Left Multiplicative Generalized Derivations on Lie Ideals in Prime Rings

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ABSTRACT

Let R be a ring. A map $F: R \to R$ is called a left multiplicative generalized derivation, if $F(xy) = g(x)y + xF(y)$ is fulfilled for all x, y in R, where $g: R \to R$ is any map (not necessarily derivation or additive map). In this paper we prove that Let R be a 2-torsion free prime ring and U be a nonzero square closed Lie ideal of R. If $F: R \to R$ is a left multiplicative generalized derivation associated with the map $g: R \to R$ such that

1) $F(uv) \pm uv = 0$, *then* $g(U) = (0)$ *and* $F(u) = \mp u$, *for all* $u \in U$.
2) $F(uv) \pm vu = 0$, *then* $U \subseteq Z(R)$, $g(U) = (0)$ *and* $F(u) = \mp u$, *for all* $u \in U$.
3) $F(u)F(v) \pm uv = 0$, *then* $g(U) = (0)$ *and* $[F(u), u] = 0$, *for all* $u \in U$.
4) $F(u)F(v) \pm vu = 0$, *then* $U \subseteq Z(R)$ *and* $g(U) = (0)$.

Keywords: *Prime ring, Derivation, Generalized derivation, Multiplicative generalized derivation, Left multiplicative generalized derivation, Ideals, Lie ideals.*

1. INTRODUCTION

Ashraf and Rehman[3], proved that if R is a prime ring with a nonzero ideal I and d is a derivation of R such that either $d(xy) - xy \in Z(R)$ for all $x, y \in I$ or $d(xy) + xy \in Z(R)$ for all $x, y \in I$, then R is commutative. Being inspired by this result, recently Ashraf et al. [4] have studied the situation obtained by replacing derivation d with a generalized derivation F. Moreover, Ali et al.[5]proved similar results by considering a square closed Lie ideal of a prime ring R with generalized derivation F. Motivated by these results Ali et al.[2] considered the similar situation with multiplicative generalized derivation on a square closed Lie ideal of a prime ring. Being inspired by this result, we consider the similar situation with left multiplicative generalized derivation on a square closed Lie ideal of a prime ring.

Preliminaries: Throughout this paper R will denote an associative ring with centre $Z(R)$. For any x, y in R the symbol $[x, y] = xy - yx$ is called commutator. We recall that R is prime if for any a, b in R, $aRb = 0$ implies $a = 0$ or $b = 0$. Center $Z(R)$ is defined as

$Z(R) = \{z \in R \,/\, [z,R] = o\}$. An additive map $d: R \to R$ is called a derivation of R if $d\,(xy) = d(x)\,y + x\,d(y)$ for all x, y in R. An additive map $F: R \to R$ is called a generalized derivation if there exists a derivation$d: R \to R$such that$F(xy) = F(x)\,y + x\,d(y)$for all $x, y \in R$.A map $F: R \to R$ (not necessarily additive) is called a multiplicative generalized derivation if$F(xy) = F(x)\,y + x\,d(y)$ for all $x, y \in R$,where$d$is any map (not necessarily derivation or additive map). A map $F: R \to R$ (not necessarily additive) is called left multiplicative generalized derivation if $F(xy) = d(x)y + x\,F(y)$for all $x\,, y \in R$, where $d$is any map (not necessarily derivation or additive map). And an additive subgroup U of R is said to be a Lie ideal of Rif $[u,r] \in U$ for all $u \in U$ and $r \in R$.U is said to be a square closed Lie ideal of R if $u^2 \in U$ for all $u \in U$. Moreover if U is a square closed Lie ideal of Rthen$2uv \in U$ for all $u\,, v \in U$.

We use the following basic identities:

$[xy,z] = x[y,z] + [x,z]y$ and$[x,yz] = [x,y]z + y[x,z]$ for all $x, y, z \in R$.

2. MAIN RESULTS

Lemma 1: Let R be a 2- torsion free prime ring and U be a Lie ideal of R. If $U \not\subseteq Z(R)$ then $C_R(U) = Z(R)$.

