Prefibers and the cartesian product of metric spaces

Claude Tardif

Département de Mathématiques et de Statistique, Université de Montréal, Montréal, Qué., Canada H3C 3P8

Received July 1990
Revised 26 April 1991

Dedicated to Gert Sabidussi, on the occasion of his 60th birthday.

Abstract


The properties of certain sets called prefibers in a metric space are used to show that the algebraic properties of the cartesian product of graphs generalize to metric spaces.

1. Introduction

Many proofs have already been given of the fact that finite connected graphs enjoy unique factorization under the cartesian product (see [6, 8, 11, 13]). The introduction of the weak cartesian product of graphs in [11] allows to generalize this result to infinite connected graphs (see [6, 8]), while disconnected graphs do not enjoy unique cartesian factorization even in the finite case (see [15]). In this respect, one feature of the cartesian product of connected graphs is that it can be defined by setting the shortest path metric in the product as the sum of shortest path metrics in the factors. This may suggest that the algebraic properties of this product pertain only to its definition as an operator on metric spaces. The purpose of this note is to give a new proof of the unique cartesian factorization property which is valid for metric spaces in general. In this perspective, we introduce the following terminology.

Definition 1.1. (i) Let $X = \prod_{i \in I} X_i$, for $i \in I$, $pr_i$ denotes the projection on $X_i$; for $x \in X$, the $i$-fiber of $X$ through $x$ is the set

$$X(i, x) = \left\{ y \in \prod_{i \in I} X_i : pr_j(y) = pr_j(x) \text{ for all } j \in I \setminus \{i\} \right\}.$$
Let \( q_{i,x} : X_i \rightarrow X(i, x) \) denote the natural bijection between these two sets.

(ii) The cartesian product \( \prod_{i \in I} (X_i, \delta_i) \) of a finite family \( \{(X_i, \delta_i)\}_{i \in I} \) of metric spaces is the space \( (X, \delta) \) where \( X = \prod_{i \in I} X_i \), and \( \delta = \sum_{i \in I} \delta_i \circ (pr_i \times pr_i) \).

(iii) A nontrivial metric space \( (X, \delta) \) is said to be indecomposable if for every isomorphism \( \varphi : (X, \delta) \rightarrow (X_1, \delta_1) \times (X_2, \delta_2) \), either \( X_1 \) or \( X_2 \) is a singleton.

The fibers of a cartesian product of metric spaces are endowed with an invariant structure which will be discussed in the next section. Note that fibers are incomparable as long as none of the sets \( X_i \) is a singleton.

For any infinite family \( \{(X_i, \delta_i)\}_{i \in I} \) of nontrivial metric spaces, many subsets of \( \prod_{i \in I} X_i \) induce metric spaces with each space \( (X_i, \delta_i) \) as a factor. Thus, an extension of Definition 1.1(ii) similar to the weak cartesian product of \([11]\) is conceivable; however, some examples of \( L_1 \) spaces show that it is too ambitious to ask for an extension of this definition to an operator for which both existence and unicity of a factorization into indecomposable factors hold. Our main result is the following.

**Theorem 1.2.** Let \( (X, \delta) = \prod_{i \in I} (X_i, \delta_i), (Y, \lambda) = \prod_{i \in J} (Y_i, \lambda_i) \), where \( \{(X_i, \delta_i)\}_{i \in I} \) and \( \{(Y_i, \lambda_i)\}_{i \in J} \) are finite families of nontrivial indecomposable metric spaces; let \( \varphi : (X, \delta) \rightarrow (Y, \lambda) \) be an isomorphism. Then there is a bijection \( \psi : I \rightarrow J \) and a family \( \{\varphi_i : (X_i, \delta_i) \rightarrow (Y_{\psi(i)}, \lambda_{\psi(i)})\}_{i \in I} \) of isomorphisms such that \( \varphi = \prod_{i \in I} \varphi_i \).

