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Optimal maintenance of two stochastically deteriorating machines with an intermediate buffer

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Abstract

We consider a manufacturing system in which an input generating installation transfers a raw material to a subsequent production unit. Both machines deteriorate stochastically with usage and may fail. For each machine the deteriorating process is described by some known transition probabilities between different degrees of deterioration. A buffer has been built between the two machines in order to cope with unexpected failures of the installation. A discrete-time Markov decision model is formulated for the optimal preventive maintenance of both machines. The maintenance times are geometrically distributed and the cost structure includes operating costs, storage costs, maintenance costs and costs due to the lost production. It is proved that for fixed buffer content and for fixed deterioration degree of one machine, the average-cost optimal policy initiates a preventive maintenance of the other machine if and only if its degree of deterioration exceeds some critical level. We study, by means of numerical results, the effect of the variation of some parameters on the optimal policy and on the minimum average cost. For the case in which the maintenance times follow continuous distributions, an approximate discrete-time Markov decision model is proposed.

1. Introduction

The optimal maintenance or replacement of one or many deteriorating machines has been an interesting research subject for a long time. The deterioration of a machine is due to its operation (see e.g. Douer and Yechiali [6], Yeh [27]) or to shocks from its environment (see e.g. Stadje and Zuckerman [22], Hu [8]). Various maintenance optimization models have been introduced and analysed in order to find the optimal balance between the costs and benefits of performing preventive and corrective maintenance on deteriorating machines. The papers of Scarf [20], Wang [25], Cho and Parlar [2] and Chapter 9 of the recent book of Hu and Yue [9] give a summary of some part of the research done in this area. The most widely used optimization criteria are the minimization of the long-run expected total discounted cost and the minimization of the long-run expected average cost per unit time. In some models it is possible to compute the optimal policy explicitly (see e.g. Jansen and Van Der Duyn Schouten [11], Stadje and Zuckerman [22]) and in other models it is possible to develop a suitable algorithm for its determination (see e.g. Maillart and Fang [14]). In recent years a considerable number of maintenance models for production-inventory systems (see Van Der Duyn Schouten and Vanneste [24], Iravani and Duenyas [10], Sloan [21], Yao et al. [26], Kyriakidis and Dimitrakos [13], Dimitrakos and Kyriakidis [5], Karamatsoukis and Kyriakidis [12]) have appeared in the literature, in which the preventive maintenance depends on the working condition of a machine and on the level of a subsequent buffer. The buffer has been built in order to avoid frequent interruptions of the production process when the machine has failed. The capacity of the buffer in these models was assumed to be fixed and the main problem was to find, under suitable cost structures, the states at which it is preferable to perform a preventive maintenance of the machine. Using techniques from Markov decision theory, analytical results concerning the form of the average-cost optimal policy were derived (see [24,21,26,13]), or efficient algorithms were developed for its determination (see [10,5]). Note that Chelbi and Ait-Kadi [1] and Rezg et al. [18] studied production-inventory systems in which the buffer size is variable. The optimum values of the buffer size and of the instants at which the preventive maintenances of the machine are initiated were obtained by trading off the maintenance cost, the inventory holding cost and the shortage cost. Meller and Kim [15] also studied a similar model and determined the optimal buffer inventory level, that triggers the preventive maintenance of the machine. A cost model was developed and the average cost was calculated as a function of the critical buffer level. Diamantidis and Papadopoulos [4] introduced a production system...
with two workstations and an intermediate buffer. Each workstation consists of multiple parallel machines with different processing, failure and repair times. Various performance measures of the system, as the throughput, were evaluated.

In this paper, we introduce a maintenance model for a manufacturing system that consists of two deteriorating machines with an intermediate buffer. The first machine is an input generating installation that transfers a raw material to the buffer and the second machine is a production unit that pulls the raw material from the buffer. The rate at which the installation fills the buffer with the raw material is greater than the rate at which the production unit pulls the raw material from the buffer. As soon as the buffer is filled up, these rates become equal. Both machines deteriorate as they operate and they may reach an inoperative condition with non-zero probability. If this happens, a corrective maintenance must be initiated. It is assumed that the corrective maintenance brings the failed machine to its perfect condition after some random time which is geometrically distributed. We also suppose that it is possible to initiate a preventive maintenance of the installation and/or of the production unit whenever they are in operative conditions. The preventive maintenance brings the machines to their perfect working conditions after some random times that are geometrically distributed with smaller means than the means of the times needed for the corresponding corrective maintenances. If a corrective or a preventive maintenance is performed on the installation and the buffer is not empty, the production unit can operate normally by pulling the raw material from the buffer until it is evacuated. If a corrective or a preventive maintenance is performed on the production unit and the buffer is not full, the installation can operate normally by transferring the raw material to the buffer until it is filled up. For both machines we introduce transition probabilities between their different working conditions. We also introduce operating costs and maintenance costs for both machines, costs for storing the raw material in the buffer and costs due to lost production that are incurred when the production unit does not operate. A discrete-time Markov decision model is formulated, in which the goal is to find a policy that minimizes the expected long-run average cost per unit time among all possible policies. The states of this model consist of the working conditions of the two machines and the buffer content. We specify the one-step transition probabilities and the one-step transition costs of this model and we prove that the average-cost optimal policy has the following control-limit form: For fixed buffer content and for fixed deterioration degree of the installation (production unit), the optimal policy initiates a preventive maintenance of the production unit (installation) if and only if the degree deterioration of the production unit (installation) exceeds some critical level. This structural property of the optimal policy is proved through the corresponding finite-horizon and infinite-horizon discounted cost problems. We also consider the problem when the maintenance times follow some known continuous distributions and we propose an approximate discrete-time finite state Markov decision model.

The model that we study in this paper is closely related to the models introduced by Van Der Dyun Schouten and Vanneste [24], Kyriakidis and Dimitrakos [13] and Pavitos and Kyriakidis [17]. In [24] it was assumed that, as time evolves, only the installation deteriorates and the production unit is always in operative condition. The states of that model consist of the age of the installation and the buffer content. The cost structure included costs due to lost production that were incurred when a preventive or a corrective repair was performed on the installation and the buffer was empty. The average-cost optimal policy of the installation was assumed to be geometrically distributed. It was proved that, for fixed buffer content, the average-cost optimal policy initiates a preventive maintenance of the installation if and only if its age is greater or equal to a critical value. Kyriakidis and Dimitrakos [13] generalized the model introduced in [24] by supposing that the states consist of the working condition of the installation and the buffer content. The cost structure included not only costs due to lost production but also operating and maintenance costs for the installation and costs for storing the raw material in the buffer. It was proved that, for fixed buffer content, the average-cost optimal policy initiates a preventive maintenance of the installation if and only if its working condition is worse than a critical level. In Pavitos and Kyriakidis [17] it was assumed that, as time evolves, only the production unit deteriorates and the installation is always in operative condition. The states of that model consist of the working condition of the production unit and the buffer content. It was supposed that the maintenance times of the production unit are geometrically distributed. The cost structure included operating costs and maintenance costs for the production unit, storage costs, costs due to lost production and penalty costs that are incurred when a preventive or a corrective maintenance is performed on the production unit and the buffer is full. It was proved that, for fixed buffer content, the average-cost optimal policy initiates a preventive maintenance of the production unit if and only if its degree of deterioration exceeds some critical value. The model that we study in the present paper can be considered as a generalization of the models studied in [13,17].

