New methods of approach related to the Riemann Hypothesis.

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Abstract. In this paper we develop techniques related to the Riemann Hypothesis that are based on the series expansion of the Riemann Xi function, and the asymptotic behavior of $\xi^{(2n)}(\frac{1}{2})$.

1. Introduction.

In this article I present a formal treatment of a special limit process associated with an infinite series, such that when we add a new term of the series, some of the previous terms also slightly change, but in such a manner that the limit can be precisely defined (mainly section 3 and theorem 2).

I also present a sufficient condition for an infinite series, in order to take only positive values (propositions 3 and 5).

In section 2 I present a theorem related to convexity that will be useful later in the article.

The tools presented here allow us to attack Riemann’s Hypothesis in a completely new manner. The presentation is informal but the results are clearly stated and the proofs given in full. I believe that these tools will allow the discovery of a solution to this 150 year old unsolved problem.

2. A theorem related to convexity.

Theorem 1. We are given a complex function with real positive values that does not vanish identically on the domain specified. We are given such a positive, analytic, convex function $F(\sigma + it)$ (convex when seen as a function of the real variable $\text{Re} (\sigma + it) = \sigma$), defined on the strip $0 < \sigma < 1$. If this function satisfies the functional equation $F(s) = F(1 - s)$, then this function must have all its zeros on the vertical $\text{Re} (\sigma + it) = \sigma = \frac{1}{2}$.

Proof. We assume that the function $F(s)$ has a zero at $x + it$, where $x < \frac{1}{2}$. Then from the functional equation $F(s) = F(1 - s)$, the function also has a zero at $1 - x - it$. Since the complex conjugate of $1 - x - it$ is $1 - x + it$, the function $F(s)$ will also have a zero at $1 - x + it$.

We define the real function with real values (for fixed $t$):

$\varphi(\sigma) := F(\sigma + it)$.

From the assumptions of the theorem, the function $\varphi(\sigma)$ is convex for $0 < \sigma < 1$. For any $x_1$ and $x_2$ we have:

$\varphi(\sigma) \leq \frac{x_2 - \sigma}{x_2 - x_1} \cdot \varphi(x_1) + \frac{\sigma - x_1}{x_2 - x_1} \cdot \varphi(x_2)$ for $x_1 < \sigma < x_2$. 

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We take \( x_1 \) to be the real part of a zero of \( F(s) \), and \( x_2 = 1 - x_1 \) (which is the real part of another zero, for a fixed \( t \)). That means that \( \varphi(\sigma) \leq 0 \) for \( x_1 < \sigma < x_2 \). Since the hypothesis of the theorem states that \( F(s) \) is positive, that means that \( \varphi(\sigma) \geq 0 \) for \( x_1 < \sigma < x_2 \). The conclusion is that for that fixed \( t \), we have \( \varphi(\sigma) = F(\sigma + it) = 0 \) for \( x_1 < \sigma < x_2 \).

We know that if a function is analytic in a region, and vanishes at all points of any smaller region included in the given region, or along any arc of a continuous curve in the region, then it must vanish identically. Since we see that \( F(\sigma + it) \) vanishes on the segment joining the two zeroes of \( F(s) \), then \( F(\sigma + it) \) would have to vanish identically of the domain under consideration. We reached a contradiction, since we assumed that \( F(\sigma + it) \) does not vanish identically. Our assumption, that the function \( F(s) \) has a zero at \( x + it \), where \( x < \frac{1}{2} \) is false.

The function \( F(s) \) has all its zeroes on the vertical Re\( (s) = \frac{1}{2} \). The horizontal segment joining the two zeroes must collapse to a point. QED.

3. The main method of approach and basic calculations.

We consider the Riemann Xi function defined as:

\[
\xi(s) := \frac{1}{2} \cdot s \cdot (s - 1) \cdot \Gamma\left(\frac{1}{2} s\right) \cdot \pi^{-\frac{s}{2}} \cdot \zeta(s)
\]

For the Riemann Xi function \( \xi(s) \) we have the following series expansion:

\[
\xi(s) = a_0 + a_2 \cdot (s - \frac{1}{2})^2 + a_4 \cdot (s - \frac{1}{2})^4 + a_6 \cdot (s - \frac{1}{2})^6 + \ldots \ldots , \tag{1}
\]

where all the coefficients \( a_{2n} \) are positive real numbers. This statement is proved in [1], page 17.

We define the following functions. We define:

\[
F_{2N}(s) := a_0 + a_2 \cdot (s - \frac{1}{2})^2 + a_4 \cdot (s - \frac{1}{2})^4 + a_6 \cdot (s - \frac{1}{2})^6 + \ldots + a_{2N} \cdot \left( s - \frac{1}{2} \right)^{2N} \tag{2}
\]

We have then: \( |F_{2N}(\sigma + it)|^2 \to |\xi(\sigma + it)|^2 \), when \( N \to \infty \) (more general, we have \( F_{2N}(\sigma + it) \to \xi(\sigma + it) \) when \( N \to \infty \)).

We also define (for a fixed \( t \)):

\[
f_{2N}(\sigma) := |F_{2N}(\sigma + it)|^2 . \tag{3}
\]

In the following, we write \( \beta = \sigma - \frac{1}{2} \).

We start with the identities:

\[
(\beta + it)^2 = \left( \begin{array}{c} 2 \\
0
\end{array} \right) \cdot \beta^2 - \left( \begin{array}{c} 2 \\
1
\end{array} \right) \cdot t^2 + i \cdot \left( \begin{array}{c} 4 \\
2
\end{array} \right) \cdot \beta t
\]

\[
(\beta + it)^4 = \left( \begin{array}{c} 4 \\
0
\end{array} \right) \cdot \beta^4 - \left( \begin{array}{c} 4 \\
2
\end{array} \right) \cdot \beta^2 t^2 + \left( \begin{array}{c} 4 \\
1
\end{array} \right) \cdot t^4 + i \cdot \left( \begin{array}{c} 6 \\
3
\end{array} \right) \cdot \beta^3 t - \left( \begin{array}{c} 4 \\
3
\end{array} \right) \cdot \beta t^3
\]

\]

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\[(\beta + it)^6 = \binom{6}{0} \cdot \beta^6 - \binom{6}{2} \cdot \beta^4 t^2 + \binom{6}{4} \cdot \beta^2 t^4 + \binom{6}{6} \cdot t^6 + i \cdot \left(\binom{6}{1} \cdot \beta^5 t - \binom{6}{3} \cdot \beta^3 t^3 + \binom{6}{5} \cdot \beta t^5\right)\]

\[(\beta + it)^8 = \binom{8}{0} \cdot \beta^8 - \binom{8}{2} \cdot \beta^6 t^2 + \binom{8}{4} \cdot \beta^4 t^4 + \binom{8}{6} \cdot \beta^2 t^6 + \binom{8}{8} \cdot t^8 + i \cdot \left(\binom{8}{1} \cdot \beta^7 t - \binom{8}{3} \cdot \beta^5 t^3 + \binom{8}{5} \cdot \beta^3 t^5 - \binom{8}{7} \cdot \beta t^7\right)\]

It is clear how to continue this sequence of identities up to \((\beta + it)^{2N}\).