The existence of a bijection \( \psi \) such that \( (X_i, \delta_i) \) is isomorphic to \( (Y_{\psi(i)}, \lambda_{\psi(i)}) \) is the unique factorization property in the usual sense (see \([7]\)). The additional statement that \( \varphi = \prod_{i \in I} \varphi_i \) generalizes the results of \([6, 9, 11]\) on the automorphism group of a cartesian product of graphs.

The proof of Theorem 1.2 relies on the invariant structure of the fibers in a cartesian product of metric spaces, which motivates the definition of prefibers in the next section.

### 2. Prefibers and projection maps in metric spaces

**Definition 2.1.** A subset \( A \) of a metric space \( (X, \delta) \) is a **prefiber** of \( (X, \delta) \) if for any \( x \in X \), there exists a (necessarily unique) \( p_A(x) \in A \) such that for all \( y \in A \),

\[ \delta(x, y) = \delta(x, p_A(x)) + \delta(p_A(x), y). \]

The map \( p_A : X \rightarrow A \) is the projection on \( A \).

This set structure is relevant to several topics. Under various motivations, prefibers were introduced independently as *Chebychev sets* in \([4, 5]\), *gated sets* in \([2, 3]\) and *J-convex sets* in \([14]\). In the present setting, projection maps are an invariant generalization of the maps \( q_{i,x} \circ pr_i \) in a cartesian product of metric spaces (see Corollary 3.2.)
This section presents some basic results about the general structure of prefibers in metric spaces; we provide only some sketches of the proofs, since many of these results are also contained in [2].

In an arbitrary metric space $(X, \delta)$, the set $X$ and all one element sets are prefibers; also, a connected graph is bipartite if and only if each set of two adjacent vertices is a prefiber. Prefibers are convex sets in the sense of [10], and projection maps are idempotent and non-expanding.

**Proposition 2.2.** Let $A$, $B$ be prefibers of a metric space such that $A \cap B \neq \emptyset$. Then $p_A(B) = p_B(A) = A \cap B$ is a prefiber of $X$ and $p_{A \cap B} = p_A \circ p_B = p_B \circ p_A$.

This result can be shown by a straightforward application of the definition of a prefiber (see [2, 4, 5]); another proof is given by the alternative definition of a $J$-convex set in [14]. For $x, y \in X$, define $I(x, y) = \{z \in X : \delta(x, y) = \delta(x, z) + \delta(z, y)\}$, and $J(x, y) = \{z \in X : I(z, x) \cap I(z, y) = \{z\}\};$ a set $A \subseteq X$ is $J$-convex if $J(x, y) \subseteq A$ for all $x, y \in A$. It can be shown that in a complete metric space, the prefibers are precisely the non-empty $J$-convex sets, and the property of closure under non-empty intersection follows immediately from this fact.

Proposition 2.2 also provides the inductive step for the proof of the following statement. If a finite collection $\{A_i\}_{i \in I}$ of prefibers has pairwise non-empty intersection, then $\bigcap_{i \in I} A_i \neq \emptyset$. Hence, as noted in [1], any graph is endowed with a natural convexity satisfying the Helly property.

The following results generalize the partial interpretation of the composition of projection maps given in Proposition 2.2.

**Proposition 2.3.** Let $A$ and $B$ be two prefibers of a metric space $(X, \delta)$. Then the following properties are equivalent:

(i) $p_B$ is injective on $A$,
(ii) $p_A$ is surjective on $B$,
(iii) $p_A \circ p_B$ is the identity function on $A$,
(iv) $\delta(a_1, a_2) = \delta(p_B(a_1), p_B(a_2))$ for any $a_1, a_2 \in A$,
(v) $\delta(a, p_B(a))$ is constant on $A$.

