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Strongly Gorenstein Flat Dimensions

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Abstract This article is concerned with the strongly Gorenstein flat dimensions of modules and rings. We show this dimension has nice properties when the ring is coherent, and extend the well-known Hilbert's syzygy theorem to the strongly Gorenstein flat dimensions of rings. Also, we investigate the strongly Gorenstein flat dimensions of direct products of rings and (almost) excellent extensions of rings.

Keywords strongly Gorenstein flat module; strongly Gorenstein flat dimension; coherent ring; direct product; (almost) excellent extension.

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1. Introduction

Unless stated otherwise, throughout this paper all rings are associative with identity and all modules are unitary modules. Let R be a ring. Denote by $\mathcal{P}(\mathcal{R})$, $\mathcal{I}(\mathcal{R})$ and $\mathcal{F}(\mathcal{R})$ the class of all projective, injective and flat R-modules respectively.

Let R be a ring and M a right R-module. M is said to be Gorenstein flat (G-flat for short) if there is an exact sequence of flat right R-modules

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and such that $- \otimes_R I$ leaves the sequence exact whenever I is an injective left R-module. G-flat modules have been studied extensively by many authors [3–5, 7, 11, 13, 14, 19]. In particular, it was proved that these modules share many nice properties of the classical homological modules: flat modules.

In 2009, Ding, Li and Mao introduced in [8] the notion of strongly Gorenstein flat modules, which lie strictly between projective and Gorenstein flat modules over coherent rings. A right R-module M is said to be strongly Gorenstein flat (SG-flat for short) if there exists an exact sequence of projective right R-modules

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

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with $M \cong \mathrm{Im}(P_0 \to P^0)$ and such that $\mathrm{Hom}_R(\text{-}, \mathcal{F}(\mathcal{R}))$ leaves the sequence exact. Note this definition is different from the concept of strongly Gorenstein flat modules studied in [11, 19]. Then they defined the strongly Gorenstein flat dimension $SGfd(M)$ of M as $\inf\{n|$ there exists an exact sequence $0 \to G_n \to \cdots \to G_0 \to M \to 0$ of right R-modules, where each G_i is SG-flat$\}$. If no such sequence exists for any n, set $SGfd(M) = \infty$. So M is SG-flat if $SGfd(M) = 0$. It is clear that $SGfd(M)$ measures how far away a right R-module M is from being SG-flat. The right strongly Gorenstein flat dimension $rSGFD(R)$ of R is defined as $\sup\{SGfd(M)|\ M$ is any right R-module$\}$ and measures how far away a ring R is from being QF ring [8, Proposition 2.16].

Following [5], a right R-module M is said to be Gorenstein projective (G-projective for short) if there exists an exact sequence of projective right R-modules

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M \cong \mathrm{Im}(P_0 \to P^0)$ and such that $\mathrm{Hom}_R(\text{-}, \mathcal{P}(\mathcal{R}))$ leaves the sequence exact. By definition, every SG-flat right R-module is G-projective. The Gorenstein projective dimension, $Gpd_R M$, of a right R-module M is defined by declaring that $Gpd_R M \leq n$ if and only if M has a Gorenstein projective resolution of length n.

In Section 2, with the additional assumption of coherence and FP-id$({}_RR) < \infty$, we show that the SG-flat dimension has the properties of a "dimension". Let R be a left coherent ring and FP-id$({}_RR) < \infty$. It is shown that $rSGFD(R) = \sup\{SGfd(M)|\ M$ is a cyclic right R-module$\} = \sup\{SGfd(M)|\ M$ is any f.g. right R-module$\} = \sup\{\mathrm{id}(M)|\ M$ is a flat right R-module$\} = \sup\{\mathrm{id}(M)|\ M$ is a right R-module with fd(M) $< \infty\}$. As corollaries, we have that $rSGFD(R) \leq 1$ if and only if every submodule of an SG-flat right R-module is SG-flat. For a right semi-Artinian left coherent ring R with FP-id$({}_RR) < \infty$, we prove that $rSGFD(R) = \sup\{SGfd(M)|\ M$ is a simple right R-module$\}$. We also extend the equalities of the well-known Hilbert′s syzygy theorem to the strongly Gorenstein flat dimension.

In Sections 3 and 4, we are mainly interested in computing the strongly Gorenstein flat dimensions of direct products of commutative rings and (almost) excellent extensions of rings.

Let R be a ring and M a right R-module. We use $pd(M)$, $id(M)$, and $fd(M)$ to denote the projective, injective and flat dimensions of M, respectively. $\mathcal{SGF}(\mathcal{R})$ and $\mathcal{GP}(\mathcal{R})$ denote the class of all strongly Gorenstein flat and Gorenstein projective right R-modules, respectively. General background materials can be found in Ding, Li and Mao (2009), Bennis and Mahdou (2007), Enochs and Jenda (2000), and Holm (2004).

2. General results of Strongly Gorenstein flat dimensions

Recall the definition of strongly Gorenstein flat dimension.

