Reverse Derivations in Prime Rings

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ABSTRACT

In this paper, we prove some results on reverse derivations in prime rings. We prove that let R be a prime ring and d a reverse derivation of R such that $d(a)\ a - ad\ (a) = 0$*, for all* $a \in R$*. Then R is Commutative or d is zero.*

Keywords: *Derivation, Reverse Derivation, Prime Ring, Center.*

INTRODUCTION

Posner [6], Herstein [4], Felzenszwalb [3], Daif and Bell [2] have investigated the properties of prime (or) semi prime rings with derivations, Bresar and Vukman [1] have introduced the notion of a reverse derivation, Samman and Alyamani [7] and Jaya Subba Reddy [5] have studied some properties of prime(or) semi prime rings with reverse derivations. In this paper, we prove some results on reverse derivations in prime rings.

PRELIMINARIES

We know that an additive map d from a ring R to R is called a derivation on R if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A ring R is called prime if $xay = 0$ implies $x = 0$ or $y = 0$, for all x, a, y in R. An additive mapping d from a ring R into itself satisfying $d(xy) = d(y)x + yd(x)$, for all $x, y \in R$, is called a reverse derivation on R. Throughout this paper R will denote a prime ring and Zits center. We use the following elementary identities in our work.

In any ring R, $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ hold for all $x, y, z \in R$. If R is any ring and d any derivation, we have $d(Z) \subseteq Z$, $d([x, y]) = [x, d(y) + [d(x), y]$.

Now, we present some results on reverse derivations in prime rings.

Lemma 1: Let d be a reverse derivation of a prime ring R and a be an element of R. If $ad(x) = 0$, for all $x \in R$, then either $a = 0$ (or) d is zero.

Proof: In $ad(x)=0$, for all $x \in R$, we replace x by yx then,

$$\Rightarrow ad(yx)=0$$

$$\Rightarrow ad(x)y+axd(y)=0$$

$$\Rightarrow axd(y)=0, \text{ for all } x, y \in R.$$

If d is not zero, that is, if $d(y)\neq 0$,forsome$y \in R$,then by the definition no f a prime ring, $a=0$.

The following Lemma has some independent interest.

Lemma 2: Let R be a prime ring and let p, q, r be elements of R such that $paqar=0$, for all a in R. Then one, atleast, of p, q, r is zero.

Proof: In $paqar=0$, we replace $a$by$a+b$.

By using $paqar=pbqbr=0$, we find $paqbr+pbqar=0$, for all $a,b \in R$.

If now $pa=0$, then, for all b in R, $pbqar=0$, so that $p=0$, (or) else $qar=0$.

But if $pa=0$, then $pat=0$, for all $t\in R$, so that $p=0$(or) $qatr=0$, for all t in R. Again $r=0$, or else $qa=0$.So,$p=0$(or)$r=0$(or) qa is zero whenever $p\ a$ is zero. We replace a by $aqar$.

Since $p\ (aqar)=0$, for all $a \in R$,we see that $p=0$(or)$r=0$(or) $qaqar=0$, for all $a \in R$. Similarly, $p=0$ (or) $r=0$(or) $qaqaq=0$, for all $a \in R$.

Therefore by assuming that $p\neq 0, r\neq 0$ and replacing a by $a+b$ in $qaqaq=0$,we find as before that $qaqbq+qbqaq=0$.

In this equation, we replace b by aqb to find $(qaqaq)bq+qaqbqaq=0$,

$$\Rightarrow (qaq)b(qaq)=0, \text{ for all } a,b \in R.$$

So, $qaq=0$, for all $a \in R$,$q=0$, if$p\neq 0, r\neq 0$.

Theorem 1: Let R be a prime ring and d a reverse derivation of R such that $d(a)\ a-ad(a)=0$, for all $a\in R$. Then R is commutative or d is zero.

Proof: From the hypothesis, we have,$d(a)\ a-ad(a)=0$, for all $a\in R$.

We replace a by $(a+b)$ in the above equation, then we get,

$$d(a)a+d(a)b+d(b)a+d(b)b-ad(a)-ad(b)-bd(a)-bd(b) \ldots..(1)$$

By subtracting $d(a)a+d(b)b-ad(a)-bd(b)=0$from equ.(1), then we get,

$$d(a)b+d(b)a-ad(b)-bd(a)=0, \text{ for all } a,b\in R.$$

We write this as$d(a)b-ad(b)=bd(a)-d(b)a$.

By adding $d(a)b+ad(b)=d(ba)$ to the above equation, then we find that,

$$d(a)b-ad(b)+d(a)b+ad(b)=bd(a)-d(b)a+d(ba)$$

$$\Rightarrow 2d(a)b=bd(a)-d(b)a+d(ba)\text{, for all } a,b\in R\ldots\ldots (2)$$

