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LIE IDEAL AND GENERALIZED JORDAN REVERSE DERIVATIONS ON SEMIPRIME RINGS

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Abstract

Let R be a 2-torsion free semiprime ring in which $x^2 = 0$ implies $x = 0$. Let $g : R \to R$ be a generalized Jordan reverse derivation associated with the Jordan reverse derivation $d : R \to R$. Then g is a generalized reverse derivation on R. Thus, there exists $q \in Q_r(S)$. The Martindale quotient ring of S, such that $g(x) = qx + d(x)$, for all $x \in R$.

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1. Preliminaries

Throughout this paper, R denotes a 2-torsion free semiprime ring. A ring R is said to be *semiprime* if $xRx = 0$ with $x \in R$ implies $x = 0$. An additive subgroup U of R is to a Lie ideal of R if $[U, R] \subseteq U$, where $[x, y]$ denotes the Lie product $xy - yx$ of x and y. A mapping $A : R \times R \to R$ is said to be *biadditive* if it is additive in both the variables. Let C be the extended centroid of R and $S = RC$ be the central closure of R. The notion of extended centroid and central closure of semiprime ring R are given by Amitsur in [2]. Let $Q_r(S)$ be the Martindale quotient ring corresponding to S. An additive mapping $t : R \to R$ is said to be a *left centralizer* if $t(xy) = t(x)y$ (resp. $t(x^2) = t(x)x$) for all $x, y \in R$. An additive mapping $d : R \to R$ is said to be a *derivation* (resp. *Jordan derivation*) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) for all $x, y \in R$. An additive mapping $d : R \to R$ is said to be a *reverse derivation* (resp. *Jordan reverse derivation*) if $d(xy) = d(y)x + yd(x)$ (resp. $d(x^2) = d(x)x + xd(x)$) for all $x, y \in R$. An additive mapping $d : R \to R$ is said to be a *right reverse derivation* if $d(xy) = d(y)x + d(x)y$ for all $x, y \in R$. An additive mapping $g : R \to R$ is said to be a *generalized derivation* (resp. *generalized Jordan derivation*) associated with the derivation d if $g(xy) = g(x)y + xd(y)$ (resp. $g(x^2) = g(x)x + xd(x)$) for all $x, y \in R$. An additive mapping $g : R \to R$ is said to be a *generalized reverse derivation* (resp. *generalized Jordan reverse derivation*) associated with the Jordan reverse derivation $d : R \to R$ if $g(xy) = g(y)x + yd(x)$ (resp. $g(x^2) = g(x)x + xd(x)$) for all $x, y \in R$. We shall make use of commutator identities; $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = [x, z]y + x[y, z]$, for all $x, y, z \in R$.

2. Introduction

Bresar [3] defined generalized derivation on rings. Hvala [7] studied the properties of generalized derivation semiprime rings. Herstein [5] have introduced the concept of reverse derivation for prime ring and studied the notation of reverse derivation and some properties of reverse derivation. During the last thirty years, a lot of work about commutativity of prime or semiprime rings with derivation or generalized derivation or reverse derivation etc. Also, the notion of generalized reverse derivation was defined by Aboubakr and González [1]. Golbasi [4] extended some well known results concerning derivations of prime rings to the right generalized derivations and nonzero left ideal of a prime ring which is a semi prime ring. Sharma and Prajapati [9] studied Lie ideal and generalized Jordan left derivations on semiprime rings. Reddy et al. [8] studied Lie ideals and Jordan generalized reverse derivations of prime rings, we have generalized the previous results to the Lie ideals of semiprime rings. In this article, the authors investigated some results on Lie ideals of semiprime rings with generalized Jordan reverse derivation.

Lemma 1 [5]**.** *Let R be a prime ring and suppose that d is a nonzero reverse derivation of R. Then R is a commutative integral domain and d is an ordinary derivation of R.*

Lemma 2. *Let R be a* 2*-torsion free ring and U be a Lie ideal of R such that* $x^2 \in U$, *for all* $x \in U$. *If* $d : R \to R$ *is an additive mapping satisfying* $d(x^2) = d(x)x + xd(x)$ *for all* $x \in U$, *then*

(i) $d(xy + yx) = d(y)x + yd(x) + d(x)y + xd(y)$, *for all* $x, y \in U$.

