi'_{3c}(L(H')) = \psi'_{3c}(L(H))$. Now, attach a path of length $2k - 4$ to $x'$. By Proposition 9, for the graph $H''$ obtained, $\psi'_{3c}(L(H'')) = \psi'_{3c}(L(H')) + k - 2$ holds. Hence, $\psi'_{3c}(L(H)) \geq 2$ if and only if $\psi'_{3c}(L(H'')) \geq k$.

Observe that in (vi) and (vii), it suffices to prove $\mathsf{NP}$-hardness for $\epsilon < 1/2$. On the other hand, to get the degree bound $\Delta = 5$ and restrict to line graphs of bipartite graphs, we need to recall the result of [10] in a greater detail. It is shown there that STABLE CUTSET is $\mathsf{NP}$-complete on line graphs of bipartite graphs $B$ such that in one vertex class of $B$ all degrees are equal to 3 while in the other class each vertex has degree 2 or 4, moreover the number $p$ of vertices and $q$ of edges in $L(B)$ satisfy $q \leq (2 + \epsilon)p$ where $\epsilon > 0$ is any fixed real. Since
the line graphs of (3, 4)-biregular bipartite graphs are 5-regular and hence satisfy $q = 5p/2$, the present bound $q \leq (2 + \epsilon)p < 5p/2$ implies that some vertex of $B$ has degree 2.

Let us start with a graph $G$, which is the line graph of a bipartite graph $B$ complying with the above conditions. Choosing a vertex $x$ of degree 2 in $B$, we denote by $B''$ the graph obtained from $B$ by attaching a path of length $2k - 3$ to $x$. We will prove that $B''$ is in the graph class of (vii), and apply the observations made on line graphs for $H = B$ and $H'' = B''$.

Since the edges incident to $x$ meet just three other edges of $B$, path attachment at $x$ does not create any vertices of degree higher than 4 in the line graph, and so the maximum degree remains 5 in $L(B'')$. Clearly, $B''$ is bipartite, and the number of edges and vertices in the line graph have been increased by $2k - 2$ and $2k - 3$, respectively. Thus, $q \leq (2 + \epsilon)p$ remains valid in $L(B'')$, and hence $L(B'')$ belongs to (vii). Moreover, by what has been said about $H$ and $H''$, $\psi_{3c}(L(B)) \geq 2$ if and only if $\psi_{3c}(L(B'')) \geq k$. This proves the theorem for (vii) and implies NP-completeness for (iv)–(vi).

\[\square\]

6 Polynomial algorithms

As we have seen, the problem of computing $\psi_{3c}'$ is algorithmically hard, and even the distinction between $\psi_{3c}' = 1$ and $\psi_{3c}' = 2$ is intractable. On the other hand, there are some well-structured graph classes for which this can be done efficiently. We begin with the class of trees, for which one can determine $\psi_{3c}'$ efficiently.

**Theorem 28.** For trees, the 3-consecutive edge coloring number can be determined and a 3-consecutive edge coloring with maximum number of colors can be obtained in linear time.

**Proof.** According to Theorem 2, $\psi_{3c}'(G)$ is equal to the largest number of components in an induced subgraph which can be obtained from $G$ deleting a stable vertex set. First, we describe a dynamic programming procedure which calculates this largest number $\psi_{3c}'(G)$ for trees.

Consider a tree $T = (V, E)$ with a fixed root vertex $v$. For each $x \in V$, $T(x)$ will denote the subtree rooted in $x$; that is, the subgraph of $T$ induced by all vertices $y$ for which the $y - v$ path involves vertex $x$. The following two values are defined for every $v \in V$:

- $a(x)$ denotes the maximum number of components in $T(x) - S$, over all stable sets $S$ containing $x$.
- $b(x)$ denotes the maximum number of components in $T(x) - S$, over all stable sets $S$ not containing $x$.

These values will be calculated for the vertices of $T$ proceeding in postorder.
• If \( x \) is a leaf then we set \( a(x) = 0 \) and \( b(x) = 1 \). Clearly, these values correspond to the defined meanings of \( a(x) \) and \( b(x) \).

