A Fibrational Framework for Possible-World Semantics of ALGOL-like Languages

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1 Introduction

Pioneering work by John Reynolds and Frank Oles [Rey81a, Ole82, Ole85, Ole97, OT92] showed how block-structured storage management in ALGOL-like languages [OT97] may be explicated using a semantics based on functor categories \( W \Rightarrow S \), where \( W \) is a suitable category of “worlds” characterizing local aspects of storage structure, and \( S \) is a conventional semantic category of sets or domains. Every programming-language type \( \theta \) is interpreted as a functor \( \llbracket \theta \rrbracket : W \to S \) and every programming-language term-in-context \( \pi \vdash X : \theta \) is interpreted as a natural transformation \( \llbracket \pi \vdash X : \theta \rrbracket : \llbracket \pi \rrbracket \to \llbracket \theta \rrbracket \).

This functor-category framework was later exploited to analyze the non-interference predicate in Reynolds’s specification logic [Rey81b, Ten90, O’H90, OT93a], block expressions in ALGOL-like languages [Ten85], and the concept of passivity in a variant of Reynolds’s Syntactic Control of Interference [Rey78, OP+99].

O’Hearn and Tennent [OT93b, OT95] obtained a more precise analysis of block structure by internalizing additional uniformity constraints along the lines of Reynolds’s relational parametricity [Rey83]. This work also uses structures of the form \( W \Rightarrow S \) but \( W \) and \( S \) are now reflexive graphs, with appropriate binary-relational categories above the usual categories of worlds and of sets (or domains). This framework was developed further by Reddy [Red97] and Dunphy [Dun02] by imposing additional conditions on \( W \) and \( S \).

O’Hearn and Reynolds [OR00] describe an alternative approach to the semantics of local storage: the source language is translated into a polymorphic linear lambda calculus, which is then interpreted using a semantics with relational parametricity constraints. For example, \( (\theta_0 \to \theta_1)^*(\alpha) \), the translation of type \( \theta_0 \to \theta_1 \) in world \( \alpha \), is defined to be \( \forall \beta. \theta_0^*(\alpha \otimes \beta) \to \theta_1^*(\alpha \otimes \beta) \); here, \( \beta \) may be thought of as the “new” storage allocated between the definition of the procedure and an application. Possible worlds (states) are thus modelled by tensor products of free type variables, and phrase types (as in the example above) are coded so as to be meaningful on extensions of the state (a further tensoring of free type variables), allowing for the possibility that a procedure is invoked in an expanded state (extra variables) from that in which it is defined.

Here, we use the categorical concept of fibration (or fibered category) to provide a general framework within which these kinds of semantics may be expressed and compared. [Jac99] provides a fairly comprehensive account of fibrations in categorical logic and type theory.
Fibrations are used here for three purposes: first, as a framework to model indexing by worlds; second, as a framework for categories of “relations” above categories, as in Hermida’s analysis of logical relations [Her93] above cartesian closed categories; and, third, as a framework for models of polymorphic languages. All of these are standard applications of fibrations, discussed in, for example, [Jac99]. When these uses of fibrations are combined, one obtains fibrations of fibrations, that is to say, fibrations in the 2-category $\mathbf{Fib}$ of fibered categories, a sub-2-category of the arrow 2-category $\mathbf{Cat}^{\to}$.

The relevant theory of fibrations over a fibration is developed in [Her03], where the purpose is to provide a framework for logical systems over polymorphic type theories (such as that of [PA93]). Since polymorphic type theories are modelled categorically as fibrations with certain structure, and a logical system over a type theory is organised as a fibration over such, the resulting construction must be a fibration over the given fibration with structure corresponding to the type theory. The extra structure at the level of predicates or relations reveals the meaning of so-called logical relations.

Recapping, the sources for this work are threefold:

- the functor-category approach to semantics of ALGOL-like languages enhanced with reflexive graphs of relations to impose parametricity constraints [OT93b, OT95);
- the interpretation of ALGOL in polymorphic linear lambda calculus [OR00], which motivates our construction in Section 2 of a fibration from a functor category;
- the theory of fibrations in $\mathbf{Fib}$ from [Her03], which allows us to extend this construction to the construction of a fibration over a fibration from a reflexive graph over functor categories, thereby bringing the relational framework over functor categories into the realm of fibered categorical type theory.

## 2 From Functor Categories to Fibrations Using Slices

We begin by showing how an arbitrary functor category $W^{\text{op}} \Rightarrow S$ may be turned into a fibration on $W$. Note that henceforth we will consistently work with contravariant functors on worlds; that is, we consider $W$ to be a small category whose morphisms are, typically, projections, rather than “expansions.” This is the opposite of the convention established by Reynolds and Oles, but fits better with mathematical practice. In categorical logic, re-indexing contravariantly along projections corresponds to weakening and quantifiers are explained as adjoints to weakening functors.