(ii) $d(xyx) = xyd(x) + xd(y)x + d(x)yx$, *for all* $x, y \in U$.

(iii)

$$d(xyz + zyx) = xyd(z) + xd(y)z + d(x)yz + zyd(x) + zd(y)x + d(z)yx.$$

(iv) $A(x, y)[x, y] = 0$, *for all* $x, y \in U$.

Proof. (i) Since $xy + yx = (x + y)^2 - x^2 - y^2$, we find that $xy + yx \in U$, for all $x, y \in U$:

$$\begin{aligned} d(xy + yx) &= d((x + y)^2) - d(x^2) - d(y^2) \\ &= d(x + y)(x + y) + (x + y)d(x + y) \\ &\quad - d(x)x - xd(x) - d(y)y - yd(y) \\ &= (d(x) + d(y))(x + y) + (x + y)(d(x) + d(y)) \\ &\quad - d(x)x - xd(x) - d(y)y - yd(y) \end{aligned}$$

$$d(xy + yx) = d(y)x + yd(x) + d(x)y + xd(y), \text{ for all } x, y \in U. \quad (1)$$

(ii) Since $xy + yx \in U$, replacing y by $xy + yx$ in equation (1), we get

$$\begin{aligned} &d(x(xy + yx) + (xy + yx)x) \\ &= d(xy + yx)x + (xy + yx)d(x) + d(x)(xy + yx) + xd(xy + yx) \\ &= (d(y)x + yd(x) + d(x)y + xd(y))x + (xy + yx)d(x) \\ &\quad + d(x)(xy + yx) + x(d(y)x + yd(x) + d(x)y + xd(y)) \\ &= d(y)x^2 + yd(x)x + d(x)yx + xd(y)x + xyd(x) + yxd(x) \\ &\quad + d(x)xy + d(x)yx + xd(y)x + xyd(x) + xd(x)y + x^2d(y). \end{aligned} \quad (2)$$

On the other hand, we have

$$\begin{aligned} &d(x(xy + yx) + (xy + yx)x) \\ &= d(x^2y + 2xyx + yx^2) \\ &= d(y)x^2 + yd(x^2) + d(x^2)y + x^2d(y) + 2d(xyx) \\ &= d(y)x^2 + yd(x)x + yxd(x) + d(x)xy + xd(x)y \\ &\quad + x^2d(y) + 2d(xyx). \end{aligned} \quad (3)$$

