Gamma function estimates via completely monotonicity arguments

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ABSTRACT. The aim of this paper is to improve some recent results of Sandor and Debnath [On certain inequalities involving the constant e and their applications J. Math. Anal. Appl. 249 (2000) 569-582] and Batir [Sharp inequalities for factorial n Proyecciones 27 (2008) 97-102] about the problem of approximation of gamma function. In proving the results in this paper, the proofs are not completely analytic but essentially based on formal computations obtained by means of software packages.

1. INTRODUCTION

The gamma function $\Gamma$ is widely studied in almost all branches of mathematics, and it is defined for every $x > 0$ by

$$\Gamma (x) = \int_0^\infty t^{x-1} e^{-t} dt.$$ 

The problem of estimating the large factorials $\Gamma (n + 1) = 1 \cdot 2 \cdot 3 \cdots n$, for every $n \in \mathbb{N}$ has attracted the attention of many authors. Maybe the most known and most used formula is the following

$$\Gamma (n + 1) \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n,$$

now called Stirling’s formula, but more precise results of the form

$$\Gamma (n + 1) \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\lambda_n}$$

were introduced. More exactly, Robbins [8] proved

$$\frac{1}{12n + 1} < \lambda_n < \frac{1}{12n}$$

and recently Shi, Liu and Hu [10] showed

$$\frac{1}{12n} - \frac{1}{360n^3} < \lambda_n < \frac{1}{12n} - \frac{1}{360n (n + 1)(n + 2)}.$$

Now by truncation of only a few terms, the asymptotic series [1]

$$\Gamma (x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \exp \left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \cdots\right),$$

as $x \to \infty$

provides approximations of any desired accuracy $x^{-k}$, where $k \geq 1$. 

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Stirling’s formula (1.1) is in fact the first approximation of the following Stirling asymptotic series as \( n \to \infty \),

\[
(1.3) \quad n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \left( 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{517}{2488320n^4} + \cdots \right).
\]

In terms of Bernoulli numbers, (1.2) can be written in the form

\[
\Gamma (x + 1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \exp \left( \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \left( \frac{1}{2k-1} \right) \frac{x^{2k-1}}{x^{2k-1}} \right), \quad \text{as } x \to \infty
\]

and Alzer [2] proved the inequalities for all \( x > 0 \) and integers \( m, n \geq 1 \),

\[
(1.4) \quad \exp \left( \sum_{k=1}^{2m} \frac{B_{2k}}{2k} \left( \frac{1}{2k-1} \right) \frac{x^{2k-1}}{x^{2k-1}} \right) < \Gamma (x + 1) \sqrt{2\pi x} \left( \frac{x}{e} \right)^x < \exp \left( \sum_{k=1}^{2n-1} \frac{B_{2k}}{2k} \left( \frac{1}{2k-1} \right) \frac{x^{2k-1}}{x^{2k-1}} \right).
\]

In fact, Alzer proved that for all integers \( m, n \geq 1 \), the functions

\[
F_m (x) = \ln \Gamma (x) - \left( x - \frac{1}{2} \right) \ln x + x - \frac{1}{2} \ln 2\pi - \sum_{k=1}^{2m} \frac{B_{2k}}{2k} \left( \frac{1}{2k-1} \right) x^{2k-1}
\]

\[
G_n (x) = -\ln \Gamma (x) + \left( x - \frac{1}{2} \right) \ln x - x + \frac{1}{2} \ln 2\pi + \sum_{k=1}^{2n-1} \frac{B_{2k}}{2k} \left( \frac{1}{2k-1} \right) x^{2k-1}
\]

are completely monotonic on \((0, \infty)\), i.e. they have derivatives of all orders on \((0, \infty)\) and

\[
(-1)^k F_m^{(k)} (x) \geq 0, \quad \text{and} \quad (-1)^k G_n^{(k)} (x) \geq 0,
\]

for every \( x \in (0, \infty) \) and \( k = 0, 1, 2, 3, \ldots \). See [11, Chapter 4]. In particular, \( F_n > 0 \), \( G_n > 0 \) imply (1.4).