A standard way to construct a fibration on a category $W$ is to define categories indexed on $W$, that is, to define a (pseudo) functor $S$ from $W^{\text{op}}$ to the category $\mathbf{Cat}$ of (locally small) categories, and apply the Grothendieck construction to it; in fact, every fibration arises in this way.
In our case, given categories $W$ and $S$, we define the functor $S: W^{\text{op}} \to \mathbf{Cat}$ as follows:
for any world $w$, the relevant fiber category $S(w)$ will be the category $(W/w)^{\text{op}} \Rightarrow S$ of all contravariant functors from $W/w$ to $S$, where the slice category $W/w$ has as objects all $W$-morphisms into $w$ and, as morphisms from $f: x \to w$ to $f': x' \to w$, all commuting diagrams of the form

\[
\begin{array}{ccc}
  w & \xrightarrow{f} & f' \\
  \downarrow & & \downarrow \\
  x & \xrightarrow{g} & x'
\end{array}
\]

This construction retains in $S(w)$ information about the behaviour of any functor $F: W^{\text{op}} \to S$ in possible future worlds derived from $w$, bearing in mind that we think of a morphism $f: x \to w$ as a projection from an expanded world $x$ to $w$, the contravariant action of $F$ on $f$ being a “logical weakening” of the object to the expanded context. This is consistent with the philosophy behind possible-world semantics. In fact, from the perspective of world $w$, all that matters about a functor $F$ is its behaviour in the “restricted” universe $W/w$.

For functors $(W/w)^{\text{op}} \xrightarrow{F} S$, the morphisms from $F$ to $G$ are, of course, the natural transformations $(W/w)^{\text{op}} \xrightarrow{\eta} S$. Note that, if $W$ has a terminal object $1$, $S(1)$ is just the familiar functor category $W^{\text{op}} \Rightarrow S$.

**Proposition 1** For any small category $W$, if $S$ is cartesian closed and complete, so is $S(w)$ for every $w \in W$.

**Proof.** See Appendix A.1.

To complete the definition of $S$, we must define, for every $h: w \to w'$, a re-indexing functor $S(h): S(w') \to S(w)$. Note that any $h: w \to w'$ induces by composition a functor $\Sigma_h: (W/w) \longrightarrow (W/w')$, taking the diagram above to

\[
\begin{array}{ccc}
  w' & \xrightarrow{f ; h} & f' ; h \\
  \downarrow & & \downarrow \\
  x & \xrightarrow{g} & x'
\end{array}
\]

where $; \; \text{denotes composition in diagrammatic order}$. So, for any $F: (W/w')^{\text{op}} \longrightarrow S$ in $S(w')$, we define $S(h)(F)$ in $S(w)$ to be $\Sigma_h^{\text{op}} ; F$. Similarly, for any morphism $\eta: F \to G$ in $S(w')$, $S(h)(\eta)$ in $S(w)$ is $\Sigma_h^{\text{op}} ; \eta$, so that, for any $f: x \to w$, $S(h)(\eta)(f) = \eta(\Sigma_h^{\text{op}}(f)) = \eta(f ; h)$. In short, the functorial action for $h$ is simply precomposition with the functor $\Sigma_h$ induced by $h$. 

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These definitions make $S$ a functor from $W^{op}$ to $Cat$; we may then use the Grothendieck construction to obtain a split fibration on $W$, which we portray as follows:

\[
\begin{array}{ccc}
\text{Slices}(W, S) & \xrightarrow{p} & W \\
\end{array}
\]

**Proposition 2** If $S$ is complete and cartesian closed, the fibration $p: \text{Slices}(W, S) \to W$ is fibrewise cartesian closed and this structure is preserved by re-indexing.

*Proof.* See Appendix A.2.

**Proposition 3** If $S$ is complete, the fibration $p: \text{Slices}(W, S) \to W$ admits products; dually, if $S$ is co-complete, it admits co-products.

*Proof.* See Appendix A.3.

### 3 Relations

We now consider (binary) relations on pairs $(w_0, w_1)$ and $(s_0, s_1)$ of objects in $W$ and $S$, respectively. Suppose we have two categories $RW$ and $RS$ with two functors $rw: RW \to W \times W$ and $rs: RS \to S \times S$. An object $W$ of $RW$ is regarded as being a kind of relation on $(w_0, w_1) = rw(W)$; similarly, an object $S$ of $RS$ is typically a binary relation on $s_0 \times s_1$ where $(s_0, s_1) = rs(S)$. Morphisms in $RW$ and $RS$ may be thought of as morphisms in $W \times W$ or $S \times S$ that preserve these relations; see Example 5.1. We assume the functors $rw$ and $rs$ are fibrations.

We may now repeat the slices construction in the 2-category $Cat^{Sp}$ of categorical spans, where $Sp$ is the graph $\Rightarrow$. We will be defining a category $\text{Slices}(rw, rs)$ which will be fibered over $RW$ by construction, but also fibered over $\text{Slices}(W, S) \times \text{Slices}(W, S)$, with commutativity as follows:

\[
\begin{array}{ccc}
\text{Slices}(rw, rs) & \xrightarrow{\tilde{rw}} & \text{Slices}(W, S) \times \text{Slices}(W, S) \\
\text{RW} & \xrightarrow{rw} & W \times W \\
\end{array}
\]

Consider any “relation” $W$ in $RW$ with $rw(W) = (w_0, w_1)$. We define the fiber over $W$ as follows:

- objects are triples $(\tilde{F}, F_0, F_1)$ of functors such that
commutes, where \( rw/W \) is the functor obtained from \( rw \) by applying \( rw \) to objects and morphisms of \( RW/W \);

- morphisms from \((\bar{F}, F_0, F_1)\) to \((\bar{G}, G_0, G_1)\) are triples \((\bar{\eta}, \eta_0, \eta_1)\) of natural transformations:

In particular, if 1 is a terminal object in \( RW \), the fiber over 1 is just \( \text{Cat}^{\Sigma}(rw^{\text{op}}, rs) \), which we will write \( rw^{\text{op}} \Rightarrow rs \). This corresponds to the category of “parametric” functors and natural transformations of [OT93b, OT95].

