A substantially improvement of the Stirling formula

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Abstract: We introduce the approximation formula

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} + \frac{1}{12en} \right)^n \]

for the factorial function. Finally, some numerical computations are made to prove the superiority over other well-known formulas.

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Keywords: Factorial function; gamma function; Stirling formula; rate of convergence; approximations

1. Introduction

The factorial function \( n! = 1 \cdot 2 \cdot 3 \cdots n \) defined for positive integers \( n \), and its extension gamma function

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \]

to the real and complex values \( z \), except the non-positive integers play an important role in pure mathematics and in other branches of science, as probability theory, statistical physics as well as combinatorics. In consequence, these functions have caught the interest of many authors.

The problem of extending the factorial function to non-integer arguments was first considered by the Swiss mathematician Daniel Bernoulli (1700-1782) and the Prussian mathematician Christian Goldbach (1690-1764) in the 1720s, and was solved at the end of the same decade by the Swiss mathematician Leonhard Euler (1707-1783), who gave the following definition as an infinite product

\[ n! = \prod_{k=1}^{\infty} \left( \frac{1 + \frac{1}{k}}{1 + \frac{n}{k}} \right)^n. \]

The German mathematician Karl Friederich Gauss (1777-1855) rewrote this formula as

\[ \Gamma(z) = \lim_{m \to \infty} \frac{m^z m!}{z(z+1)(z+2)\cdots(z+m)} \]

and used it to discover new properties of the gamma function as the multiplication formula and investigated the connection between the gamma function and elliptic integrals.
One of the most known and most used formula for approximating the large factorials is Stirling’s formula

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n = \sigma_n, \]

first discovered by the Scottish mathematician James Stirling (1692–1770) and the French mathematician Abraham de Moivre (1667-1754). For proofs and details, see [1, 12, 14].

While in probabilities, applied statistics, or statistical physics, the approximation given by Stirling’s formula is satisfactory, in pure mathematics, more precise estimations are necessary. As a consequence, there have been a lot of variety of approaches to Stirling’s formula, ranging from elementary to advanced methods. We mention the estimation

\[ n! \sim \sqrt{2\pi} \left( \frac{n + 1/2}{e} \right)^{n+1/2}, \]

due to W. Burnside, whose superiority over Stirling’s formula was proved in [5]. N. Batir [4] proved the superiority of its approximation formula

\[ n! \sim \frac{n^{n+1}e^{-n}2\sqrt{\pi}}{\sqrt{n-1/6}}, \]

over Burnside’s formula. A much better approximation which gives good results, having a simple form, is the following, due to R. W. Gosper [6]:

\[ n! \sim \sqrt{2\pi} \left( n + \frac{1}{6} \right) \left( \frac{n}{e} \right)^n = \gamma_n. \]

As a recent example, giving similar results with Gosper’s formula, we mention the under- and upper-approximations

\[ \sqrt{2\pi} e \cdot e^{-\omega} \left( \frac{n + \omega}{e} \right)^{n+1/2} < n! < \sqrt{2\pi} e \cdot e^{-\zeta} \left( \frac{n + \zeta}{e} \right)^{n+1/2}, \]

established in [8], where \( \omega = (3 - \sqrt{3})/6 \) and \( \zeta = (3 + \sqrt{3})/6 \).

Furthermore, many other increasingly accurate approximations can be constructed, but with a sacrifice of simplicity. We introduce the new approximation formula

\[ n! \sim \sqrt{2\pi} n \left( \frac{n}{e} + \frac{1}{12en} \right)^n = \mu_n \quad (1.1) \]

which has great superiority over all the previous formulas. In fact, it is comparable only with the Ramanujan formula [13]:

\[ n! \sim \sqrt{\pi} \left( \frac{n}{e} \right)^n \sqrt{n \left( 1 + 4n (1 + 2n) \right) + \frac{1}{30}} = \rho_n \]
but our new formula (1.1) has the advantage of simplicity.

The Stirling’s formula is in fact the first approximation to the following asymptotic series:

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \left( 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \ldots \right) \]  

(1.2)

also named the Stirling series, e.g., [1, p. 257]. Our formula (1.1) is surprisingly connected with the Stirling series in the sense that if we consider the first three terms of the binomial expansion

\[
\left( \frac{n}{e} + \frac{1}{12en} \right)^n = \left( \frac{n}{e} \right)^n + \left( \frac{n}{e} \right)^{n-1} \frac{1}{12en} + \left( \frac{n}{e} \right)^{n-2} \left( \frac{1}{12en} \right)^2 + \ldots
\]

then the following approximation is obtained:

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \left( 1 + \frac{1}{12n} + \frac{n-1}{288n^3} + \ldots \right), \]

and we remark that the first three coefficients are the coefficients from the Stirling series (1.2). We believe that also this connection opens new directions of discovering other properties and further improvements.