The rest of the paper is organized as follows. In Section 2 we describe the problem and we define its parameters. We also impose some conditions on the maintenance times, on the transition probabilities and on the operating and maintenance costs. In Section 3 we specify the one-step transition probabilities and the one-step transition costs of the Markov decision model, and we prove the structure of the optimal policy. The effect on the optimal policy by varying some specific parameters is studied in Section 4. In Section 5 we examine the case in which the maintenance times are continuous random variables. In the last section we give a summary of the results of the paper and we propose topics for future research. The proof of a crucial result of Section 3 is presented in the Appendix.

2. The model

We consider a manufacturing system consisting of an input generating installation (I), a production unit (P) and an intermediate buffer (B). The installation transfers the raw material to the buffer and the production unit pulls the raw material from the buffer. The capacity of the buffer is equal to K units of the raw material. As long as the buffer is not full and both machines are in operative condition, during one unit of time the installation supplies the buffer with p units of raw material, the production unit pulls d < p units of raw material from the buffer and p - d units of raw material are stored in the buffer. The numbers p and d are assumed to be positive integers. As soon as the buffer capacity is reached, the installation slows down from speed p to d. The three components of the system are presented in Fig. 1. An example of this system could be an automated workcenter which consists of an automated lathe or milling machine, an automated part feeder and an intermediate buffer.

We suppose that both machines deteriorate as time evolves. They are inspected at discrete, equidistant time epochs τ = 0, 1, ... (say every day) and a decision must be made at each time epoch. After each inspection, the installation is classified into one of the m + 2 working conditions 0, 1, ..., m + 1, which describe increasing levels of deterioration. State 0 denotes a new installation (or functioning as good as
new), while state $m+1$ denotes a failed (inoperative) installation that cannot transfer the raw material to the buffer. The intermediate states $1,\ldots,m$ are operative. Similarly, after each inspection, the production unit is classified into one of the $n+2$ working conditions: 0, 1,\ldots,n+1, which represent increasing degrees of deterioration. State 0 represents a new production unit (or functioning as good as new), while state $n+1$ denotes a failed (inoperative) production unit that cannot pull the raw material from the buffer. We suppose that the deterioration of the installation is independent of the deterioration of the production unit.

It is assumed that if at time epoch $\tau$ the state of the installation is $i \in [0,\ldots,m]$ then the state of the installation at time epoch $\tau+1$ will be $j \in [0,\ldots,m+1]$ with probability $p_{ij}$ or $q_{ij}$, if action $a_i = 0$ (leave the installation to operate) or action $a_i = 3$ (leave the installation idle) is selected, respectively. Similarly, if at a time epoch $\tau$ the state of the production unit is $i \in [0,\ldots,n]$ then the state of the production unit at time epoch $\tau+1$ will be $j \in [0,\ldots,n+1]$ with probability $p_{ij}$ or $q_{ij}$, if action $a_p = 0'$ (leave the production unit to operate) or action $a_p = 3'$ (leave the production unit idle) is selected, respectively. If at time epoch $\tau$ the working conditions of the installation and the production unit are $i \leq m$ and $i \leq n$, respectively, the buffer contains $x < K$ units of raw material, and the actions $a_i = 0$ and $a_p = 0'$ are chosen, then the buffer content of the production unit at time epoch $\tau+1$ will be $x \sim (x + p - d)K$. If the installation at time epoch $\tau$ is found to be at state $n+1$, a corrective maintenance must be started immediately. We denote this action with $a_i = 2$. The corrective repair time (expressed in time units) is random and we assume that it is geometrically distributed with probability of success $b$, i.e. the probability that it will last $t$ $\geq 1$ time units is equal to $(1-b)^{t-1}b$. If the installation at a time epoch $\tau$ is found to be at any state $i$, $i < m$, a preventive maintenance may be started. We denote the action of initiating a preventive maintenance of the installation with $a_i = 1$. The installation does not operate when a preventive maintenance is performed. The preventive repair time (expressed in time units) is also random and we assume that it is geometrically distributed with probability of success $a$, i.e. the probability that it will last $t$ $\geq 1$ time units is equal to $(1-a)^{t-1}a$. Similarly, if the production unit at time epoch $\tau$ is found to be at state $n+1$ or at state $i' \in [0,\ldots,n]$, a corrective maintenance (action $a_p = 2'$) must be commenced or a preventive maintenance (action $a_p = 1'$) may be started. The corrective repair time and the preventive repair time of the production unit are geometrically distributed with probability of success $b'$ and $a'$, respectively. We assume that the preventive and the corrective maintenance of the installation and the production unit are nonpreemptive, i.e. they cannot be interrupted, and they bring them to the perfect state 0. Note that, if at a time epoch $\tau$ the buffer content is $x < K$ and the actions $a_i = 0$ and $a_p = 1$ are selected, then the buffer content at epoch $\tau+1$ will be $x \sim (x+p)K$. If at a time epoch $\tau$ the buffer content is $x > 0$ and the actions $a_i \in [1,2]$ and $a_p = 0'$ are selected for the installation and the production unit, respectively, then the buffer content at time epoch $\tau+1$ will be $x \sim (x-d)K$. The action $a_i = 3$ for the installation may be chosen only at those time epochs in which the buffer is full and the action $a_p = 1'$ or $a_p = 2'$ is selected for the production unit. In this case the other possible action for the installation is the action $a_i = 1$. Similarly, the action $a_p = 3'$ for the production unit may be chosen only at those time epochs in which the buffer is empty and the action $a_i = 1$ or $a_i = 2$ is selected for the installation. In this case the possible action for the production unit is the action $a_p = 1'$.

If at a time epoch the installation is found to be at state $i \in [0,\ldots,m]$, and the action $a_i = 0$ is selected, an operating cost is incurred until the next time epoch, which is equal to $c_i$ if the buffer is not full, or to $c_i$ if the buffer is full. Similarly, if at a time epoch the production unit is found to be at state $i' \in [0,\ldots,n]$, and the action $a_p = 0'$ is selected, an operating cost is incurred until the next time epoch, which is equal to $c_i$. It is natural to assume that no operating costs are incurred if the actions $3,3'$ are selected. As long as a preventive or a corrective maintenance is performed on the production unit, the production process is interrupted. The same happens when the production unit is idle. We assume that we incur a cost, which is equal to $C > 0$, during a unit of time in which the production unit is idle or it is under a corrective or a preventive maintenance. When a preventive maintenance is performed on the installation or on the production unit, a repair cost is incurred, which is equal for each unit of time to $c_p$ or $c_i$, respectively. Similarly, when the installation or the production unit are under corrective maintenance, a repair cost is incurred, which is equal for each unit of time to $c_p$ or $c_i$, respectively. We also assume that the cost of holding a unit of raw material in the buffer for one unit of time is equal to $h > 0$.