To define the re-indexing functors, consider any map \( \tilde{h}: W \to W' \) in \( RW \) and suppose \( rw(\tilde{h}) = (w_0, w_1) \xrightarrow{(h_0, h_1)} (w_0', w_1') \); then any object \((\bar{G}, G_0, G_1)\) of the fiber on \( W' \) is mapped to a triple of functors obtained by pre-composition with \( \Sigma \) functors induced by \( \bar{h}, h_0, \) and \( h_1 \), respectively, as follows:

and similarly for fiber morphisms. This defines a functor from \( RW^{\text{op}} \) to \( \text{Cat} \), and applying the Grothendieck construction to it gives the desired split fibration on \( RW \):

\[ \text{Slices}(rw, rs) \]

\[ q \]

\[ RW \]
We may now define the forgetful functor $\tilde{\mathcal{w}}$ from $\text{Slices}(\mathcal{w}, \mathcal{S})$ to $\text{Slices}(\mathcal{W}, \mathcal{S}) \times \text{Slices}(\mathcal{W}, \mathcal{S})$ that maps an object $(\tilde{F}, F_0, F_1)$ to $(F_0, F_1)$, and similarly for morphisms. This gives us the following commuting diagram:

$$
\begin{array}{ccc}
\text{Slices}(\mathcal{w}, \mathcal{S}) \times \text{Slices}(\mathcal{W}, \mathcal{S}) & \xrightarrow{\tilde{\mathcal{w}}} & \text{Slices}(\mathcal{W}, \mathcal{S}) \\
q \downarrow & & \downarrow p \times p \\
\text{RW} & \xrightarrow{\tilde{\mathcal{w}}} & \text{W} \times \text{W}
\end{array}
$$

**Proposition 4** The preceding diagram, $(\tilde{\mathcal{w}}, \mathcal{w}) : q \rightarrow p \times p$, is a fibration in $\mathcal{Fib}$.

*Proof.* See Appendix A.4. □

As we will see in the next section, the examples of categories of worlds and relations in the literature all involve a functor $\mathcal{w}$ which is a fibration; however, it may be worthwhile to note that it is actually the fibrational nature of $\mathcal{S}$ that is of crucial importance.

**Corollary 5** If $\mathcal{S}$ is a fibration, so is $\mathcal{w}^{\text{op}} \Rightarrow \mathcal{S}$ for any functor $\mathcal{w}$.

*Proof.* See Appendix A.5. □

**Proposition 6** If $\mathcal{S}$ is a fibration of complete cartesian closed categories, the preceding diagram is a morphism of fibered cartesian closed categories with products.

*Proof.* See Appendix A.6. □

4 **Equality**

Because of the apparent significance of equality to relational parametricity [PA93], we briefly examine the status of equality relations in this fibrational framework. Assume our basic fibration $\mathcal{S} : \mathcal{R} \mathcal{S} \rightarrow \mathcal{S} \times \mathcal{S}$ admits an equality relation in the sense of Lawvere [Law68]; i.e., a least reflexive relation. Then all (fiber) fibrations $\text{Slices}(\mathcal{W}, \mathcal{S})$ admit equality, built pointwise, just like their cartesian liftings. Likewise, such equality relations for the various fibrations are preserved by re-indexing.

By Proposition 4, we have a fibration in $\mathcal{Fib}$, $(\tilde{\mathcal{w}}, \mathcal{w}) : q \rightarrow p \times p$. Taking the pullback of $p \times p$ along $\mathcal{w}$ we get a fibration $(p \times p)^{\mathcal{w}} : \text{Slices}(\mathcal{w}, \mathcal{S}) \rightarrow \text{RW}$. Factorising the morphism $(\tilde{\mathcal{w}}, \mathcal{w}) : q \rightarrow p \times p$ through this pullback, we obtain a functor $\tilde{\mathcal{w}} : \text{Slices}(\mathcal{w}, \mathcal{S}) \rightarrow \text{Slices}(\mathcal{w}, \mathcal{S})$, which, by the characterisation of fibrations in $\mathcal{Fib}$ in [Her03], is a fibration $\tilde{\mathcal{w}} : q \rightarrow (p \times p)^{\mathcal{w}}$ between fibrations (over $\text{RW}$). We have noticed in Propositions 2 and 3 that functors preserving cartesian closure and limits induce structure-preserving morphisms between the fibrations of slices. Summing up the above observations on equality, we obtain the following:
Corollary 7 Assume the fibration \( rs: \mathcal{RS} \rightarrow \mathcal{S} \times \mathcal{S} \) admits equality so that the resulting equality-relation functor \( Eq: \mathcal{S} \rightarrow \mathcal{RS} \) preserves cartesian closure and limits. The induced fibration \( \tilde{w}: q \rightarrow (p \times p)_{rs} \) admits (fibered) equality and the resulting equality-relation functor \( Eq_{rs}: \text{Slices}(rw, rs)_{rs} \rightarrow \text{Slices}(rw, rs) \) is a morphism of fibered cartesian closed categories with products.

5 Examples

Example 5.1 The category of worlds introduced in [Ten90] may be described as follows.

