Probabilistic Teleportation and Entanglement Matching

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Teleportation may be taken as sending and extracting quantum information through quantum channels. In this report, it is shown that to get the maximal probability of exact teleportation through partially entangled quantum channels, the sender (Alice) need only to operate a measurement which satisfy an “entanglement matching” to this channel. An optimal strategy is also provided for the receiver (Bob) to extract the quantum information by adopting general evolutions.

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Quantum teleportation, the process that transmits an unknown qubit from a sender (Alice) to a receiver (Bob) via a quantum channel with the help of some classical information, is originally concerned by Bennett, Brassard, Crepeau, Jozsa, Peres, and Wootters (BBCJPW) [1]. In their scheme, such a quantum channel is represented by a maximally entangled pair (any of the Bell states) and the original state could be deterministically transmitted to Bob.

The process of teleportation may be regarded as sending and extracting quantum information via the quantum channel. We will apply this picture to investigate a partially entangled quantum channel. Because a mixed state could never provide a new quantum channel to carry out a teleportation. (even with some probability). For the reason of Schmidt disposition [5], a partially entangled pair may be expressed as
\[|\Phi\rangle_{2,3} = a|00\rangle_{2,3} + b|11\rangle_{2,3} \quad (|a|^2 + |b|^2 = 1, \ |a| > |b|). \tag{1}\]

(hereafter, we assume particle 2 is at Alice’s site and particle 3 at Bob’s site) Absolute value of the Schmidt coefficient |b| is an invariant under local operations, and it corresponds to the entanglement entropy \(E\) of the state [6]. Such a state can be concentrated to a Bell state [6, 7] with the probability of \(2|b|^2\) and the concentrated pair may be used as a new quantum channel to carry out a teleportation.

In this report, Alice performs a Von-Neumann measurement on her side while Bob performs a corresponding general evolution to reestablish the initial state with a certain probability. We will give a measure of the entanglement degree to Alice’s measurement and show that the optimal probability of an exact teleportation is determined by the less one of the entanglement degrees of Alice’s measurement and the quantum channel. Thus the matching of these entanglement degrees should be considered and the entanglement degree of the measurement is endowed a meaning of Alice’s ability to send quantum information.

First, we consider the case Alice operates a Bell measurement and give Bob’s proper general evolution to reestablish the initial state with an optimal probability. Considering the previously shared pair shown in Eq. (1) and the unknown state (which is to be send) of particle 1 \(|\phi\rangle_1 = \alpha|0\rangle_1 + \beta|1\rangle_1\), the total state could be written as \(|\Psi\rangle_{1,2,3} = |\phi\rangle_1 |\Phi\rangle_{2,3} = \alpha a |00\rangle_{1,2,3} + \alpha b |01\rangle_{1,2,3} + \beta a |10\rangle_{1,2,3} + \beta b |11\rangle_{1,2,3}\). If Alice operates a Bell measurement, Bob will get the corresponding unnormalized states as the following:
\[
\begin{align*}
\langle\Phi^+_{1,2} | \Psi\rangle_{1,2,3} &= \sqrt{2} \left(\alpha a |0\rangle_3 + \beta b |1\rangle_3\right), \\
\langle\Psi^+_{1,2} | \Psi\rangle_{1,2,3} &= \sqrt{2} \left(\alpha a |0\rangle_3 - \beta b |1\rangle_3\right), \\
\langle\Psi^+_{1,2} | \Phi\rangle_{1,2,3} &= \sqrt{2} \left(\beta a |0\rangle_3 + \alpha b |1\rangle_3\right), \\
\langle\Phi^+_{1,2} | \Phi\rangle_{1,2,3} &= \sqrt{2} \left(\beta a |0\rangle_3 - \alpha b |1\rangle_3\right). \tag{2}
\end{align*}
\]

where \(\left\{ |\Phi^\pm_{1,2}\rangle = \sqrt{2} \left(|00\rangle_{1,2} \pm |11\rangle_{1,2}\right), \ |\Psi^\pm_{1,2}\rangle = \sqrt{2} \left(|01\rangle_{1,2} \pm |10\rangle_{1,2}\right)\) are Bell states of particle 1 and particle 2. Alice informs Bob her measurement result, for example \(|\Phi^+_1\rangle\) (with the corresponding collapsed state of particle 3