Definition 2.1 ([8]) *Given a right R-module M. Let $SGfd(M)$ denote $\inf\{n|$ there exists an exact sequence of right R-modules $0 \to G_n \to \cdots \to G_0 \to M \to 0$, where each G_i is SG-flat$\}$ and call $SGfd(M)$ the strongly Gorenstein flat dimension of M. If no such n exists, set $SGfd(M) = \infty$.*

Put $rSGFD(R) = \sup\{SGfd(M)|M$ is any right R-module$\}$ and call $rSGFD(R)$ the right *strongly Gorenstein flat dimension of R. Similarly, we have $lSGFD(R)$ (when R is a commutative ring, we drop the unneeded letters r and l).*

Remark 2.2 (1) By the definitions of strongly Gorenstein flat dimension and strongly Gorenstein flat module, M is an SG-flat right R-module whenever $SGfd(M) = 0$.

(2) For a right R-module M, $Gpd(M) \leq SGfd(M) \leq pd(M)$ (From the trivial fact that $\mathcal{P}(\mathcal{R}) \Rightarrow \mathcal{SGF}(\mathcal{R}) \Rightarrow \mathcal{GP}(\mathcal{R})$).

Lemma 2.3 ([8, Lemma 3.4]) *Let R be a left coherent ring with FP-id$(_RR) < \infty$, M a right R-module with finite strongly Gorenstein flat dimension. Then the following are equivalent for a fixed nonnegative integer n:*

(1) $SGfd(M) \leq n$;

(2) For any exact sequence $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with each P_i projective, K_n is SG-flat,

(3) $\mathrm{Ext}^{n+i}(M, F) = 0$ for any flat right R-module F and any $i \geq 1$.

Consequently, the $SGfd(M)$ is determined by the formula:

$$SGfd(M) = \sup\{i | \exists \text{ flat left } R\text{-module } F, s.t. \mathrm{Ext}^i(M, F) \neq 0\}.$$

Lemma 2.4 ([8, Proposition 2.10, Remark 2.2(2)]) *Let R be a left coherent ring. The class $\mathcal{SGF}(\mathcal{R})$ is projectively resolving. Furthermore, $\mathcal{SGF}(\mathcal{R})$ is closed under arbitrary direct sums and direct summands.*

In general, $\mathcal{SGF}(\mathcal{R})$ is not injectively resolving. But we have the following result.

Proposition 2.5 *Let R be a left coherent ring, $0 \to G' \to G \to M \to 0$ a short exact sequence where G and G' are SG-flat right R-modules. If $\mathrm{Ext}^1(M, Q) = 0$ for all projective right R-module Q, then M is SG-flat.*

Proof Since $SGfd(M) \leq 1$, [8, Theorem 4.1] gives the existence of an exact sequence $0 \to Q \to \widetilde{G} \to M \to 0$, where Q is projective, and $\widetilde{G}$ is SG-flat. By our assumption $\mathrm{Ext}^1(M, Q) = 0$, this sequence splits, and hence M is SG-flat by Lemma 2.4. □

The next result generalizes the Schanuel's lemma involving projective modules.

Proposition 2.6 *Let R be a left coherent ring and M a right R-module. Consider two exact sequences,*

$$0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

$$0 \to \widetilde{K_n} \to \widetilde{P_{n-1}} \to \cdots \to \widetilde{P_1} \to \widetilde{P_0} \to M \to 0,$$

where each P_i and $\widetilde{P_i}$ are projective right R-modules. Then K_n is SG-flat if and only if $\widetilde{K_n}$ is SG-flat.

Proof By Lemma 2.4. □

The proof of the next theorem is standard homological algebra.

Theorem 2.7 *Let R be a left coherent ring, $0 \to M' \to M \to M'' \to 0$ a short exact sequence* *of right R-modules. If any two of $SGfd(M')$, $SGfd(M)$ and $SGfd(M'')$ are finite, then so is the third. Moreover,*

(1) $SGfd(M') \leq \sup\{SGfd(M), SGfd(M'')-1\}$ with equality if $SGfd(M) \neq SGfd(M'')$;

(2) $SGfd(M) \leq \sup\{SGfd(M'), SGfd(M'')\}$ with equality if $SGfd(M'') \neq SGfd(M')+1$;

(3) $SGfd(M'') \leq \sup\{SGfd(M), SGfd(M')+1\}$ with equality if $SGfd(M) \neq SGfd(M')$.

Corollary 2.8 *Let R be a left coherent ring. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of right R-modules, where $0 < SGfd(M') < \infty$ and M is SG-flat, then $SGfd(M'') = SGfd(M')+1$.*

Another dimension which is closely related to the SG-flat dimension is defined as follows [8]:

$$rFID(R) = \sup\{\text{id}(M) | M \text{ is a flat right } R\text{-module}\}.$$

Theorem 2.9 *Let R be a left coherent ring and FP-id$(_RR) < \infty$. Then the following are identical:*

(1) $rSGFD(R)$;

(2) $\sup\{SGfd(M)|M$ is a cyclic right R-module$\}$;

(3) $\sup\{SGfd(M)|M$ is any f.g. right R-module$\}$;

(4) $rFID(R)$;

(5) $\sup\{\text{id}(M)|M$ is a right R-module with $fd(M) < \infty\}$.