2. Result and numerical computations

Unlike most formulas which are variations of the asymptotic expansion of the Stirling formula, our formula (1.1) has an original construction mode. The idea is to improve the factor \((n/e)^n\) in the Stirling’s formula, i.e., to find the real constant \(a\) which produces the best approximation of the form

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} + \frac{a}{n} \right)^n \]  

(2.1)

One method to measure the accuracy of an approximation formula of the form (2.1) is to use the following

Lemma 2.1. If \((x_n)_{n \geq 1}\) is convergent to zero and

\[ \lim_{n \to \infty} n^k (x_n - x_{n+1}) = l \in [-\infty, \infty], \]

with \(k > 1\), then

\[ \lim_{n \to \infty} n^{k-1} x_n = \frac{l}{k-1}. \]

Recently, this result was used by Mortici [9] to establish new sharp approximations of the gamma function in terms of the digamma function, which is a
refinement of a previous result of Alzer and Batir [3]. See also [2, 7]. Lemma 2.1 is also a powerful tool for accelerating some convergences. For complete proof and other details, see [9, 11, 10].

One way to measure the accuracy of an approximation of type (2.1) is to define the sequence \((w_n)_{n \geq 1}\) by the relations

\[
 n! = \sqrt{2\pi n} \left(\frac{n}{e} + \frac{a}{n}\right)^n \exp w_n, \quad n \geq 1
\]

and to consider an approximation of type (2.1) to be better when the sequence \((w_n)_{n \geq 1}\) faster converges to zero. In fact, we have

\[
 w_n = \ln n! - \ln \sqrt{2\pi} - \frac{1}{2} \ln n - n \ln \left(\frac{n}{e} + \frac{a}{n}\right)
\]

and

\[
 w_n - w_{n+1} = -\frac{1}{2} \ln n (n+1) - n \ln \left(\frac{n}{e} + \frac{a}{n}\right) + (n+1) \ln \left(\frac{n+1}{e} + \frac{a}{n+1}\right).
\]

As we wish to use Lemma 2.1, it is natural to expand \(w_n - w_{n+1}\) in power series in \(n^{-1}\). Using a mathematical and analytical software such as MAPLE, we obtain

\[
 w_n - w_{n+1} = \left(-ae + \frac{1}{12}\right) \frac{1}{n^2} + \left(ae - \frac{1}{12}\right) \frac{1}{n^3} + \left(\frac{3}{2}a^2e^2 - ae + \frac{3}{40}\right) \frac{1}{n^4} - \left(3a^2e^2 - ae + \frac{1}{15}\right) \frac{1}{n^5} - \left(\frac{5}{3}a^3e^3 - 5a^2e^2 + ae - \frac{5}{84}\right) \frac{1}{n^6} + O\left(\frac{1}{n^7}\right). \tag{2.2}
\]

We are in position to give the following

**Theorem 2.1.** i) If \(a \neq \frac{1}{12e}\), then the rate of convergence of the sequence \((w_n)_{n \geq 1}\) is \(n^{-1}\), since

\[
 \lim_{n \to \infty} nw_n = -ae + \frac{1}{12} \neq 0.
\]

ii) If \(a = \frac{1}{12e}\), then the rate of convergence of the sequence \((w_n)_{n \geq 1}\) is \(n^{-3}\), since

\[
 \lim_{n \to \infty} n^3 w_n = \frac{1}{1440}.
\]

The proof follows from Lemma 2.1 and using the relation (2.2). In case \(a = \frac{1}{12e}\), the relation (2.2) becomes

\[
 w_n - w_{n+1} = \frac{1}{480n^4} - \frac{1}{240n^5} + \frac{361}{36288n^6} + O\left(\frac{1}{n^7}\right),
\]

which justifies ii).

Our new approximation formula \(n! \sim \mu_n\) gives surprisingly good results. It is much better than the Gosper’s approximation \(n! \sim \gamma_n\) (which is already stronger
than Stirling, Burnside, and Batir formulas) and it has the same order of accuracy with the Ramanujan approximation \( n! \sim \rho_n \). See the next table, where also the Stirling’s approximation \( n! \sim \sigma_n \) was included for sake of completeness.

<table>
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<th>( n! - \sigma_n )</th>
<th>( n! - \gamma_n )</th>
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</table>

As we can see from the last two columns of this table, Ramanujan’s approximation \( n! \sim \rho_n \) and the approximation \( n! \sim \mu_n \) from the present paper are comparable; the errors of these estimates are much smaller than the errors given by Stirling’s and Gosper’s formulas.

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**References**


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