• If \( x \) is not a leaf, let its children be denoted by \( x_1, \ldots, x_d \). When a stable set \( S \) of \( T(x) \) involves the vertex \( x \), \( S \) can contain none of the children of \( x \), and the maximum number \( a(x) \) of components can be calculated as

\[
a(x) = \sum_{i=1}^{d} b(x_i).
\]

When \( x \not\in S \), for each integer \( 1 \leq i \leq d \) both cases \( x_i \in S \) and \( x_i \not\in S \) are possible. Beginning the counting with one component, if a child \( x_i \) is chosen to be in \( S \), the number of possible components can be increased by at most \( a(x_i) \). In the other case when \( x_i \not\in S \), the number of components increases by at most \( b(x_i) - 1 \), since in this case \( x_i \) and \( x \) belong to the same component. For each child we can choose the more advantageous possibility, and hence,

\[
b(x) = 1 + \sum_{i=1}^{d} \max(a(x_i), b(x_i) - 1).
\]

Assuming that \( a(x_i) \) and \( b(x_i) \) are appropriate values for each child \( x_i \), the above formulas yield integers \( a(x) \) and \( b(x) \) corresponding to the definitions.

It is clear that the 3-consecutive edge coloring number of the tree is

\[
\psi'_{3c}(T) = \max(a(v), b(v)),
\]

and the algorithm outputs this value in \( O(p) \) steps if \( T \) is a tree of order \( p \).

Also, a stable set \( S \), for which the number of components in \( T - S \) equals \( \psi'_{3c}(T) \), can be obtained in linear time. We define a function \( s(x) \) on \( V \) in preorder.

• For the root vertex \( v \):

\[
s(v) = \begin{cases} 
1 & \text{if } a(v) \geq b(v), \\
0 & \text{if } a(v) < b(v).
\end{cases}
\]

• For a vertex \( x \neq v \), denote by \( p(x) \) the parent vertex of \( x \). We define:

\[
s(x) = \begin{cases} 
0 & \text{if } s(p(x)) = 1, \\
0 & \text{if } s(p(x)) = 0 \text{ and } a(x) < b(x) - 1, \\
1 & \text{if } s(p(x)) = 0 \text{ and } a(x) \geq b(x) - 1.
\end{cases}
\]
The set \( S = \{ x \in V \mid s(x) = 1 \} \) is a stable set of \( T \) and \( T - S \) has \( \psi'_{3c}(T) \) components.

Having this set \( S \) in hand, a \( 3 \)-consecutive edge coloring \( \varphi \) can be constructed in linear time. Denote the \( k = \psi'_{3c}(T) \) components of \( T - S \) by \( C_1, \ldots, C_k \) and define the color of the edge \( e = xy \) as follows:

\[
\varphi(e) = i \iff x \in C_i \text{ or } y \in C_i.
\]

We have shown in the proof of Proposition 2 that this definition yields a \( 3 \)-consecutive edge coloring of \( T \) with exactly \( k = \psi'_{3c}(T) \) colors. \( \square \)

Deciding whether \( \psi'_{3c}(G) \geq 2 \), i.e. whether \( G \) admits a \( 3 \)-consecutive edge coloring different from the trivial monochromatic one, can be done in polynomial time on further graph classes.

Given a graph \( F \), by \( F \)-free graph we mean one with no induced subgraphs isomorphic to \( F \). The claw is the star graph with three edges, that is \( K_{1,3} \).

**Proposition 29.** It can be decided in polynomial time if the \( 3 \)-consecutive edge coloring number of a graph is at least \( 2 \), on each of the following classes:

(a) graphs with \( \Delta \leq 3 \);

(b) line graphs with \( \Delta = 4 \);

(c) (claw, \( K_4 \))-free graphs;

(d) claw-free planar graphs;

(e) \( 2K_2 \)-free graphs;

(f) graphs with \( q \leq 2p - 4 \) (where \( p \) and \( q \) denote the number of vertices and edges, respectively).

Moreover, if the answer is affirmative, a \( 3 \)-consecutive edge coloring with at least two colors can be obtained in polynomial time on each class (a)–(f).