We write for the real and imaginary part of \(F_{2N}(\sigma + it)\) as \(\text{Re}(F_{2N}(\sigma + it))\) and \(\text{Im}(F_{2N}(\sigma + it))\).

It is clear that we have:

\[\text{Re}(F_{2N}(\sigma + it)) = B_0 + B_2 \cdot \beta^2 + B_4 \cdot \beta^4 + B_6 \cdot \beta^6 + \cdots + B_{2N} \cdot \beta^{2N}. \tag{4}\]

\[\text{Im}(F_{2N}(\sigma + it)) = B_1 \cdot \beta + B_3 \cdot \beta^3 + B_5 \cdot \beta^5 + \cdots + B_{2N-1} \cdot \beta^{(2N-1)}. \tag{5}\]

The coefficients will depend on \(t\) and they will have the form:

\[B_0 = a_0 - a_2 \cdot \binom{2}{0} \cdot t^2 + a_4 \cdot \binom{4}{0} \cdot t^4 \pm \cdots + (-1)^N \cdot a_{2N} \cdot \binom{2N}{2N} \cdot t^{2N}\]

\[B_2 = a_2 \cdot \binom{2}{0} - a_4 \cdot \binom{4}{2} \cdot t^2 + a_6 \cdot \binom{6}{4} \cdot t^4 \pm \cdots + (-1)^{N+1} \cdot a_{2N} \cdot \binom{2N}{2N-2} \cdot t^{2N-2}\]

\[B_4 = a_4 \cdot \binom{4}{0} - a_6 \cdot \binom{6}{2} \cdot t^2 + a_8 \cdot \binom{8}{4} \cdot t^4 \pm \cdots + (-1)^{N+2} \cdot a_{2N} \cdot \binom{2N}{2N-4} \cdot t^{2N-4}\]

\[B_6 = a_6 \cdot \binom{6}{0} - a_8 \cdot \binom{8}{2} \cdot t^2 + a_{10} \cdot \binom{10}{4} \cdot t^4 \pm \cdots + (-1)^{N+3} \cdot a_{2N} \cdot \binom{2N}{2N-6} \cdot t^{2N-6}\]

\[B_{2N-2} = a_{2N-2} \cdot \binom{2N-2}{0} - a_{2N} \cdot \binom{2N}{2} \cdot t^2\]

\[B_{2N} = a_{2N} \cdot \binom{2N}{0}\]

In general we have:

\[B_{2i} = \sum_{k=0}^{N} (-1)^{k+i} \cdot a_{2k} \cdot \binom{2k}{2k-2i} \cdot t^{2k-2i} \tag{6}\]

In the same way, the odd order coefficients will have the form:

\[B_1 = a_2 \cdot \binom{2}{1} \cdot t - a_4 \cdot \binom{4}{3} \cdot t^3 + a_6 \cdot \binom{6}{5} \cdot t^5 \pm \cdots + (-1)^{N+1} \cdot a_{2N} \cdot \binom{2N}{2N-1} \cdot t^{2N-1}\]

\[B_3 = a_4 \cdot \binom{4}{1} \cdot t - a_6 \cdot \binom{6}{3} \cdot t^3 + a_8 \cdot \binom{8}{5} \cdot t^5 \pm \cdots + (-1)^{N+2} \cdot a_{2N} \cdot \binom{2N}{2N-3} \cdot t^{2N-3}\]

\[B_5 = a_6 \cdot \binom{6}{1} \cdot t - a_8 \cdot \binom{8}{3} \cdot t^3 + a_{10} \cdot \binom{10}{5} \cdot t^5 \pm \cdots + (-1)^{N+3} \cdot a_{2N} \cdot \binom{2N}{2N-5} \cdot t^{2N-5}\]
\[ B_{2N-3} = a_{2N-2} \cdot \binom{2N-2}{1} \cdot t - a_{2N} \cdot \binom{2N}{3} \cdot t^3 \]

\[ B_{2N-1} = a_{2N} \cdot \binom{2N}{1} \cdot t \]

In general we have:

\[ B_{2i+1} = \sum_{k=0}^{N} (-1)^{k+i+1} \cdot a_{2k} \cdot \binom{2k}{2k-2i-1} \cdot t^{2k-2i-1} \quad (7) \]

In relations (6) and (7) except for the usual conventions, we make the following conventions about the binomial coefficients:

**Conventions.** For \( x, y > 0 \) we have \( \binom{0}{0} = \frac{x}{0} = 1, \binom{0}{-y} = \frac{x}{y} = 0 \), and if \( x < y \) then \( \frac{x}{y} = 0 \).

Relations (6) and (7) can be written in the unified form:

\[ B_m = \sum_{k=0}^{N} (-1)^{k+m-\left[ \frac{m}{2} \right]} \cdot a_{2k} \cdot \binom{2k}{2k-m} \cdot t^{2k-m} \quad (8) \]

Here \( \left[ \frac{m}{2} \right] \) represents the integer part of \( \frac{m}{2} \) and we use the conventions about the binomial coefficients above. Also \( m \) takes values from 0 to \( 2N \). It would be better if we wrote \( B_{m,2N} \) instead of \( B_m \) but we use the latter notation for simplicity.