Regarding the following approximation formula

\[
\Gamma (n + 1) \approx \sqrt{\pi} \left( \frac{n}{e} \right)^n \sqrt[6]{8n^3 + 4n^2 + n + \frac{1}{30}}
\]

and the inequalities

\[
(1.5) \quad \sqrt[6]{8x^3 + 4x^2 + x + \frac{1}{100}} \leq \frac{\Gamma (x + 1)}{\sqrt{\pi} \left( \frac{x}{e} \right)^x} < \sqrt[6]{8x^3 + 4x^2 + x + \frac{1}{30}}, \quad x \geq 1
\]

recorded in ”The lost notebook and other unpublished papers” of Srinivasa Ramanujan”, Anderson et al. [3] conjectured that the function \( h : [1, \infty) \to \left[ \frac{1}{100}, \frac{1}{30} \right] \) is increasing monotonically, where

\[
h (x) = \left( \frac{\Gamma (x + 1)}{\sqrt{\pi} \left( \frac{x}{e} \right)^x} \right)^6 - \left( 8x^3 + 4x^2 + x \right).
\]

Recently, Karatsuba [5] solved this open problem using a complicated method and proved that \( \frac{1}{100} \) in (1.5) can be replaced by the sharp constant \( h (1) = \frac{e^6}{\pi^3} - 13 = 0.011198\ldots \), since

\[
h (1) \leq h (x) \leq h (\infty) = \frac{1}{30}.
\]
A direct method for proving the previous conjecture and for obtaining new refinements were proposed in [6], where the more accurate formula was introduced

\[
\Gamma (x + 1) \approx \sqrt{\pi} \left( \frac{x}{e} \right)^x \sqrt[8]{16x^4 + \frac{32}{3} x^3 + \frac{32}{9} x^2 + \frac{176}{405} x - \frac{128}{1215}}.
\]

It is proved in [6] that the corresponding function

\[
g(x) = \left( \frac{\Gamma (x + 1)}{\sqrt{\pi} \left( \frac{x}{e} \right)^x} \right)^8 - \left( 16x^4 + \frac{32}{3} x^3 + \frac{32}{9} x^2 + \frac{176}{405} x \right)
\]

is strictly decreasing on \([3, \infty)\) and consequently, for all \(x \geq 3\),

\[
\sqrt[8]{16x^4 + \frac{32}{3} x^3 + \frac{32}{9} x^2 + \frac{176}{405} x + g(\infty)} < \frac{\Gamma (x + 1)}{\sqrt{\pi} \left( \frac{x}{e} \right)^x} < \sqrt[8]{16x^4 + \frac{32}{3} x^3 + \frac{32}{9} x^2 + \frac{176}{405} x + g(3)}.
\]

2. Completely monotonic arguments

The starting point of our study is the work of Sandor and Debnath [9], where the factorial \(n\) was positioned between Stirling’s formula and a new upper limit

\[
n^{n+1}e^{-n}\sqrt{2\pi} \leq n! \leq \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1}}.
\]

Subsequently, Batir [4] studied the variation of the function

\[
h(x) = \frac{2\pi x^2 e^{-2x}}{\Gamma^2 (x)} - x
\]

to prove

\[
n^{n+1}e^{-n}\sqrt{2\pi} \leq n! \leq \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-a}},
\]

where the constants \(a = 1 - 2\pi e^{-2} = 0.14966\ldots\) and \(b = 1/6 = 0.16666\ldots\) are the best possible.

Remark 2.1. Numerical computations show that for large values of \(n\), the approximation

\[
n! \approx \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1/6}}
\]

gives increasingly better results than

\[
n! \approx \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-(1-2\pi e^{-2})}}.
\]

In consequence, if we are interested to produce good approximations of the form

\[
n! \approx \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\theta(n)}},
\]

then \(\theta (n)\) should become closer to \(1/6\), as \(n\) approaches infinity.
One way to attack approximations of type (2.7) is to attach the corresponding function
\[ f(x) = \ln \frac{\Gamma(x + 1)}{\sqrt{x + 1} e^{-x} \sqrt{2\pi}}, \quad x > \frac{1}{6}, \]
and to study its (completely) monotonicity properties. The famous Bernstein-Widder theorem [11, p. 160] states that a function \( z \) is completely monotonic if and only if \( z \) is a Laplace transform of the measure \( \alpha(t) \), that is
\[ z(x) = \int_0^\infty e^{-xt} d\alpha(t), \]
for all \( x \in I \) (\( \alpha \) is a bounded and non-decreasing function).

