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PRIME NEAR RINGS WITH GENERALIZED JORDAN DERIVATIONS

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ABSTRACT

In this paper we proved that N be a 2, 3-torsion free prime near ring. If N admits a nonzero generalized Jordan derivation f such that $f(N) \subseteq Z$, then N is a commutative ring and if D is a derivation and $f^2 = 0$, then $D^3 = 0$. If N is 2-torsion free, then $D(Z) = \{0\}$. Moreover f satisfies $f(x^2) = D(x)x + xf(x)$, for all $x \in N$, then N is a commutative ring.

KEYWORDS: Prime Ring, Near Ring, Center, Derivation, Generalized Derivation, Jordan Derivation, Generalized Jordan Derivation

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Original Article

INTRODUCTION

In [1] Bell. H. E and Mason. G had Proved the following statements:

(A): If N is 3-prime and 2-torsion free and D is a derivation such that $D^2 = 0$, then $D = 0$.

(B): If N is a 3-prime 2-torsion free near-ring which admits a nonzero derivation D for which

$D(N) \subseteq Z$, then N is a Commutative ring.

(C):If N is 3-prime 2-torsion free near-ring admitting a nonzero derivation D such that $D(x)D(y) = D(y)D(x)$, for all $x, y \in N$, then N is a commutative ring. After that Bell.H.E [2] replaced D by Generalized derivation$f$. Motivated by the above work we replaced Generalized derivation f by Generalized Jordan derivation$f$.

Preliminaries: Throughout this paper N stands for a prime near ring with multiplicative center Z. An additive map $D: N \to N$ is a derivation (resp. Jordan derivation), if $D(xy) = D(x)y + xD(y)$ (resp. $D(x^2) = D(x)x + xD(x)$, holds for all $x, y \in N$.An additive map $f: N \to N$ is generalized derivation (resp.generalized Jordan derivation), if $f(xy) = f(x)y + xd(y)$ (resp.$f(x^2) = f(x)x + xd(x)$), holds for all $x, y \in N$.According to [2], a near ring N is said to be prime if $xNy = 0$, for all $x, y \in N$, implies $x = 0$ or $y = 0$ .While the symbol $[x, y]$ will denote the commutator $xy - yx$.

We need the following lemmas:

Lemma 1: [1, Lemma 3] Let N be a 3-prime near ring.

- If $z \in Z\backslash\{0\}$, then z is not a zero divisor.
- If $Z\backslash\{0\}$ contains an element z such that $z + z \in Z$, then (N, +) is abelian.
- If D is a nonzero derivation and $x \in N$ is suc that $xD(N) = \{0\}$ or $D(N)x = \{0\}$,

then $x = 0$.

Lemma 2: [3, Proposition 1] If N is an arbitrary near ring and D is a derivation on N, then $D(xy) = D(x)y + xD(y)$, for all $x, y \in N$.

Lemma 3: Let N be an arbitrary near ring and let f be a generalized Jordan derivation on N with associated derivation D.Then $\big(f(a)a + aD(a)\big)b = f(a)ab + aD(a)b$, for all $a, b, c \in N$.

Proof: Clearly $f(a^2b) = f(a^2)b + a^2D(b)$. (1)

$f\big(a(ab)\big) = f(a)ab + aD(ab)$

$= f(a)ab + a(D(a)b + aD(b))$

$= f(a)ab + aD(a)b + a^2D(b)$. (2)

Comparing (1) & (2), we get

$\big(f(a)a + aD(a)\big)b = f(a)ab + aD(a)b$, for all $a, b, c \in N$.

Lemma 4: Let R be a 3-prime near ring, and let f be a generalized Jordan derivation with associated derivation $D \neq 0$. If $D(f(N)) = \{0\}$, then $f\big(D(N)\big) = \{0\}$.

Proof: We are assuming that $D\big(f(x)\big) = 0$, for all $x \in N$.

We have to find out $D(f(x^2)$, for all $x \in N$

For that we take

$D(f(xy) = D\big(f(x)y + xD(y)\big) = 0$

$\Longrightarrow D(f(x)y) + D\big(xD(y)\big) = 0$

$\Rightarrow D\big(f(x)\big)y + f(x)D(y) + D(x)D(y) + xD^2(y) = 0$

$\Rightarrow\ f(x)D(y) + D(x)D(y) + xD^2(y) = 0$

$D(f(xy)) = f(x)D(y) + D(x)D(y) + xD^2(y) = 0$. (3)

Put $x = y$ in equation (3), we get

$D(f(x^2) = f(x)D(x) + D(x)D(x) + xD^2(x) = 0$.

Applying D again, we get

$D(f(x)D(x) + D(x)D(x) + xD^2(x)) = 0$

$D(f(x))D(x) + f(x)D^2(x) + D^2(x)D(x) + D(x)D^2(x) + D(x)D^2(x) + xD^3(x) = 0$

$f(x)D^2(x) + D^2(x)D(x) + D(x)D^2(x) + D(x)D^2(x) + xD^3(x) = 0$. (4)

Substitute y for $D(x)$ in equation (3), we get

$f(x)D^2(x) + D(x)D^2(x) + xD^3(x) = 0$. (5)

Substitute (5) in (4), we get

$D^2(\mathrm{x})D(x) + \mathrm{D}(\mathrm{x})D^2(\mathrm{x}) = 0.$ (6)

Substitute x in $D(x)$ in equation (3), we get

$f(D(x))D(y) + D^2(x)D(y) + D(x)D^2(y) = 0.$ (7)

Substitute y in x in equation (7), we get

$f(D(x))D(x) + D^2(x)D(x) + D(x)D^2(x) = 0.$

Substitute (6) in above equation, we get

$f(D(x))D(x) = 0$, for all $x \in N$.

Thus, by lemma1 (iii), $f(D(x)) = 0$, for all $x \in N$.

The Main Theorems

Theorem 1: (i) Let N be a 3-prime 2-torsion free near ring. If N admits a non zero generalized Jordan derivation f such that $f(N) \subseteq Z$, then N is a commutative ring.

(ii) If N is 3-prime 2-torsion free near-ring admitting a nonzero derivation D such that $D(x)D(y) = D(y)D(x)$, for all $x, y \in N$, then N is a commutative ring.

Proof: (i): From lemma 4, we have if $(f(N)) = \{0\}$, then $f\big(D(N)\big) = \{0\}$.

Since $f \neq 0, \exists\ x \in N$ such that $0 \neq f(x) \in Z$

Since $f(x) + f(x) = f(x + x) \in Z$,

(N,+) is abelian by lemma1(ii)

To complete the proof, we show that N is multiplicatively commutative.

First, consider the case $D = 0$.

So that $f(x^2) = f(x)x + xD(x) = f(x)x \in Z$, for all $x \in N$.

Then $f(x)xw = wf(x)x$

$f(x)xw - f(x)wx = 0$

$f(x)(xw - wx) = 0.$

$f(x)[x, w] = 0.$

$f \neq 0$, then $[x, w] = 0$.

Now assume that $D \neq 0$, and let $x \in Z\backslash\{0\}$

Then $f(x^2) = f(x)x + xD(x) \in Z$.

$(f(x)x + xD(x))y = y(f(x)x + xD(x))$, for all $x, y \in N$, and lemma 3 we see tat

Since both $f(x)$ in Z, we have

$D(x)(xy - yx) = 0$, for all $x, y \in N$.

Provided that $D(Z) \neq \{0\}$, we can conclude that N is commutative.

Proof: (ii) Assume that $D \neq 0$ and $D(Z) = \{0\}$.

In particular $D\big(f(x)\big) = 0$, for all $x \in N$.

Note that for $x \in N$ such that $f(x) = 0$.

$f(x^2) = f(x)x + xD(x) = xD(x) \ \in Z$

Hence by lemma 2, $D(x)D(y) \in Z$ and $D(y)D(x) \in Z$, for each $x, y \in N.$ If one of these is zero, the other is a central element squaring to 0, hence is also 0.The remaining possibility is that $D(x)D(y)$ and $D(y)D(x)$ are non-zero central elements, in which case $D(x)$ is not a zero diviser. Thus $D(x)D(x)D(y) = D(x)D(y)D(x)$ yields

$D(x)(D(x)D(y) - D(y)D(x)) = 0.$

$D(x)D(y) - D(y)D(x) = 0$. Consequently, N is commutative.

Theorem2: Let N be a 3- prime near ring and let f be a generalized Jordan derivation on N with associated derivation D.If $f^2 = 0$,then $D^3 = 0$.Moreover, if N is 2-torsion free,then $D(z) = \{0\}$.

Proof: We have $f^2(x^2) = 0$.

$f(f(x^2) = f\big(f(x)x + xD(x)\big) = 0$

$\Rightarrow f^2(x)D(x) + f(x)D^2(x) + f(x)D^2(x) + xD^3(x) = 0.$ (8)

Now we have to find $f^2(xy) = 0$

$f(f(xy) = 0$

$f\big(f(x)y + xD(y)\big) = 0$

$f(f(x)y) + f\big(xD(y)\big) = 0$

$f^2(x)y + f(x)D(y) + f(x)D(y) + xD^2(y) = 0$

$f(x)D(y) + f(x)D(y) + xD^2(y) = 0.$ (9)

substitute $D(x)$ for y in equation (9) gives

$f(x)D^2(x) + f(x)D^2(x) + xD^3(x) = 0.$ (10)

From (8) & (10) we get $f(x)D^2(x) = 0$, for all $x \in N$. (11)

It now follows from (10) that $xD^3(x) = 0$, for all $x \in N$ and since N is 3-prime, $D^3 = 0$.

Suppose now that N is 2-torsion free and that $D(z) \neq \{0\}$ and let $z \in Z$ be such that $D(z) \neq 0$.

Then if $x, y \in N$ and $f(N)x = \{0\}$, then

$f(xz)x = f(x)zx + xD(z)x = 0$

$\Rightarrow xD(z)x = 0$ and since N is 3-prime and $D(z)$ is not a zero devisor,$x = 0$.

It follows from equation (10) that $D^2 = 0$ and hence by (A) that $D = 0$.

But this contradicts our assumption that $D(z) \neq \{0\}$.

Hence $D(z) = \{0\}$.

Theorem 3: Let N be a 2, 3-torsion free prime near ring with 1. If f is a generalized Jordan derivation on N such that $f^2 = 0$ and $f(1) \in Z$, then $f = 0$.

Proof: $f(x) = f(1.x) = f(1)x + 1D(x)$.

So $f(x) = cx + D(x), c \in z$. (12)

If $c = 0$, then $f = D$ and $D^2 = 0$, so $D = 0$ by (A) and therefore $f = 0$.

If $c \neq 0$, then c is not a zero divisor, hence by equation (11) $D^2 = 0$ and $D = 0$.

But then $f(x) = cx$ and $f^2(x) = c^2x = 0$, for all $x \in N$.

Since c^2 is not a zero divisor, we get $N = \{0\}$ is a contradiction.

Thus $c = 0$.

Let us we introduced a generalized derivation f also satisfied the following property

$f(x^2) = D(x)x + xf(x)$, for all $x \in N$, where D is derivation. (*)

Using the above property we prove the following theorem:

Theorem 4: Let N be a 2, 3-torsion free prime near ring which admits a generalized Jordan derivation f with non zero associated derivation D such that f satisfies (*).Then N is a commutative ring.

Proof: From lemma 4 says that $f(N) \subseteq Z$ or $D(f(N) = \{0\}$, then$f(D(N) = 0$

Now we calculate $f(D(x)D(x))$ in two ways

Using the defining property of f, we obtain

$f\big(D(x)D(x)\big) = f\big(D(x)\big)D(x) + D(x)D^2(x) = D(x)D^2(x)$. (13)

And using (*), we obtain$f\big(D(x)D(x)\big) = D^2(x)D(x) + D(x)f\big(D(x)\big) = D^2(x)D(x)$

Thus $D^2(x)D(x) = D(x)D^2(x)$, for all $x \in N$.

But from lemma4, equation (6), we get

$D(x)D^2(x) = 0$, for all $x \in N$.

Hence by lemma 1(iii) and using (A), we have $D^2 = 0$, thus $D = 0$.

Then N is commutative.

CONCLUSIONS

In this paper we proved that Nbe a 2,3-torsion free prime near ring. If N admits a nonzero generalized Jordan derivation f such that $f(N) \subseteq Z$, then N is a commutative ringand if D is a derivation and $f^2 = 0$, then $D^3 = 0$. If N is 2-torsion free, then $D(Z) = \{0\}$. Moreover f satisfies $f(x^2) = D(x)x + xf(x)$, for all $x \in N$, then N is a commutative ring

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