We have then:

\[ |F_{2N}(\sigma + it)|^2 = \left( B_0 + B_2 \cdot \beta^2 + B_4 \cdot \beta^4 + B_6 \cdot \beta^6 + \ldots + B_{2N} \cdot \beta^{2N} \right)^2 + (B_1 \cdot \beta + B_3 \cdot \beta^3 + B_5 \cdot \beta^5 + \ldots + B_{2N-1} \cdot \beta^{(2N-1)})^2 \]

\[ |F_{2N}(\sigma + it)|^2 = B_0^2 + \beta^2 \cdot \left( B_1^2 + 2 \cdot B_0 \cdot B_2 \right) + \beta^4 \cdot \left( B_2^2 + 2 \cdot B_0 \cdot B_4 + 2 \cdot B_1 \cdot B_3 \right) + \beta^6 \cdot \left( B_3^2 + 2 \cdot B_0 \cdot B_6 + 2 \cdot B_1 \cdot B_5 + 2 \cdot B_2 \cdot B_4 \right) + \ldots + \beta^{4N} \cdot B_{2N}^2 \quad (9) \]

We remember that \( \beta = \left( \sigma - \frac{1}{2} \right) \).

From relation (8), we have then:

\[ \frac{d^2}{d\sigma^2} f_{2N}(\sigma) = \frac{d^2}{d\sigma^2} |F_{2N}(\sigma + it)|^2 = 2 \cdot \left( B_1^2 + 2 \cdot B_0 \cdot B_2 \right) + 12 \cdot \left( \sigma - \frac{1}{2} \right)^2 \cdot \left( B_2^2 + 2 \cdot B_0 \cdot B_4 + 2 \cdot B_1 \cdot B_3 \right) + 30 \cdot \left( \sigma - \frac{1}{2} \right)^4 \cdot \left( B_3^2 + 2 \cdot B_0 \cdot B_6 + 2 \cdot B_1 \cdot B_5 + 2 \cdot B_2 \cdot B_4 \right) + \ldots + (4N) \cdot (4N - 1) \cdot \left( \sigma - \frac{1}{2} \right)^{4N-2} \cdot B_{2N}^2 \quad (10) \]

From (10) we see that we have to calculate the quantities:

\[ D'_{2n,2N} := \sum_{p+q=2n, \ 0\leq p,q\leq 2N} B_p \cdot B_q. \]

Using relation (8) we have:
\[ D'_{2n,2N} = \sum_{p+q=2n} a_{2i} \cdot a_{2j} \cdot \left( \frac{2i}{2i-p} \right) \cdot \left( \frac{2j}{2j-q} \right) \cdot t^{2k-2n} \] 

We also define the quantities:

\[ D_{2n,2N} := \sum_{p+q=2n} a_{2i} \cdot a_{2j} \cdot \left( \frac{2i}{2i-p} \right) \cdot \left( \frac{2j}{2j-q} \right) \cdot t^{2k-2n} \] 

We see that \( 0 \leq 2n \leq 4N \), and we have:

\[ |F_{2N}(\sigma + it)|^2 = D'_{0,2N} + \beta^2 \cdot D'_{2,2N} + \beta^4 \cdot D'_{4,2N} + \beta^6 \cdot D'_{6,2N} + \ldots + \beta^{4N} \cdot D'_{4N,2N} \] 

We also note that when \( n \) is greater than \( N \) the quantities \( D'_{2n,2N} \) will be incomplete (will not contain all its terms), but as \( N \) increases the number of complete \( D'_{2n,2N} \)'s in (13) will increase. The difference between \( D'_{2n,2N} \) and \( D_{2n,2N} \) is that in the third sum the condition \( 0 \leq p, q \leq 2N \) is discarded. For large \( N \), the quantities \( D'_{2n,2N} \) will be equal to \( D_{2n,2N} \), for \( 2n \leq 2N \), but will start to differ for \( 2n \geq 2N \). As \( N \) increases though, more and more terms in (13) will have their coefficients \( D'_{2n,2N} \) equal to \( D_{2n,2N} \).

We write \( D_{2n} \) for the quantities:

\[ D_{2n} := \sum_{k=0}^{\infty} \left( \sum_{i+j=k} \frac{a_{2i} \cdot a_{2j} \cdot \left( \frac{2i}{2i-p} \right) \cdot \left( \frac{2j}{2j-q} \right)}{t^{2k-2n}} \right) \] 

We write then:

\[ C_{2k,2n} := \sum_{i+j=k} \frac{a_{2i} \cdot a_{2j} \cdot \left( \frac{2i}{2i-p} \right) \cdot \left( \frac{2j}{2j-q} \right)}{t^{2k-2n}} \] 

Relation (14) can then be written as:

\[ D_{2n} = \sum_{k=0}^{\infty} C_{2k,2n} \cdot t^{2k-2n} \] 

We note that in (16) we used the conventions mentioned before, in fact (16) can also be written:

\[ D_{2n} = \sum_{k=n}^{\infty} C_{2k,2n} \cdot t^{2k-2n} \]

We see that the important relation seems to be (15). This is a relation that involves only the coefficients \( a_{2n} \) that are involved in (1).

We also define \( \alpha_{2n} := a_{2n} \cdot (2n)! = \xi^{(2n)}(\frac{1}{2}) \).

After these calculations and definitions we are ready to state the main theorem on which the rest of the results will be based.
Theorem 2. If for any $n \geq 0$ we have $\alpha_{2n} \leq (\ln(n + 2))^{2n}$ (in other words, if $\xi^{(2n)}(\frac{1}{2}) \leq (\ln(n + 2))^{2n}$), then the following series are absolutely convergent (also the series (16) and (16')) and the following relations are well defined and valid:

$$\sum \leq 2n \cdot \left( \sigma - \frac{1}{2} \right)^{2n-1} \cdot D_{2n} + \cdots \cdots$$

(18)

$$\frac{d}{d\sigma} |\xi(\sigma + it)|^2 = 2 \cdot \left( \sigma - \frac{1}{2} \right) \cdot D_2 + 4 \cdot \left( \sigma - \frac{1}{2} \right)^3 \cdot D_4 + 6 \cdot \left( \sigma - \frac{1}{2} \right)^5 \cdot D_6 + \cdots \cdots$$

(19)

$$\frac{d^2}{d\sigma^2} |\xi(\sigma + it)|^2 = 2 \cdot D_2 + 12 \cdot \left( \sigma - \frac{1}{2} \right)^2 \cdot D_4 + 30 \cdot \left( \sigma - \frac{1}{2} \right)^4 \cdot D_6 + \cdots \cdots$$

(20)

Proof. We start from relation (15) and we use the definition (17).