**Proof.** The implications $(iii) \Rightarrow (i)$, $(iii) \Rightarrow (ii)$ and $(iv) \Rightarrow (i)$ are obvious; $(i) \Rightarrow (iii)$, $(ii) \Rightarrow (iii)$ and $(v) \Rightarrow (iii)$ follow from the fact (easily verifiable) that for any pair $A, B$ of prefibers, the mapping $p_B \circ p_A$ is always idempotent. $(iii) \Rightarrow (iv)$ and $(v)$ is deduced from the inequalities

$$\delta(p_A \circ p_B(a_1), p_A \circ p_B(a_2)) \leq \delta(p_B(a_1), p_B(a_2)) \leq \delta(a_1, a_2) - |\delta(a_1, p_B(a_2)) - \delta(a_1, p_B(a_2))|,$$

which follow from [2, Lemma 1]: For any prefiber $C$ of $X$ and $x, y \in X$,

$$\delta(p_C(x), p_C(y)) + |\delta(x, p_C(x)) - \delta(y, p_C(y))| \leq \delta(x, y).$$

\[\square\]
When \( p_A \) is bijective on \( B \), statements (iii), (iv), (v) generalize the relation between two distinct \( i \)-fibers in a cartesian product of metric spaces. Statements (i)–(v) may also be seen as an extension of properties which are trivially verified when \( A \subseteq B \); in this perspective, the following is a generalization of the property of closure under non-empty intersection.

**Proposition 2.4** [2]. Let \( A \) and \( B \) be prefibers of a metric space \( (X, \delta) \). Then \( p_A(B) \) is a prefiber of \( (X, \delta) \) and \( p_{p_A(B)} = p_A \circ p_B \circ p_A ; p_A(B) \) and \( p_B(A) \) are maximal prefibers, contained respectively in \( A \) and \( B \), on which both \( p_A \) and \( p_B \) are injective; \( p_A \) and \( p_B \) induce isomorphisms, inverse to each other, between \( p_A(B) \) and \( p_B(A) \).

In some classes of metric spaces with additional structure (hypermetricity, for example) the identity \( p_{p_A(B)} = p_A \circ p_B \) holds for all pairs of distinct prefibers; however, this is not the case in general, as is shown by any two disjoint prefibers of the graph \( K_{2,3} \).

The generalization of Proposition 2.2 to infinite families of prefibers with non-empty intersection is dependent of some weak completeness conditions, as is the equivalence between prefibers and the \( J \)-convex sets of [14].

### 3. Proof of Theorem 1.2

**Lemma 3.1.** Let \( (X, \delta) = \prod_{i \in I} (X_i, \delta_i) \), where \( I \) is finite.

(i) If \( A_i \) is a prefiber of \( X_i, i \in I \), then \( A = \prod_{i \in I} A_i \) is a prefiber of \( X \), and

\[
 p_A = \prod_{i \in I} p_{A_i}.
\]

(ii) If \( C \subseteq X \) is a prefiber, then for all \( i \in I \), \( \text{pr}_i(C) \) is a prefiber of \( X_i \), and

\[
 p_{\text{pr}_i(C)} \circ \text{pr}_i = \text{pr}_i \circ p_C.
\]

(iii) If \( C \subseteq X \) is a prefiber, then \( C = \prod_{i \in I} \text{pr}_i(C) \).

**Proof.** The statements (i) and (ii) can easily be verified using definitions and the triangle inequality; (iii) is deduced from (ii) as follows. Take \( x \in \prod_{i \in I} \text{pr}_i(C) \), for all \( i \in I \), \( \text{pr}_i(x) \in \text{pr}_i(C) \), thus \( \text{pr}_i(x) = p_{\text{pr}_i(C)} \circ \text{pr}_i(x) = \text{pr}_i \circ p_C(x) \), so \( x = p_C(x) \in C \).

Since each \( (X_i, \delta_i) \) has \( X_i \) and singletons as trivial prefibers, statement (i) of this lemma gives a formal proof to the following statement.

**Corollary 3.2.** Each fiber \( X(i, x) \) of \( (X, \delta) \) is a prefiber, and \( p_{X(i,x)} = q_{i,x} \circ \text{pr}_i \).

**Lemma 3.3.** Let \( (X, \delta) = \prod_{i \in I} (X_i, \delta_i) \), \( (Y, \lambda) = \prod_{i \in J} (Y_i, \delta_i) \), where \( \{(X_i, \delta_i)\}_{i \in I} \) and \( \{(Y_j, \lambda_j)\}_{j \in J} \) are finite families of nontrivial indecomposable metric spaces; let
$\varphi: (X, \delta) \rightarrow (Y, \lambda)$ be an isomorphism and $x$ be an element of $X$. Then there is a bijection $\psi: I \rightarrow J$ such that for all $i \in I$, $\varphi(X(i, x)) = Y(\psi(i), \varphi(x))$.

**Proof.** By Lemma 3.1(iii), for $i \in I$, $\varphi(X(i, x)) = \prod_{j \in J} p_{r_j}(\varphi(X(i, x)))$; since this set with its induced metric is isomorphic to $(X_i, \delta_i)$, which is indecomposable, there exists a unique $j \in J$ such that $p_{r_j}(\varphi(X(i, x))) = \{\varphi(x)\}$ for all $k \in J \setminus \{j\}$, i.e., $\varphi(X(i, x)) \subseteq Y(j, \varphi(x))$. Thus we can define two mappings $\psi: I \rightarrow J$, $\psi': J \rightarrow I$ by setting $\varphi(X(i, x)) \subseteq Y(\psi(i), \varphi(x))$, $\varphi^{-1}(Y(j, \varphi(x)) \subseteq X(\psi'(j), x)$. Accordingly, $X(i, x) \subseteq X(\psi' \circ \psi(i), x)$; since fibers are incomparable this means that $\psi' = \psi^{-1}$, thus $\psi$ is bijective and $\varphi(X(i, x)) = Y(\psi(i), \varphi(x))$. \[\square\]

**Corollary 3.4.** For all $i \in I$, $\varphi_i = q_{\psi^{-1}(i), \psi(x)} \circ \varphi \circ q_{l, x} : (X_i, \delta_i) \rightarrow (Y_{\psi(i)}, \lambda_{\psi(i)})$ is an isomorphism.

**Proof.** It remains to check that $\varphi = \prod_{i \in I} \varphi_i$; by definition, $\prod_{i \in I} \varphi_i$ is characterized by the conditions

$$p_{r_{\varphi(i)}} \circ \prod_{i \in I} \varphi_i = \varphi_i \circ p_{r_i}$$

for each $i \in I$. The result follows from the identities

$$P_{\varphi(X(i, x))} \circ \varphi = \varphi \circ p_X(i, x),$$

i.e., the invariance of projections on prefibers. According to Lemma 3.3, $\varphi(X(i, x)) = Y(\psi(i), \varphi(x))$; replacing $p_X(i, x)$ and $p_Y(\psi(i), \varphi(x))$ by their equivalent expressions in Corollary 3.2, we get

$$q_{\psi(i), \psi(x)} \circ p_{r_{\psi(i)}} \circ \varphi = \varphi \circ q_{l, x} \circ p_{r_i}$$

thus

$$p_{r_{\psi(i)}} \circ \varphi = q_{\psi^{-1}(i), \psi(x)} \circ \varphi \circ q_{l, x} \circ p_{r_i} = \varphi_i \circ p_{r_i}. \quad \square$$

Since the finiteness of the sets $I$ and $J$ is not an essential argument in any part of this proof. Theorem 1.2 also holds for a generalization of Definition 1.1(ii) which preserves finiteness of distances. Some elements of the proof can also be used to show that any two factorizations of a metric space admit a common refinement. In particular our methods imply that connected graphs have unique prime factorization with respect to the cartesian product. This was conjectured by Sabidussi [11] and first shown by Miller [8] and Imrich [6]. Imrich’s approach is similar to ours, his concept of layers being the same as that of our fibers for products of graphs.

**Acknowledgements**

The author wishes to thank G. Sabidussi for many useful discussions on this subject, and H.J. Bandelt for introducing me to the earlier literature on prefibers.
References