Proof (2)≤(3)≤(1) and (4)≤ (5) are obvious.

(1) = (4). See [8, Corollary 3.5(1)].

(4) ≤ (2). We may assume $\sup\{SGfd(M)|\ M$ is a cyclic right R-module$\} = n < \infty$. Let N be arbitrary flat right R-module and I any right ideal. Then $SGfd(R/I) \leq n$, by Lemma 2.3, $\text{Ext}^{n+1}(R/I, N) = 0$, and so $\text{id}(N) \leq n$.

(5) ≤ (4). By dimension shifting. □

Corollary 2.10 *Let R be a left coherent ring and FP-id$(_RR) < \infty$. Then the following are equivalent:*

(1) $rSGFD(R) \leq 1$;

(2) Every submodule of an SG-flat right R-module is SG-flat;

(3) Every right ideal of R is SG-flat.

Proof (1)⇒(2). Let N be a submodule of an SG-flat right R-module M. Then, for any flat right R-module F, we get an exact sequence

$$0 = \text{Ext}^1(M, F) \to \text{Ext}^1(N, F) \to \text{Ext}^2(M/N, F).$$

Note that the last term is zero by (1). Hence $\text{Ext}^1(N, F) = 0$ and (2) follows.

(2)⇒(3) is trivial.

(3)⇒(1). Let I be any right ideal of R. The exact sequence $0 \to I \to R \to R/I \to 0$ implies $SGfd(R/I) \leq 1$ by Lemma 2.3. So (1) follows from Theorem 2.9(2). □

It is well known that for any finitely generated right R-module M, the dual module Hom(M, R) is finitely generated projective. Here we have the following

Corollary 2.11 *If R is a left coherent ring with $rSGFD(R) \leq 1$ and FP-$\mathrm{id}(_RR) < \infty$, then the dual module $\mathrm{Hom}(M, R)$ of any finitely generated right R-module M is SG-flat.*

Proof Let M be a finitely generated right R-module. Then there exists an exact sequence $P \to M \to 0$ with P finitely generated projective. So we have a right R-module exact sequence $0 \to \mathrm{Hom}(M, R) \to \mathrm{Hom}(P, R)$. Note that $\mathrm{Hom}(P, R)$ is projective, therefore $\mathrm{Hom}(M, R)$ is SG-flat by Corollary 2.10. □

Corollary 2.12 *Let R be a commutative hereditary ring. Then $\mathrm{Tor}_1(M, N)$ is SG-flat for any R-module M and any SG-flat R-module N.*

Proof For any R-module M, there is an exact sequence $0 \to P_1 \to P_0 \to M \to 0$, with P_0 and P_1 projective by hypothesis, which induces an exact sequence $0 \to \mathrm{Tor}_1(M, N) \to P_1 \otimes N$. It is easy to see that $P_1 \otimes N$ is SG-flat (for N is SG-flat). Thus $\mathrm{Tor}_1(M, N)$ is SG-flat by Corollary 2.10. □

A ring R is called right semi-Artinian if every nonzero cyclic right R-module has a nonzero socle. The following proposition shows that we may compute the strongly Gorenstein flat dimension of a semi-Artinian coherent ring using just the SG-flat dimensions of simple modules.

Proposition 2.13 *If R is a left coherent right semi-Artinian ring with FP-$\mathrm{id}(_RR) < \infty$, then $rSGFD(R) = \sup\{SGfd(M) | M$ is a simple right R-module$\}$.*

Proof It suffices to show that $rSGFD(R) \leq \sup\{SGfd(M) | M$ is a simple right R-module$\}$. We may assume that $\sup\{SGfd(M) | M$ is a simple right R-module $\} = n < \infty$. Let F be a flat right R-module and I a maximal right ideal of R.

Consider the injective resolution of F.

$$0 \to F \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to \cdots$$

Write $L = \mathrm{coker}(E^{n-2} \to E^{n-1})$. Then $\mathrm{Ext}^1(R/I, L) = \mathrm{Ext}^{n+1}(R/I, F) = 0$ by Lemma 2.3. Therefore L is injective by [9, Lemma 4], since R is right semi-Artinian. So $\mathrm{id}(F) \leq n$, and hence $rSGFD(R) \leq n$ by Theorem 2.9. □

It is well known that if R is a right coherent ring, then $fd(M) = pd(M)$ for any finitely presented right R-module M (see [12, Lemma 5]). Now we have

Proposition 2.14 *If M is an SG-flat right R-module, then $fd(M) = pd(M)$.*

Proof It is clear that $fd(M) \leq pd(M)$. Conversely, we may suppose that $fd(M) = n < \infty$. There is an exact sequence

$$0 \to F_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

with P_0, P_1, …, P_{n-1} projective. Since $fd(M) = n$, F_n is flat. Note that F_n is also SG-flat and hence projective by [8, Corollary 2.5]. So $pd(M) \leq n$ as desired. □

At end of this section, we give the strongly Gorenstein flat dimensions of commutative polynomial rings. The proof of the next main result requires a lemma.