Relation (15) can then be written:

$$C_{2k,2n} = \sum_{i+j=k} \sum_{p+q=2n} (-1)^k \left( \frac{q}{2} - \frac{1}{2} \right) \cdot \frac{\alpha_{2i} \cdot \alpha_{2j}}{(2i)! \cdot (2j)!} \cdot \frac{1}{(2i-p) \cdot (2j-q)}$$

Now we calculate (we take into account the fact that $i = k$ and $p + q = 2n$):

$$\frac{1}{(2i)!} \cdot \frac{1}{(2j)!} \cdot \frac{1}{(2i-p)!} \cdot \frac{1}{(2j-q)!} \cdot \frac{1}{p! \cdot q!} = \frac{1}{(2k)!} \cdot \frac{1}{(2k-2n)! \cdot (2n)!} \cdot \frac{1}{(2k-2n)! \cdot (2n)!} \cdot \frac{1}{p! \cdot q!} = \frac{1}{(2k)!} \cdot \frac{1}{(2k-2n)! \cdot (2n)!} \cdot \frac{1}{(2k-2n)! \cdot (2n)!} \cdot (2n)! \cdot (2n)! = \frac{1}{(2k)!} \cdot \frac{1}{(2k-2n)! \cdot (2n)!}$$

Using the relations above, we can write (15) in the form (where $i + j = k$ and $p + q = 2n$):

$$C_{2k,2n} = \frac{1}{(2k)!} \cdot \frac{1}{(2k-2n)! \cdot (2n)!} \cdot \sum_{s=0}^{\infty} \sum_{s=0}^{\infty} (-1)^k \left( \frac{p}{2} - \frac{1}{2} \right) \cdot \frac{\alpha_{2i} \cdot \alpha_{2j}}{(2i)! \cdot (2j)!} \cdot \frac{1}{(2i-p) \cdot (2j-q)}$$

(21)

From hypothesis we note that we have $\alpha_{2i} \cdot \alpha_{2j} \leq (\ln(i + 2))^{2i} \cdot (\ln(j + 2))^{2j} \leq (\ln(k + 2))^{4k}$

From (21) we have then (for $s \geq 0$):

$$\frac{\ln(n + s + 2)^{4n+4s}}{(2n)! \cdot (2s)!} \cdot 2^{2s-1} \cdot 2^n = \frac{2^{2n}}{(2n)!} \cdot 2^{2s-1} \cdot (\ln(n + s + 2))^{4n+4s}$$

(22)

As a consequence, using (16), (16') and the relation above we have:

$$|D_{2n}| = \sum_{s=0}^{\infty} \sum_{s=0}^{\infty} (-1)^k \left( \frac{p}{2} - \frac{1}{2} \right) \cdot \frac{\alpha_{2i} \cdot \alpha_{2j}}{(2i)! \cdot (2j)!} \cdot \frac{1}{(2i-p) \cdot (2j-q)}$$

(23)
From (22) we see that the series defined by (16) and (16)’ are absolutely convergent (the coefficients decrease very fast), so the series on the right side of relations (18), (19) and (20) are well defined.

We write the proof for (18), the other two are similar. We consider the expression:

\[ H(\sigma + it) = D_0 + (\sigma - \frac{1}{2})^2 \cdot D_2 + (\sigma - \frac{1}{2})^4 \cdot D_4 + (\sigma - \frac{1}{2})^6 \cdot D_6 + \cdots + \left(\sigma - \frac{1}{2}\right)^{2n} \cdot D_{2n} + \cdots \]

We know that for any \( \varepsilon \geq 0 \) there is a \( n_0 \) such that for all \( n \geq n_0 \) we have:

\[ |D_0 - D'_{0,2n}| \leq \varepsilon, \ |D_2 - D'_{2,2n}| \leq \varepsilon, \ |D_4 - D'_{4,2n}| \leq \varepsilon, \ \ldots, \ |D_{2n} - D'_{4n,2n}| \leq \varepsilon, \ |D_{4n+2} | \leq \varepsilon, \ |D_{4n+4} | \leq \varepsilon, \ |D_{4n+6} | \leq \varepsilon, \ \ldots \]

We also know that \( |\sigma - \frac{1}{2}| \leq \frac{1}{2} \). As a consequence, using (13) we can write:

\[ |H(\sigma + it) - |F_{2n}(\sigma + it)|^2| \leq \varepsilon + \frac{1}{2^n} \cdot \varepsilon + \frac{1}{4n} \cdot \varepsilon + \cdots + \frac{1}{4^n} \cdot \varepsilon + \frac{1}{2^{4n+2}} \cdot \varepsilon + \cdots = \varepsilon \cdot \left(1 + \frac{1}{2^n} + \frac{1}{4^n} + \cdots + \frac{1}{4^n} + \frac{1}{2^{4n+2}} + \cdots \right) < 2\varepsilon. \]

In other words, we have proved that \( |H(\sigma + it) - |F_{2n}(\sigma + it)|^2| \to 0 \) as \( N \to \infty \).

That basically means that \( H(\sigma + it) = |E(\sigma + it)|^2 \)

Relations (19) and (20) can be proved in a similar manner. \textbf{QED.}

\textbf{Proposition 1.} We consider the analytic function \( f(s) \) defined on the strip \( 0 \leq \text{Re}(s) \leq 1 \), given by series of the type (Taylor series at \( \frac{1}{2} \)):

\[ f(s) = b_0 + b_2 \cdot (s - \frac{1}{2})^2 + b_4 \cdot (s - \frac{1}{2})^4 + b_6 \cdot (s - \frac{1}{2})^6 + \cdots + b_{2n} \cdot (s - \frac{1}{2})^{2n} + \cdots , \]

We also define \( \mu_{2n} := b_{2n} \cdot (2n)! = f^{(2n)}(\frac{1}{2}) \)

If for any \( n \geq 0 \), the coefficients \( b_{2n} \) are real and satisfy the relations \( b_{2n} \geq 0 \) and for any \( n \geq 1 \) we have:

\[ f^{(1)}(\frac{1}{2}) \cdot f^{(2n)}(\frac{1}{2}) = f^{(2)}(\frac{1}{2}) \cdot f^{(2n-2)}(\frac{1}{2}) = f^{(4)}(\frac{1}{2}) \cdot f^{(2n-4)}(\frac{1}{2}) = \cdots = f^{(2u)}(\frac{1}{2}) \cdot f^{(2n-2u)}(\frac{1}{2}) = \cdots \]

then the function \( f(s) \) has all its zeros on the vertical \( \text{Re}(s) = \frac{1}{2} \).