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as \(\langle \Phi^+_1,2 | \Psi\rangle_{1,2,3} = \frac{\sqrt{2}}{2} (aa |0\rangle_3 + bb |1\rangle_3)\), which is unnomalized, and Bob gives a corresponding general evolution. To carry out a general evolution, an auxiliary qubit with the original state \(0\rangle_{aux}\) is introduced. Under the basis \(\{0\rangle_3 |0\rangle_{aux}, |1\rangle_3 |0\rangle_{aux}, |0\rangle_3 |1\rangle_{aux}, |1\rangle_3 |1\rangle_{aux}\}\), a collective unitary transformation

\[
U_{\text{sim}} = \begin{pmatrix}
\frac{b}{a} & 0 & \sqrt{1 - \frac{b^2}{a^2}} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
\frac{1}{a} & 0 & -\frac{b}{a} & 0
\end{pmatrix},
\]

transforms the unnomalized product state \(\frac{\sqrt{2}}{2} (aa |0\rangle_3 |0\rangle_{aux} + bb |1\rangle_3 |0\rangle_{aux})\) to the result:

\[
|\Phi\rangle_{3,aux} = \frac{\sqrt{2}}{2} \left[ b (\alpha |0\rangle_3 + \beta |1\rangle_3) |0\rangle_{aux} + a \sqrt{1 - \frac{b^2}{a^2}} \alpha |1\rangle_3 |1\rangle_{aux} \right],
\]

which is also unnomalized. Then a measurement to the auxiliary particle follows. If the measurement result is \(0\rangle_{aux}\), the teleportation is successfully accessed, while if the result is \(1\rangle_{aux}\), teleportation fails to the state of qubit 3 transformed to a blank state \(|1\rangle_3\) and no information about the initial qubit 1 left (thus an optimal probability of teleportation is accessed). The contribution of this unnomalized state to the probability of successful teleportation may be expressed by the probabilistic amplitude of \(\alpha |0\rangle_3 + \beta |1\rangle_3\) in Eq. (4) as \(\left| \frac{\sqrt{2}}{2} b \right|^2 = \frac{1}{2} |b|^2\).

Other states in Eq. (2) could be discussed in the same way, and their contributions to the probability of successful teleportation may be calculated directly by using a general method: if the unnomalized state in Eq. (2) is written as \(\alpha |0\rangle_3 + \beta |1\rangle_3\) or \(\alpha |1\rangle_3 + \beta |0\rangle_3\), after Bob’s optimal operation, it gives a contribution to the whole successful probability as

\[
p = (\min \{|x|, |y|\})^2.
\]

Adding up all the contributions, the optimal probability of successful teleportation is obtained as \(P = \frac{1}{2} |b|^2 \times 4 = 2 |b|^2\).

Then consider more general cases: Alice operates a measurement with such eigenstates:

\[
|\Phi\rangle_{1,2} = a |0\rangle_{1,2} + b |1\rangle_{1,2}, \\
|\Phi\rangle_{1,2}^2 = b |0\rangle_{1,2} - a |1\rangle_{1,2}, \\
|\Phi\rangle_{1,2}^3 = a |0\rangle_{1,2} + b |1\rangle_{1,2}, \\
|\Phi\rangle_{1,2}^4 = b |0\rangle_{1,2} - a |1\rangle_{1,2}.
\]

For the reason of Schmidt disposition, this basis has represented all possible Von-Neumann measurements of two particles when \((a', b')\) varies. The four states above are orthogonal and have the same entanglement entropy, so the measurement’s entanglement degree \(E\) can be defined as that of any of the four states. Collapsed states of particle 3 corresponding to the four measurement result could be written as:

\[
\langle \Phi^+_1,2 | \Psi\rangle_{1,2,3} = \alpha a a' |0\rangle_3 + \beta b b' |1\rangle_3, \\
\langle \Phi^+_2,2 | \Psi\rangle_{1,2,3} = \alpha a b' |0\rangle_3 - \beta b a' |1\rangle_3, \\
\langle \Phi^+_3,2 | \Psi\rangle_{1,2,3} = \beta a a' |0\rangle_3 + \alpha b b' |1\rangle_3, \\
\langle \Phi^+_4,2 | \Psi\rangle_{1,2,3} = \beta a b' |0\rangle_3 - \alpha b a' |1\rangle_3,
\]

which is unnomalized. The general evolution to particle 3 is similar to what is shown in Eq. (3). From the result of Eq. (6), the probability of successful teleportation could be considered directly in the following two cases:

1. \(|a| \geq |a'| \geq |b'| \geq |b|\)

   In this case, because \(|a b'|^2 = |a|^2 \left(1 - |a'|^2\right)\) and \(|b a'|^2 = |a'|^2 \left(1 - |a|^2\right)\), inequality \(|a| \geq |b|\) is established, and \(|a a'| \geq |b b'|\) is obvious, so the whole probability of successful teleportation may be written as

\[
P = \left| (bb')^2 \right| + \left| (ba')^2 \right| + \left| (bb')^2 \right| + \left| (ba')^2 \right| = 2 |b|^2,
\]
which is just the same as the case Alice operates a Bell measurement.