We introduce the state $PM$ to denote the situation that a preventive maintenance is performed on the installation or on the production unit, respectively. The state of the production system is described in terms of three variables $(i,i',x)$, where $i$ and $i'$ denote the working conditions of the installation and of the production unit, respectively, at a time epoch, and $x$ is the content of the buffer at that epoch. The state space $S$ of the system is the following set:

$$S = \{0,\ldots,m+1,PM\} \times \{0,\ldots,n+1,PM\} \times \{0,\ldots,K\}.$$  

A policy is any rule for choosing actions at each time epoch $\tau = 0,1,\ldots$. A policy is said to be stationary, if at each time epoch $\tau = 0,1,\ldots$, it chooses one decision which depends only on the current state of the system. We consider a discrete-time Markov decision process in which we aim to find a policy which minimizes the long-run expected average cost per unit of time. The following conditions on the cost structure and on the transition probabilities are assumed to be valid:

**Condition 1.** The sequences $\{c_i\}$, $\{c_i\}$, $0 \leq i \leq m$, and $\{c_i\}$, $0 \leq i \leq n$, are nondecreasing. That is, as the working conditions of the installation or of the production unit deteriorate, the operating costs increase.

**Condition 2.** $c_i \leq c_i$, $0 \leq i \leq m$. That is, the reduction of the replenishment rate from $p$ to $d$, as soon as the buffer is filled up, causes a reduction of the operating cost of the installation.

**Condition 3.** $0 < b < a < 1$ and $0 < b' < a' < 1$. That is, the expected times required for the preventive maintenances of the installation and the production unit are smaller than the expected times required for the corrective maintenances.
Condition 4. \( c_0 \leq c_i \) and \( c_0 \leq c_j \). That is, the cost rates of the preventive maintenances of the installation and the production unit do not exceed the cost rates of the corrective maintenances.

Condition 5 (Increasing Failure Rate Assumptions). For \( k \in \{0, \ldots, m + 1\} \) and for \( k' \in \{0, \ldots, n + 1\} \), the quantities \( \sum_{j=1}^{n+1} p_{ij}, \sum_{j=1}^{n+1} q_{ij} \), \( 0 \leq i \leq m \), and \( \sum_{j=1}^{n} p_{i'}, \sum_{j=1}^{n} q_{i'} \), \( 0 \leq i' \leq n \), are nondecreasing in \( i \) and \( i' \), respectively.

A consequence of Condition 5 is that \( I_i, I_{i'}, 0 \leq i \leq m, \) and \( I_{i'}, 0 \leq i' \leq n \), where \( I_i \) and \( I_{i'} \) are random variables representing the next working condition of the installation and the production unit, respectively, if their present working conditions are \( i \) and \( i' \), respectively. It can be shown (see Derman [3], pp. 122–123) that this condition is equivalent to the following one:

Condition 6. For any nondecreasing functions \( h(j), 0 \leq j \leq m + 1 \), and \( h'(j'), 0 \leq j' \leq n + 1 \), the quantities \( \sum_{i=0}^{n+1} p_{ij} h(j), \sum_{i=0}^{n+1} q_{ij} h(j), 0 \leq i \leq m \), and \( \sum_{j=0}^{n} p_{i'} h'(j'), \sum_{j=0}^{n} q_{i'} h'(j'), 0 \leq i' \leq n \), are nondecreasing in \( i \) and \( i' \), respectively.

We summarize the notation introduced in this section.

\[ \begin{align*}
p & \quad \text{input rate} \\
d & \quad \text{pull rate} \\
K & \quad \text{buffer capacity} \\
m + 1 & \quad \text{failure state of the installation} \\
n + 1 & \quad \text{failure state of the production unit} \\
p_{ij}, q_{ij} & \quad \text{transition probabilities from working condition } i \text{ of the installation to working condition } j, \text{if it operates or if it is idle, respectively} \\
p_{i'j'}, q_{i'j'} & \quad \text{transition probabilities from working condition } i' \text{ of the production unit to working condition } j', \text{if it operates or it is idle, respectively} \\
c_i, c_i' & \quad \text{operating cost for the installation if the buffer is not full or full, respectively} \\
c_j, c_j' & \quad \text{operating cost for the production unit} \\
a, a' & \quad \text{probabilities of success of the preventive maintenances of the installation and the production unit, respectively} \\
b, b' & \quad \text{probabilities of success of the corrective maintenances of the installation and the production unit, respectively} \\
c_p, c_p' & \quad \text{preventive maintenance costs per unit time for the installation and the production unit, respectively} \\
c_j, c_j' & \quad \text{corrective maintenance costs per unit time for the installation and the production unit, respectively} \\
C & \quad \text{cost rate due to lost production} \\
h & \quad \text{inventory holding cost rate per unit or raw material} \\
S & \quad \text{state space} \\
\end{align*} \]

3. The optimal policy

Since the state space \( S \) of the system is finite and the state \( (0, 0, 0) \) is accessible from every state under any stationary policy it follows that an average-cost optimal stationary policy exists (see Corollary 2.5 in Ross [18]). We will prove that the optimal policy has a particular structure using the corresponding finite-horizon and infinite-horizon discounted-cost problems. Before considering these problems we have to specify the one-step transition probabilities and the one-step transition costs. Let \( P_{ij}(a, a_j) \) be the probability that the state of the system at the next time epoch will be \( s \in S \), given that the present state is \( s \) and the actions \( a_i \in \{0, 1, 2, 3\} \) and \( a_j \in \{0, 1, 2, 3\} \) are selected for the installation and for the production unit, respectively. Let also \( C(s, (a_i, a_j)) \) be the expected cost until the next time epoch if the present state is \( s \) and the actions \( a_i \in \{0, 1, 2, 3\} \) and \( a_j \in \{0, 1, 2, 3\} \) are selected for the installation and the production unit, respectively. The quantities \( P_{ij}(a, a_j), C(s, (a_i, a_j)), s \in S, a_i \in \{0, 1, 2, 3\}, a_j \in \{0, 1, 2, 3\} \), can be specified by taking into account the cost structure of the problem and the possible transitions under the permissible actions. We give below some of these quantities.

\[ \begin{align*}
P_{(i, j), (0, 0)}(0.0') & = p_{i0} p_{j0}, 0 \leq i \leq m, 0 \leq j \leq n, 0 \leq j' \leq n + 1, 0 \leq j'' \leq n + 1, 0 \leq x \leq K, \\
P_{(i, j), (PM_{ij})}(0.0') & = (1 - a) p_{ij}, 0 \leq i \leq m, 0 \leq j \leq n, 0 \leq j' \leq n + 1, 1 \leq x \leq K, \\
P_{(i, j), (1, 0)}(0.0') & = (1 - a) q_{ij}, 0 \leq i \leq m, 0 \leq i' \leq n, 0 \leq j' \leq n + 1, \\
P_{(i, j), (1, 1)}(0.0') & = q_{ij}, 0 \leq i \leq m, 0 \leq i' \leq n, 0 \leq x \leq K, \\
P_{(i, j), (PM_{ij})}(2, 1') & = p_{ij} b, 0 \leq i \leq m, 0 \leq j \leq m + 1, 0 \leq x \leq K - 1, \\
P_{(i, j), (PM_{ij})}(0, 2) & = p_{ij} b, 0 \leq i \leq m, 0 \leq j \leq m + 1, 0 \leq x \leq K - 1, \\
C((i, j), (0.0')) & = c_i + c_j' + h x, 0 \leq i \leq m, 0 \leq j' \leq n, 0 \leq x \leq K - 1, \\
C((i, j), (0.0')) & = c_i + c_j' + h x, 0 \leq i \leq m, 0 \leq i' \leq n, 0 \leq j' \leq n, 1 \leq x \leq K, \\
C((i, j), (1, 0)) & = c_p + c_j + h x, 0 \leq i \leq m, 0 \leq i' \leq n, 0 \leq j \leq n, 0 \leq x \leq K, \\
C((i, j), (1, 3')) & = c_p + C, 0 \leq i \leq m, 0 \leq i' \leq n, \\
C((i, j), (1, 1')) & = c_p + c_j' + h x + C, 0 \leq i \leq m, 0 \leq j \leq n, 0 \leq i' \leq n, 0 \leq x \leq K, \\
C((i, j), (0.2')) & = c_i + c_j + h x + C, 0 \leq i \leq m, 0 \leq x \leq K - 1, \\
C((m + 1, j), (2, 1')) & = c_j + c_j' + h x + C, 0 \leq x \leq K. \\
\end{align*} \]