Completely monotonic functions \( z \) involving gamma function are of great interest, since they produce sharp bounds for \( z \) and their derivatives.

We also recall that the logarithmic derivative of \( \Gamma(x) \), denoted by \( \psi(x) = \frac{d}{dx} \ln \Gamma(x) \), is known as digamma function, while the derivatives \( \psi^{(i)}(x), i = 1, 2, 3, \ldots \) are called the polygamma functions.

We prove the following

**Theorem 2.1.** Let \( f: \left(\frac{1}{6}, \infty\right) \to \mathbb{R} \) given by
\[ f(x) = \ln \frac{\Gamma(x + 1)}{\sqrt{x + 1} e^{-x} \sqrt{2\pi}}. \]
Then \( -f \) is completely monotonic.

In particular, \( f \) is strictly increasing, so \( f(1) \leq f(x) < f(\infty) \), for all \( x \in [1, \infty) \). As \( f(1) = \ln \frac{e^{\sqrt{5}}}{\sqrt{12\pi}} \) and \( f(\infty) = 0 \), we can state the following

**Theorem 2.2.** For all \( x \in [1, \infty) \), we have
\[ q \cdot \frac{x^{x+1} e^{-x} \sqrt{2\pi}}{\sqrt{x - 1/6}} \leq \Gamma(x + 1) < \frac{x^{x+1} e^{-x} \sqrt{2\pi}}{\sqrt{x - 1/6}}, \]
where the constant \( q = \frac{e^{\sqrt{5}}}{\sqrt{12\pi}} = 0.98995 \ldots \) is the best possible. Equality in the left-hand side of (2.9) holds if and only if \( x = 1 \).

In order to prove Theorem 2.1, we need the following

**Lemma 2.1.** Let
\[ a_n = 3n \cdot 7^{n-1} - (n - 1) 6^n - 3n. \]
Then \( a_n > 0 \), for all integers \( n \geq 4 \).

**Proof.** Inequality \( a_n > 0 \) can be written as \( 3n \cdot 7^{n-1} - (n - 1) 6^n > 0 \) and it follows from
\[ 3n \left(\frac{13}{2}\right)^{n-1} > (n - 1) 6^n. \]
The first inequality in (2.10) is true for \( n = 4 \), and it is true for every \( n \geq 4 \) because \( 7n-1 - \left(\frac{13}{2}\right)^{n-1} \) is strictly increasing function of \( n \).

The second inequality in (2.10) is equivalent to
\[
\left(\frac{13}{12}\right)^n > \frac{13(n-1)}{6n}
\]
and it follows by induction, using
\[
\frac{13}{12} \frac{13(n-1)}{6(n+1)} = \frac{13(n^2-13)}{72n(n+1)} > 0.
\]

Proof of Theorem 2.1. We have
\[
f(x) = \ln(\Gamma(x+1)) + x - (x+1)\ln x - \ln\sqrt{2\pi} + \frac{1}{2}\ln\left(x - \frac{1}{6}\right)
\]
and using the recursion formula \( \psi(x+1) = \psi(x) + 1/x \), see [1, p. 258], direct computations lead to
\[
f''(x) = \psi'(x) - \frac{1}{x} - \frac{1}{2\left(x - \frac{1}{6}\right)^2}.
\]
Making appeal to the formula [1, p. 255]
\[
\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1}e^{-xt}dt
\]
for \( x > 0 \) and \( r > 0 \) and to the integral representations [1, p. 260]
\[
\psi^{(n)}(x) = (-)^{n+1} \int_0^\infty \frac{t^{n-1}e^{-xt}}{1-e^{-t}}dt
\]
for \( x > 0 \) and \( n = 1, 2, 3, \ldots \), we get
\[
f''(x) = \int_0^\infty \frac{e^{-tx}}{e^t - 1} \varphi\left(\frac{t}{6}\right)dt,
\]
where
\[
\varphi(t) = 3te^t - e^{6t} + 6te^{6t} - 3te^{7t} + 1 = -\sum_{n=4}^{\infty} \frac{a_n}{n!} t^n < 0.
\]
In consequence, \( -f'' \) is completely monotonic, in particular \( f' \) is strictly decreasing. But \( f'(\infty) = 0 \), so \( f'>0 \).

Now \( f \) is strictly increasing, with \( f(\infty) = 0 \), thus \( f < 0 \).