- Objects are sets \( W, X, Y, \ldots \); these are regarded as sets of local states.
- Morphisms from \( X \) to \( W \) are pairs \( (V, X \xleftarrow{m} W \times V) \) where \( V \) is a set (of values for “new” local variables) and \( m \) is a monic function (to impose constraints on the local states).
- The identity on \( W \) is \( (1, (\text{id}_W, !_W)) \) and the composite of \( (V, m): Y \rightarrow X \) and \( (V', m'): X \rightarrow W \) is \( (V' \times V, m; (m' \times \text{id}_V)) \).

This description makes it clear that we can construct a category \( T(C) \) of “worlds” not only from a category \( C \) of sets and functions, but from any category with finite products; furthermore, any finite-product and mono preserving functor \( rs: D \rightarrow C \) between such categories induces a functor \( T(rs): T(D) \rightarrow T(C) \) on the categories of worlds; that is, \( T \) is a functor from categories with finite products and finite-product and mono preserving functors between them to \( \text{Cat} \).

For example, let

- \( \text{Set} \) be a small category of sets and all functions between them,
- \( \text{Rel} \) be the category whose objects are binary relations \( R: W_0 \leftarrow \rightarrow W_1 \) on pairs of \( \text{Set} \)-objects and whose morphisms are relation-preserving pairs of functions:

\[(f, g): (R: W_0 \leftarrow \rightarrow W_1) \longrightarrow (S: X_0 \leftarrow \rightarrow X_1)\]

such that \( w_0[R]w_1 \) implies \( f w_0[S]g w_1 \) and

- \( r: \text{Rel} \rightarrow \text{Set} \times \text{Set} \) be the functor such that \( r(R: W_0 \leftarrow \rightarrow W_1) = (W_0, W_1) \), and similarly for morphisms.

The category \( \text{Rel} \) has products: given \( R: W_0 \leftarrow \rightarrow W_1 \) and \( S: X_0 \leftarrow \rightarrow X_1 \), their product is \( R \times S: W_0 \times X_0 \leftarrow \rightarrow W_1 \times X_1 \) such that

\[(w_0, x_0)[R \times S](w_1, x_1) \text{ iff } (w_0[R]w_1 \text{ and } x_0[S]x_1)\]

Since monos in \( \text{Rel} \) are simply pairs of monomorphisms in \( \text{Set} \), \( r \) preserves products and monos. So, if we let \( W = T(\text{Set}) \) be our category of worlds (based on \( \text{Set} \)), \( RW = T(\text{Rel}) \) is a category of relations above \( W \times W \). In fact, an object of \( RW \) is a binary relation \( R: W_0 \leftarrow \rightarrow W_1 \) and a morphism from \( S: X_0 \leftarrow \rightarrow X_1 \) to \( R: W_0 \leftarrow \rightarrow W_1 \) is a relation.
shown that their domains and a morphism as above to the underlying $W$ that is, if $x_0[S]x_1$ then $m_0(x_0)(R \times T)m_1(x_1)$.

Now $r w = T(r): RW \rightarrow W \times W$ is the forgetful functor that map relations to their domains and a morphism as above to the underlying $W$-morphisms. It is easily shown that $r w$ is a fibration: for any $RW$-object $R: W_0 \rightarrow W_1$ and $(W \times W)$-morphism $((V_0, X_0) \rightarrow W_0 \times V_0), (V_1, X_1) \rightarrow W_1 \times V_1))$, a cartesian lifting has as domain the relation $S: X_0 \rightarrow X_1$ defined by $x_0[S]x_1$ iff $w_0[R]w_1$, where $m_0(x_0) = (w_0, v_0)$ and $m_1(x_1) = (w_1, v_1)$, and $v_0[T]v_1$ is true for all $v_0 \in V_0$ and $v_1 \in V_1$.

Let us also point out that $r: Rel \rightarrow Set \times Set$ admits an equality (the usual diagonal relation on a set). Rel is cartesian closed and (co-)complete and both $r$ and $Eq: Set \rightarrow Rel$ preserve this structure. Hence, our Corollary 7 applies in this setting.

**Example 5.2** The Oles category of worlds [Ole82, Ole85, Ole97] and the O’Hearn-Tennent category of relations above this [OT93b] are obtained by restricting the function components of the morphisms for Tennent’s categories to isomorphisms, and so they are similarly the functorial image of a subobject fibration (an observation noted in [OT95, Section 10.1] and attributed there to Andy Pitts) and the relevant “domains” functor is again a fibration.

**Example 5.3** The category of worlds originally proposed for interpreting Algol is described by Reynolds [Rey81a] using sets and partial functions. We rephrase his construction using total maps from an arbitrary cartesian closed category $C$.

The category $R(C)$ has the same objects as $C$. A morphism $(g, G): X \rightarrow Y$ consists of a pair of $C$-morphisms $g: Y \rightarrow X$ and $G: (X \Rightarrow X) \rightarrow (Y \Rightarrow Y)$ satisfying the following:


2. $G; (Y \Rightarrow g) = (g \Rightarrow X)$

3. $D_X; G = (g \Rightarrow G); D_Y$, where for any object $X$, the diagonalisation morphism $D_X: X \Rightarrow (X \Rightarrow X) \rightarrow (X \Rightarrow X)$ is defined by $D_X = \theta_X; (\delta_X \Rightarrow X)$, with $\theta_X: X \Rightarrow (X \Rightarrow X) \cong (X \times X \Rightarrow X)$ and $\delta_X: X \rightarrow X \times X$.