We explain how (1) is derived. The other expressions are derived in a similar manner. Suppose that the present state of the installation is \( i \in \{0, \ldots, m\} \), the present state of the production unit is \( j \in \{0, \ldots, n\} \) and the present buffer content is \( x \in \{1, \ldots, K\} \). If we initiate a preventive
maintenance of the installation, the state of the installation at the next time epoch will be PM with probability 1 − a. If we leave the production unit to operate, its working condition at the next time epoch will be j with probability p_j. Therefore, the state of the system at the next time epoch will be \( (PM, j, \max (x - d, 0)) \) with probability \( (1 - a)p_j \), since the state of the production unit is independent of the state of the installation, and the installation does not transfer the raw material to the buffer.

Let \( \alpha, 0 < \alpha < 1 \), be a discount factor. The minimum \( n \)-step expected discounted cost \( V^n_\alpha(s), n \geq 0 \), where \( s \in S \) is the initial state of the process, can be found recursively from the following equation (see Eq. (1.2) in Ross [19]):

\[
V^n_\alpha(s) = \min_{a \in A} \left\{ C(s, (a, a_P)) + \sum_{n=3}^{\infty} P_{\text{wa}}(a_i, a_P) V_{n-1}(u) \right\}, \quad s \in S, \ n \geq 1.
\]

The minimum in the right-hand-side of (2) is taken among all admissible combinations of the actions \( a_i \in \{0, 1, 2, 3\} \) and \( a_P \in \{0', 1', 2', 3'\} \) for each state \( s \in S \). The Eq. (2) take the following forms for some specific states that belong to \( S \):

For \( 0 \leq i \leq m, 0 \leq x < K \),

\[
V^n_i(i, n + 1, x) = \min \left\{ c_i + c_{ij} + h + C + x \sum_{j=0}^{n-1} p_{ij} V_{n-1}^a(j, 0, \min(x + p, K)) + \alpha(1 - b) \sum_{j=0}^{n-1} p_{ij} V_{n-1}^a(j, 0, \min(x + p, K)) \right\},
\]

\[
+ \alpha(1 - b)a_{ij} V_{n-1}^a(0, 1, x), \quad 0 \leq i \leq m, 0 \leq x < K,
\]

where \( \{x + d\} = \max (x - 0, 0) \). In Eq. (3), the first term inside the curly brackets corresponds to actions \( a_i = 2 \) and the second term corresponds to \( a_i = 2 \) and \( a_P = 3 \). In Eq. (4), the first term inside the curly brackets corresponds to actions \( a_i = 0 \) and \( a_P = 2 \) and the second term corresponds to actions \( a_i = 1 \) and \( a_P = 2 \). The right side of (5) corresponds to actions \( a_i = 1 \) and \( a_P = 2 \). In (6) the first term inside the curly brackets corresponds to actions \( a_i = 0 \) and \( a_P = 0 \), the second term corresponds to actions \( a_i = 1 \) and \( a_P = 0 \), the third term corresponds to actions \( a_i = 1 \) and \( a_P = 1 \), and the fourth term corresponds to actions \( a_i = 1 \) and \( a_P = 1 \). The quantity \( V^n_i(i, \tilde{i}, x) \) coincides with the right side of (6) with \( x = K \) if, in the first term inside the brackets, we replace \( c_i \) with \( \tilde{c}_i \), and in the third term inside the curly brackets, we replace \( c_i \) with \( \tilde{c}_i \) and we replace \( p_{ij} \) with \( q_{ij} \).

According to the following lemma, \( V^n_i(i, \tilde{i}, x), 0 \leq i \leq m + 1, 0 \leq \tilde{i} \leq n + 1, 0 \leq x < K \), is nondecreasing in \( \tilde{i}(i) \) for fixed \( i(i) \) and fixed \( x \). Its proof is presented in the Appendix.

**Lemma 1.** For each \( n = 0, 1, \ldots \) the minimum \( n \)-step expected discounted cost \( V^n_\alpha(s), s \in S \), has the following properties:

(i) \( V^n_\alpha(PM, 1, x) \leq V^n_\alpha(m + 1, 1, x), i \in \{0, 1, \ldots, n + 1, PM\}, 0 < x < K \),

(ii) \( V^n_\alpha(i, 1, x) \leq V^n_\alpha(i + 1, 1, x), 0 < i \leq m, \tilde{i} \in \{0, \ldots, n + 1, PM\}, 0 < x < K \),

(iii) \( V^n_\alpha(i, PM, x) \leq V^n_\alpha(i, n + 1, x), i \in \{0, \ldots, m + 1, PM\}, 0 < x < K \),

(iv) \( V^n_\alpha(i, 1, x) \leq V^n_\alpha(i, \tilde{i} + 1, x), 0 < \tilde{i} \leq n, i \in \{0, \ldots, m + 1, PM\}, 0 < x < K \).
Since the state space is finite and the state $(0,0,0)$ is accessible from every other state under any stationary policy it follows that there exist numbers $a_j(x)(x), x \in S$, and a constant $g$ that satisfy the average- cost optimality equations (see Corollary 2.5 in Ross [19], p. 98). For the states $(i,i,i)$ and $(PM, i, x)$, $0 \leq i \leq m$, $0 \leq i \leq n$, $0 \leq x < K$, the optimality equations take the following forms:

$$v(i,i,x) = \min \left\{ c_i + c'_i + hx - g + \sum_{j=0}^{m-1} p_{ij} \sum_{j=0}^{n-1} p_{ij} \min(x + p - d, K), c_i + c_i + hx - g + (1 - a_i) \sum_{j=0}^{m-1} p_{ij} v(j, PM, (x + p, K)) + a_i \sum_{j=0}^{m-1} p_{ij} v(j, 0, \min(x + p, K)), v(PM, i, x) \right\}$$

$$v(PM, i, x) = \min \left\{ c_i + c'_i + hx - g + a_i \sum_{j=0}^{m-1} p_{ij} v(j, 0, \min(x + p, K)) + (1 - a_i) \sum_{j=0}^{m-1} p_{ij} v(j, PM, (x + p, K)) + a_i (1 - a_i) v(0, PM, x) + (1 - a_i) v(0, 0, x) \right\}.$$

In view of Parts (ii) and (iv) of the above lemma and Theorem 2.2 in Ross [19], we have the following result:

**Corollary 1.**

(i) $v(i,i,x) \leq v(i+1,i,x), 0 \leq i \leq m, i \in \{0, \ldots, n + 1, PM\}, 0 \leq x < K$.