Lemma 2.15 *The following inequalities hold:*

$$rG-gldim(R)\leq rSGFD(R)\leq gldim(R),$$

where equalities hold if $wgldim(R)<\infty$.

Proof From Remark 2.2(2), for any right R-module M, $Gpd(M)\leq SGfd(M)\leq pd(M)$. So $rG-gldim(R)\leq rSGFD(R)\leq gldim(R)$, and hence the equalities hold if $wgldim(R)<\infty$ by [4, Corollary 1.2(2)]. □

From [2, Theorem 4.3], if R is a commutative coherent ring of global dimension two, then the polynomial ring $R[X_1,X_2,\dots,X_n]$ in n indeterminates over R is coherent. Now we have

Theorem 2.16 *Let R be a commutative coherent ring of global dimension two, $R[X_1,X_2,\dots,X_n]$ the polynomial ring in n indeterminates over R. Then: $SGFD(R[X_1,X_2,\dots,X_n])=SGFD(R)+n$.*

Proof By [2, Theorem 4.3], the polynomial ring $R[X_1,X_2,\dots,X_n]$ is coherent, and $gldim(R[X_1,X_2,\dots,X_n])=gldim(R)+n=n+2$ by Hilbert's syzygy theorem. From Lemma 2.15, $SGFD(R[X_1,X_2,\dots,X_n])=gldim(R[X_1,X_2,\dots,X_n])=n+2=gldim(R)+n=SGFD(R)+n$. □

Corollary 2.17 *Let R be a commutative coherent ring of global dimension two, $R[X_1,X_2,\dots,X_n,\dots]$ the polynomial in infinity of indeterminates over R. Then: $SGFD(R[X_1,X_2,\dots,X_n,\dots])=\infty$.*

3. Strongly Gorenstein flat dimensions of direct products of rings

The aim of this section is to compute the strongly Gorenstein flat dimensions of direct products of commutative ring, which generalizes the classical equality: $gldim(\prod_{i=1}^m R_i)=\sup\{gldim(R_i)|\,1\leq i\leq m\}$, where $\{R_i\}_{i=1,\dots,m}$ is a family of rings [6, Chapter VI, Exercise 8, page 123].

To prove the main result, we need the following concept and some results:

Definition 3.1 *A right R-module M is called SSG-flat if there exists an exact sequence of projective right R-modules*

$$\mathbb{P}=\cdots\xrightarrow{f}P\xrightarrow{f}P\xrightarrow{f}P\xrightarrow{f}\cdots$$

with $M\cong\operatorname{Ker}f$ and such that $\operatorname{Hom}_R(\text{-},\mathcal{F}(\mathcal{R}))$ leaves the sequence $\mathbb{P}$ exact.

The class of all SSG-flat right R-modules is denoted by $\mathcal{SSGF}(\mathcal{R})$. Using the definition, $\mathcal{SSGF}(\mathcal{R})$ is closed under any direct sums.

The next result gives a simple characterization of SSG-flat right R-modules.

Theorem 3.2 *For any right R-module M, the following are equivalent*

(1) M is SSG-flat;

(2) There exists an exact sequence $0\to M\to P\to M\to 0$ with P projective, such that $\operatorname{Hom}_R(\text{-},F)$ leaves the sequence exact whenever F is a flat right R-module;

(3) There exists an exact sequence $0\to M\to P\to M\to 0$ with P projective, such that $\operatorname{Hom}_R(\text{-},F)$ leaves the sequence exact whenever F is a right R-module with finite flat dimension;

(4) There exists an exact sequence $0 \to M \to P \to M \to 0$ *with* P *projective, and* $\mathrm{Ext}^i_R(M,F)=0$ *for all flat right* R*-modules* F *and all* $i \geq 1$*;*

(5) There exists an exact sequence $0 \to M \to P \to M \to 0$ *with* P *projective, and* $\mathrm{Ext}^i_R(M,F)=0$ *for all right* R*-modules* F *with finite flat dimension and all* $i \geq 1$.

Proof Using standard arguments, this follows immediately from the definition of SSG-flat modules. □

Now we give a new characterization of SG-flat modules by SSG-flat modules.

Theorem 3.3 *Let* R *be a left coherent ring. A right* R*-module* M *is SG-flat if and only if it is a direct summand of an SSG-flat right* R*-module.*

Proof By Lemma 2.4, it remains to prove the "only if" part.

Let M be an SG-flat right R-module. Then, there exists an exact sequence P of projective right R-modules

$$\cdots \to P_1 \xrightarrow{d_1^{\mathbb{P}}} P_0 \xrightarrow{d_0^{\mathbb{P}}} P_{-1} \xrightarrow{d_{-1}^{\mathbb{P}}} P_{-2} \to \cdots$$

with $M \cong \mathrm{Im}\, d_0^P$ and such that $\mathrm{Hom}_R(\text{-}, \mathcal{F}(\mathcal{R}))$ leaves the sequence P exact.