Composition of morphisms is given componentwise, $(g, G) ; (h, H) = (h ; g, G ; H)$, and the identity is $(id, id)$. Intuitively, $g: Y \rightarrow X$ projects out the small state embedded in a larger one, whereas $G: (X \Rightarrow X) \rightarrow (Y \Rightarrow Y)$ maps any command on small states to the corresponding command on large states that preserves the values of new variables (Condition 2 above). The third condition is relevant to the object-oriented view of variables in Algol [Rey81a].
In fact, the category of possible worlds used by Oles [Ole82, Ole85] is isomorphic to the Reynolds category. For any category with finite products $C$, the category $O(C)$ has the same objects as $C$, while a morphism $(g, \rho): X \rightarrow Y$ consists of a pair of $C$-morphisms $g: Y \rightarrow X$ and $\rho: X \times Y \rightarrow Y$ satisfying the following:

1. $\rho;g = \pi_0$, with $\pi_0: X \times Y \rightarrow X$ being the first projection
2. $(g, \text{id}) \cdot \rho = \text{id}$
3. $(X \times \rho) \cdot \rho = \pi_{0,2}; \rho: X \times X \times Y \rightarrow X \times Y$, where $\pi_{0,2}: X \times X \times Y \rightarrow X \times Y$ selects the first and third components of the triple.

Composition of morphisms involves diagonalisation. Since we are interested in the case when $C$ is cartesian closed, we present a simplified version using adjoint transposition. Given morphisms $(g, \rho): X \rightarrow Y$ and $(h, \rho'): Y \rightarrow Z$, consider $\hat{\rho}: X \rightarrow (Y \Rightarrow Y)$, the adjoint transpose of $\rho$, and similarly for $\rho'$. The composite $(g, \rho); (h, \rho')$ is $(h; g, \rho'')$ where $\rho''$ is the adjoint transpose of $\hat{\rho}; (h; g) \Rightarrow \hat{\rho}'; D_Z: X \rightarrow (Z \Rightarrow Z)$.

Intuitively, $g: Y \rightarrow X$ is, again, a projection, and $\rho: X \times Y \rightarrow Y$ replaces the $X$-part of a large state, leaving the values of the new variables invariant.

We may now describe the isomorphism between these categories of worlds. Any $\mathcal{R}(C)$-morphism $(g, G)$ may be mapped to the $O(C)$-morphism $(g, \rho_G)$ such that $\rho_G$ applies $G$ to a “constant” $X$-command that, for all input states, outputs the desired new state; more precisely, $\rho_G$ is the adjoint transpose of $\kappa_X; G: X \rightarrow (Y \Rightarrow Y)$, where for any object $X$, $\kappa_X: X \rightarrow (X \Rightarrow X)$ is the adjoint transpose of the first projection $\pi_0: X \times X \rightarrow X$. Intuitively, $\kappa_X$ takes an element $x \in X$ to the constant $x$-valued function on $X$.

In the other direction, any $O(C)$-morphism $(g, \rho)$ may be mapped to the $\mathcal{R}(C)$-morphism $(g, G_\rho)$ such that $G_\rho$ uses $g$ to project out the $X$-part of a $Y$-state, applies the relevant $X$-command to it, and then uses $\rho$ to replace the $X$-part of the original state. In detail: $G_\rho = (g \Rightarrow \hat{\rho}); D_Y$.

We leave to the reader the detailed calculations needed to verify that these constructions are mutually inverse.

**Proposition 8** For any cartesian closed category $C$, the categories $\mathcal{R}(C)$ and $O(C)$ are isomorphic.

Because of this isomorphism, a category of relations fibered over the Reynolds category may be constructed as in Example 5.2 (or, more directly, by applying $\mathcal{R}$ to a subobject fibration on $C \times C$).

**Example 5.4** Several authors [Mog90, OT92, OT93b, PS93, Sie96, Dun02] have used the category $\text{Loc}$ of finite sets (of “locations”) and injections (or inclusions) as a category of worlds. A systematic way of constructing a suitable category of relations fibered over $\text{Loc} \times \text{Loc}$ is to pull back the fibration $r: \text{RW} \rightarrow W \times W$ along the functor $f: \text{Loc}^{op} \rightarrow W$ that interprets a set $\{\ell_1, \ldots, \ell_n\}$ of locations as the cartesian product.
$S = V(\ell_1) \times \cdots \times V(\ell_n)$, where $V(\ell_k)$ is the set of values storable at location $\ell_k$. An injection $i: \{\ell_1, \ldots, \ell_n\} \hookrightarrow \{\ell'_1, \ldots, \ell'_m\}$ is mapped to the pair $(g, \rho)$, where $g: V(\ell'_1) \times \cdots \times V(\ell'_m) \to V(\ell_1) \times \cdots \times V(\ell_n)$ projects out the components that are not in the image of $i$, and $\rho: S \times S' \to S'$ substitutes the $S$-part of an $S'$-tuple, leaving the remaining components unchanged.