**Proof.** If the given graph is not connected, and it has at least two components not being edgeless, then \( \psi'_{3c}(G) \geq 2 \) automatically holds and the required coloring can be immediately obtained. For connected graphs, polynomial-time decision and search algorithms were given for the problem \textsc{Stable Cutset} on each of the above classes: on classes (a) and (b) it can be found in [10]; on (c) and (d) in [9]; and on (f) in [6]. In connection with (e) note that \( G \) is \( 2K_2 \)-free if and only if \( L(G) \) has diameter at most two. The problem \textsc{Stable Cutset} has been shown to be polynomial-time solvable in [1] for graphs whose line graph has diameter at most two. These algorithms equivalently solve the problem \textsc{3-Consecutive Edge 2-Colorability} and an appropriate coloring with precisely two colors can be constructed easily if a stable cutset is known. \( \square \)
7 Concluding remarks

Concerning Proposition 7 the following natural question arises.

**Problem 30.** Find tight estimates on $\psi_{3c}'(G)$ for graphs in which all blocks are complete graphs.

There seem to be relations with the tree of blocks, and also with the intersection graph of the clique hypergraph, but the correspondence is not quite clear.

Also concerning algorithmic complexity, many problems can be raised. Below we mention some of them.

**Problem 31.** Give further efficiently checkable sufficient conditions for $\psi_{3c}'(G) = 1$ and for $\psi_{3c}'(G) \geq 2$.

**Problem 32.** Describe further ‘nice’ classes of graphs for which $\psi_{3c}'$ can be determined in polynomial time.

**Problem 33.** On which classes of graphs is the equality $\psi_{3c}'(G) = \psi_{3c}'(G - v)$ testable in polynomial time?

**Problem 34.** Let $k \geq 2$. Is the 3-Consecutive Edge $k$-Colorability problem NP-complete on the class of graphs with connectivity number $\kappa$, for every fixed $\kappa$?

The case $\kappa = 2$ has been confirmed in Theorem 27 for all $k$. In general, for $\kappa \geq 2$ we have the following related result.

**Proposition 35.** If the Stable Cutset problem is NP-complete on the class of $\kappa$-connected graphs, then also 3-Consecutive Edge $k$-Colorability is NP-complete on $\kappa$-connected graphs.

**Proof.** We make the following reduction. Let $G = (V, E)$ be $\kappa$-connected, $\kappa \geq 2$. Fix a set $Z \subset V$ of cardinality $|Z| = \kappa$. We take two further sets $Z'$ and $Z''$ with $|Z'| = |Z''| = \kappa$, such that $V, Z', Z''$ are mutually disjoint. Let $Z'$ be independent and let $Z''$ induce a complete graph $K_\kappa$. Insert further edges forming a perfect matching between $Z$ and $Z'$, and a complete bipartite graph $K_{\kappa, \kappa}$ whose two vertex classes are $Z'$ and $Z''$. Formally, we denote by $zz'$ the edges of the $Z-Z'$ matching ($z \in Z$, $z' \in Z'$).

The graph $G'$ derived from $G$ in this way clearly is $\kappa$-connected. We claim that $G'$ has a stable $k$-separator if and only if $G$ has a stable $(k - 1)$-separator. Indeed, if $S$ is a stable $(k - 1)$-separator of $G$, then the set

$$S' := S \cup \{z' \mid z \in Z \setminus S\}$$

is a stable $k$-separator of $G'$ because $Z'' \cup (Z' \setminus S')$ is a component in $G' - S'$, separated from all components of $G - S$.  

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Conversely, suppose that some $S'$ is a $k$-separator of $G'$. We set $S := S' \setminus (Z' \cup Z'')$. Note that $|S' \cap Z''| \leq 1$ holds, as $Z''$ induces a complete graph; hence, $|Z'' \setminus S'| \geq k - 1 \geq 1$ and the entire $(Z' \cup Z'') \setminus S'$ belongs to the same component in $G' - S'$. Consequently, the number of components in $G \setminus S$ is not smaller than that in $G' \setminus S'$ minus 1. Thus, $S$ is a stable $(k - 1)$-separator of $G$.

Applying this extension sequentially $k - 2$ times, the assertion follows. □

The above reduction also yields a different proof for part (iii) of Theorem 27.

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