(ii) $v(i,i,x) \leq v(i,i+1,x), 0 \leq i \leq n, i \in \{0, \ldots, m + 1, PM\}, 0 \leq x < K$.

The following proposition gives a characterization of the form of the optimal policy.

**Proposition 1.**

(i) For fixed buffer content $x, 0 \leq x < K$, and for fixed working condition $i \in \{0, \ldots, n + 1, PM\}$ of the production unit, there exists a critical working condition $i(i,i,x)$ of the installation such that the optimal policy initiates a preventive maintenance of the installation if and only if its working condition $i \in \{0, \ldots, m\}$ is greater than or equal to $i(i,i,x)$.

(ii) For fixed buffer content $x, 0 \leq x < K$, and for fixed working condition $i \in \{0, \ldots, m + 1, PM\}$ of the installation, there exists a critical working condition $i(i,i,x)$ of the production unit such that the optimal policy initiates a preventive maintenance of the production unit if and only if its working condition $i \in \{0, \ldots, n\}$ is greater than or equal to $i(i,i,x)$.

**Proof.** We will prove only Part (i) since Part (ii) can be proved similarly. Suppose that, for $0 \leq x < K$ and $i \in \{0, \ldots, n\}$, the optimal policy initiates a preventive maintenance of the installation at the state $(i, i, x)$, where $i \in \{0, \ldots, m - 1\}$. This implies that

$$v(PM, i, x) \leq \min \left\{ c_i + c_i + hx - g + \sum_{j=0}^{m-1} p_{ij} \sum_{j=0}^{n-1} p_{ij} \min(x + p - d, K), c_i + c_i + hx - g + (1 - a_i) \sum_{j=0}^{m-1} p_{ij} v(j, PM, (x + p, K)) + a_i \sum_{j=0}^{m-1} p_{ij} v(j, 0, \min(x + p, K)) \right\}$$

To show that the optimal policy prescribes a preventive maintenance on the installation at the state $(i+1, i, x)$ it is enough to verify that

$$v(PM, i, x) \leq \min \left\{ c_i + c_i + hx - g + \sum_{j=0}^{m-1} p_{ij} \sum_{j=0}^{n-1} p_{ij} \min(x + p - d, K), c_i + c_i + hx + C - g + (1 - a_i) \sum_{j=0}^{m-1} p_{ij} v(j, PM, (x + p, K)) + a_i \sum_{j=0}^{m-1} p_{ij} v(j, 0, \min(x + p, K)) \right\}.$$

From Conditions 1 and 6 and Part (i) of the above Corollary, it follows that the right-hand side of (8) is greater than or equal to the right-hand side of (7). Hence (7) implies (8). The same result is obtained similarly when $x = K$ or $i \in \{n + 1, PM\}$. □

**Remark 1.** In the above proposition, if for a particular buffer content $x, 0 \leq x < K$, and for a fixed working condition $i \in \{0, \ldots, n, n + 1, PM\}$ of the production unit, the critical working condition $i(i,i,x)$ of the installation is equal to $m + 1$, then the optimal policy never initiates a preventive maintenance of the installation whenever the buffer contains $x$ units of the raw material and the working condition of the production unit is $i$. Furthermore, if for a particular buffer content $x, 0 \leq x < K$, and for a fixed working condition $i \in \{0, \ldots, m, m + 1, PM\}$ of the installation, the critical working condition $i(i,i,x)$ of the production unit is equal to $n + 1$, then the optimal policy never initiates a preventive maintenance of the production unit whenever the buffer contains $x$ units of the raw material and the working condition of the installation is $i$.

**Remark 2.** In the formulation of our model we assumed that the installation transfers the raw material to the buffer at a constant rate that is equal to $p$ units per unit time and the production unit pulls the raw material from the buffer at a constant rate that is equal to $d$ units per unit of time. This assumption can be relaxed by assuming that these rates depend on the working condition of the installation and on the working condition of the production unit. Specifically, we could suppose that, when the buffer is not full, the input rate is $p_i$ and the pull rate is $d_i$ if $i \in \{0, \ldots, m\}$ and $i \in \{0, \ldots, n\}$ are the working conditions of the installation and the production unit, respectively. It is also reasonable to assume that $p_0 > p_1 > \cdots > p_m > d_0 > d_1 > \cdots > d_m$, $p_m > d_0$ and that the input rate $p_i$ is reduced to $d_i$, as soon as the buffer is filled up. In this more general model it is possible to find numerically the optimal policy but is difficult to prove its structure.
Remark 3. In the literature, a wide variety of distributions for the repair time of a machine have been used (see Chapter 11 of Morse’s book [16]). In a specific system this distribution can be estimated from the data on lengths of time for repair. In the present paper, the assumption of geometric maintenance times is crucial for the derivation of the analytical results concerning the structure of the optimal policy. This assumption can be considered as realistic in situations in which the repair of the machine consists of successive attempts that aim to return the machine to its perfect condition. For instance, the repair could consist of successive attempts to replace a component of the machine that broke down or to fix a component that has removed from its regular position during the operation of the machine or to assemble two parts of the machine that were separated or to restore an adjustment that has been changed. It is possible to assume that the repair times follow the geometric distribution if the probability of a successful attempt is constant. In our model the memoryless property of the geometric distribution enables us to formulate a suitable discrete-time Markov decision model and to have usable forms of the minimum n-step expected discounted cost given by Eqs. (3)–(6). Note that geometric maintenance times were also assumed in the models studied in [11–13,17,24]. In these models the structure of the average-cost optimal policy was proved following the approach that we follow in the present paper, i.e. through the corresponding discounted-cost problem. Hatoyama [7] considered geometric maintenance times in which the probability of success depends on the degree of deterioration of the machine. If we consider other distributions for the maintenance times it seems difficult to derive analytical structural results for the optimal policy. In Section 4 we will see how we can find the optimal policy approximately when the maintenance times are continuous.

4. Numerical results

The optimal policy can be found numerically by implementing the policy-iteration algorithm or the value-iteration algorithm or the linear programming algorithm. We refer to Chapter 3 of Tijms’s [23] book for the description of these algorithms. Using numerical results, that are obtained by the value-iteration algorithm, we will examine the effect of varying some parameters of the model on the optimal policy and on the minimum average cost.

The value-iteration algorithm converges since the states \((m + 1, n + 1, x)\), \(0 \leq x \leq K\), are aperiodic (see Theorem 3.4.2 in Tijms [23]). We present below five numerical examples in which \(p_{ij} = (m + 2 - i)^{-1}\), \(0 \leq i \leq m\), \(i \leq j \leq m + 1\), \(p_{ij} = (n + 2 - i)^{-1}\), \(0 \leq i \leq n\), \(i \leq j \leq n + 1\), \(q_{0i} = 0.95\), \(q_{i,i+1} = 0.05\), \(0 \leq i \leq m\), \(q_{i,i'} = 0.95\), \(q_{i,i+1} = 0.05\), \(0 \leq i \leq n\). It can be easily checked that these probabilities satisfy Condition 5.