Lemma 2: If $U \not\subset Z(R)$is a Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ are such that$aUb = (0)$, then$a = 0$ or $b = 0$.

Lemma 3: Let R be a 2-torsion free prime ring and U be Lie ideal of R. If U is a commutativeLie ideal of R,then$U \subseteq Z(R)$.

Theorem 1: Let R be a 2-torsion free prime ring and$U$ be a nonzero square closed Lie ideal of R. If R admits a left multiplicative generalized derivation $F: R \to R$ associated with the map $g: R \to R$ such that $F(uv) \pm uv = 0$ for all $u, v \in U$, then $g(U) = (0)$ and $F(u) = \mp u$ for all $u \in U$

Proof: Assume first that $F(uv) - uv = 0 \;\forall uv \in U$ (1)

We replace$u$ by $2wu$ in equation (1) we get

$F(2wuv) - 2wuv = 0$

$F\big(w(2uv)\big) - 2wuv = 0$

$g(w)2uv + wF(2uv) - 2wuv = 0$

$2w(F(uv) - uv) + 2g(w)uv = 0$

From equation (1) we get

$2g(w)uv = 0$

Since R is 2- torsion free ring

$g(w)uv = 0 \;\forall\; u, v, w \in U$ (2)

We replace$v$ by$[v, r], r \in R$ in equation (2) we get

$g(w)u[v, r] = 0$

$g(w)uvr - g(w)urv = 0$

$g(w)urv = 0$
$\Longrightarrow g(w)uRv = 0 \; \forall u, v, w \in U$
Since R is prime ring and U is a nonzero Lie ideal of R
$g(w)u = 0 \; \forall \, u, v, w \in U$ (3)

We replace u by $[u, r]$ in equation (3) we get
$g(w)[u, r] = 0$
$g(w)ur - g(w)ru = 0$
$g(w)ru = 0$
$\Longrightarrow g(w)Ru = 0$
Since R is prime ring U is nonzero ideal then
$g(w) = 0 \; \forall \, w \in U$
$g(u) = 0 \; \forall \, u \in U$ (4)

Now $F(uv) - uF(v) - uv$ (5)

$u(F(v) - v) = 0$ (6)
We replace u by $[u, r]$ in equation (6) we get
$[u, r](F(v) - v) = 0$
$ur(F(v) - v) - ru(F(v) - v) = 0$
From equation (6) we get
$ur(F(v) - v) = 0$
$\Rightarrow uR(F(v) - v) = 0 \;\; \forall \, u, v \in R$
Using the primeness of R we get
$(F(v) - v) = 0$
$F(v) = v \;\; \forall v \in U$
$\longrightarrow F(u) = u$
Hence the theorem.

Theorem 2: Let R be a 2- torsion free prime ring and U be a nonzero square closed Lie ideal of R. If R admits a left multiplicative generalized derivation $F\colon R \to R$ associated with the map $g\colon R \to R$ such that $F(uv) \pm vu = 0$ for all $u, v \in U$, then $U \subseteq Z(R)$, $g(u) = 0$ and $F(u) = \mp u \;\; \forall \; u \in U$.

Proof: Suppose $U \nsubseteq Z(R)$
We have $F(uv) - vu = 0 \; \forall \, u, v \in U$ (7)
We replace u by $2wu$ in equation (7) we get
$F(2wuv) - v2wu = 0$
$F(w(2uv)) - v2wu = 0$
$g(w)2uv + wF(2uv) - 2vwu - 0$
$2w(F(uv) - vu) + 2wvu + g(w)2uv - 2vwu = 0$
From equation (7) we get above expression
$2wvu + g(w)2uv - 2vwu = 0$
$2(wv - vw)u + 2g(w)uv = 0$
Using the fact 2-torsion free ring we get

$(wv - vw)u + g(w)uv = 0$

$[w, v]u + g(w)uv = 0 \; \forall \; u, v, w \in U$ (8)

Substitute w by v in the equation (8) we obtain

$g(v)uv = 0 \; \forall \; u, v \in U$ (9)

We replace u by $[u, r]$ in equation (9) we get

This implies $g(v)[u, r]v = 0$

$g(v)urv - g(v)ruv = 0 \; \forall \; u, v \in U, r \in R$

Taking r by $g(v)$ the above expression becomes

$g(v)ug(v)v - g(v)g(v)uv = 0$

From equation (9) we get

$g(v)ug(v)v = 0$

We replace u by $2vu$ and using the 2- torsion freeness of R we get the above expression

$g(v)vug(v)v = 0$

$\Rightarrow g(v)vUg(v)v = 0$

By Lemma2 we obtain

$g(v)v = 0 \;\; \forall \; v \in U$ (10)

We replace w by u in equation (8) we get

$[u, v]u + g(u)uv = 0$

From equation (10) the above expression becomes

$[u, v]u = 0 \; \forall \; u, v \in U$ (11) We replace v by $2vu$ in equation (10) we get

$[u, 2vw]u = 0$

$2v[u, w]u + 2[u, v]wu = 0$

Using 2-torsion freeness

$v[u, w]u + [u, v]wu = 0$

From equation (11) the above expression becomes

$[u, v]wu = 0$

$\Longrightarrow [u, v]U[u, v] = 0$

From Lemma 2 $[u, v] = 0$

Again using Lemma3 we get $U \subseteq Z(R)$ contradiction

We must have $U \subseteq Z(R)$

Then our assumption become $F(uv) - uv = 0 \; \forall u, v \in U$

By Theorem 1 we get $g(u) = 0$ and $F(u) = \mp u$

In similar manner we can prove our conclusion when $F(uv) + uv = 0 \; \forall \; u, v \in U$.

Theorem 3: Let R be a 2-torsion free prime ring U be a nonzero square closed Lie ideal of R. If R admits a left multiplicative generalized derivation $F: R \rightarrow R$ associated with the map $g: R \rightarrow R$ such that $F(u)F(v) \pm uv = 0 \;\; \forall \; u, v \in U$, then $g(U) = 0$ and $[F(u), u] = 0 \forall \; u \in U$.

Proof: First consider case $F(u)F(v) - uv = 0 \;\; \forall \; u, v \in U$ (12)

We replace u by $2wu$ in equation (12) we get

$F(2wu)F(v) - 2wuv = 0$

$\big(g(w)2u + wF(2u)\big)F(v) - 2wuv = 0$

$\big(g(w)2uF(v) + wF(2u)\big)F(v) - 2wuv = 0$

$2w(F(u)F(v) - uv) + 2g(w)uF(v) = 0$

From equation (12) the above expression becomes

$2g(w)uF(v) = 0$

Since R is 2-torsion free ring

$g(w)uF(v) = 0 \ \forall\, w, u, v \in U$ (13)

Right multiplying equation (13) by $F(t)$ we get

$g(w)uF(v)F(t) = 0 \ \forall\, u, v, w, t \in U$

From equation (12) we have

$g(w)uvt = 0$ (14)

We replace t by $[t, r]$ $r \in R$ in equation (14) we get

$g(w)uv[t, r] = 0 = 0$

$g(w)uvtr - g(w)uvrt = 0$

From equation (14) the above expression becomes

$g(w)uvrt = 0$

$\Longrightarrow g(w)uvRt = 0$

Since R is a prime ring and U is a nonzero Lie ideal of R

$g(w)uv = 0$ (15)

We replace v by $[v, r]$ $r \in R$ in equation (15) we get

$g(w)u[v, r] = 0$

$g(w)uvr - g(w)urv = 0$

From equation (15) the above expression becomes

$g(w)urv = 0$

$\Longrightarrow g(w)uRv = 0$

Prime ring R, U is nonzero ideal of R

$g(w)u = 0$ (16)

We replacing u by $[u, r]$ $r \in R$ in equation (16) we get

$g(w)[u, r] = 0$

$g(w)ur - g(w)ru = 0$

From equation (16) the expression becomes

$g(w)ru = 0$

$g(w)Ru = 0$

By primeness of R, U is nonzero,

$g(w) = 0 \ \forall\ w \ \in U$

$\Longrightarrow g(u) = 0 \ \forall\ u \in U$ (17)

Thus for any $u, v \in U$ we have

$F(uv) = uF(v)$ (18)

On replacing v by $2uv$ in equation (12)

$F(u)F(2uv) = u2uv$

$F(u)uF(2v) - 2u^2v = 0$

Since r is 2-torsion free ring we obtain

$$F(u)uF(v) - u^2v = 0 \quad (19)$$

Left multiplying u in equation (12) we obtain

$$uF(u)F(v) - u^2v = 0 \quad (20)$$

Subtract equation (19)and (20) we get

$F(u)uF(v) - F(u)uF(v) = 0$

$$[F(u), u]F(v) = 0 \quad (21)$$

Substitute v by $2wv$ in (21) we obtain

$[F(u), u]F(2wv) = 0$

$[F(u), u]2wF(v) = 0$

Using 2-torsion free ring we have

$[F(u), u]wF(v) = 0$

It follows

$[F(u), u]w[F(u), u] = 0$

If $U \not\subset Z(R)$, lemma2 gives $[F(u), u] = 0 \ \forall \ u, v \in U$ and the same condition is obtained if $U \subseteq Z(R)$.

In a similar manner, we can prove that the same conclusion holds for $F(u)F(v) + uv = 0 \ \forall \ u, v \in U$.

Hence the theorem.

Theorem 4: Let R be a 2-torsion free prime ring and U be a nonzero square closed Lie ideal of R. If R admits a left multiplicative generalized derivation $F: R \to R$ associated with the map $g: R \to R$ such that $F(u)F(v) \pm vu = 0 \ \ \forall \ u, v \in U$, then $U \subseteq Z(R)$ and $g(u) = 0$.

Proof: Suppose on contrary that $U \nsubseteq Z(R)$

Assume that

$F(u)F(v) - vu = 0 \ \forall \ u, v \in U$(22)

We replace u by $2vu$ in equation (22) we obtain

$F(2vu)F(v) - v2vu = 0 \ \forall \ u, v \in U$

$2\big(g(v)u + vF(u)\big)F(v) - 2v^2u = 0$

The above expression is 2- torsion free ring

$g(v)u + vF(u)F(v) - v^2u = 0$

$$v(F(u)F(v) - vu) + g(v)uF(v) = 0 \quad (23)$$

From equations (22) & (23), we obtain

$$g(v)uF(v) = 0 \ \forall \ u, v \in U \quad (24)$$

Right multiplying equation (24) with $F(w)$ we get

$g(v)uF(v)F(w) = 0$

From equation (22) we get

$$g(v)uwv = 0 \ \forall \ u, v, w \in U \quad (25)$$

We replace u by v in equation (25) we obtain

$g(v)vwv = 0$

$\Rightarrow g(v)vUv = 0 \ \forall \ w \in U$

By lemma2 we have for each $u \in U$ $either$ $v = 0$ or $g(v)v = 0$

Since $v \neq 0 \Rightarrow g(v)v = 0 \ \forall u \in U$

Now replacing u by u^2 in equation (22)

$F(u^2)F(v) - vu^2 = 0$

$(g(u)u + uF(u))F(v) - vu^2 = 0$

Since $g(u)u = 0$ we obtain

$$uF(u)F(v) - vu^2 = 0 \tag{26}$$

Left multiplying equation (22) by u we get

$$uF(u)F(v) - uvu = 0 \tag{27}$$

Subtract equations (26) & (27) we get

$(uv - vu)u = 0$

$[u, v]u = 0 \ \forall \ u, v \in U$

Then same argument as given in the proof of Theorem 2 we have $U \subseteq Z(R)$ a contradiction. We must have $U \subseteq Z(R)$.

In a similar manner, we can prove that the same conclusion holds for $F(u)F(v) + vu = 0 \ \forall \ u, v \in U$.

There by the proof of the theorem is completed.

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