For all $m \in Z$, denote by $P[m]$ the exact sequence obtained from P by increasing all indexes by m:

$$P[m]_i = P_{i-m} \text{ and } d_i^{P[m]} = d_{i-m}^P \text{ for all } i \in Z.$$

Consider the exact sequence

$$\oplus P[m] = \cdots \to Q = \oplus P_i \xrightarrow{\oplus d_i^P} Q = \oplus P_i \xrightarrow{\oplus d_i^{\mathbb{P}}} Q = \oplus P_i \to \cdots$$

Since $\mathrm{Im}(\oplus d_i^P) \cong \oplus \mathrm{Im}\, d_i^P$, M is a direct summand of $\mathrm{Im}(\oplus d_i^P)$.

Moreover, from [1, Proposition 20.2(1)]

$$\mathrm{Hom}(\bigoplus_{m\in Z}(P[m]), L) \cong \prod_{m\in Z} \mathrm{Hom}(P[m], L).$$

Since M is SG-flat, $\mathrm{Hom}(P[m], L)$ is exact for any flat right R-module L. So $\mathrm{Hom}(\bigoplus_{m\in Z}(P[m]), L)$ is exact. Thus, M is a direct summand of the SSG-flat right R-module $\mathrm{Im}(\oplus d_i^P)$, as desired. □

Now we consider the strongly Gorenstein flat dimensions of direct products of commutative rings. From [10, Theorem 2.4.3], let $\{R_i\}_{i=1,\ldots,m}$ be a family of commutative coherent rings. Then $\prod_{i=1}^m R_i$ is a commutative coherent ring. So, we have the following

Theorem 3.4 *Let* $\{R_i\}_{i=1,\ldots,m}$ *be a family of commutative coherent rings. Then*

$$SGFD(\prod_{i=1}^m R_i) = \sup\{SGFD(R_i)|\ 1 \leq i \leq m\}.$$

To prove this theorem, we need the following two lemmas.

Lemma 3.5 *Let* R *and* S *be coherent rings,* $R \to S$ *a ring homomorphism such that* S *is a projective left* R*-module. If* M *is a (strongly) SG-flat right* R*-module, then* $M \otimes_R S$ *is a (strongly) SG-flat right* S*-module.*

Namely, we have $SGfd_S(M \otimes_R S) \leq SGfd_R(M)$.

Proof Assume at first M is an SSG-flat right R-module. Then, there exists a short exact sequence of right R-modules $0 \to M \to P \to M \to 0$, where P is a projective right R-module, and $\mathrm{Ext}_R(M, F) = 0$ for any flat right R-module F. Since S is a projective (then flat) left R-module, we have a short exact sequence of right S-modules $0 \to M \otimes_R S \to P \otimes_R S \to M \otimes_R S \to 0$ such that $P \otimes_R S$ is a projective right S-module, and for any flat right S-module (then flat right R-module) N, $\mathrm{Ext}_S(M \otimes_R S, N) = \mathrm{Ext}_R(M, N)) = 0$ (see [13, Theorem 11.65]). This implies that $M \otimes_R S$ is an SSG-flat right S-module.

Now, let M be any SG-flat right R-module. Then, it is a direct summand of an SSG-flat right R-module N. Then, $M \otimes_R S$ is a direct summand of the S-module $N \otimes_R S$ which is, from the reason above, SSG-flat. Therefore, $M \otimes_R S$ is an SG-flat right S-module. □

Lemma 3.6 *Let* $\{R_i\}_{i=1,\ldots,m}$ *be a family of commutative coherent rings such that all flat* R_i*-modules have finite injective dimension for* $i = 1, \ldots, m$. *Let* M_i *be an* R_i*-module for* $i = 1, \ldots, m$. *If each* M_i *is a (strongly) SG-flat* R_i*-module, then* $\prod_{i=1}^m M_i$ *is a (strongly) SG-flat* $\prod_{i=1}^m R_i$*-module.*

Namely, we have $SGfd_{\prod_{i=1}^m R_i}(\prod_{i=1}^m M_i) \leq \sup\{SGfd_{R_i}(M_i)|\ 1 \leq i \leq m\}$.

Proof By induction on m, it suffices to prove the assertion for $m = 2$.

We assume at first that M_i is an SSG-flat R_i-module for $i = 1, 2$. Then, there exists a short exact sequence of R_i-modules $0 \to M_i \to P_i \to M_i \to 0$, where P_i is projective R-modules. Hence, we have a short exact sequence of $R_1 \times R_2$-modules $0 \to M_1 \times M_2 \to P_1 \times P_2 \to M_1 \times M_2 \to 0$, where $P_1 \times P_2$ is a projective $R_1 \times R_2$-module ([15, Lemma 2.5.(2)]).