Finally, \( f < 0, f' > 0 \) and \( -f'' \) is completely monotonic, which means that \( -f \) is completely monotonic. \( \Box \)

3. Asymptotic Denominator of Stirling’s Formula

Motivated by Remark 2.1, we propose the approximations
\[
\Gamma(x+1) \approx \frac{x^{x+1}e^{-\frac{x}{6}}\sqrt{2\pi}}{\sqrt{x - \frac{1}{6} + \frac{1}{72x}}} \quad \text{and} \quad \Gamma(x+1) \approx \frac{x^{x+1}e^{-x}\sqrt{2\pi}}{\sqrt{x - \frac{1}{6} + \frac{1}{72x} + \frac{31}{6480x^2}}}
\]
with \( \theta(x) = 1 - \frac{1}{6} - \frac{1}{72x} \), respective \( \theta(x) = 1 - \frac{1}{6} - \frac{31}{6480x^2} \), and we state the following
Theorem 3.3. For every $x \geq 1$, we have

\[
\frac{x^{x+1} e^{-x} \sqrt{2\pi}}{\sqrt{x - \frac{1}{6} + \frac{1}{72x} + \frac{31}{6480x^2}}} < \Gamma(x + 1) < \frac{x^{x+1} e^{-x} \sqrt{2\pi}}{\sqrt{x - \frac{1}{6} + \frac{1}{72x}}}
\]

Obviously, approximations (3.11) already improve much the results of Sandor and Deb-nath [9] and Batir [4], but we propose in the next section much stronger formulas obtained by truncation of the following new asymptotic series

\[
\Gamma(x + 1) \approx \frac{x^{x+1} e^{-x} \sqrt{2\pi}}{\sqrt{x - \frac{1}{6} + \frac{1}{72x} + \frac{31}{6480x^2} - \frac{139}{155520x^3} - \frac{9871}{6531840x^4} + \ldots}}, \text{ as } x \to \infty.
\]

The first terms of this series are computed below, but a systematically method for con-structing (3.13) can be given using the ideas described in the recent paper [7].

By letting $m = 1$ and $n = 2$ in (1.4), we get

\[
\exp\left(\frac{1}{12x} - \frac{1}{360x^3}\right) < \frac{\Gamma(x + 1)}{\sqrt{2\pi x (\frac{x}{e})^x}} < \exp\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}\right).
\]

Proof of Theorem 4. Inequality (3.12) can be written as

\[
\frac{1}{\sqrt{1 - \frac{1}{6x} + \frac{1}{72x^2} + \frac{31}{6480x^3}}} < \frac{\Gamma(x + 1)}{\sqrt{2\pi x (\frac{x}{e})^x}} < \frac{1}{\sqrt{1 - \frac{1}{6x} + \frac{1}{72x^2}}},
\]

and if we take into account (3.14), it suffices to show that

\[
\exp\left(\frac{1}{12x} - \frac{1}{360x^3}\right) > \frac{1}{\sqrt{1 - \frac{1}{6x} + \frac{1}{72x^2} + \frac{31}{6480x^3}}}
\]

and

\[
\frac{1}{\sqrt{1 - \frac{1}{6x} + \frac{1}{72x^2}}} > \exp\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}\right),
\]

or $f > 0$ and $g < 0$, on $[1, \infty)$, where

\[
f(x) = \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{2} \ln\left(1 - \frac{1}{6x} + \frac{1}{72x^2} + \frac{31}{6480x^3}\right)
\]

and

\[
g(x) = \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} + \frac{1}{2} \ln\left(1 - \frac{1}{6x} + \frac{1}{72x^2}\right).
\]

We have

\[
f'(x) = -\frac{1390x^2 - 90x - 31}{120x^4 (6480x^3 - 1080x^2 + 90x + 31)} < 0
\]

and

\[
g'(x) = \frac{1302x^4 - 252x^3 - 699x^2 + 120x - 10}{2520x^6 (72x^2 - 12x + 1)} > 0.
\]

Now $f$ is strictly decreasing, $g$ is strictly increasing on $[1, \infty)$, with $f(\infty) = g(\infty) = 0$, so

$f(x) > 0$ and $g(x) < 0$, for all $x \in [1, \infty)$. The proof is complete. □
For real $b$, we introduce the following class of representations

$$n! = \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{n-b} \cdot e^{w_n(b)},$$

which can be considered as a generalization of Batir’s formula (2.6). We consider an approximation

$$n! \approx \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{n-b}$$

better when the sequence $w_n(b)$ converges to zero quickly. As we can see from the following Theorem 5, only $b = 1/6$ value provides a superior rate of convergence for the exponential remainder $w_n(b)$.