**Example 5.5** The category of worlds and relations described by Dunphy [Dun02] is constructed less uniformly: the base category of worlds is the preorder of finite sets and inclusions, but the category of relations is defined by applying the Reynolds construction $\mathcal{R}(\cdot)$ to $r: \text{Rel} \to \text{Set} \times \text{Set}$. This is isomorphic to the dual of the fibration in Example 5.2 above, by Proposition 8. Pulling it back along the functor $f: \text{Loc}^{\text{op}} \to W$ yields a fibration on $\text{Loc} \times \text{Loc}$. To endow this fibration with an equality relation, Dunphy adds relations between state transformers $R_t: (W_0 \Rightarrow W_0) \leftrightarrow (W_1 \Rightarrow W_1)$ satisfying axioms analogous to those for morphisms in $\mathcal{R}(\text{Set})$.

**Example 5.6** The main semantic innovation in [OR00] is the use of binary relations that may relate states to “undefined” states. We will show how these may be re-constructed in our framework.

First, note that the fibration $r: \text{Rel} \to \text{Set} \times \text{Set}$ may be obtained by the change-of-base construction from the fibration $c: \text{Sub}(\text{Set}) \to \text{Set}$ along the product functor $\times: \text{Set} \times \text{Set} \to \text{Set}$. In more detail: the category $\text{Sub}(\text{Set})$ has as its objects all subobjects $P \xrightarrow{m} X$ (thought of as predicates on $X$) and a morphism between $P \xrightarrow{m} X$ and $Q \xrightarrow{n} Y$ is a function $f: X \to Y$ taking $P$ to $Q$. Taking the codomains of subobjects yields a functor $c: \text{Sub}(\text{Set}) \to \text{Set}$. Clearly this construction can be performed on any category $C$ in place of $\text{Set}$. The resulting functor $c: \text{Sub}(C) \to C$ is a fibration whenever $C$ admits pullbacks of monos along arbitrary morphisms.

We now show that this construction yields a fibration when applied to certain categories of partial maps. When command executions may be non-terminating, the most natural model for command meanings is as partial functions on the relevant set of states. Let $\text{Setp}$ be a small category of sets and all partial functions between them. It turns out that the objects of the category $\text{Sub}(\text{Setp})$ are the same as the objects of $\text{Sub}(\text{Set})$.\footnote{A subobject in any bicategory of partial maps is always total, i.e., is the same as a subobject in the category of total maps.} $\text{Sub}(\text{Setp})$ is fibered over $\text{Setp}$: the resulting re-indexing functor $c^\ast$ for any partial function $c: S \hookrightarrow S'$ is given as the weakest (liberal) precondition;\footnote{This property holds more generally for bicategories of partial maps $\text{Ptl}(C)$ when the subobject fibration $c: \text{Sub}(C) \to C$ admits products along monomorphisms. See [Her02a] for further analysis of fibrations over relations and partial maps.} $c^\ast(Q)(s)$ iff $Q(c(s))$ whenever $c(s)$ is defined. The morphisms from $P$ to $Q$ are the valid Hoare triples $P\{c\}Q$.

A fibration of appropriate binary relations (on possibly distinct sets of states) may now be obtained by change of base along the product functor $\times$. Note that the categorical product $S \times S'$ in $\text{Setp}$ is the set $S \oplus (S \otimes S') \oplus S'$, where $\oplus$ denotes disjoint union and $\otimes$ is the conventional cartesian product of sets. The projection $\pi_0: S \times S' \to S$ is defined by cases
on \( S \oplus (S \otimes S') \oplus S' \) as follows: \( \pi_0(s) = s \) for \( s \in S \), \( \pi_0(s,s') = s \), and \( \pi_0(s') \) is undefined for \( s' \in S' \), and similarly for the other projection \( \pi_1: S \times S' \to S' \). This construction yields binary relations which may be “preserved” by a pair \((f,f')\) of partial functions, even if one is undefined on the relevant component of a related pair of arguments.

In practice, particularly when working with concrete examples, it is more convenient to make the “undefineds” explicit and work with the equivalent category \( \text{Set}_\perp \) of pointed sets and \( \perp\)-preserving total functions [Red97], but the objects are nonetheless \( \text{sets} \), and not “flat domains.” In particular, the analogous equivalence fails for categories of domains [Fio96]. In treating an \( \text{ALGOL}\)-like language, states become involved in elements of domains only as arguments or results of (possibly partial) \( \text{functions} \). A set of partial functions (or \( \perp\)-preserving total functions) on sets may be ordered in the obvious way to form a domain. The following is a more accurate presentation of the type system used for the target language in [OR00]:

\[
\begin{align*}
\sigma &::= \alpha | I | \sigma \otimes \sigma & \text{Level 1} \\
A &::= \sigma \to \sigma | A \to A | A & A | \forall \alpha. A & \text{Level 2}
\end{align*}
\]

where \( \alpha \) ranges over variables for Level 1 types. The Level 1 types should denote sets and the Level 2 types should denote domains.

From such a category of binary relations on state sets, it is then possible to construct categories of worlds and relations on worlds in any of the ways discussed above. There do not appear to be any impediments to using the new relations on states in treating such features as passive expressions, non-interference predicates in a specification logic, and block expressions.

6 Discussion

In functor-category models, the semantic categories are typically cartesian closed and complete, but note that these properties are not required of the categories of worlds. Similarly, in the framework we have presented here, we have not tried to impose on \( W \) all of the properties of \( S \). For example, the equality-relation functor \( \text{Eq}: S \to R S \) gives \( S \) a reflexive-graph structure, and it may be that requiring reflexive-graph structure (and not just span structure) would be useful on worlds as well, but in the absence of any compelling evidence for this, we have adopted here the simpler structure. For the same reason, we have not imposed parametricity constraints on \( W \).