\textbf{Proof.} We define, as in the case of the Xi function the following quantities (the calculations are similar):

\[ c_{2k,2n} = \frac{1}{(2k)!} \cdot \frac{(2k)}{2n} \cdot \sum_{0 \leq s \leq k} \sum_{0 \leq p \leq 2n} (-1)^k \cdot [\frac{k}{2}] \cdot [\frac{g}{3}] \cdot \mu_{2i} \cdot \mu_{2j} \cdot \left(\frac{2k-2n}{2i-p}\right)! \cdot \left(\frac{2n}{p}\right)! \]

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\[D_{2n} = \sum_{k=n}^{\infty} C_{2k,2n} \cdot t^{2k-2n}\]

Let's see how the expression \(\left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor\) behaves, for various values for \(n\) (where \(p + q = 2n\)).

For \(n = 0\).

\[
\begin{array}{|c|c|c|}
\hline
p & q & \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor \\
\hline
0 & 0 & 0 \\
\hline
\end{array}
\]

For \(n = 1\).

\[
\begin{array}{|c|c|c|}
\hline
p & q & \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor \\
\hline
0 & 2 & 1 \\
1 & 1 & 0 \\
2 & 0 & 1 \\
\hline
\end{array}
\]

For \(n = 2\).

\[
\begin{array}{|c|c|c|}
\hline
p & q & \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor \\
\hline
0 & 4 & 2 \\
1 & 3 & 1 \\
2 & 2 & 2 \\
3 & 1 & 1 \\
4 & 0 & 2 \\
\hline
\end{array}
\]

We note that when \(p\) is even, then \(n - \left\lfloor \frac{p}{2} \right\rfloor - \left\lfloor \frac{q}{2} \right\rfloor = 0\)

In general, we get the general pattern (the alternating even–odd when \(p\) takes values from 0 to \(2n\) is important).

We consider \(n \geq 1\).

We consider the case \(2k = 2n\). In this case, the expression \(2i - p\) in (21, or the similar relation applied for our function) can only take the value 0 (with the conventions mentioned before, all the other terms are 0).

In this case we have then:

\[C_{2n,2n} = \frac{1}{(2n)!} \cdot \left( \mu_0 \cdot \mu_{2n} \cdot \binom{2n}{0} + \mu_2 \cdot \mu_{2n-2} \cdot \binom{2n}{2} + \mu_4 \cdot \mu_{2n-4} \cdot \binom{2n}{4} + \cdots + \mu_{2n} \cdot \mu_0 \cdot \binom{2n}{2n} \right)\].
We note that if all the quantities $\mu_{2i} \cdot \mu_{2j}$ with $i + j = k = n$, take a constant value $\mu_0 \cdot \mu_{2n} = \mu_2 \cdot \mu_{2n+2} = \mu_4 \cdot \mu_{2n-4} = \ldots = M_n$, then $C_{2n,2n}$ would take the value: 

$$C_{2n,2n} = \frac{M_n}{(2n)!} \left( \binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \cdots + \binom{2n}{2n} \right) = \frac{M_n \cdot 2^{2n-1}}{(2n)!}.$$ 

We consider now the case $2k = 2n + 2$. In this case we have $0 \leq 2i - p \leq 2$.

In this case we have:

$$C_{2n+2,2n} = \frac{1}{(2n+2)!} \cdot \left( \binom{2n+2}{2n} \right) \left( \left( \binom{2n}{0} \cdot \binom{2n}{0} \cdot \binom{2n}{0} \cdot \binom{2n}{0} \right) + \mu_2 \cdot \mu_{2n+2} \cdot \binom{2n}{2} + \mu_4 \cdot \mu_{2n} \cdot \binom{2n}{4} \right) - \binom{2n}{0} \cdot \binom{2n}{0} \cdot \binom{2n}{0} \cdot \binom{2n}{0} - \mu_2 \cdot \mu_{2n+2} \cdot \binom{2n}{2} + \mu_4 \cdot \mu_{2n} \cdot \binom{2n}{4}.$$ 

We note that if all the quantities $\mu_{2i} \cdot \mu_{2j}$ with $i + j = k = n + 1$ take a constant value $\mu_0 \cdot \mu_{2n+2} = \mu_2 \cdot \mu_{2n} = \mu_4 \cdot \mu_{2n-2} = \ldots = M_{n+1}$, then $C_{2n+2,2n}$ would be zero, because in this case $C_{2n+2,2n} = \frac{1}{(2n+2)!} \cdot \left( \binom{2n+2}{2n} \right) \cdot \left( \binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \binom{2n}{6} \right) = 0$.

We consider now the case $2k = 2n + 4$. In this case we have $0 \leq 2i - p \leq 4$.

In this case we have:

$$C_{2n+4,2n} = \frac{1}{(2n+4)!} \cdot \left( \binom{2n+4}{2n} \right) \left( \left( \mu_0 \cdot \mu_{2n+4} \cdot \binom{2n}{0} \right) + \left( \mu_2 \cdot \mu_{2n+2} \cdot \binom{2n}{2} \right) + \left( \mu_4 \cdot \mu_{2n} \cdot \binom{2n}{4} \right) + \left( \mu_6 \cdot \mu_{2n-2} \cdot \binom{2n}{6} \right) - \left( \mu_2 \cdot \mu_{2n+2} \cdot \binom{2n}{2} \right) + \left( \mu_4 \cdot \mu_{2n} \cdot \binom{2n}{4} \right) + \left( \mu_6 \cdot \mu_{2n-2} \cdot \binom{2n}{6} \right) \right).$$ 

We note that if all the quantities $\mu_{2i} \cdot \mu_{2j}$ with $i + j = k = n + 2$ take a constant value $\mu_0 \cdot \mu_{2n+4} = \mu_2 \cdot \mu_{2n+2} = \mu_4 \cdot \mu_{2n} = \ldots = M_{n+2}$, then $C_{2n+4,2n}$ would be zero, because in this case $C_{2n+4,2n} = \frac{1}{(2n+4)!} \cdot \left( \binom{2n+4}{2n} \right) \cdot 8 \cdot \frac{M_{n+2}}{(2n+2)\cdot(2n+2)!} \left( \binom{2n}{0} - \binom{2n}{2} + \binom{2n}{4} - \binom{2n}{6} \right) = \frac{2n+2}{(2n+2)!} = 0$.

The calculations for $C_{2n+6,2n}$, $C_{2n+8,2n}$, $\ldots$, $C_{2n+2s,2n}$ can be done in a similar way. I will not write the general form because the calculations are similar (and the general form will look like (21)).