**Theorem 3.4.** i) For every $b \in \mathbb{R} \setminus \{1/6\}$, the sequence $w_n(b)$ converges like $n^{-1}$, since

$$\lim_{n \to \infty} n w_n(b) = \frac{1}{12} - \frac{b}{2} \neq 0.$$  

ii) The sequence $w_n(1/6)$ converges like $n^{-2}$, since

$$\lim_{n \to \infty} n^2 w_n(1/6) = -\frac{1}{144}.$$  

The following lemma is an useful tool for measuring the rate of convergence.

**Lemma 3.2.** If $x_n$ is a sequence convergent to $x$ and there exists the limit

$$\lim_{n \to \infty} n^k (x_n - x_{n+1}) = l \in \mathbb{R},$$

with $k > 1$, then there exists the limit:

$$\lim_{n \to \infty} n^{k-1} (x_n - x) = \frac{l}{k-1}.$$  

For proof and other details, see [7]. Remark that the convergence rate of the sequence $x_n$ is even higher as the value $k$ satisfying (3.16) is greater, and this fact explains the sharpness of the constant $1/6$ in Theorem 5.

**Proof of Theorem 5.** From (3.15), we have:

$$w_n = \sum_{k=1}^{n} \ln k + \frac{1}{2} \ln (n - b) - (n + 1) \ln n + n - \ln \sqrt{2\pi}.$$  

Then

$$w_n - w_{n+1} = -\ln (n + 1) + \frac{1}{2} \ln \frac{n-b}{n+1-b} - (n + 1) \ln n + (n + 2) \ln (n + 1) - 1,$$

or using Maple software,

$$w_n - w_{n+1} = \left(\frac{1}{12} - \frac{1}{2}b\right) \frac{1}{n^2} + \left(\frac{1}{2}b - \frac{1}{2}b^2 - \frac{1}{12}\right) \frac{1}{n^3} + O\left(\frac{1}{n^4}\right).$$

If $b \neq 1/6$, then

$$\lim_{n \to \infty} n^2 (w_n - w_{n+1}) = \frac{1}{12} - \frac{1}{2}b \neq 0$$

and i) follows by Lemma 6. If $b = 1/6$, then

$$\lim_{n \to \infty} n^3 (w_n - w_{n+1}) = -\frac{1}{72}$$

and Theorem 5 is proved. □
4. FURTHER IMPROVEMENTS

We show in this section how can be improved Batir’s estimates (2.7)-(2.8) and (3.11). In this sense, we find the best constants $a, b, c, d$ such that the sequence $\lambda_n$ given by the relations

\begin{equation}
(4.17) \quad n! = \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n - \frac{1}{6} + \frac{a}{n} + \frac{b}{n^2} + \frac{c}{n^3} + \frac{d}{n^4}}} \cdot e^{\lambda_n}, \quad n \geq 1
\end{equation}

has the fastest possible convergence rate. By (4.17) and using Maple software, we get

\begin{equation}
(4.18) \quad \lambda_n - \lambda_{n+1} = \left( a - \frac{1}{72} \right) \frac{1}{n^3} + \left( \frac{3}{2}b - \frac{5}{4}a + \frac{11}{1080} \right) \frac{1}{n^4} + \left( -a^2 + \frac{14}{9}a - \frac{8}{3}b + 2c - \frac{89}{12960} \right) \frac{1}{n^5} + \left( \frac{305}{72}b - \frac{775}{432}a - \frac{55}{12}c + \frac{5}{2}d - \frac{5}{2}ab + \frac{25}{12}a^2 + \frac{223}{54432} \right) \frac{1}{n^6} + \left( \frac{431}{216}a - \frac{217}{36}b + \frac{53}{6}c - 7d + \frac{13}{2}ab - 3ac - \frac{31}{8}a^2 + a^3 - \frac{3}{2}b^2 - \frac{1357}{653184} \right) \frac{1}{n^7} + O \left( \frac{1}{n^8} \right).
\end{equation}