This approach is consistent with the fibration-over-fibration formulation of polymorphic logical relations in [Her03]: when formulating Reynolds’s relational parametricity [Rey83] (which amounts to a property of the equality relation with respect to generic objects and type-quantification) only the \( \text{types} \) are endowed with an equality, not the \( \text{kinds} \). This is evident in the syntactic formal framework of [PA93], which expresses Reynolds’s relational parametricity as an additional axiom to allow formal derivation of the expected consequences, such as the existence of initial algebras and dinaturality.

In contrast, Dunphy’s thesis [Dun02] develops further the reflexive-graph approach of O’Hearn and Tennent [OT93b, OT95], exploiting the cartesian closure of the 2-category of
reflexive graphs of categories. Dunphy succeeds in capturing the relationally-parametric
type quantifier (as formulated in [OR00, Section 7]) as a “small product” (right adjoint to
a diagonal); nevertheless, in our opinion, a full-fledged categorical account of “relational
parametricity” which would reconcile these various approaches is still lacking.

A Appendix: Proofs

A.1 Proof of Proposition 1

For any small category \( W \), if \( S \) is cartesian closed and complete, so is \( S(w) \) for every \( w \in W \).

Proof. Consider any functors \( F, G : W^{op} \to \mathbf{Set} \); \((F \Rightarrow G)(w)\) for any \( w \) in \( W \) is the set of all natural
transformations from \( \text{dom}_w;F \) to \( \text{dom}_w;G \), where \( \text{dom}_w : (W/w)^{op} \to W^{op} \) is the forgetful functor
that maps \( x \to x' \). It is easily verified that this is equivalent to the usual Yoneda-derived formula.

As shown in [Nel81], this may be generalized to any complete and cartesian closed category
\( S \) in place of \( \mathbf{Set} \). For \( F, G : W^{op} \to S \), the natural transformations from \( F \) to \( G \) may be internalized
in \( S \) as the following object:

\[
\int_{f : x \to w} [(\text{dom}_w ; F)(f) \Rightarrow (\text{dom}_w ; G)(f)]
\]

Products and other limits in \( W^{op} \Rightarrow S \) are constructed pointwise.

Finally we note that \( S(w) \) is of the form \( W^{op} \Rightarrow S \) where \( W \) is itself a slice category.

Corollary 9 If \( S \) and \( T \) are complete and cartesian closed categories and functor \( F : S \to T \) preserves this
structure, the post-composition functor \( \mathbf{Cat}(W, F) : \mathbf{Cat}(W, S) \to \mathbf{Cat}(W, T) \) preserves completeness and
cartesian closure for any small category \( W \).

A.2 Proof of Proposition 2

If \( S \) is complete and cartesian closed, the fibration \( p : \mathbf{Slices}(W, S) \to W \) is fibrewise cartesian
closed and this structure is preserved by re-indexing.

Proof. Each fiber is cartesian closed (Proposition 1). It remains to show that, for every \( h : w \to w' \)
in \( W \), the corresponding re-indexing functor preserves this structure.

Lemma 10 If \( S \) is a complete and cartesian closed category and \( p : W' \to W \) is a discrete fibration (i.e., ev-
ery fiber is discrete), precomposition with \( p \)

\[
p ; - : \mathbf{Cat}(W, S) \to \mathbf{Cat}(W', S)
\]
preserves cartesian closure.
Proof of Lemma 10. Given functors $G, H: \mathbf{W} \to \mathbf{S}$, and an object $w'$ of $\mathbf{W}'$, we must show that the canonical comparison between

$$[G \Rightarrow H](pw') = \int_{f,x\to pw'} [Gx \Rightarrow Hx]$$

and

$$[(p;G) \Rightarrow (p;H)](w') = \int_{g:x\to w'} [(p;G)(x') \Rightarrow (p;H)(x')]$$

is an isomorphism. But because $p$ is a discrete fibration, $p/w': \mathbf{W}'/w' \to \mathbf{W}/pw'$ is an isomorphism of categories, so that both ends are computed over isomorphic categories and on isomorphic diagrams.

To complete the proof of Proposition 2, we have, for any $h: w \to w'$, the following isomorphism of functors into the slice $\mathbf{W}/w'$:

$$\xymatrix@C=4em{\mathbf{W}/w \ar[r]^\sim & (\mathbf{W}/w')/h \ar[dl]_{\Sigma_h} \ar[dr]^{\text{dom}_h} \\
\mathbf{W}/w'}$$

Because $\text{dom}_x: \mathbf{C}/x \to \mathbf{C}$ is always a discrete fibration (obtained by applying the Grothendieck construction to the representable $\mathbf{C}(-,x): \mathbf{C}^{\text{op}} \to \mathbf{Set}$), so is $\Sigma_{h'}$ and Lemma 10 then yields the result that precomposition with $\Sigma_h$ preserves cartesian closure.

A.3 Proof of Proposition 3

If $\mathbf{S}$ is complete, the fibration $p: \text{Slices}(\mathbf{W}, \mathbf{S}) \to \mathbf{W}$ admits products; dually, if $\mathbf{S}$ is co-complete, it admits co-products.