We note that if all the quantities $\mu_{2i} \cdot \mu_{2j}$ with $i + j = k = n + s$ take a constant value $\mu_0 \cdot \mu_{2n+2s} = \mu_2 \cdot \mu_{2n+2s-2} = \mu_4 \cdot \mu_{2n+2s-4} = \ldots = M_{n+s}$, then $C_{2n+2s,2n}$ would be zero, for $s \geq 1$.

We reach the conclusion that $D_{2n} \geq 0$ for any $n \geq 1$.

From the relation (similar to (20) but applied for our function):
\[
\frac{d^2}{d\sigma^2} |f(\sigma + it)|^2 = 2 \cdot D_2 + 12 \cdot (\sigma - \frac{1}{2})^2 \cdot D_4 + 30 \cdot (\sigma - \frac{1}{2})^4 \cdot D_6 + \cdots + 2n \cdot (2n-1) \cdot (\sigma - \frac{1}{2})^{2n-2} \cdot D_{2n} + \cdots
\]

we see that \( \frac{d^2}{d\sigma^2} |f(\sigma + it)|^2 \geq 0 \) for \( 0 \leq \sigma \leq 1 \).

Our function \( |f(\sigma + it)|^2 \), seen as a function of \( \sigma \) is a convex function, and from theorem 1 we conclude that all its zeros are on the vertical \( \sigma = \frac{1}{2} \). QED.

**Proposition 2.** We consider the analytic function \( f(s) \), where \( s = \sigma + i \cdot t \), defined on the strip \( 0 \leq \text{Re}(s) \leq 1 \), given by series of the type (Taylor series at \( \frac{1}{2} \)):

\[
f(s) = b_0 + b_2 \cdot (s - \frac{1}{2})^2 + b_4 \cdot (s - \frac{1}{2})^4 + b_6 \cdot (s - \frac{1}{2})^6 + \cdots + b_{2n} \cdot (s - \frac{1}{2})^{2n} + \cdots
\]

We define \( \mu_{2n} := b_{2n} \cdot (2n)! = f^{(2n)}(\frac{1}{2}) \).

We define the quantities:

\[
C_{2k,2n} = \frac{1}{(2k)!} \cdot \binom{2k}{2n} \cdot \sum_{s \leq k} \sum_{t \leq 2n} (-1)^{k-s-t} \cdot \frac{1}{2k-2n-s-t} \cdot \mu_{2i} \cdot \mu_{2j} \cdot \binom{2k-2n}{2i-p} \cdot \binom{2n}{p}
\]

\[
D_{2n} = \sum_{k=n}^{\infty} C_{2k,2n} \cdot t^{2k-2n}
\]

If for any \( n \geq 0 \) we have \( \mu_{2n} \leq (\ln(n+2))^{2n} \), and if \( D_{2n} \geq 0 \) for any \( n \geq 1 \) and for any \( t \), then our function has all its zeros on the vertical \( \text{Re}(s) = \frac{1}{2} \).

**Proof.** The proof is immediate from theorem 1 and the relation:

\[
\frac{d^2}{d\sigma^2} |f(\sigma + it)|^2 = 2 \cdot D_2 + 12 \cdot (\sigma - \frac{1}{2})^2 \cdot D_4 + 30 \cdot (\sigma - \frac{1}{2})^4 \cdot D_6 + \cdots + 2n \cdot (2n-1) \cdot (\sigma - \frac{1}{2})^{2n-2} \cdot D_{2n} + \cdots \text{ QED.}
\]

Many other interesting results can be based on theorem 2, our main result.

**Proposition 3.** We consider the analytical function \( f(s) \) on the whole complex plane with the Taylor series at \( \frac{1}{2} \) of the form:

\[
f(s) = c_0 + c_2 \cdot (s - \frac{1}{2})^2 + c_4 \cdot (s - \frac{1}{2})^4 + c_6 \cdot (s - \frac{1}{2})^6 + \cdots
\]

We also define:

\[
\mu_{2n} := c_{2n} \cdot (2n)! = f^{(2n)}(\frac{1}{2})
\]

We consider the series:
\( S(t) = b_0 + b_2 \cdot t^2 + b_4 \cdot t^4 + b_6 \cdot t^6 + \cdots \), where we take:

\[
\begin{align*}
b_0 &= \mu_0^2, b_2 = -\frac{1}{2!} \cdot \left( \binom{2}{0} \cdot \mu_0 \cdot \mu_2 + \binom{2}{2} \cdot \mu_2 \cdot \mu_0 \right), \quad b_4 = \frac{1}{4!} \cdot \left( \binom{4}{0} \cdot \mu_0 \cdot \mu_4 + \binom{4}{2} \cdot \mu_2^2 + \binom{4}{4} \cdot \mu_4 \cdot \mu_0 \right), \quad \ldots, \quad b_{2k} = (-1)^k \cdot \frac{1}{(2k)!} \cdot \left( \binom{2k}{0} \cdot \mu_0 \cdot \mu_{2k} + \binom{2k}{2} \cdot \mu_2 \cdot \mu_{2k-2} + \cdots \right) \binom{2k}{2k} \cdot \mu_{2k} \cdot \mu_0.
\end{align*}
\]

Then \( S(t) \geq 0 \) for any real \( t \), in other words, \( S(t) \) takes only positive values. We note here that when the coefficients \( c_{2n} \) are positive, the series \( S(t) \) will have terms alternating in sign (and the proof of the proposition is not obvious).

**Proof.** The proof is immediate from:

\[
|f(\sigma + it)|^2 = D_0 + (\sigma - \frac{1}{2})^2 \cdot D_2 + (\sigma - \frac{1}{2})^4 \cdot D_4 + (\sigma - \frac{1}{2})^6 \cdot D_6 + \cdots \geq 0.
\]

When observing that our series \( S(t) \) is exactly \( D_0 \). That means that:

\[
S(t) = D_0 = \left| f\left( \frac{1}{2} + it \right) \right|^2 \geq 0. \text{ QED.}
\]

**Proposition 4.** (see reference [2] for the proof). Let \( \xi(z) \) be the Riemann Xi function and \( n \) a positive integer. Then, as \( n \to \infty \) we have:

\[
\ln \left( \xi^{(2n)} \left( \frac{1}{2} \right) \right) = 2n \cdot \ln(\ln(n)) - 2 \left( \ln 2 + \frac{1}{\ln(n)} \right) \cdot n + \frac{9}{4} \cdot \ln(2n) - \frac{3}{4} \cdot \ln(\ln(n)) + O(1). \tag{24}
\]

We also know that (Stirling):

\[
\ln((2n)!) = \left( 2n - \frac{1}{2} \right) \cdot \ln(2n) - 2n + \ln \sqrt{2\pi} + o(1). \tag{25}
\]

From (24) and (25) we conclude that:

\[
\ln \left( \frac{\xi^{(2n)}(\frac{1}{2})}{(2n)!} \right) = -2n \cdot (\ln(2n) - \ln(\ln(n))) - 2n \cdot \left( \ln 2 + \frac{1}{\ln(n)} - 1 \right) + \frac{11}{4} \cdot \ln(2n) - \frac{3}{4} \cdot \ln(\ln(n)) + O(1). \tag{26}
\]

From (24) we see that the conditions of theorem 2 are satisfied by the Riemann Xi function, so the Riemann Xi function indeed satisfied relations (18), (19) and (20).