Example 1. Let \(m = n = 7\), \(C = 10\), \(a = 0.5\), \(b = 0.3\), \(c = 0.8\), \(\beta = 8.5\), \(h = 0.2\), \(c = 0.7(i + 1)\), \(0 \leq i \leq 7\), \(\beta = 0.1(i + 1)\), \(0 \leq i \leq 7\), \(\alpha = 0.4\), \(\beta = 0.2\), \(\beta = 3\), \(\beta = 3.5\), \(\beta = 4\), \(0 \leq i \leq 7\). For \(C = 2\), \(5\), \(8\) the minimum average costs are found to be 8.91, 10.95, 10.94, respectively. As expected, we see that as \(C\) increases, the corresponding minimum average cost increases. In Tables 1 and 2 we present for \(C = 2, 5, 8\) the critical points \(i(i, x)\). \(i \in \{0, …, m + 1, PM\}\). \(x \in \{0, …, 10\}\) and \(i(i, x) \in \{0, …, m, PM\}\). \(x \in \{0, …, 10\}\) that characterize the optimal policy for the preventive maintenance of the installation and of the production unit, respectively. In each cell of these Tables the first number corresponds to \(C = 2\), the second to \(C = 5\), and the third to \(C = 8\). In Table 2 we observe that, for fixed \(i\) and \(x\), the critical number \(i(i, x)\) increases as \(C\) increases. This can be explained from the fact that as \(C\) increases it is preferable to initiate a preventive maintenance of the production unit at worse working conditions in order to avoid the cost due to the lost production. Consequently, it can be deduced that, when the optimal policy is employed, the long-run proportion of time during which a preventive maintenance is performed on the production unit seems to decrease as the cost due to lost production increases.

Table 1

The critical working conditions \(i(i, x)\) for the installation for \(C = 3, 5, 8\). The nth \((n = 1, 2, 3)\) number at each cell corresponds to \(C = 3, 5, 8\).

<table>
<thead>
<tr>
<th>(i, x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
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<tr>
<td>0</td>
<td>3, 2, 1</td>
<td>2, 1, 0</td>
<td>5, 4, 3</td>
<td>6, 5, 5</td>
<td>6, 6, 6</td>
<td>6, 6, 6</td>
<td>7, 6, 6</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3, 2, 1</td>
<td>3, 1, 0</td>
<td>5, 4, 3</td>
<td>6, 5, 5</td>
<td>6, 6, 6</td>
<td>6, 6, 6</td>
<td>7, 6, 6</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3, 2, 1</td>
<td>3, 1, 0</td>
<td>5, 4, 3</td>
<td>6, 5, 5</td>
<td>6, 6, 6</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3, 2, 1</td>
<td>3, 1, 0</td>
<td>5, 4, 3</td>
<td>6, 5, 5</td>
<td>6, 6, 6</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3, 2, 1</td>
<td>4, 2, 0</td>
<td>5, 5, 4</td>
<td>6, 5, 5</td>
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<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3, 2, 1</td>
<td>5, 3, 1</td>
<td>5, 5, 5</td>
<td>6, 6, 6</td>
<td>6, 6, 6</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3, 2, 0</td>
<td>5, 4, 2</td>
<td>5, 5, 5</td>
<td>6, 6, 6</td>
<td>6, 6, 6</td>
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<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4, 2, 0</td>
<td>5, 4, 3</td>
<td>5, 5, 5</td>
<td>6, 6, 6</td>
<td>6, 6, 6</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td></td>
</tr>
<tr>
<td>(n + 1)</td>
<td>5, 4, 3</td>
<td>5, 5, 4</td>
<td>5, 6, 5</td>
<td>6, 6, 6</td>
<td>6, 6, 6</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td></td>
</tr>
<tr>
<td>PM</td>
<td>3, 2, 0</td>
<td>5, 4, 3</td>
<td>5, 5, 5</td>
<td>6, 6, 6</td>
<td>6, 6, 6</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td>7, 7, 7</td>
<td>6, 6, 6</td>
<td></td>
</tr>
</tbody>
</table>
The minimum average cost, as expected, increases as \( c_p \) or \( c_0 \) increases. Furthermore, we observe that the minimum average cost is sensitive to the variation of \( c_p \), since a small increase of \( c_p \) causes a considerable increase of the minimum average cost. However, the effect of the variation of \( c_0 \) on the minimum average cost seems to be weaker than the effect of the variation of \( c_p \). This implies that the cost rate of the preventive repair of the installation affects significantly the minimum average cost while the influence of the cost of the preventive maintenance of the production unit is less significant. Consequently, it seems reasonable that the manufacturing system must be designed in such a way that the preventive maintenance of the installation causes relatively low costs.

**Example 3.** Let \( m = 5, n = 6, K = 5, b = 0.3, c_p = 3, c_f = 3.5, c_i = 0.05(i + 1), c_i = 0.025(i + 1), 0 \leq i \leq 5, a' = 0.5, b' = 0.3, c_p = 6, c_f = 6.5, c_i = 0.1(i + 1), 0 \leq i \leq 6. \) In Fig. 3 we present the graphs of the minimum average cost \( g(h) \) as a function of \( h \) when \( a \) is equal to 0.8, 0.6, 0.4.
From Fig. 4 we see that, for all values of $a$, the minimum average cost increases as $h$ increases. We also observe that, if $h$ is constant, the minimum average cost increases as $a$ increases. This can be explained from the fact that, as $a$ increases, the preventive maintenances of the installation do not last long and, therefore, the installation replenishes the buffer with the raw material causing high storage costs.

**Example 4.** Let $m = 5$, $n = 6$, $K = 5$, $b = 0.3$, $c_p = 3$, $c_f = 3.5$, $c_i = 0.05(i + 1)$, $c_{f_i} = 0.025(i + 1)$, $0 \leq i \leq 5$, $a = 0.6$, $b' = 0.3$, $c_p' = 6$, $c_f' = 6.5$, $C = 0.7$, $c_i' = 0.1(i + 1)$, $0 \leq i \leq 6$. In Fig. 4 we present the graphs of the minimum average cost $g(h)$ as a function of $h$ when $a'$ is equal to 0.9, 0.7, 0.5.

From Fig. 4 we see that, for all values of $a'$, the minimum average cost increases as $h$ increases. We also observe that, if $h$ is constant, the minimum average cost decreases as $a'$ increases. This can be explained from the fact that, as $a'$ increases, the expected times of the preventive maintenance of the production unit decreases, and therefore, we avoid costs due to lost production.

### 5. An approximate model when the maintenance times are continuous

We modify the problem that we introduced in Section 2 by assuming that the preventive and the corrective repair times for the installation and for the production unit are continuous random variables with probability density functions $f_{RM}(t), f_{CM}(t)$ and $f_{PRM}(t), f_{PCM}(t)$ respectively. As noted in Remark 3 above, the structural results of Section 2 depend heavily on the “memoryless” property of the geometric distribution. Nevertheless, we can obtain valuable, approximate results by modifying Condition 3 as follows:

**Condition 3’**. The expected preventive repair times for the installation and for the production unit are smaller than the corresponding expected corrective times.