Proof. Recall that a fibration is said to admit products if re-indexing functors admit right adjoints, and these are pullback-stable (Beck-Chevalley condition). If $\mathbf{S}$ is complete, every re-indexing functor admits a right adjoint (right Kan extension). These are pullback stable along fibrations [Her02b, Prop. 2.4], and a commuting square $\xymatrix{\mathbf{W}/p \ar[r]^{\Sigma_{h'}} \ar[d]_{\Sigma_{\eta}} & \mathbf{W}/w' \ar[d]_{\Sigma_f} \\
\mathbf{W}/w \ar[r]_{\Sigma_{\pi}} & \mathbf{W}/x}$ is a pullback in $\mathbf{Cat}$.

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A.4 Proof of Proposition 4

The diagram

\[
\begin{array}{ccc}
\text{Slices}(rw,rs) & \xrightarrow{\tilde{rw}} & \text{Slices}(W,S) \times \text{Slices}(W,S) \\
\downarrow q & & \downarrow p \times p \\
\text{RW} & \xrightarrow{rw} & W \times W
\end{array}
\]

is a fibration in \(\mathcal{F}_{\text{ib}}\) whenever \(p\) and \(rw\) are fibrations.

Proof. When \(rw\) is a fibration, it suffices [Her03] to show that, for each \(W\) in \(\text{RW}\) with \(rw(W) = (w_0,w_1)\), the fiber functor \(\tilde{rw}_W: \text{Slices}(rw,rs)_W \to \text{Slices}(W,S)_{w_0} \times \text{Slices}(W,S)_{w_1}\), taking objects \((\tilde{F},F_0,F_1)\) (in the fiber above \(W\)) to \((F_0,F_1)\), is a fibration, and that such fibrations are preserved by the re-indexing functors induced by morphisms in \(\text{RW}\).

To show that \(\tilde{rw}_W\) is a fibration, consider any object \((\tilde{G},G_0,G_1)\) of \(\text{Slices}(rw,rs)_W\):

\[
\begin{array}{ccc}
(W/w_0)^{op} \times (W/w_1)^{op} & \xrightarrow{G_0 \times G_1} & S \times S \\
\downarrow (rw/W)^{op} \quad & \downarrow & \quad \downarrow(F_0 \times F_1) \\
(W/w_0)^{op} \times (W/w_1)^{op} & \xrightarrow{\eta_0 \times \eta_1} & S \times S
\end{array}
\]

and any morphism \((\eta_0:F_0 \to G_0, \eta_1:F_1 \to G_1)\) into the underlying pair \((G_0,G_1) = \tilde{rw}(\tilde{G},G_0,G_1)\):

\[
\begin{array}{ccc}
(W/w_0)^{op} \times (W/w_1)^{op} & \xrightarrow{G_0 \times G_1} & \mathcal{R}S \\
\downarrow (rw/W)^{op} \quad & \downarrow & \quad \downarrow(\eta_0 \times \eta_1) \\
(W/w_0)^{op} \times (W/w_1)^{op} & \xrightarrow{F_0 \times F_1} & \mathcal{R}S \times \mathcal{R}S
\end{array}
\]

We will construct the \(\tilde{\eta}\) component (with co-domain \(\tilde{G}\)) of a cartesian lifting \((\tilde{\eta},\eta_0,\eta_1)\) of \((\eta_0,\eta_1)\) pointwise, using the fact that \(rs\) is a fibration. Consider any object \(f:X \to W\) of \(\text{RW}/W\) with \(rw(f) = (f_0:x_0 \to w_0, f_1:x_1 \to w_1)\); then \(\tilde{\eta}(f)\) is defined as a cartesian lifting of \(\eta_0(f_0) \times \eta_1(f_1)\) with respect to fibration \(rs\). The domain of \(\tilde{\eta}\) is a contravariant functor from \(\text{RW}/W\) to \(\mathcal{R}S\) whose action on objects yields the relevant domain of the cartesian lifting and whose actions on morphisms is determined by the universality property of the cartesian lifting.

The pointwise nature of the liftings makes them stable under the action of re-indexing functors induced by morphisms \(h\) in \(\text{RW}\), since the action is given by precomposition with functors \(\Sigma_h\) between slices.

A.5 Proof of Corollary 5

If \(rs\) is a fibration, so is \(rw^{op}\) \(\Rightarrow\) \(rs\) for any functor \(rw\).

Proof. \(rw^{op}\) \(\Rightarrow\) \(rs\) is essentially \(\tilde{rw}_W\) for \(W = 1\) and the pointwise construction depends only on \(rs\) being a fibration.
A.6 Proof of Proposition 6

If $rs$ is a fibration of complete cartesian closed categories, the diagram

\[
\begin{array}{ccc}
\text{Slices}(rw, rs) & \xrightarrow{\bar{rw}} & \text{Slices}(W, S) \times \text{Slices}(W, S) \\
q & & p \times p \\
\text{RW} & \xrightarrow{\bar{rw}} & W \times W
\end{array}
\]

is a morphism of fibered cartesian closed categories with products.

Proof. By Corollary 9 on page 12, the functor $\bar{rw}_W$ preserves cartesian closed structure and completeness.

References


[Her03] C. Hermida. Fibrational logical relations for polymorphic $\lambda$-calculi. Unpublished draft, School of Computing, Queen’s University, Kingston, Canada, 2003.