**Proposition 5.** In relations (18), (19) and (20) the quantities \( D_{2n} \) (which depend on \( t \), as defined by (14) and (16)) take positive values for all values of \( t \), in other words \( D_{2n}(t) \geq 0 \) for all any \( t \).

**Proof.** We write \( D_{2n} \) in the form:

\[
D_{2n} = D_{2n}(t) = \sum_{s=0}^{\infty} c_{2n+2s,2n} \cdot t^{2s}.
\]

We will use proposition 3, and we claim that there are real numbers \( \mu_0, \mu_2, \mu_4, \ldots, \mu_{2n}, \ldots \) such that the following system of equations is satisfied:
\[ C_{2n,2n} = \mu_0^2 \; ; \; C_{2n+2,2n} = -\frac{1}{2!} \cdot \binom{2}{0} \cdot \mu_0 \cdot \mu_2 + \binom{2}{2} \cdot \mu_2 \cdot \mu_0 \; ; \; C_{2n+4,2n} = \frac{1}{4!} \cdot \binom{4}{0} \cdot \mu_0 \cdot \mu_4 + \binom{4}{2} \cdot \mu_2^2 + \binom{4}{4} \cdot \mu_4 \cdot \mu_0 \; ; \; \ldots \ldots \; ; \; C_{2n+2s,2n} = (-1)^s \cdot \frac{1}{(2s)!} \cdot \binom{2s}{0} \cdot \mu_0 \cdot \mu_{2s} + \binom{2s}{2} \cdot \mu_2 \cdot \mu_{2s-2} + \cdots + \binom{2s}{2s} \cdot \mu_{2s} \cdot \mu_0 \] (27)

The proof of the claim follows from recursively solving this system of equations.

\[ \mu_0 = \sqrt{C_{2n,2n}} \cdot \] We note that \( \mu_0 \) is a real number because we can prove that \( C_{2n,2n} \) is a positive quantity.

\[ \mu_2 = \frac{1}{2 \sqrt{C_{2n,2n}}} \cdot (-2 \cdot C_{2n+2,2n}) \] (28)

\[ \mu_4 = \frac{1}{2 \cdot C_{2n,2n} \cdot \sqrt{C_{2n,2n}}} \cdot (24 \cdot C_{2n,2n} \cdot C_{2n+4,2n} - 6 \cdot C^2_{2n+2,2n}) \] (29)

\[ \mu_6 = \frac{1}{2 \cdot C_{2n,2n} \cdot \sqrt{C_{2n,2n}}} \cdot (-720 \cdot C^2_{2n,2n} \cdot C_{2n+6,2n} + 360 \cdot C_{2n,2n} \cdot C_{2n+2,2n} \cdot C_{2n+4,2n} - 90 \cdot C^3_{2n+2,2n}) \] (30)

We have written them in this form because we can easily see the pattern here. The results of the recursive calculations are as follows. When we calculate \( \mu_{2s} \) the factor outside the brackets will always be \( \frac{1}{2 \cdot C^{s-1}_{2n,2n} \cdot \sqrt{C_{2n,2n}}} \). Inside the brackets the first term will always be \( (-1)^s \cdot (2s)! \cdot C^{s-1}_{2n,2n} \cdot C_{2n+2s,2n} \). The other terms will consist of products of \( s \) factors of the form

\[ A \cdot C_{2n+2k_1,2n} \cdot C_{2n+2k_2,2n} \cdot C_{2n+2k_3,2n} \cdots \cdots \cdot C_{2n+2k_s,2n}, \] where \( 2k_1 + 2k_2 + 2k_3 + \cdots + 2k_s = 2s \), and \( A \) will be a number that depends on \( s \) (that can be calculated).

The number of terms inside the brackets (each composed of \( s \) factors multiplied by a number) will be the number of distinct ways (regardless order) in which \( 2s \) can be written as a sum of \( s \) even numbers, not all necessarily distinct (this is a problem of partition theory). For example, in (28) the number 2 can be written in only one way 2, so we will have only one term inside the brackets. In (29) the number 4 can be written in two ways as a sum of two even numbers, \( 0 + 4 = 2 + 2 \), so we will have two terms inside the brackets. In (30), the number 6 can be written in three ways as a sum of 3 even numbers, \( 0 + 0 + 6 = 0 + 2 + 4 = 2 + 2 + 2 \), corresponding to the three terms inside the brackets. If we continue, \( \mu_8 \) will have the form:

\[ \mu_8 = \frac{1}{2 \cdot C^3_{2n,2n} \cdot \sqrt{C_{2n,2n}}} \cdot (8! \cdot C^3_{2n,2n} \cdot C_{2n+8,2n} + B \cdot C^2_{2n,2n} \cdot C_{2n+2,2n} \cdot C_{2n+6,2n} + C \cdot C^2_{2n,2n} \cdot C^2_{2n+4,2n} + D \cdot C_{2n,2n} \cdot C^2_{2n+2,2n} \cdot C_{2n+4,2n} + E \cdot C^4_{2n+2,2n}) \] , so we will have 5 terms inside the brackets. This corresponds to the fact that 8 can be written in 5 ways as a sum of 4 even numbers, \( 0 + 0 + 0 + 8 = 0 + 0 + 2 + 6 = 0 + 0 + 4 + 4 = 0 + 2 + 2 + 4 = 2 + 2 + 2 + 2 \). The coefficient for the \( 0 + 0 + 0 + 8 \) term is \( 8! \), and the other ones can be calculated recursively.
In general we see that from the relations

\[ C_{2n+2s,2n} = (-1)^s \cdot \frac{1}{(2s)!} \cdot \left( \binom{2s}{0} \cdot \mu_0 \cdot \mu_{2s} + \binom{2s}{2} \cdot \mu_2 \right) \]

we can recursively find the value of \( \mu_{2s} \) as a function of the \( C_{2n,2n}, C_{2n+2,2n}, C_{2n+4,2n}, \ldots \ldots, C_{2n+2k-2,2n}, C_{2n+2s,2n} \).