Note that Conditions 1, 2, 4 and 5 are not affected by the form of the distributions of the repair times. Thus we can consider an approximate discrete-time Markov decision model if we assume that there exists a relatively large positive integer $L$ such that the preventive and the corrective maintenances of both machines do not last more than $L$ time units. The state space of the approximate model is:

$$
\mathcal{S} = \{0, \ldots, m + 1, (m + 1)_{1}, \ldots, (m+1)_{L}, PM_{1}, \ldots, PM_{L}\} \times \{0, \ldots, n + 1, (n + 1)_{1}, \ldots, (n+1)_{L}, PM_{1}, \ldots, PM_{L}\} \times \{0, \ldots, K\},
$$

where $(m + 1)_{i}, (n + 1)_{i}$, $r = 1, \ldots, L$, are the situations at which a corrective maintenance is performed on the installation or on the production unit, respectively, for $r$ time units and $PM_{r}, r = 1, \ldots, L$, are the situations at which a preventive maintenance is performed on the installation or on the production unit for $r$ time units. The one-step transition costs for this model coincide with those of the original model. The one-step transition probabilities can be easily determined. We give below some of these quantities.

\[
\begin{align*}
    p_{i, (i, X)(0, 0, 0)}(1, 1') &= \int_{0}^{\alpha} f_{RM}(t)dt \int_{0}^{\alpha} f_{CM}(t)dt, \quad 0 \leq i \leq m, \ 0 \leq i' \leq n, \ 0 \leq x \leq K, \\
    p_{i, (0, PM, f, 0)}(1, 3') &= q_{y} \int_{0}^{\alpha} f_{RM}(t)dt, \quad 0 \leq i \leq m, \ 0 \leq i' \leq n, \ 0 \leq f' \leq n + 1, \\
    p_{i, (n + 1, X)(0, min[x, n, K])}(0, 2) &= p_{y} \int_{0}^{\alpha} f_{CM}(t)dt, \quad 0 \leq i \leq m, \ 0 \leq j \leq m + 1, \ 0 \leq x \leq K, \\
    p_{i, (i, X)(0, PM, K)}(3, 1') &= q_{y} \int_{0}^{\alpha} f_{CM}(t)dt, \quad 0 \leq i \leq m, \ 0 \leq j \leq m + 1, \ 0 \leq i' \leq n, \\
    p_{(PM, PM, X)(0, 0, 0)}(1, 1') &= \frac{\alpha}{L} \int_{1}^{\alpha} f_{RM}(t)dt \int_{1}^{\alpha} f_{CM}(t)dt, \quad 1 \leq r \leq L, \ 1 \leq s \leq L, \ 0 \leq x \leq K, \\
    p_{((m + 1), (n + 1), X)(0, 0, 0)}(2, 2') &= 1, \quad 0 \leq x \leq K, \\
    p_{(PM, f, 0)(0, f, 0)}(1, 3') &= q_{f'}, \quad 0 \leq i' \leq n, \ 0 \leq f' \leq n + 1.
\end{align*}
\]

The average-cost optimal policy can be found by using any standard algorithm for discrete-time Markov decision models.
6. Conclusions and future research

In the literature a great number of mathematical models have been developed and analysed for the optimal preventive or corrective maintenance of systems that are subject to deterioration with usage and age. The corrective maintenance is the maintenance that occurs when a system fails, while the preventive maintenance is the maintenance that occurs when a system is operating. The optimal strategy for the maintenance can result in substantial saving in operation and, also, in increased availability of the system.

In the present paper, we considered a production system consisting of a input generating installation, a production unit and an intermediate buffer. The installation supplies the buffer with the raw material. The production unit pulls the raw material from the buffer at a constant rate. Both machines deteriorate with usage and may fail. For each of them there are different degrees of deterioration. If a machine fails, a corrective maintenance is compulsory. If a machine is in an operative condition a preventive maintenance is optional. It is assumed that both maintenances bring the machines to their perfect condition. We formulated a discrete-time Markov decision model in which we included operating costs for both machines, storage costs, maintenance costs and cost due to lost production when the production unit does not operate. If the maintenance times are geometrically distributed we proved that the policy that minimizes the long-run expected average cost per unit time is characterized by critical numbers such that the preventive maintenances of the machines are initiated only if their degrees of deterioration are greater or equal to these numbers. If the maintenance times are continuous, an approximate discrete-time Markov decision model is formulated.

A topic for future research could be the investigation of a more general problem in which the preventive maintenances do not bring the machines necessarily to their perfect condition but to a better (intermediate) condition. Another topic for future research could be the investigation of the problem in which the production unit does not pull the raw material from the buffer at a constant rate but in a random manner.

Acknowledgment

The authors thank the referee whose valuable suggestions improved the presentation of the paper.

Appendix

Proof of Lemma 1: We prove the lemma by induction on \(n\). The lemma holds for \(n = 0\) since \(V_{2}^{a}(s) = 0, s \in S\). We assume that the lemma holds for \(n - 1\). We will first prove that Part (i) holds for \(n\) and then we will prove that Part (ii) holds for \(n\). We will omit the proofs of Parts (iii) and (iv) for \(n\), since they are similar to the proofs of Parts (i) and (ii).

Part (i): For \(x = 0\) and \(0 \leq i \leq n\) we have:

\[
V_{n}^{a}(PM, i, 0) = \min \left\{ c_{p} + c'_{p} + C + xa(1 - a')V_{n-1}^{a'}(0, PM, 0) + xaV_{n-1}^{a}(0, 0, 0) + \alpha(1 - a)(1 - a')V_{n-1}^{a}(PM, PM, 0) \right\}
\]

\[
+ \alpha(1 - a)a'V_{n-1}^{a'}(PM, 0, 0) \cdot c_{p} + C + xa \sum_{j=0}^{n-1} q_{ij}V_{n-1}^{a'}(0, j', 0) + \alpha(1 - a) \sum_{j=0}^{n-1} q_{ij}V_{n-1}^{a}(PM, j', 0) \right\}
\]

\[
\leq \min \left\{ c_{j} + c'_{j} + C + xa(1 - a')V_{n-1}^{a'}(0, PM, 0) + xaV_{n-1}^{a}(0, 0, 0) + \alpha(1 - a)(1 - a')V_{n-1}^{a}(m + 1, PM, 0) \right\}
\]

\[
+ \alpha(1 - a)a'V_{n-1}^{a'}(m + 1, 0, 0) \cdot c_{j} + C + xa \sum_{j=0}^{n-1} q_{ij}V_{n-1}^{a'}(0, j', 0) + \alpha(1 - a) \sum_{j=0}^{n-1} q_{ij}V_{n-1}^{a}(m + 1, j', 0) \right\}
\]

\[
= \min \left\{ c_{j} + c'_{j} + C + \alpha(1 - a')\alpha'aD_{1} + \alpha(1 - a')V_{n-1}^{a'}(m + 1, PM, 0) - xaD_{2} + xaV_{n-1}^{a}(m + 1, 0, 0), c_{j} \right\}
\]