We replace a by bx in equ.(2), then,

$$2d(bx)b=bd(bx)-d(b)bx+d(bbx)$$

$$\Rightarrow 2d(x)bb+2xd(b)b=bd(x)b+bxd(b)-d(b)bx+d(b^2x)$$

$$\Rightarrow 2d(x)b^2+2xd(b)b=bd(x)b+bxd(b)-d(b)bx+d(x)b^2+x2bd(b)$$

Since $d(b^2)=2bd(b)$

$$d(x)b^2=bd(x)b+bxd(b)-d(b)bx\text{, for all } b,x\in R\ldots\ldots (3)$$

Now, we replace a by xb in equ.(2), then we get,

$$2d(xb)b=bd(xb)-d(b)xb+d(bxb)$$

$$\Rightarrow 2d(b)xb+2bd(x)b=bd(b)x+bbd(x)-d(b)xb+d(xb)b+xbd(b)$$

$$\Rightarrow b^2d(x)=d(b)xb+bd(x)b-xbd(b)\text{, for all } b,x\in R\ldots. (4)$$

By adding equ.'s (3) and (4), we get,

$$d(x)b^2+b^2d(x)=bd(x)b+bd(x)b\text{, for all } b,x\in R\ldots\ldots (5)$$

(or)

$$(d(x)b-bd(x)b)b=b(d(x)b-bd(x))\text{, for all } b,x\in R\ldots\ldots (6)$$

We replace b by $b+d(x)$ in equ.(6).

We find that $d(x)$ commutes with $d(x)b-bd(x)$, for all b, x in R. This shows that the square of the inner derivation by x is zero, for all $x\in R$.

Let R is of char.$\neq 2$. Then by [5, Theorem:2], $d(x)$ is central, for all x in R.
Let b be an element of R and B denote inner derivation by b.

Now $bd(x)=d(x)b$, (or) $Bd(x)=0$, for all $x\in R$. Again [5, Theorem:2], shows that $d=0$ (or) if not, then B is zero, every b in R is central, R is commutative.

But if R is of char.2, then equ.(5) says that for all $x\in R$, $d(x)$ commutes with all squares of elements of R. Let R be a prime ring of char.2, and let $e\in R$ commute with b^2, for all $b\in R$. That is,

$$b^2e=eb^2\text{, for all } b\in R\ldots. (7)$$

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we replace b by $b+a$ and using $eb^2=b^2e$, $ea^2=a^2e$,

then $(ba+ab)e=e(ba+ab)$, for all $a, b \in R$….. (8)

we replace a by be and commute e and b^2.

Then $b^2e^2+bebe=eb^2e+ebeb$ and $b^2e^2=ebe$

$\Rightarrow bebe=ebeb$, for all $b\in R$…… (9)

We replace a by e in equ.(8), then we get,

$(be+eb)e= e(be+eb)$

$\Rightarrow be^2+ ebe= ebe+ e^2b$

So, e^2 is in the center of R…… (10)

We consider $(be+eb)^2= bebe+ ebeb+ 2beeb$

$=bebe+ebeb+2be^2b$

$=bebe+ebeb+be^2b+be^2b$

By equ.(13), we have, $bebe+ebeb=0$

$\Rightarrow be^2b+be^2b=e^2b^2+e^2b^2=0$

By equ. (10)and(7), we have,

$(be+eb)^2=0$, for all $b\in R$….. (11)

Let x,y be elements of R with $xy=0$.

By equ. (8), we have, $(yx+xy)e=e(yx+xy)$

So, $yx=0$ implies $xye=exy$…. (12)

Now $y^2x=0$, so equ.(12) also becomes,

$xy^2e=exy^2$ and $xy^2e=xey^2$, since e commutes with all squares.

Thus, $yx=0$ implies $(xe+ex)y^2=0$ ..... (13)

But $(by)x=0$, for all $b\in R$. Then we can replace y by bx in equ.(13) to obtain $(xe+ex)byby=0$, for all $b\in R$, whenever $yx=0$. Lemma: 2 implies either $y=0$ (or) $xe+ex=0$, in fact since $y(xv)=0$, for all $v\in R$. Again Lemma: 2 also implies either $y=0$ (or) $xve+(ex)v=0$, for all $v\in R$. Since $xe=ex$, f $y\neq 0$, then $y=0$ (or) $xve+xev=0$, for all $v\in R$, $x(ve+ev)=0$, for all $v\in R$. By applying Lemma:1 to the inner derivation by e shows that either $y=0$, $x=0$ (or) e is central. But by equ. (11), $(be+eb)(be+eb)=0$, for all $b\in R$.

By putting $y = be + eb$, $x = be + eb$, we find that for all $b \in R$, $be + eb = 0$,(or)e is central. That is, for all $b \in R$,$be + eb = 0$,e is central if e commutes with all squares in R, $d(x)$ is central for all $x \in R$. Let$d(a) = 0$, for all $b \in R$, $d(ab) = d(b)a + bd(a) = d(b)a$. $d(ab)$ is central, so, $d(b)$ a is central for all $b \in R$, if $d(a) = 0$ .Now if d is not zero, so that $d(a) \neq 0$ for some $a \in R$, we have, $d(a)bx = xd(a)b$, $d(a)$ is central. So, $xd(a)b = d(a)xb$. Hence $d(a)(bx + xb) = 0$,for all $x \in R$, if $d(b) = 0$. But no non-zero element of the centroid of R has non-zero Kernel, since we are assuming $d(a) \neq 0$, and also $d(a)$ is central, we have proved that b is central, whenever $d(b) = 0$. But for all $c \in R$, $d(c^2) = d(c)c + cd(c) = 2d(c)c = 0$,so,$c^2$commutes with all x in R, for all $c \in R$.

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