2. \[ |a'\rangle \geq |a| \geq |b| \geq |b'|. \]

In this case, \[ \langle ba'\rangle \geq \langle ab'\rangle, \]
and the probability of successful teleportation is

\[
P = \left| \langle bb'\rangle \right|^2 + \left| \langle ab'\rangle \right|^2 + \left| \langle ab\rangle \right|^2 = 2 \left| b'\right|^2.
\]

From the analysis above, the probability of successful teleportation is determined by the less one of \[ |b| \] and \[ |b'|, \]
and may be regarded as being determined by the less entanglement degree of Alice’s measurement and the quantum channel.

Just as what is mentioned above, teleportation may be regarded as the quantum channel’s preparation and quantum information’s sending and extraction. The result above may be explained clearly by using this picture. The entanglement degree of Alice’s measurement could be considered as Alice’s sending ability and the entanglement degree of the quantum channel could be taken as the width of it. Then the amount of transmitted quantum information is determined by the lower one of these two bounds: the width of the quantum channel \[ 2 |b|^2 \] and the sending ability of Alice \[ 2 |b'|^2 \]. If they are just the same, an “entanglement matching” is satisfied. If Bob always reestablish the to-be-sent state in an optimal probability (which means he always extract all the quantum information he received), an exact teleportation will be performed with the probability equal to the amount of the quantum information transmitted, just as what is shown in Eqs. (8), (9).

Though Bell measurement is an essential task of quantum teleportation, it is very difficult to be fully accessed and it has been shown that Bell states cannot be distinguished completely by using linear devices \[ [8, 9] \], while this difficulty can be seen in some teleportation experiments \[ [10] \]. Von-Neumann Measurements with less entangled eigenstates may be more efficient. From the result above, if a partial entanglement state \[ |\Phi\rangle_{2,3} = a |00\rangle_{2,3} + b |11\rangle_{2,3} \] is adopted as the quantum channel, the same optimal probability of successful teleportation could be accessed if only Alice’s measurement satisfied the “entanglement matching”, while a Bell measurement or a POVM is not necessary. The matching here is essential to get an optimal probability, and it could also be regarded as the matching between the quantum channel’s width and Alice’s sending ability. Without such a matching, a waste of quantum information either at Alice’s site or through the quantum channel will be caused.

The result of entanglement matching can be generalized to the teleportation of multi-particle system. Considering a \[ k \]-particle system \[ P \] at Alice’s side with the state \[ |\Psi\rangle_P = \alpha_0 |00\cdots0\rangle_{P_1,\cdots,P_k} + \alpha_1 |00\cdots01\rangle_{P_1,\cdots,P_k} + \cdots + \alpha_{2^k-1} |11\cdots11\rangle_{P_1,\cdots,P_k} \]. Without loss of generality, the quantum channel between Alice and Bob is \[ k \] independent entangled pairs with the state \[ \prod_{i=1}^{k} (a_i |00\rangle_{A_i,B_i} + b_i |11\rangle_{A_i,B_i}) \] (any other pure quantum channel could be transformed to this by local operations). Alice draws \[ k \] collective measurements, each of which is Von-Neumann measurement with the following eigenvectors:

\[
\begin{align*}
|\Phi\rangle_{i,1}^{1} &= a_i |00\rangle_{P_i,A_i} + b_i |11\rangle_{P_i,A_i}, \\
|\Phi\rangle_{i,2}^{1} &= b_i |00\rangle_{P_i,A_i} - a_i |11\rangle_{P_i,A_i}, \\
|\Phi\rangle_{i,3}^{1} &= a_i |01\rangle_{P_i,A_i} + b_i |01\rangle_{P_i,A_i}, \\
|\Phi\rangle_{i,4}^{1} &= b_i |01\rangle_{P_i,A_i} - a_i |01\rangle_{P_i,A_i}.
\end{align*}
\]

where \[ i = 1, 2, \cdots, k \]. Then Bob reestablishes the original state as \[ |\Psi\rangle^B \] with a certain probability by adopting a proper general evolution. Using similar methods as the case of mono-qubit teleportation, we may show that there also exists an entanglement matching in multi-qubit teleportation: If \[ c_i \] is defined as \[ \min \left\{ \left| b_i \right|^2, \left| b_i \right|^2 \right\} \], the optimal probability of successful teleportation could be expressed as \[ 2^k \prod_{i=1}^{k} c_i^2 \].

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