Combining the above equations (2) and (3), we get

$$2d(xyx) = d(x)yx + xd(y)x + xyd(x) + d(x)yx + xd(y)x + xyd(x)$$

$$= 2xyd(x) + 2xd(y)x + 2d(x)yx$$

$$2d(xyx) = 2(xyd(x) + xd(y)x + d(x)yx).$$

Since R is a 2-torsion free ring, we get

$$d(xyx) = xyd(x) + xd(y)x + d(x)yx, \text{ for all } x,\ y \in U.$$

(iii) Replacing x by $x + z$ in Lemma 2(ii), we get

$$d((x+z)y(x+z))$$

$$= (x+z)yd(x+z) + (x+z)d(y)(x+z) + d(x+z)y(x+z)$$

$$= (x+z)yd(x) + (x+z)yd(z) + (x+z)(d(y)x + d(y)z)$$

$$+ (d(x) + d(z))(yx + yz)$$

$$= xyd(x) + zyd(x) + xyd(z) + zyd(z) + xd(y)x + xd(y)z$$

$$+ zd(y)x + zd(y)z + d(x)yx + d(x)yz + d(z)yx + d(z)yz. \quad (4)$$

On the other hand, we have

$$d((x+z)y(x+z))$$

$$= d(xyx + xyz + zyx + zyz)$$

$$= d(xyx) + d(zyz) + d(xyz + zyx)$$

$$= xyd(x) + xd(y)x + d(x)yx + zyd(z) + zd(y)z + d(z)yz$$

$$+ d(xyz + zyx). \quad (5)$$

Combining equations (4) and (5), we get

$$d(xyz + zyx) = xyd(z) + xd(y)z + d(x)yz + zyd(x) + zd(y)x + d(z)yx.$$

(iv) Replacing $z = xy$ in Lemma 2(iii), we get

$$W = d(xy(xy) + (xy)yx)$$

$$W = xyd(xy) + xd(y)xy + d(x)yxy + xyyd(x) + xyd(y)x + d(xy)yx$$

$$W = xyd(xy) + xyyd(x) + xd(y)xy + xyd(y)x + d(x)yxy + d(xy)yx. \quad (6)$$

On the other hand, we have

$$W = d((xy)^2 + xy^2x) = d((xy)^2) + d(xy^2x)$$

$$W = (xy)d(xy) + d(xy)xy + d(x)y^2x + xd(y^2)x + xy^2d(x)$$

$$W = (xy)d(xy) + d(xy)xy + d(x)yyx + xd(y)yx + xyd(y)x + xyyd(x). \quad (7)$$

Combining (6) and (7), we have

By using Lemma 1, we get

$$(d(xy) - d(y)x - yd(x))[x, y] = 0,$$

$$A(x, y)[x, y] = 0,$$

where $A(x, y) = d(xy) - d(y)x - yd(x)$.

We can easily see that

$$A(x, y) + A(y, x) = 0.$$

We have

$$d(xy) + d(yx) = d(y)x + yd(x) + d(x)y + xd(y),$$

$$d(xy) - d(y)x - yd(x) + d(yx) - d(x)y - xd(y) = 0,$$

$$A(x, y) + A(y, x) = 0.$$

Also, if d is a right reverse derivation, then $A(x, y) - A(y, x) = d([x, y])$.

Consider

$$A(x, y) - A(y, x) = d(xy) - d(y)x - d(x)y - d(yx) + d(x)y + d(y)x$$

$$= d(xy) - d(yx)$$

$$= d([x, y]).$$

Lemma 3. *Let R be a* 2*-torsion free semiprime ring in which* $x^2 = 0$ *implies* $x = 0.$ *Let U be a Lie ideal of R such that* $x^2 \in U,$ *for all* $x \in U.$ *Let* $A, B : U \times U \to U$ *be biadditive mappings such that* $A(x, y)B(x, y) = 0,$ *for all* $x, y \in U.$ *Then* $A(x, y)B(u, v) = 0,$ *for all* $x, y, u, v \in U.$

Proof. Assume that

$$A(x, y)B(x, y) = 0, \text{ for all } x, y, u \in U. \tag{8}$$

Replacing x by $x + u$ in equation (8), we get

$$A(x + u, y)B(x + u, y) = 0,$$

$$(A(x, y) + A(u, y))(B(x, y) + B(u, y)) = 0,$$

$$A(x, y)B(x, y) + A(x, y)B(u, y) + A(u, y)B(x, y) + A(u, y)B(u, y) = 0.$$

Using equation (8), we get

$$A(x, y)B(u, y) + A(u, y)B(x, y) = 0$$

$$A(x, y)B(u, y) = -A(u, y)B(x, y).$$

Now, by above relation, we have

$$\{A(x, y)B(u, y)\}^2 = A(x, y)B(u, y)A(x, y)B(u, y)$$

$$= -A(x, y)B(x, y)A(u, y)B(u, y).$$

Thus, by given hypothesis, we get $\{A(x, y)B(u, y)\}^2 = 0,$ for all $x, y, u \in U.$ We get $A(x, y)B(u, y) = 0.$

Again using the same argument, we are given that

$$A(x, y)B(u, y) = 0, \text{ for all } x, y, u \in U. \tag{9}$$

We shall replace y by $y + v$:

$$A(x, y+v)B(u, y+v) = 0$$

$$(A(x, y) + A(x, v))(B(u, y) + B(u, v)) = 0$$

$$A(x, y)B(u, y) + A(x, y)B(u, v) + A(x, v)B(u, y) + A(x, v)B(u, v) = 0.$$

Using equation (9), we get

$$A(x, y)B(u, v) + A(x, v)B(u, y) = 0$$

$$A(x, y)B(u, v) = -A(x, v)B(u, y).$$

Now, by above relation, we have

$$\{A(x, y)B(u, v)\}^2 = A(x, y)B(u, v)A(x, y)B(u, v)$$

$$= -A(x, y)B(u, y)A(x, v)B(u, v).$$

Thus, by given hypothesis, we get $\{A(x, y)B(u, v)\}^2 = 0$, for all $x, y, u, v \in U$. We get $A(x, y)B(u, v) = 0$.

Lemma 4. *Let R be a semiprime ring and $x, y \in R$ such that $A(x, y) \cdot [u, v] = 0$, for all $u, v \in R$. Then $A(x, y) \in Z(R)$.*

Proof. Let $z, w \in R$ be arbitrary elements. Then

$$[A(x, y), z]w[A(x, y), z]$$

$$= A(x, y)zw[A(x, y), z] - zA(x, y)w[A(x, y), z]$$

$$= A(x, y)[zwA(x, y), z] - A(x, y)[zw, z]A(x, y)$$

$$- zA(x, y)[wA(x, y), z] + zA(x, y)[w, z]A(x, y).$$

By hypothesis, we get

$$\Rightarrow [A(x, y), z]w[A(x, y), z] = 0, \text{ for all } w, z \in R.$$

Since R is semiprime, we get $[A(x, y), z] = 0,$ for all $z \in R$ and so $A(x, y) \in Z(R).$

Theorem 1. *Let R be a 2-torsion free semiprime ring in which* $x^2 = 0$ *implies* $x = 0.$ *Let U be a Lie ideal of R such that* $x^2 \in U,$ *for all* $x \in U.$ *If* $d : R \to R$ *is an additive mapping satisfying* $d(x^2) = d(x)x + xd(x),$ *for all* $x \in U,$ *then* $d(xy) = d(y)x + yd(x),$ *for all* $x, y \in U.$

Proof. By Lemma 2(iv), $A(x, y)[x, y] = 0,$ for all $x, y \in U.$ Since both $A(x, y)$ and $[x, y]$ are biadditive mappings, $A(x, y)[u, v] = 0,$ for all $x, y, u, v \in U,$ by Lemma 3.

Hence, we have $A(x, y) \in Z(U),$ for x, y in U by Lemma 4, and this is true for all $x, y \in U.$

Now, we have

$$\begin{aligned} 2(A(x, y))^2 &= A(x, y)(A(x, y) + A(x, y)) \\ &= A(x, y)(A(x, y) - A(y, x)), \text{ since } A(x, y) = -A(y, x) \\ &= A(x, y)d[x, y], \text{ since } A(x, y) - A(y, x) = d[x, y] \end{aligned}$$

$$2(A(x, y))^2 = A(x, y)d[x, y]. \tag{10}$$

Since $A(x, y)[x, y] = 0$ and $A(x, y) \in Z(R),$ we have

$$A(x, y)[x, y] + [x, y]A(x, y) = 0.$$

By Lemma 2(i), we get

$$\begin{aligned} &d([x, y])A(x, y) + [x, y]d(A(x, y)) + d(A(x, y))[x, y] \\ &+ A(x, y)d([x, y]) = 0. \end{aligned}$$

Since $A(x, y) \in Z(R)$ by Lemma 4, we get

$$2A(x, y)d[x, y] + [x, y]d(A(x, y)) + d(A(x, y))[x, y] = 0.$$

Left multiplying the above equation with $A(x, y)$, we get

$$2(A(x, y))^2 d[x, y] + A(x, y)[x, y]d(A(x, y))$$
$$+ A(x, y)d(A(x, y))[x, y] = 0.$$

Using $A(x, y) \in Z(R)$ and $A(x, y)[x, y] = 0$, we arrive at

$$2A(x, y)^2 d([x, y]) = 0.$$

Again using equation (10), we get

$$4A(x, y)^3 = 0$$

and so $A(x, y)^4 = 0$, for all $x, y \in R$.

By the assuming $x^2 = 0$ implies $x = 0$, we have $A(x, y) = 0$. Hence, we obtain that $d(xy) = d(y)x + yd(x)$, for all $x, y \in R$.

Corollary 1. *Let R be a* 2*-torsion free semiprime ring with unit element. Then every Jordan reverse derivation is a reverse derivation.*

Proof. Since R is semiprime ring with unity, the assumption $x^2 = 0$ implies $x = 0$ is clearly satisfied. Hence, we obtain that the required result by Theorem 1.

Corollary 2. *Let R be a* 2*-torision free semisimple ring in which* $x^2 = 0$ *implies* $x = 0$ (*or R has unity*). *Then every Jordan reverse derivation is a reverse derivation*.

Proof. Every semisimple ring is semiprime. It is clearly by Theorem 1.

Lemma 5 [20, Proposition 1.4]**.** *Let R be a* 2*-torsion free semiprime ring and* $T : R \to R$ *be an additive mapping which satisfies* $T(x^2) = T(x)x$, *for all* $x \in R$. *Then T is a left centralizer*.

Lemma 6. *An additive mapping* $g : R \to R$ *is a generalized reverse derivation if and only if* g *is of the form* $g = d + t$, *where* d *is reverse derivation and* t *is a left centralizer.*

Proof. Let g be a generalized reverse derivation on R, i.e., $g(xy) = g(y)x + yd(x)$, where d is a reverse derivation on R.

Suppose $t = g - d$. Hence, we have

$$\begin{aligned} t(xy) = (g - d)(xy) = g(xy) - d(xy) \\ = g(y)x - d(y)x \\ = (g - d)(y)x \\ = t(y)x, \text{ for all } x, y \in R. \end{aligned}$$

This shows that t is a left centralizer. Hence $g = d + t$.

Conversely, suppose $g = d + t$. Then

$$\begin{aligned} g(xy) = (d + t)xy \\ = d(y)x + yd(x) + t(y)x \\ = d(y)x + t(y)x + yd(x) \\ = g(y)x + yd(x), \text{ for all } x, y \in R. \end{aligned}$$

Thus, g is generalized reverse derivation. The proof is completed.

Theorem 2. *Let* R *be a* 2*-torsion free semiprime ring in which* $x^2 = 0$ *implies* $x = 0$. *Let* $g : R \to R$ *be a generalized Jordan reverse derivation associated with Jordan reverse derivation* $d : R \to R$. *Then* g *is a generalized reverse derivation on* R.

Proof. Since g is a generalized Jordan reverse derivation, we have

$$g(x^2) = g(x)x + xd(x), \text{ for all } x \in R.$$

Since d is the Jordan derivation, it is also a reverse derivation on R by Theorem 1.

Let us denote $g - d$ by t. Then we have

$$\begin{aligned} t(x^2) &= (g - d)(x^2) \\ &= g(x)x + xd(x) - d(x)x - xd(x) \\ &= g(x)x - d(x)x \\ &= (g - d)(x)x \\ &= t(x)x. \end{aligned}$$

This shows that t is a Jordan left centralizer.

Hence, g is of the form $g = d + t$, where d is a Jordan reverse derivation and t is a left centralizer.

By Lemma 6, we have g is a generalized reverse derivation on R. This proves the theorem.

Corollary 3. *Let R be a 2-torsion free semiprime ring with unity. Then every generalized Jordan reverse derivation is a generalized reverse derivation.*

Corollary 4. *Let R be a semisimple ring in which $x^2 = 0$ implies $x = 0$ (or R has unity). Then every generalized Jordan reverse derivation is a generalized reverse derivation.*

Lemma 7 [15, Lemma 2]**.** *Let $f : R \to S$ be an additive map satisfying $f(xy) = f(x)y$, for all $x, y \in R$. Then there exists $q \in Q_r(S)$ such that $f(x) = qx$, for all $x \in R$.*

Theorem 3. *Let R be a 2-torsion free semiprime ring in which $x^2 = 0$ implies $x = 0$.*

Let $g : R \to R$ *be a generalized Jordan reverse derivation associated with Jordan reverse derivation* $d : R \to R$. *Then there exists* $q \in Q_r(S)$, *the Martindale quotient ring of S, such that* $g(x) = qx + d(x)$, *for all* $x \in R$.

Proof. Since g is a generalized Jordan reverse derivation associated with the Jordan reverse derivation d on R, by Theorem 2, g is a generalized reverse derivation on R. Hence, g is of the form $g = d + t$, where d is a Jordan reverse derivation and t is a left centralizer by Lemma 6. That is, $(g - d)(xy) = t(xy) = t(x)y$. Now, by Lemma 7, there exists $q \in Q_r(S)$ such that $(g - d)(x) = t(x) = qx$, for all $x \in R$. Thus, $g(x) = qx + d(x)$, for all $x \in R$.

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