On the other hand, let Q be a flat $R_1 \times R_2$-module. We have

$$Q = Q \otimes_{R_1 \times R_2} (R_1 \times R_2) = Q \otimes_{R_1 \times R_2} (R_1 \times 0 \oplus 0 \times R_2) = Q_1 \times Q_2,$$

where $Q_i = Q \otimes_{R_1 \times R_2} R_i$ for $i = 1, 2$. From [10, Chapter 2, Exercise 9, page 43], Q_i is a flat R_i-module for $i = 1, 2$. Hence, by hypothesis, $\mathrm{id}_{R_i}(Q_i) < \infty$ for $i = 1, 2$, and from [6, Chapter VI, Exercise 10, page 123], $\mathrm{id}_{R_1 \times R_2}(Q_i) \leq \mathrm{id}_{R_i}(Q_i) < \infty$ for $i = 1, 2$. Thus, $\mathrm{id}_{R_1 \times R_2}(Q_1 \times Q_2) < \infty$, so $\mathrm{Ext}^k_{R_1 \times R_2}(M_1 \times M_2, Q_1 \times Q_2) = 0$ for some positive integer k. This implies, from Theorem 3.2, that $M_1 \times M_2$ is SSG-flat $R_1 \times R_2$-module.

Now, let M_i be any SG-flat R_i-module $i = 1, 2$. Then, there exists an R_i-module G_i and an SSG-flat R_i-module N_i for $i = 1, 2$ such that $M_i \oplus G_i = N_i$. Then, $(M_1 \times M_2) \oplus (G_1 \times G_2) = (M_1 \oplus G_1) \times (M_2 \oplus G_2) = N_1 \times N_2$. Since, by the reason above, $N_1 \times N_2$ is an SSG-flat $R_1 \times R_2$-module, and from Theorem 3.3, $M_1 \times M_2$ is an SG-flat $R_1 \times R_2$-module. □

Proof of Theorem 3.4 By induction on m, it suffices to prove the equality for $m = 2$. To this end, it is equivalent to prove, for any positive integer d, the following equivalence:

$$SGFD(R_1 \times R_2) \leq d \Leftrightarrow SGFD(R_1) \leq d \text{ and } SGFD(R_2) \leq d.$$

Then, assume that $SGFD(R_1 \times R_2) \leq d$ for some positive integer d.

Let M_i be an R_i-module for $i = 1, 2$. Since each R_i is a flat $R_1 \times R_2$-module, and from Lemma

3.5, we have $SGfd_{R_i}(M_i) = SGfd_{R_i}((M_1 \times M_2) \otimes_{R_1\times R_2} R_i) \leq SGfd_{R_1\times R_2}(M_1 \times M_2) \leq d$. This implies that $SGFD(R_i) \leq d$ for $i = 1, 2$. Conversely, assume that $SGFD(R_i) \leq d$ for $i = 1, 2$, where d is a positive integer, and consider an $R_1 \times R_2$-module M. We may write $M = M_1 \times M_2$, where $M_i = M \otimes_{R_1\times R_2} R_i$ for $i = 1, 2$. By hypothesis and from Lemma 3.6, so $SGfd_{R_1\times R_2}(M_1 \times M_2) \leq \sup\{SGfd_{R_1}(M_1), SGfd_{R_2}(M_2)\} \leq d$. Therefore, $SGFD(R_1 \times R_2) \leq d$. □

4. Strongly Gorenstein flat dimensions of (almost) excellent extensions of rings

In this section, we study the strongly Gorenstein flat dimensions under (almost) excellent extensions of rings.

A ring S is said to be an almost excellent extension of a ring R [16, 17] if the following conditions are satisfied:

(1) S is a finite normalizing extension of a ring R (see [20]), that is, R and S have the same identity and there are elements $s_1, \ldots, s_n \in S$ such that $S = \sum_{i=1}^{n} s_iR$ and $s_iR = Rs_i$ for all $i = 1, \ldots, n$.

(2) ${}_RS$ is flat and S_R is projective.

(3) S is right R-projective, that is, if M_S is an S-submodule of N_S, and $M_R|N_R$, then $M_S|N_S$. For example, every $n \times n$ matrix ring $M_n(R)$ is right R-projective.

Further, S is an excellent extension of R if S is an almost excellent extensions of R and S is free with basis $\{s_1, \ldots, s_n\}$ as both a right and a left R-module with $s_1 = 1_R$. The concept of excellent extension was introduced by Passman [21] and named by Bonami [22]. The notion of almost excellent extension was introduced and studied in [16, 17] as a non-trivial generalization of excellent extension. Examples include $n \times n$ matrix rings [21], and crossed products $R * G$ where G is a finite group with $|G|^{-1} \in R$ (see [23]).

Lemma 4.1 ([17, Theorem 1.9]) *Let S be an almost excellent extension of R. Then S is right (left) coherent if and only if R is right (left) coherent.*

Lemma 4.2 *Let R be right coherent ring and S an almost excellent extensions of R. M_S is a right S-module. Then $M_R \in \mathcal{SSGF}(\mathcal{R})$ if and only if $M \otimes_R S \in \mathcal{SSGF}(\mathcal{S})$.*

Proof (⇒). If M_R is an SSG-flat R-module, then there exists a short exact sequence of right R-modules $0 \to M \to P \to M \to 0$ with P projective, and $\mathrm{Ext}_R(M, F) = 0$ for any flat R-module F. Since ${}_RS$ is flat, we have a short exact sequence of right S-modules

$$0 \to M \otimes_R S \to P \otimes_R S \to M \otimes_R S \to 0.$$

Note that $P \otimes_R S$ is a projective S-module, and for any flat S-module (then flat R-module) N, $\mathrm{Ext}_S(M \otimes_R S, N) = \mathrm{Ext}_R(M, N) = 0$ (see [13, Theorem 11.65]). This implies that $M \otimes_R S$ is an SSG-flat S-module.

(⇐). Assume $M \otimes_R S$ is an SSG-flat S-module. Then there exists an exact sequence of right S-modules $0 \to M \otimes_R S \to \bar{P} \to M \otimes_R S \to 0$ with $\bar{P}$ projective. Then there is a projective right S-module $\bar{P}'$ such that $\bar{P} \oplus \bar{P}' = S \otimes_R \bar{P}$. Set $L = (\bar{P} \oplus \bar{P}')^{(\mathbb{N})}$. Consider the exact sequence $0 \to (M \otimes_R S) \oplus L \to \bar{P} \oplus L \to (M \otimes_R S) \oplus L \to 0$. Then $0 \to (M \oplus \bar{P}^{(\mathbb{N})}) \otimes_R S \to \bar{P}^{(\mathbb{N})} \otimes_R S \to (M \oplus \bar{P}^{(\mathbb{N})}) \otimes_R S \to 0$ is exact, and so $0 \to M \oplus \bar{P}^{(\mathbb{N})} \to \bar{P}^{(\mathbb{N})} \to M \oplus \bar{P}^{(\mathbb{N})} \to 0$ is exact sequence of right R-modules with $\bar{P}^{(\mathbb{N})}$ projective since S is a faithfully flat left R-module. Let Q be any flat right R-module. Then $Q \otimes_R S$ is a flat right S-module. Thus $0 = \mathrm{Ext}_S^i(M \otimes_R S, Q \otimes_R S) \cong \mathrm{Ext}_R^i(M, Q \otimes_R S)$ by [13, Theorem 11.65], and so $\mathrm{Ext}_R(M, Q) = 0$ since Q is isomorphic to a summand of $Q \otimes_R S$. It follows that $M \in \mathcal{SSGF(R)}$. □

Proposition 4.3 *Let R be right coherent ring and S an almost excellent extensions of R. M_S is a right S-module. Then the following are equivalent:*

(1) M_R is SG-flat;

(2) $(M \otimes_R S)_R$ is SG-flat;

(3) $(M \otimes_R S)_S$ is SG-flat;

(4) M_S is SG-flat.

Proof Since $S = \sum_{i=1}^n s_i R$ is an almost excellent extensions of R, there exists an integer $t > 0$, such that $R_R | S_R^t$ and $S_R | R_R^t$.

(1) ⇒ (3). Let M_R be an SG-flat right R-module. Then, it is a direct summand of an SSG-flat R-module N_R. Then, $M \otimes_R S$ is a direct summand of the S-module $N \otimes_R S$ which is SSG-flat by Lemma 4.2. Therefore, $M \otimes_R S$ is an SG-flat S-module.

(3) ⇒ (4). Since M_S is isomorphic to a direct summand of $(M \otimes_R S)_S$.

(4) ⇒ (1). If M_S is SG-flat, then there exists an exact sequence of projective right S-modules

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with $M_S \cong \mathrm{Im}(P_0 \to P^0)$ and such that $\mathrm{Hom}_S(\text{-}, \mathcal{F(S)})$ leaves the sequence exact. Note that each P_i and P^i are projective right R-modules. Let F be any flat right R-module. Then $\mathrm{Hom}_R(S, F) | F^t$. While $\mathrm{Hom}_Z(\mathrm{Hom}_R(S, F), Q/Z) \cong S \otimes_R \mathrm{Hom}(F, Q/Z)$ which is injective left R-module by [13, Lemma 3.59], so $\mathrm{Hom}_R(S, F)$ is a flat right R-module, then $\mathrm{Hom}_R(S, F)$ is a flat right S-module. Thus

$$\mathrm{Hom}_R(\text{-}, F) \cong \mathrm{Hom}_R(- \otimes_S S, F) \cong \mathrm{Hom}_S(\text{-}, \mathrm{Hom}_R(S, F))$$

is exact. It follows that M_R is SG-flat.

(3) ⇒ (2). By (1) ⇒ (4). □

Theorem 4.4 *Let R be right coherent ring and S an almost excellent extensions of R. Then $SGfd_S(M) = SGfd_S(M \otimes_R S) = SGfd_R(M)$ for any right S-module M_S.*

Proof By Lemma 3.6, $SGfd_S(M) \le SGfd_S(M \otimes_R S)$ since M_S is isomorphic to a direct summand of $M \otimes_R S_S$.

Now we prove that $SGfd_S(M \otimes_R S) \le SGfd_R(M)$. If $SGfd_R(M) = n < \infty$, then there exists an exact sequence of right R-modules

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0,$$

where each G_i is SG-flat right R-module. Since ${}_RS$ is flat, we have the following exact sequence of right S-modules

$$0 \to G_n \otimes_R S \to G_{n-1} \otimes_R S \to \cdots \to G_0 \otimes_R S \to M \otimes_R S \to 0.$$

Note that each $G_i \otimes_R S$ is SG-flat right S-module by Proposition 4.3, and so $SGfd_S(M \otimes_R S) \leq n$.

At last we prove that $SGfd_R(M) \leq SGfd_S(M)$. If $SGfd_S(M) = n < \infty$. Then, there exists an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0,$$

where each G_i is SG-flat right S-module. Note that each G_i is also SG-flat right R-module by Proposition 4.3, and hence $SGfd_R(M) \leq n$. □

Corollary 4.5 *Let R be right coherent ring.*

(1) If S is an almost excellent extensions of R, then $rSGFD(S) \leq rSGFD(R)$.

(2) If S is an excellent extensions of R, then $rSGFD(S) = rSGFD(R)$.

Proof (1) It follows from Theorem 4.4.

(2) Since S is an excellent extensions of R, R is a direct summand of R-bimodule S. Let ${}_RS_R = R \oplus T$, and M_R be any right R-module. Note that $M \otimes_R S \cong M_R \oplus (M \otimes_R T)$. Therefore by Theorem 4.4, we have

$$SGfd_R(M) \leq SGfd_R(M \otimes_R S) = SGfd_S(M \otimes_R S) \leq rSGFD(S)$$

and hence $rSGFD(R) \leq rSGFD(S)$. So we have the desired equality by (1). □

Theorem 4.6 *Let S be an almost excellent extensions of a ring R. If R is right coherent and $rSGFD(R) < \infty$, then $rSGFD(S) = rSGFD(R)$.*

Proof It is enough to show that $rSGFD(R) \leq rSGFD(S)$ by Corollary 4.5. Let $rSGFD(R) = n < \infty$. There exists a right R-module M such that $SGfd_R(M) = n$. Define a right R-homomorphism $\alpha : M \to M \otimes_R S$ via $\alpha(m) = m \otimes 1$ for any $m \in M$. Note that the exact sequence $0 \to \ker(\alpha) \to M$ gives rise to the exactness of the sequence $0 \to \ker(\alpha) \otimes_R S \to M \otimes_R S$ since ${}_RS$ is flat. So $\ker(\alpha) \otimes_R S = 0$, and hence $\ker(\alpha) = 0$. Thus α is monic, and so we have a right R-modules exact sequence $0 \to M \to M \otimes_R S \to L \to 0$. Note that

$$n = SGfd_R(M) \leq \sup\{SGfd_R(M \otimes_R S), SGfd_R(L) - 1\} \leq rSGFD(R) = n$$

by Theorem 2.5. Since $SGfd_R(L) - 1 \leq n - 1$, $SGfd_R(M \otimes_R S) = n$. On the other hand, by Theorem 4.4, we get $SGfd_R(M \otimes_R S) = SGfd_S(M \otimes_R S) \leq rSGFD(S)$. Therefore $rSGFD(R) \leq rSGFD(S)$, as desired. □

Corollary 4.7 *Let R be right coherent ring and $R * G$ a crossed product, where G is a finite group with $|G|^{-1} \in R$. Then $SGfd_{R*G}(M) = SGfd_{R*G}(M \otimes_R (R * G)) = SGfd_R(M)$ for any right $R * G$-module M.*

Moreover, we have $rSGFD(R * G) = rSGFD(R)$.

Corollary 4.8 *Let R be a right coherent ring and n be any positive integer. Then for any right $M_n(R)$-module M, we have $SGfd_{M_n(R)}(M) = SGfd_{M_n(R)}(M \otimes_R (M_n(R))) = SGfd_R(M)$.*

Moreover, we have $rSGFD(M_n(R)) = rSGFD(R)$.

Let R be graded by a finite group G. The smash product $R\sharp G$ is a free right and left R-module with basis $\{p_a|a \in G\}$ and multiplication determined by $(rp_a)(sp_b) = rs_{ab^{-1}}p_b$ where $s_{ab^{-1}}$ is the ab^{-1} component of s.

Corollary 4.9 *Let R be a right coherent ring and $R\sharp G$ a smash product, where R is graded by a finite group G with $|G|^{-1} \in R$. Then $SGfd_{R\sharp G}(M) = SGfd_{R\sharp G}(M \otimes_R (R\sharp G)) = SGfd_R(M)$ for any right $R\sharp G$-module M.*

Moreover, we have $rSGFD(R\sharp G) = rSGFD(R)$.

Proof By [24, Theorem 4.1], $(R\sharp G) * G \cong M_n(R)$. □

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