By means of Lemma 6, the highest convergence rate of the sequence $\lambda_n$ is obtained when the first four coefficients in (4.18) vanish, that is

\begin{equation*}
\begin{cases}
  a - \frac{1}{72} = 0 \\
  \frac{3}{2}b - \frac{5}{4}a + \frac{11}{1080} = 0 \\
  -a^2 + \frac{14}{9}a - \frac{8}{3}b + 2c - \frac{89}{12960} = 0 \\
  \frac{305}{72}b - \frac{775}{432}a - \frac{55}{12}c + \frac{5}{2}d - \frac{5}{2}ab + \frac{25}{12}a^2 + \frac{223}{54432} = 0
\end{cases}
\end{equation*}

Hence $a = \frac{1}{72}, b = \frac{31}{6480}, c = -\frac{139}{155520}, d = -\frac{9871}{6531840}$, and consequently,

\begin{equation*}
\lambda_n - \lambda_{n+1} = -\frac{324179}{391910400} \cdot \frac{1}{n^7} + O \left( \frac{1}{n^8} \right).
\end{equation*}

By Lemma 6, we get (3.13). Finally, we compare our formula

\begin{equation}
(4.19) \quad n! \approx \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n - \frac{1}{6} + \frac{1}{72n} + \frac{31}{6480n^2} - \frac{139}{155520n^3} - \frac{9871}{6531840n^4}}} := \mu_n
\end{equation}

and the following truncation of Stirling’s series (1.3)

\begin{equation}
(4.20) \quad n! \approx \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \left( 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{517}{2488320n^4} \right) := \sigma_n.
\end{equation}
To be precise, we prove the following

\[
\begin{align*}
\frac{n!}{\sqrt{n - \frac{1}{6} + \frac{1}{72n} + \frac{31}{6480n^2}} - \frac{139}{155520n^3} - \frac{9871}{6531840n^4}} & < n^{n+1} e^{-n\sqrt{2\pi}} \\
& < \sqrt{2\pi n} \left( \frac{ne}{n} \right)^n \left( 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{517}{2488320n^4} \right)
\end{align*}
\]  

(the first inequality holds for every \( n \geq 1 \), while the second one holds for every \( n \geq 37 \)).

The first inequality (4.21) follows from the decreasing monotonicity to 1 of the sequence

\[
t_n = \frac{1}{(n!)^2} n - \frac{1}{6} + \frac{1}{72n} + \frac{31}{6480n^2} - \frac{139}{155520n^3} - \frac{9871}{6531840n^4}.
\]

To prove this, we denote \( \frac{t_{n+1}}{t_n} = \exp f(n) \) with

\[
f(x) = (2x + 2) \ln \left( 1 + \frac{1}{x} \right) - \ln \frac{G(x+1)}{G(x)} - 2
\]

and

\[
G(x) = x - \frac{1}{6} + \frac{1}{72x} + \frac{31}{6480x^2} - \frac{139}{155520x^3} - \frac{9871}{6531840x^4}.
\]

The function \( f \) is concave on \([1, \infty)\) since

\[
f''(x) = -\frac{P(x)}{W^2(x)}
\]

where \( W \) and \( P \) are polynomials. Moreover \( P(x) > 0 \), for every \( x \in [1, \infty) \), since it can be written as powers of \((x - 1)\) with all coefficients positive,

\[
P(x) = 168639095611600732894003200(x - 1)^{15} + \ldots + 166879635666653109297872232176.
\]

Now \( f \) is strictly concave on \([1, \infty)\) with \( f(\infty) = 0 \), so \( f < 0 \) on \([1, \infty)\). Thus \( t_n \) is strictly decreasing to 1 and consequently \( t_n > 1 \).

Finally, the second inequality in (4.21) can be directly proved for \( n = 37, 38, \ldots, 70 \) by using a computer software for symbolic computation. For \( n \geq 71 \) case we have

\[
\left( 1 - \frac{1}{6n} + \frac{1}{72n^2} + \frac{31}{6480n^3} - \frac{139}{155520n^4} \right) \left( 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{517}{2488320n^4} - \frac{9871}{6531840n^5} \right)^2 - 1 = \frac{424656032149536768000n^{14}}{2442320} > 0,
\]

which concludes the second inequality (4.21) (\( T(n) \) is a tenth degree polynomial which can be written as powers of \((x - 71)\) with all coefficients positive).

\section*{References}


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