\[
+ C + \alpha \sum_{j=0}^{n-1} q_{ij}V_{n-1}^{a'}(m + 1, j', 0) - xa \sum_{j=0}^{n-1} q_{ij}D_{3} \right\}
\]

\[
\leq \min \left\{ c_{j} + c'_{j} + C + \alpha(1 - a')bD_{1} + \alpha(1 - a')V_{n-1}^{a'}(m + 1, PM, 0) - xaD_{2} + xaV_{n-1}^{a}(m + 1, 0, 0), c_{j} \right\}
\]

\[
+ C + \alpha \sum_{j=0}^{n-1} q_{ij}V_{n-1}^{a'}(m + 1, j', 0) - xb \sum_{j=0}^{n-1} q_{ij}D_{3} \right\} = V_{n}^{a}(m + 1, i', 0).
\]

where,

\[
D_{1} = V_{n-1}^{a'}(m + 1, PM, 0) - V_{n-1}^{a'}(0, PM, 0),
\]

\[
D_{2} = V_{n-1}^{a'}(m + 1, 0, 0) - V_{n-1}^{a'}(0, 0, 0),
\]

\[
D_{3} = V_{n-1}^{a'}(m + 1, j', 0) - V_{n-1}^{a'}(0, j', 0).
\]

Note that the first inequality follows from Condition 4 and from Part (i) of the induction hypothesis. The second inequality follows from Condition 3 and the inequalities \(D_{1} \geq 0, D_{2} \geq 0, D_{3} \geq 0\), which are consequences of Part (ii) of the induction hypothesis. The inequality

\[
V_{n}^{a}(PM, i', x) \leq V_{n}^{a}(m + 1, i', x), \quad 0 \leq i' \leq n, \quad 0 < x \leq K,
\]
can be proved similarly. For $x = 0, \ldots, K$ we have
\[
V_n^a(PM, PM, x) = c_p + c'_p + hx + C + \alpha(1 - a)(1 - a')V_{n-1}^x(PM, PM, x) + \alpha a(1 - a)V_{n-1}^x(0, PM, x) + \alpha a'(1 - a)V_{n-1}^x(0, PM, 0, x)
\]
\[
+ \alpha a'V_{n-1}^x(0, 0, x) \leq c'_p + c'_p + hx + C + \alpha(1 - a)(1 - a')V_{n-1}^x(m + 1, PM, x) + \alpha(1 - a')V_{n-1}^x(0, PM, x)
\]
\[
+ \alpha a'(1 - a)V_{n-1}^x(m + 1, 0, x) + \alpha a a'V_{n-1}^x(0, 0, x)
\]
\[
= c'_p + c'_p + hx + C + \alpha(1 - a')V_{n-1}^x(m + 1, PM, x) - \alpha(1 - a')aD_4 - \alpha a' D_5 + \alpha a a'V_{n-1}^x(m + 1, 0, x)
\]
\[
\leq c'_p + c'_p + hx + C + (1 - a')V_{n-1}^x(m + 1, PM, x) - \alpha(1 - a')bD_4 - \alpha b aD_5 + \alpha a a'V_{n-1}^x(m + 1, 0, x) = V_n^x(m + 1, PM, x),
\]
where,
\[
D_4 = V_{n-1}^x(m + 1, PM, x) - V_{n-1}^x(0, PM, x),
\]
\[
D_5 = V_{n-1}^x(m + 1, 0', x) - V_{n-1}^x(0', 0, x).
\]

Note that the first inequality follows from Condition 4 and from Part (i) of the induction hypothesis. The second inequality follows from Condition 3 and the inequalities $D_4 \geq 0, D_5 \geq 0$, which are consequences of Part (ii) of the induction hypothesis. The inequality
\[
V_n^x(PM, n + 1, x) \leq V_n^x(m + 1, n + 1, x), \quad 0 \leq x \leq K,
\]
can be proved similarly.

Part (ii): For $0 < x < K, i \in \{0, \ldots, m - 1\}, \hat{r} \in \{0, \ldots, n\}$ we have
\[
V_n^a(i, \hat{r}, x) = \min \left\{ c_i + c'_i + hx + C + \alpha \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(j, j', \min (x + p - d, K)), c_p + c'_p + hx + C + \alpha(1 - a) \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(PM, j', (x - d)^+) \right. \\
\left. + \alpha a \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(0, \hat{r}, (x - d)^+), c_i + c'_i + hx + (1 - a') \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(j, PM, \min (x + p, K)) \right. \\
\left. + \alpha a' \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(0, \hat{r}, (x - d)^+), c_i + c'_i + hx + (1 - a') \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(PM, j, (x - d)^+) \right. \\
\left. + \alpha a \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(0, \hat{r}, (x - d)^+), c_i + c'_i + hx + (1 - a) \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(j, PM, \min (x + p, K)) \right. \\
\left. + \alpha a' \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(0, \hat{r}, (x - d)^+), c_p + c'_p + hx + (1 - a) \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(0, PM, x) \right. \\
\left. + \alpha a' \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(0, 0), x) \right\}
\]
\[
\leq c_i + c'_i + hx + \alpha \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(j, j', \min (x + p - d, K)), c_p + c'_p + hx + \alpha(1 - a) \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(PM, j', (x - d)^+) \right. \\
\left. + \alpha a \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(0, \hat{r}, (x - d)^+), c_i + c'_i + hx + \alpha(1 - a') \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(j, PM, \min (x + p, K)) \right. \\
\left. + \alpha a' \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(0, \hat{r}, (x - d)^+), c_i + c'_i + hx + \alpha(1 - a') \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(PM, j, (x - d)^+) \right. \\
\left. + \alpha a \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(0, \hat{r}, (x - d)^+), c_i + c'_i + hx + \alpha(1 - a) \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(0, PM, x) \right. \\
\left. + \alpha a' \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(0, 0), x) \right\} = V_n^a(i + 1, \hat{r}, x).
\]

Note that the above inequality follows from Condition 1 and the inequalities
\[
\sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(j, j', \min (x + p - d, K)) \leq \sum_{j=0}^{m-1} \sum_{j'=0}^{n-1} p_{ij}^f V_{n-1}^x(j, j', \min (x + p - d, K)),
\]
\[
\sum_{j=0}^{m-1} p_j V_{n-1}^x(j, PM, \min (x + p, K)) \leq \sum_{j=0}^{m-1} p_j V_{n-1}^x(j, PM, \min (x + p, K)),
\]
\[
\sum_{j=0}^{m-1} p_j V_{n-1}^x(j, 0, \min (x + p, K)) \leq \sum_{j=0}^{m-1} p_j V_{n-1}^x(j, 0, \min (x + p, K)),
\]

which are consequences from Part (ii) of the induction hypothesis and Condition 6. For $0 < x < K$ and $\hat{r} \in \{0, \ldots, n\}$ we have
where the last inequality follows from Part (i). We have proved that Part (ii) holds for $0 < a < 1$ and $0 < T < n$. The proof for $x \in [0, K]$ and $\bar{I} \in \{n+1, PM\}$ is similar. □

References