I also present the following known results. First an inequality:

\[
\frac{2^{nH\left(\frac{r}{n}\right)}}{n+1} \leq \binom{n}{r} \leq 2^{nH\left(\frac{r}{n}\right)}, \text{ where } H(x) \text{ is the entropy}
\]

\[ H(x) = -x \cdot \log(x) - (1 - x) \cdot \log(1 - x). \]  \hspace{1cm} (31)

Second, we will need the following estimation (using (22)):

\[
| (2s)! \cdot C_{2n+2s,2n} | \leq (2s)! \cdot \frac{(\ln(n+s+2))^{4n+4s}}{(2n)! \cdot (2s)!} \cdot \sum_{0 \leq k \leq 2s} \sum_{0 \leq p \leq 2n} (-1)^k \cdot \binom{p}{k} \cdot \binom{2s}{2i-p} \cdot \binom{2n}{p} \leq \frac{2^{2n}}{(2n)!} \cdot 2^{2s-1} \cdot (\ln(n+s+2))^{4n+4s}. \]  \hspace{1cm} (32)

The expressions that depend on \( n \) can be considered as a constant in this case, because we are interested in the absolute convergence of the series \( \mu_0 + \frac{\mu_2}{2!} \cdot (z - \frac{1}{2})^2 + \frac{\mu_4}{4!} \cdot (z - \frac{1}{2})^4 + \frac{\mu_6}{6!} \cdot (z - \frac{1}{2})^6 + \ldots \). So we are interested in the factor that depends on \( s \). The calculations can be done for all the terms in the expression for \( \mu_{2s} \), and the number of terms in the expression for \( \mu_{2s} \) can be calculated using partition theory (and there are some good estimations available). The coefficients that appear in (28), (29), (30) (and so on...) will involve binomial coefficients and factorials, and that is where inequalities (31) and (32) become essential.

In other words, the system of equations mentioned above always has a solution. In order to apply proposition 3 we only have to prove that the series \( \mu_0 + \frac{\mu_2}{2!} \cdot (z - \frac{1}{2})^2 + \frac{\mu_4}{4!} \cdot (z - \frac{1}{2})^4 + \frac{\mu_6}{6!} \cdot (z - \frac{1}{2})^6 + \ldots \) is absolutely convergent on the whole complex plane (for any \( z \)). This will follow from the proved properties of \( C_{2n+2s,2n} \) (relation (22)), as briefly described above. At this point I have not been able to reach a closed form expression for the coefficients involved in the expression for \( \mu_{2s} \) (even if the calculations are straightforward and recursive). I leave this problem as a challenge for a mathematician (probably aided by some symbolic computation software) willing to finalize these calculations.

Once the calculations are finalized, from proposition 3 we could conclude that all the quantities \( D_{2n} \) are positive, in other words \( D_{2n}(t) \geq 0 \) for all any \( t \). !!!!

In the following, I will give an example that would also be a verification that the calculations (and theorem 2 in particular) are correct.

**Example.** We consider the values \( b_{2n} = \frac{1}{(2n)!} \), \( n \geq 0 \) for the coefficients \( b_{2n} \). Then we have:
\[ \cosh \left( z - \frac{1}{2} \right) = 1 + \frac{1}{2!} \cdot (z - \frac{1}{2})^2 + \frac{1}{4!} \cdot (z - \frac{1}{2})^4 + \frac{1}{6!} \cdot (z - \frac{1}{2})^6 + \cdots, \]

where we take \( z = \sigma + i \cdot t. \)

When we make the calculations we find:

\[ |\cosh(\sigma - \frac{1}{2} + i \cdot t)|^2 = \frac{1}{4} \cdot \left( e^{2 \left( \sigma - \frac{1}{2} \right)} + e^{-2 \left( \sigma - \frac{1}{2} \right)} \right) + \frac{1}{2} \cdot (\cos^2 t - \sin^2 t) \]

We can then write:

\[ |\cosh(\sigma - \frac{1}{2} + i \cdot t)|^2 = \frac{1}{2} \cdot \cosh(2 \cdot (\sigma - \frac{1}{2})) + \frac{1}{2} \cdot (1 + \cos(2t)) + \left( \sigma - \frac{1}{2} \right)^2 + \frac{1}{3} \cdot \left( \sigma - \frac{1}{2} \right)^4 + \cdots. \]

We see that in this case we have \( D_0 = \frac{1}{2} \cdot (1 + \cos(2t)) \), \( D_2 = 1 \), \( D_4 = \frac{1}{3} \), and so on for the other quantities. If we use our formulas \((15), (16), (16)'\) and \((21)\) for this particular case, we see that we find the right values.


We do not claim that we proved that the Riemann Hypothesis is true, at this point. We do emphasize the following points. Using the asymptotic results of Mark Coffey (proposition 4) we can prove that Riemann’s Xi function satisfies the conditions from theorem 2. That means that relations \((18), (19)\) and \((20)\) are valid. Using proposition 3, we sketched the proof in proposition 5 that all the quantities \( D_{2n}(t) \) are positive for any value of \( t \) (the absolute convergence of the series \( \mu_0 + \frac{\mu_2}{2!} \cdot (z - \frac{1}{2})^2 + \frac{\mu_4}{4!} \cdot (z - \frac{1}{2})^4 + \cdots \) has to be clearly established). From relation \((20)\) we see that the function \( |\xi(\sigma + it)|^2 \) (seen as a function of \( \sigma \)) is convex, and from theorem 1 we conclude that Riemann’s Xi function has all its zeros (on the strip) on the vertical \( \sigma = \frac{1}{2} \). We could then conclude that Riemann’s Hypothesis is true.

There is a second way to approach the problem that avoids convexity, but using the reformulation of Riemann’s Hypothesis (in reference \([3]\)). We see from relation \((19)\) that if all the quantities \( D_{2n}(t) \) are positive, then the function \( |\xi(\sigma + it)|^2 \) (seen as a function of \( \sigma \)) is decreasing for \( \sigma < \frac{1}{2} \) and increasing for \( \sigma > \frac{1}{2} \). Using the result in \([3]\) this would be a proof of Riemann’s Hypothesis.

References:
