The optimal layout of football players:

A case study for AC Milan

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Abstract

This paper attempts to find the optimal formation of three midfielders and three forward football players on ground, using the classic Quadratic Assignment Problem or Facility Layout problem. Players are treated as "machines", their positions as locations, and the flow of materials between machines as "flow of passes" and "flow of markings". Based on detailed statistics from four matches of AC Milan, and formulated the problem as minimum (quick strategy), maximum (slow strategy), and mixed or balanced strategies, a number of various layouts emerged. Compared to the initial formation of players, the efficiency time gains in the unconditioned layouts are between 3 and 6.8%. Also, when the manager claims that his three forwards shouldn't shift positions with the midfielders, the efficiency gains in these restricted layouts is about 14´´ to 74´´, which is about 1 to 3% of the approximately 40´ effective time spent into passes and markings from both teams.

Key words: layout, assignment, football, players, passes, markings

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1. Introduction

Football managers have very often a clear picture of the system they should play. Given their preferences and some key parameters, such as the quality of their players, the absence of some important player(s) due to injury or punishment, whether the game is played at home or away and against strong or weak teams and whether they desperately need the victory or are just satisfied with a draw, they will decide on an aggressive or defensive system. Players can be positioned on the ground in many different formations or layouts. There are at least twenty-five complete formations which have been used in various periods, by various teams. The most frequently used nowadays are the standard or modified versions of the 4-3-3 and 4-4-2, systems, with four defenders, three (or four) midfielders and three (or two forwards). For more details and facts on various formations, the interested reader is referred to the following site: http://en.wikipedia.org/wiki/Formation_%28association_football%29.

While the defenders normally, can't shift positions with midfielders and forwards, the midfielders (mainly) and also the forwards (sometimes) can shift positions which each other. When the manager has decided to place two midfielders A and B in position P_1 and P_2 respectively, hopefully he knows that it is the best matching or assignment for the whole team (but not necessarily for the individual players). In theory though, the manager should be aware that there are many different layouts and the selected one might not be superior unless all other good layouts have been considered. For instance, if each one of the four midfielders can be placed to any one of the four possible positions, there are 24 (=4!) different layouts to compare. Similarly, for six players (with all midfielders and forwards to shift position with each other), the number of layouts increases to 720.

Before a given position has been allocated to the right player, the manager must somehow know the requirements of every position and that the allocated player has indeed the appropriate qualities to meet the expectations, and consequently "optimize" the objective function for his team. In a football match, each position can require various qualities, such as "good passes", "tight marking", "runs", "dribbles", "shoots", "co-operation" etc. A good manager must therefore know how skillful each one of his players is in those functions, both individually and pair wise, in order to make the appropriate allocations.

The problem the manager faces is therefore similar to the classic Quadratic Assignment Problem (QAP), or the Facility Layout (FL) problem. In these problems, the decision maker must find out where to place the functions that interact with each other with different flows, in order to minimize the flow-distance or the flow-cost product. For instance, if the distance between two gates (*i* and *j*) at an airline terminal is d_{ij}, and there are t_{kl} travelers to be transferred between flights *k* and *l*, in which gates should these flights be placed in order to minimize the overall distance of all transferred travelers? This problem has been analyzed,

formulated, programmed in various codes and also solved for a large number of functions. The interested reader can find two excellent and updated surveys, one for the QAP by Loiola et al. (2007) and another for the FL by Drira et al. (2007).

In principle we can treat the layout of players as the layout of airlines to different gates, by equating each one of the allocated players as a "flight", his location as a "gate" and the "travelers" as passes or any other measurable flow one can think of, and find the optimal formation of players on the ground. As far as I know, the QAP or FL has never been used in the layout of football players. Thus, the aim of this paper is to find the optimal formation of players, using offensive, defensive or more balanced strategies (i.e. under different objective functions) and also using two sets of functions required by each player, "passes" and "markings" (instead of one as most researchers have used). The model presented is therefore classified as a 2QAP type. Multi-objective layout problems have been treated recently, among others, by Knowles & Corne (2002), Yang & Kuo (2003), while more complex problems, like the Quadratic 3-dimensional AP has been formulated and solved recently by Hahn et al. (2008).

In some empirical studies, the efficiency gains from optimal layouts are very high. Nahmias (1997), referring to some studies, argues that the US spent more than $ 500 billion annually on construction and modification of facilities. Effective facilities planning could reduce costs by 10 to 30 percent per year. He also argues that intelligent layout is a key factor to the Japanese production efficiency. Tompkings et al. (1996) estimated that a good layout can reduce the cost of flows in manufacturing by 10-30%. Elshafei (1977) reallocated 19 departments to 19 different physical regions in the hospital and reduced the patient travel by 38%. How high could the efficiency gains be, if a football team assigned its players optimally?

The paper consists of five sections: Section two describes the model based on a simple graph; section three discusses the problems and the collection of the appropriate data for both sets of functions; section four formulates the model as: (i) a min, and (ii) a max, both as unrestricted to all players, and restricted, by satisfying the a priori manager's constraint regarding the position of his forwards; section five presents and comments on the optimal layouts from all models; section six concludes the paper.

2. The problem as a graph

Any QAP can be presented using graphs. For instance, in the 2QAP model, one needs an undirected weighted and complete graph for the distances (or the costs) of its edges, a second graph for the flows of the edges of the first function and a third graph for the flows

of the edges of the second function. In the optimal solution, the vertices of all three graphs must coincide and the objective function is the sum of two products of the corresponding edges.

In this paper we disregard the four defenders and limit to a 3-3 formation system of three midfielders and three forwards, shown in the symmetric graph below. The central vertex, i.e. position (2), is usually assigned to the playmaker. A complete graph with 6 vertices has $\frac{(6)(5)}{2}=15$ edges, i.e. 15 pairs among all these 6 players (while for the whole team, of 11 players there are 55 pairs).
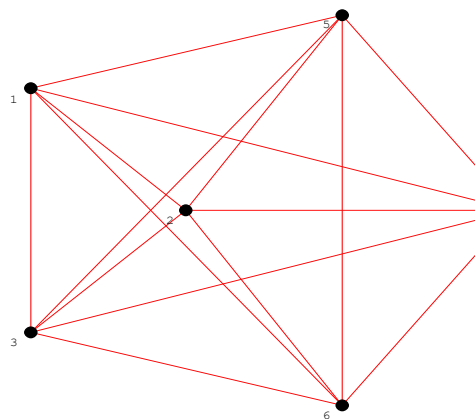
When the problem is treated as a graph, the manager needs to decide the following: (a) the symmetric or asymmetric properties of the graph, (b) its flexibility (i.e. how easily it can shift to another formation if he wishes), and (c) the optimal distance among the vertices (i.e. how close to each other the players should be). For instance, in a very large graph, the longest distances can be about 50m, given the fact that for international matches, the International Football Association Board has decided to set a fixed size of 68m wide and 105m long (see for instance http://en.wikipedia.org/wiki/Association_football_pitch). When the graph is large, the team's tactics are more "open" and offensive. In large graphs midfielders are expected to play long balls or crosses to the forwards, or try to play more from the sidelines in order to open the "closed" midfield opponents.

In general, there are two weaknesses with "open" tactics. First, the success rate of "passes" is lower. Own observations from a large number of UEFA CL and Serie A matches, shows that when the distance between two weakly marked or pressed midfielders is less than 12m, the success rates of passes is more than 95%. Moreover, when the distance is about 16m the success rate falls to 70% and for 36m it falls to about 40%. Even players of top teams, like FC Barcelona, need to be close to each other in order to keep their outstanding success rate in passes. Second, an open tactic is rather vulnerable because the opponent players can find

enough spaces to attack. Midfielders who are positioned more than 10m from each other have very little chance to defend successfully, even if they are very fast runners.

On the other hand, if more "close" tactics improve the success rates of passes and the probability of a good defense against the opponents who try to penetrate the players, the offensive play of the team deteriorates as well, due to the following reason. If the forwards should play near the midfielders too, they must play far from the opponents' defense zone and consequently have lower probability to score. If the forwards are placed far from the midfielders, they might not get enough successful passes from them. As a consequence, neither a tight concentration, nor a loose dispersion of players is the optimal size of the graph. Good teams are naturally very flexible in their tactics and can shift from larger (open) graphs when they miss the ball, to smaller (close) graphs, very fast.

3. **The measurement of functions**

(a) The time

The distance of the edges in a graph is obviously easy to measure. Frequently, the distance matrix is expressed in meters, or in time units, that the given flow, between a pair of functions located in two positions, will take. In this paper the distance is measured in time units, i.e. the number of seconds it takes for a pair of players to accomplish a specific function. I will use the same isomorphic graph but with two different time units, one for quick actions and another for slow actions. As a consequence there is one "slow" time matrix, T_1, and one "quick" matrix, T_2, which measure the number of seconds it takes for a pair of players to "*pass the ball*", or "*mark the opponent players*".

In my estimates I assume that the quick time is half of the slow time and that both T-matrices are symmetrical. Both T-matrices (in seconds) are shown below. For instance, assuming the longest distance between two players is about 40m, the slow time to pass the ball is about 5´´, if the ball speed is 8m/sec. Similarly, it will take 2.5´´ in the quick time matrix (either through a shorter distance between players and/or passing the ball quicker).

$$
T_1 = \begin{Bmatrix}
0.0 & 2.0 & 2.5 & 2.0 & 4.0 & 4.5 \\
2.0 & 0.0 & 4.5 & 2.0 & 4.0 & 2.5 \\
2.5 & 4.5 & 0.0 & 2.5 & 3.0 & 5.0 \\
2.0 & 2.0 & 2.5 & 0.0 & 2.0 & 2.5 \\
4.0 & 4.0 & 3.0 & 2.0 & 0.0 & 3.0 \\
4.5 & 2.5 & 5.0 & 2.5 & 3.0 & 0.0
\end{Bmatrix} ; \quad
T_2 = \begin{Bmatrix}
0.00 & 1.00 & 1.25 & 1.00 & 2.00 & 2.25 \\
1.00 & 0.00 & 2.25 & 1.00 & 2.00 & 1.25 \\
1.25 & 2.25 & 0.00 & 1.25 & 1.50 & 2.50 \\
1.00 & 1.00 & 1.25 & 0.00 & 1.00 & 1.25 \\
2.00 & 2.00 & 1.50 & 1.00 & 0.00 & 1.50 \\
2.25 & 1.25 & 2.50 & 1.25 & 1.50 & 0.00
\end{Bmatrix} ;
$$

Notice that the values in the matrices have been estimated from passes only (see next section for details), but are used to multiply both passes and markings as well.

(b) <u>The passes</u>

To estimate the flow of passes and markings was very hard indeed. In a previous study (Papahristodoulou, 2008), based on 814 UEFA CL matches, it was estimated that the average effective playing time is about 55 minutes. Top teams, irrespectively if they play at home or away, they keep the ball for approximately 30´. About 2/3 of that time (20´) is spent on passes and the rest in dribbles, shoots or runs with the ball.

Top teams, like FC Barcelona or AC Milan very often dominate in ball possession and passes. In an average match, these teams can achieve about 500 successful passes[1], i.e. 9 successful passes per pair of players (given all possible 55 pairs, including the goalkeeper). Obviously good players pass more to others than they receive from them and some pairs pass more than other pairs. If we divide the effective time spent on passes by the number of passes, there are approximately 25 successful passes per effective playing minute, i.e. the average successful pass, for an average distance, will take about 2.5´´. This estimate is in fact identical with the minimum time in the slow T_1-matrix and the maximum time in the quick T_2-matrix.

But how can we measure and estimate the "flow of successful passes"? Top and experienced managers have perhaps good information on these values. For instance, during the training sessions, they might test how various pair of players perform in terms of "passes", under various conditions.

Before I explain the procedure how I collected the "flow of passes", I made the following decisions, before an action is counted as a "pass":

(i) I counted only successful passes (and nicks as well). All incoming passes which have not been controlled completely by the targeted player, or disputed by the opponent players as well, did not count.

(ii) I did not pay any attention whether the player who passed and/or the player who received the ball were completely free or tightly marked.

(iii) Passes from free kicks and corners are not counted, since it is not relevant to the layout of the team (and also very difficult to decide who the targeted player is).

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[1] AC Milan during its first 15 Serie A matches (2009-10), had 7498 successful passes (i.e. 500 per match), and was leading the passes statistics, (see the following sport site: http://sport.virgilio.it/calcio/serie-a/statistiche/index.html) .

(iv) Passes intended to a teammate, but stopped by unjust actions, such as fouls committed to any involved teammate, or by hitting the ball in hands, are counted.

(v) Passes from all possible positions count, irrespectively where the midfielder(s) and the forward(s) were when they passed to each other. Long successful crosses, clearly directed to a specific player, count as well.

(vi) No passes to or from defenders are counted[2].

In order to measure "passes" and "markings" as well, I have investigated in detail four Serie A matches of my favorite Italian team, AC Milan[3], during the period September-November 2009. I recorded and also followed the matches live in the official site of the Italian Sports Journal La Gazzetta dello Sport (http://www.gazzetta.it/). In that site, one for instance can watch live match graphics and find out interesting facts, among others the number of passes. The graphics are available on the site for some hours after the end of the game, a sufficient time for somebody to examine the weighted average position of players in the field, the flow of the game, the sequential visualization of passes, the topological distribution of the offense and defense, shoots etc.

The next step was to play the four recorded matches back, many times, in order to verify to what extent the passes from Gazzetta's graphics are consistent with my definitions above. Despite the fact that I certainly spent more than two days work per match, by playing back the tricky cases many times, there might exist measurement errors. For instance, in two matches Gazzetta overestimated the passes by 3, respectively 4 units, compared to my estimates, and in two other matches it underestimated by 3 units.

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[2] In almost all matches, backs and midfields pass and mark more frequently. For instance, in the Lazio-Milan 1-2 match, Zambrotta (who is a left back) passed with Pirlo (midfielder) 32 times, and Oddo (who is a right back) passed with Pato (forward) 26 times. Two midfielders (Ambrosini and Pirlo) were first in the third place with 25 passes.

[3] In fact I planned to use statistics from more matches, but mainly due to injuries and/or punishments, the manager was forced to use different players in some, or part of matches. The four selected matches are those where the three midfielders (Pirlo, Seedorf, Ambrosini) and the three forwards (Ronaldinho, Pato, Borriello) appeared most. The investigated matches were: (1) Milan-Bologna 1-0 (Sept. 19, Seedorf); Ronaldinho and Borriello (injured) did not play and Leonardo played a 4-4-2 formation, with two forwards, Pato and Inzaghi; Huntelaar substituted Inzaghi in the 60´; (2) Milan-Roma 2-1 (Oct. 18, Ronaldinho, Pato); Borriello was still injured, and Leonardo started with 4-4-2, (with two forwards, Ronaldinho, Pato) but shifted to 4-4-3 (by setting in Inzaghi in the second half instead of the midfielder Abate); Ronaldinho played 83´; (3) Milan-Parma 2-0 (Oct. 31, Borriello, Borriello); Milan played 4-3-3, and started with Gattuso in the midfield, who was substituted by Ambrosini in 75´; Seedorf played 83´; (4) Lazio-Milan 1-2 (Nov. 8, T.Silva, Pato); Milan played 4-3-3; Borriello played 75´, Seedorf 81´and Ronaldinho 87´. Thus, as a whole, the selected players' playing time is: Pirlo and Pato played 90*4 = 360´, Seedorf, 344´, Ambrosini, 330´, Ronaldinho, 260´ and Borriello 165´.

(c) <u>The markings</u>

The statistics of the "flow of markings" is the most questionable in the paper. First of all, while the "flow of successful passes" between players is obvious, the "flow of markings" needs some explanation. Usually, people rank the defensive qualities of a player depending on how well or badly the player has defended, individually. The players of good teams though, work together and help each other in their defense. Thus, since we need a "pair of players" and not individual players, the "flow of markings" is interpreted as the respective pair of players who defended together against the opponents who attempt to pass the ball through their "edge", or dribble them. For instance, a "flow of 17 markings" for a pair of players, means that they defended together, against some opponent players, 17 times.

Obviously a team defends when the opponents keep the ball. Consequently, the time it takes to defend, is a function of the opponent team's effective playing time and not of the own playing time. Because top teams are pressing their opponents all time, their ball possession time is almost identical (or slightly less) to the own team's markings time. For instance, if the opponents keep the ball for about 25′, the own team is going to defend by at least 20′ (assuming the own team does not need to defend when the opponent players keep the ball in their defense area for, say, about 5′). Own observations from many matches show that a good team might have about 370-440 "good" markings per match. As a consequence, the average "good" marking takes about 3′′, an estimate which is rather consistent with the values in the T-matrices.

Before I estimated the "flow of markings" I had to make similar decisions, as in the "flow of passes". For instance, markings from free kicks and corners are not counted because it is not relevant to the layout of the team. Markings by defenders and defenders/midfielders or defenders/forwards are excluded. Neither pair of player who "pretended to mark" is credited with marking. If only one player defended, the second player could be either the nearest teammate who was backing him, or the nearest teammate who marked another opponent, depended upon the circumstances. On the other hand, "the flow of markings" contains both successful and sometimes unsuccessful or "unjust" actions, such as when a player commits a foul while he is marking his opponent. Such an action is reported to the adjacent pair of players, i.e. the player who committed the foul, and the teammate nearest to him.

There is a deliberately asymmetry in the criteria, because, while the unsuccessful passes are not counted, some unsuccessful markings are counted. There are two reasons for that asymmetry. First, since we concentrate on midfielders and forwards we expect them to have better qualities in passes than in markings. Second, unsuccessful passes reflect often "bad quality" of the sender and/or the receiver, while unsuccessful markings for a pair of players who tried to defend as one should expect from them, is still a marking. If, despite their hard

effort, failed, their failure might have been due to the higher quality of opponents, or to the unfavorable position they might had been in that particular case. Thus, apart from case (iv), where an unfinished pass will count, it is the outcome of successful passes which normally counts and the good defense effort that counts in markings, irrespectively of its success.

To my knowledge, markings statistics do not exist, probably due to the subjectivity that such a measurement involves. The "flow of markings" is therefore my best personal estimates, probably biased, after I investigated the four recorded matches for many hours.

Matrix N below denotes the average pair of passes and matrix R denotes the average pair of markings from AC Milan's four matches[4]. In both matrices, the raw and column players are ordered as: Pirlo[5], Ambrosini, Seedorf, Borriello, Ronaldinho, and Pato.

$$
N = \begin{Bmatrix} 0 & 26 & 23 & 19 & 24 & 15 \\ 0 & 0 & 22 & 17 & 14 & 10 \\ 0 & 0 & 0 & 23 & 17 & 7 \\ 0 & 0 & 0 & 0 & 11 & 13 \\ 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Bmatrix} ; \quad R = \begin{Bmatrix} 0 & 18 & 19 & 17 & 16 & 16 \\ 0 & 0 & 27 & 23 & 21 & 17 \\ 0 & 0 & 0 & 24 & 15 & 20 \\ 0 & 0 & 0 & 0 & 21 & 15 \\ 0 & 0 & 0 & 0 & 0 & 19 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Bmatrix} ;
$$

Before we move to the next section, let me point out one more trouble, with these values. In the classical FL or QAP problems, the flow between functions is independent from the layout of the functions. For instance, if there is a fixed flow of materials between functions F_1 and F_2, that flow will remain fixed, irrespectively if F_1 is placed in 1 and F_2 is placed in 2, or in 5.

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[4] The values are based on a 90´ match. When a player was substituted by another, the new player's passes and markings are also included in the relevant player who is selected. For instance, since the Pirlo-Gattuso pair had 17 passes in 75´, and the Pirlo-Ambrosini pair had 6 passes in 15´ (when Ambrosini replaced Gattuso in the Milan-Parma match), all 23 passes will be reported to Pirlo-Ambrosini pair (because Gattuso is not included). Similarly, the values by Inzaghi and Huntelaar to all others are credited to Borriello and all others, and those by Abate, to Ronaldinho. Given the fact that all six selected players played together in exactly one match, i.e. the last 15´ in Milan-Parma and the first 75´ in Lazio-Milan, average values from just one match, might have been worse, compared to the method used, based on 360´. In any case, the values in the N- and R-matrices are simply rough estimates and might over- or underestimate the correct ones.

[5] Since I collected data on flows per pair, it is impossible to find out the exact values attributed to (or originated from) each player. Regarding individual players, Pirlo was leading the passes statistics during the first 15 Serie A matches, with 962 useful passes, or about 64 passes per match, or about 13% of his team's passes. Consequently, he must have passed to his teammates more times than they passed to him. (For updated statistics, see for instance http://sport.virgilio.it/calcio/serie-a/statistiche/index.html). A more appropriate method to measure the values in the N- and R-matrices would be to separate the incoming flows from the outgoing flows. Such a refinement though would take a considerable amount of time.

In a real football match, neither the positions of the players, nor the flows are fixed. For instance, the number of passes depends not only on which pair of players we are looking at, but also on the distance of the players who pass, and on how free or unmarked the player who receives the ball is. As is understood from this simple observation, it is not certain whether the problem can ever be solved, when the values in all matrices change all time, or become stochastic. Raman et al. (2007) tried to address a simplified version of a similar problem in manufacturing, by considering the interaction values due to the layout's flexibility, productive area utilization and closeness gap.

In order to be able to solve the problem effectively, we need to make the simple assumption that all fifteen time costs, passes and markings, are fixed and independent from the position of the players. In fact, sometimes, there are players who like to "pass" to each other, and "mark" together, irrespectively of their positions. But most players are more "practical" and try to play with the "nearest" or the more "free" teammate. Regarding the distances among players, and consequently the time costs, it is assumed that all players shift left, right, up or down simultaneously, in order to preserve the isomorphism of the graph. The graphics from Gazzetta show that the weighted position of the players is not very symmetric, because some midfielders play more close to each other. Thus, it is not the number of passes or of markings that change, when players shift positions, but the value of the products (time*passes) and (time*markings). As a consequence, the fixed number of passes and/or markings can take place slower, or quicker, depending on the time dimension they are multiplied with.

Let us now assume initially that the team's layout is: (Ambrosini, 1), (Seedorf, 2), (Pirlo, 3), (Borriello, 4), (Ronaldinho, 5), (Pato, 6). As was mentioned earlier, Milan played the first match, using a 4-4-2 formation which was shifted to a 4-2-1-3 formation (a simple variation of the 4-3-3), as is shown in the graph. Is that layout chosen by team's manager Leonardo optimal, given the parameters in the three matrices?

4. The problem formulation

Traditionally, the QAP and the FL have been treated as finding the minimum cost allocation of facilities into locations, where costs are the sum of all possible distance-flow. Accordingly, in our problem we need to find a minimum time allocation of players into positions, where time is the sum of two products, (quick time*passes) and (quick time*markings). As a consequence, when team minimizes the sum of these two products, the manager seeks to place his players in such a way so that the both passes and markings will be completed as

soon as possible. This "fast" strategy can for instance be applied when the team plays for a victory and does not want to waste time.

In some situations though, when the team tries to "kill the game" because it attempts to keep their lead or the draw until the final whistle, it can apply instead a "slow" strategy. Such a strategy is more consistent when the sum of (slow time*passes) and (slow time*markings) is maximized. With such an objective function, the pair of players who pass often should be now placed in a longer distance from each other, in order to make their (slow time*passes) product, as large as possible. Consequently, the players might be re-positioned compared to the previous strategy.

In the paper I have considered four mixed cases as well, when teams should play carefully or more balanced. That can be achieved when the team minimizes: (1) the sum (slow time*passes) and (quick time*markings); (2) the sum of (quick time*passes) and (slow time*markings); and similarly when the team maximizes: (3) the sum of (slow time*passes) and (quick time*markings); (4) the sum of (quick time*passes) and (slow time*markings).

Other mixed cases are also possible, such as when the team needs to minimize (maximize) one product, given some upper (lower) specific value in the other product and vice versa. Also, one can assign various weights to the two sums in the objective function, if for instance the product (time*passes) is considered as more or less significant to the (time*markings) product.

Sometimes, unconstrained layouts of players can lead to very revolutionary or ridiculous players-positions assignment. For instance, if the unconstrained optimal layout should force some forward(s) to play as midfielder(s), or the playmaker has shifted to another position, the manager can set his own "strong" or "soft" constraints, to correct the layout. Obviously, such conditions will limit the number of possible layouts and the forced optimal solution is often inferior compared to unconstrained layouts.

Below I will formulate the problem as: (i) min, and (ii) max; both formulations will be unrestricted to any player to play in any position the optimal layout suggests, and restricted according to the managers' a priori strong beliefs, that his three forwards (or two in another formation) must play as forwards. Let us first look at the notation of all variables. To avoid warnings and sometimes error codes[6], I have deliberately used a large number of subscripts to identify easier the pair of passes, and markings. Thus, both passes and markings graphs have their own subscripts.

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[6] I have used the package LINGO. That package is based on the powerful feature "Sets", i.e. groups of related objects, which can be misunderstood and lead to error codes.

<u>Notation</u>:

(j,k): pair of positions for passes; $\forall\ j \neq k$

(s,q): pair of positions for markings; $\forall\ s \neq q$

(b,c): pair of passes; $\forall\ b \neq c$

(l,m): pair of markings; $\forall\ l \neq m$

$Z_{1,bjck}$*: (passes, position, passes, position),* $\forall\ b < c\ \&\ j \neq k\ ;$

$Z_{2,lsmq}$*: (markings, position, markings, position),* $\forall\ l < m\ \&\ s \neq q\ ;$

X_{bj}*,* X_{ck}*: (passes, position), (passes, position); it is a binary variable with the following meaning: If* $X_{bj} = X_{ck} = 1, \forall b, j, c, k\ \&\ (b \neq c, j \neq k); \Rightarrow both\ pairs\ are\ correctly\ placed\ ;*

If $X_{bj} = X_{ck} = 0 \Rightarrow incorrect\ pairs\ ;$

Y_{ls}*,* Y_{mq}*: (marking, position), (marking, position), it is a binary variable with the following meaning: If* $Y_{ls} = Y_{mq} = 1, \forall\ l, s, m, q\ \&\ (l \neq m, s \neq q); \Rightarrow both\ pairs\ are\ correctly\ placed;*

If $Y_{ls} = Y_{mq} = 0 \Rightarrow incorrect\ pairs\ ;$

$N_{bc}T_{jk}$*: (value of pair of passes)*(time value of pair of positions), when player b is placed in j and player c is placed in k;*

$R_{lm}T_{sq}$*: (value of pair of markings)*(time value of pair of positions), when player l is placed in s and player m is placed in q;*

(i)   Formulation as a Min (unrestricted)

$$Min\ \sum_{b,j=1}^{6}\sum_{c,k=1}^{6}\left[N_{bc}T_{jk} + N_{cb}T_{kj}\right]z_{1,bjck} + \sum_{l,s=1}^{6}\sum_{m,q=1}^{6}\left[R_{lm}T_{sq} + R_{ml}T_{qs}\right]z_{2,lsmq} \qquad 1.1$$

$$s.t.\ \sum_{j=1}^{6}x_{bj} = 1,\quad \forall b \qquad 1.2$$

$$\sum_{b=1}^{6}x_{bj} = 1,\quad \forall\ j \qquad 1.3$$

$$\sum_{s=1}^{6}y_{ls} = 1,\quad \forall l \qquad 1.4$$

$$\sum_{l=1}^{6} y_{ls} = 1, \quad \forall s \qquad\qquad 1.5$$

$$z_{1,bjck} - x_{bj} - x_{ck} \geq -1, \qquad\qquad 1.6$$

$$z_{2,lsmq} - y_{ls} - y_{mq} \geq -1, \qquad\qquad 1.7$$

$$x_{bj} - y_{ls} = 0, \qquad\qquad 1.8$$

$$x_{bj} \in {0,1}, \quad y_{ls} \in {0,1} \qquad\qquad 1.9$$

$$z_{1,bjck} \geq 0, \quad z_{2,lsmq} \geq 0 \qquad\qquad 1.10$$

Constraint 1.2 states that each one of the players who pass should be assigned to a position. An identical interpretation applies for constraint 1.4 (for players who mark).

Constraints 1.3 and 1.5 are also similar, so that each position in the "passes" graph should receive one of the players who pass and similarly, each position in the "markings" graph should receive one of the players who mark.

Constraints 1.6 and 1.7 are also similar to each other. Since our four matrices are non-negative and the objective value minimizes both Z-values, in the optimal solutions the Z-values should be low or zero. For instance constraint 1.6 ensures that if b is assigned to j and c assigned to k (for players who pass), and given the binaries, X_{bj}, X_{ck}, it is impossible for $Z_{1,bjck}$, not to be equal to unit. If both pairs are wrong (both binaries are equal to zero), or only one is wrong, there is no need for $Z_{1,bjck}$ to be higher than zero.

Notice also that from the binary constraint 1.9, it follows that the Z-values will in fact be binary, even if constraint 1.10 requires that they should be just non-negative continuous! Thus, given the fact that there are $\dfrac{(n)^2(n\text{-}1)^2}{2} = \dfrac{(6)^2(5)^2}{2} = 450$ variables for $Z_{1,bjck}$ and equally as many for $Z_{2,lsmq}$, we save many unnecessary binary variables and decrease the estimation time. From constraint 1.9, there are only 36 + 36 binary variables.

Finally, in order to force the players to be placed in the same position in both graphs, the consistency constraint 1.8 is required. That constraint ensures that it is impossible to place players C and D in, say, positions 1 and 5 respectively as far as their "passes" is concerned and not place them in exactly the same positions with respect to their "markings". Thus, the fact that we have used two different T-matrices, will not lead to inconsistencies, because the optimal layout is common for both, the players who pass and the same players who mark.

(ii)   **Formulation as a Max (unrestricted)**

Equation 1.1 changes simply into maximum while constraint *1.6* changes into the following:

$$2z_{1,bjck} \leq x_{bj} + x_{ck},$$

$$z_{1,bjck} \leq x_{bj}, \qquad\qquad 1.6´$$

$$z_{1,bjck} \leq x_{ck}$$

Similarly, constraint *1.7* changes into the following:

$$2z_{2,lsmq} \leq y_{ls} + y_{mq},$$

$$z_{2,lsmq} \leq y_{ls}, \qquad\qquad 1.7´$$

$$z_{2,lsmq} \leq y_{mq}$$

Given the maximization in the objective function, the Z-values would be unlimited leading to an unbounded solution. Equations 1.6´ and 1.7´ bound these values and ensure that it is not possible for the Z-values to be higher than the respective binary values. For instance, if both pairs are wrongly placed the Z-values should be zero. If both pairs are correctly placed, the Z-values should be equal to unit, because the left part side in the first constraint is multiplied by 2. If only one pair is correct, such as y_{ls}, there is a conflict between the first and the second constraint. According to the second constraint, $z_{2,lsmq} \leq y_{ls}$, it may be equal to unit, but definitely equal to 0.5 according to $2z_{2,lsmq} \leq y_{ls} + y_{mq}$ constraint. In order for both constraints to be satisfied, it should be at most 0.5. In that case, the respective "markings*time" product has been multiplied by 0.5 and is not maximum. Consequently, it is better if the pair y_{ls} is also incorrect, precisely as y_{mq}, and get $z_{2,lsmq} = 0$, or if it is correct, it should combined with another correct pair and consequently have $z_{2,lsmq} = 1$. And of course, it is not possible to have a value of 0.5 if the pair y_{ls} is not correct.

Restricted formulation

If the two unrestricted formulations above lead to strange formations (for instance by placing one or more forwards as midfielders) the manager will normally reject that. His restriction can be either hard or "soft". If the manager does not want any forward to shift positions with another forward, it is a hard restriction. For instance, if the three forwards are supposed to play in positions 4, 5 and 6 respectively, the hard constraint is simply formulated as:

$$x_{bj} = 1, \qquad \forall (b, j) = 4$$
$$x_{bj} = 1, \qquad \forall (b, j) = 5 \qquad\qquad (a)$$
$$x_{bj} = 1, \qquad \forall (b, j) = 6$$

If the manager did not bother whether his three players shift positions, as long as they remained forwards, that "soft" constraint could be modified to:

$$x_{bj} = 1, \qquad \forall (b, j) = 4, \, or \, 5, or \, 6 \qquad\qquad (b)$$

The manager can of course set other restrictions, on other players. Given the fact that we have a layout of 6 players only, additional constraints would lead to the initial layout. Thus, in order to increase the flexibility of the layout, no other constraints should be set and the restricted formulation should de as case (b), i.e. formulated as a "soft" constraint.

5. **The optimal formations**

The solution of all models is depicted in Tables 1a and Tables 1b. In Table 1a, all objective functions minimize; as a pure min model, (when both functions are multiplied with the quick time matrix), as a mixed 1, where passes are multiplied with the slow time matrix and markings with the quick time matrix, and as mixed 2, where passes are multiplied with the quick time matrix and markings with the slow time matrix; in Table 1b, all objective functions maximize; as a pure max model, (when both functions are multiplied with the slow time matrix), as a mixed 3 (which is similar to mixed 1 in terms of slow and quick time), and as mixed 4 (which is similar to mixed 2 in terms of quick and slow time respectively). The first two columns in both tables show the initial layout, where, 818 is the objective value of "quick" time matrices and 1636 of "slow" time matrices multiplications; the objective value 1191.5 is the same[7] in both mixed 1 and 3 and similarly, 1262.5 is the same objective value in both mixed 2 and 4. Columns "pass", "mark" and "both" show if the respective objective function optimizes only for "time*passes", for "time*markings" and for the sum of "time*passes" and "time*markings". Columns "both & (b)" show when the layout for both "pass" and "mark" must also satisfy the "soft" constraint set by the manager, regarding the position of the three forwards. Letters in bald denote the players who keep their initial positions; letters in italics are the players who change positions.

First, among the six unrestricted "both" models, the layout in (3) is identical to (5) and the layout in (11) is identical to (13). All of them will be probably rejected since either Borriello or Ronaldinho would be playmakers. Similarly among the six restricted "both & (b)" models, the layout in (4) is identical to (8), while (14) is identical to (16). Layout (12) that satisfies the "soft" condition is also almost identical to (14) and (16), since two forwards shift positions. While the unrestricted models lead to diverse layouts, the layouts that satisfy the manager's

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[7] Notice that passes and markings in mixed 1 and 3 are multiplied by "slow" and "quick" time respectively. Consequently, the objective function of the fixed initial layout is unchanged (1191.5), irrespectively if the team minimizes or maximizes. A similar argument applies for the mixed 2 and 4 that share the same value (1262.5).

"soft" condition are close to the initial layout. Thus, the "soft" condition set by the manager was sufficient enough for all six players to keep their initial positions, in models, (12), (14) and (16).

Table 1a: The optimal layout of players (min)

Initial layout		Pure Min				mixed 1		mixed 2	
Place	Player	Pass only (1)	Mark only (2)	Both (3)	Both & (b), (4)	Both (5)	Both & (b) (6)	Both (7)	Both & (b) (8)
1	**AMB**	*BOR*	*PIR*	*PIR*	*SEE*	*PIR*	*SEE*	*BOR*	*SEE*
2	**SEE**	*PIR*	*BOR*	*BOR*	*PIR*	*BOR*	*AMB*	*PIR*	*PIR*
3	**PIR**	*PAT*	*AMB*	*SEE*	*AMB*	*SEE*	**PIR**	*RON*	*AMB*
4	**BOR**	*AMB*	*SEE*	*AMB*	**BOR**	*AMB*	**BOR**	*SEE*	**BOR**
5	**RON**	*SEE*	***PAT***	**RON**	**RON**	**RON**	**RON**	*AMB*	**RON**
6	**PAT**	***RON***	***RON***	**PAT**	**PAT**	**PAT**	**PAT**	**PAT**	**PAT**
Obj.	818 1191.5	351.75	424.25	794.25	804	1146.25	1176.5	1222	1232

Table 1b: The optimal layout of players (max)

Initial layout		Pure Max				mixed 3		mixed 4	
Place	Player	Pass only (9)	Mark only (10)	Both (11)	Both & (b), (12)	Both (13)	Both & (b) (14)	Both (15)	Both & (b) (16)
1	**AMB**	*BOR*	*PAT*	*PIR*	**AMB**	*PIR*	**AMB**	*BOR*	**AMB**
2	**SEE**	*RON*	*BOR*	*RON*	**SEE**	*RON*	**SEE**	*AMB*	**SEE**
3	**PIR**	*AMB*	*SEE*	*AMB*	**PIR**	*AMB*	**PIR**	*SEE*	**PIR**
4	**BOR**	***PAT***	*AMB*	***PAT***	***PAT***	***PAT***	**BOR**	*PAT*	**BOR**
5	**RON**	*PIR*	**RON**	***BOR***	**RON**	***BOR***	**RON**	**RON**	**RON**
6	**PAT**	*SEE*	*PIR*	*SEE*	***BOR***	*SEE*	**PAT**	*PIR*	**PAT**
Obj.	1636 1262.5	832.5	919.5	1730	1710	1278	1256.75	1319.5	1308.25

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Second, there are no big differences in the objective values of mixed models, irrespectively if the manager minimizes or maximizes the sum of both products, while the differences between (4) and (12) are very large, mainly due to the different time values which have been used to multiply the respective pure min and max models. If we use the same quick-time matrix in the maximization model (12) that we used in the minimization model (4), its value reduces to 855, which is about 6.3% higher, and the layout remains as is shown in (12). Similar comparisons of (6) with (14) show a difference of about 6.9% and of (8) with (16) of about 6.2%.

Third, the overall sum from "both" in pure min, is higher than the separate sums from "passes" and "markings" by 18.25´´ (= 794.25 – 424.25 – 351.75), while the overall sum from "both" in pure max, is lower than the two separate sums by 22´´ (= 1730 – 919.5 – 832.5). This is due to the consistency constraint *1.8* that forces players to be in the same vertex of the graph, with respect to both functions.

Fourth, using only one function to optimize, either "time*passes" or "time*markings", the respective layouts are different to each other, different when they are compared to "both" functions and also different compared to the initial one. The "time*passes" layout in the maximization model (9) is more close to (11), because four players are placed in the same position, but none when the "time*markings" model (10), is maximized. On the other hand, model (2) seems to be better than model (10), because in model (10), two forwards should shift positions with midfielders and in model (2) only one.

How high are the efficiency gains of all these optimal layouts? Table 2 shows the efficiency gains when we compare the objective functions from the six unrestricted "both" models, the six restricted ones and the initial layouts. Depending upon the objective function, the efficiency is measured: (a) the quicker the time (in seconds) the higher the efficiency; this applies in the minimization models (when the functions are supposed to finish as fast as possible); (b) the slower the time, the higher the efficiency; this applies in the maximization models (when the functions are supposed to finish as late as possible, since the team wants to waste time until the final whistle). The efficiency gains in the maximization models are in parentheses.

If we compare the six unrestricted "both" models' objective function with the four initial objective functions, we find small improvements of the range of 3 - 6.8%. For instance the unrestricted pure minimum layout (3) is about 24´´ faster (compared to 818´´), and the unrestricted (5) is about 45´´ faster (compared to 1191.5). Thus, the "quick" and "balanced" strategies in the unrestricted models could have been faster by 24-45´´. Similarly, the unrestricted pure maximum (11) is about 1.5´ slower (compared to 1636´) and the unrestricted maximum (13) is also about 1.5´ slower (compared to 1191.5´). Consequently,

the "slow" and "balanced" strategies in the unrestricted models could have been delayed by 1.5´!

If we compare the six unrestricted "both" with the six restricted "both & (b)", the unrestricted layouts are about 10 to 30´´ quicker, respectively 12 to 21´´ slower. For instance, if the manager did not set a "soft" constraint, the passes and markings according to (5), will be completed 30´´ quicker, compared to (6), where his "soft" condition is satisfied. Similarly, if the manager prefers layout (14) over layout (13), the "slow" time strategy is in fact 21´´ quicker than it could have been if (13) was preferred.

Table 2: Efficiency gains in seconds for all "both" models

Min			Max		
Pure	**Mixed 1**	**Mixed 2**	**Pure**	**Mixed 3**	**Mixed 4**
Unrestricted versus Initial					
(3)	**(5)**	**(7)**	**(11)**	**(13)**	**(15)**
24´´	*45´´*	*40´´*	*(94´´)*	*(87´´)*	*(57´´)*
Unrestricted versus Restricted					
(3) – (4)	**(5) – (6)**	**(7) – (8)**	**(11) – (12)**	**(13) – (14)**	**(15) – (16)**
10´´	*30´´*	*10´´*	*(20´´)*	*(21´´)*	*(12´´)*
Restricted versus Initial					
(4)	**(6)**	**(8)**	**(12)**	**(14)**	**(16)**
14´´	*15´´*	*30´´*	*(74´´)*	*(66´´)*	*(45´´)*

And finally, the comparison between restricted and the initial ones is equal to the respective differences above. Even these restricted layouts are more efficient than the initial one. For instance, in order to achieve a "slow" strategy, it is enough if only Pato shifts position with Borriello and all others remain unchanged, i.e. layout (12). That layout is better compared with the intial layout, because both functions will be delayed by 1´and 14´´. Similarly, when team plays the mixed 2 strategy (which is supposed to be as quick as possible), the midfielders need to shift places. Pirlo should be the "playmaker", Seedorf will take Ambrosini's position and Ambrosini will take Pirlo's position, layout (8). The passes and markings in such a formation will be 30´´ quicker than in the initial one.

If the manager prefers the six restricted models, that satisfy his "soft" condition, he has some options. First, depending on the strategy he is interested in, he can select anyone of the six restricted layouts. A second option is to place the players according to how many times these models have placed them in that position. The three forwards will keep their initial positions, i.e. Ronaldinho should definitely take position 5 (placed there in all 6 models), Borriello and Pato should take positions 4 and 6 respectively (both placed in the respective positions in 5 models). But the position of the other three midfielders is less clear. Position 1 should be disputed by Ambrosini and Seedorf (each one is placed there in 3 models). If the team needs to slow down its speed (i.e it maximizes), Ambrosini is the most appropriate; on the other hand, if team minimizes, and needs to pass and mark quickly, Seedorf should be placed there. In position 3, Pirlo is placed there in 3 models (once when the team minimizes and twice when it maximizes), with Ambrosini in second place (twice when the team minimizes); consequently, Pirlo is more appropriate for position 3. Finally, in position 2, Seedorf is most appropriate (is placed there in all three max models), with Pirlo in second place (twice when team minimizes).

To summarize: If AC Milan needs to play a quick game in terms of passes and markings and its manager's "soft" condition must be satisfied, it can use the initial layout with two minor changes, namely, Seedorf and Ambrosini should shift positions with each other. On the other hand, if AC Milan needs to slow down its speed, it can return to the initial layout. Would these layouts make sense to Milan's manager, Leonardo?

6. Conclusions

In this paper I tried to find out the optimal position of football players on the ground, for offensive, defensive and mixed strategies. The optimal layouts are obviously based on the critical assumptions and the quality of data.

The most critical assumptions were the following. The shape of a team's formation during a match remains relatively stable and all players move simultaneously to the same direction where the ball is played. The pair of passes and markings is fixed and independent on the distance of players, or to any position they might be re-placed. The sum of passes and markings is divided equally to both players, irrespectively if a player sends more passes to and receive less from a teammate, or one of the teammates defends better or worse than the other. No other functions will influence a manager's decision.

Regarding the quality of data, it is assumed that the four matches investigated are representative enough and, both passes and markings have been estimated correctly.

Under these assumptions, the unrestricted layouts will be about 3 and 6.8% more efficient compared to the initial one, i.e. between 24´´ and 94´´. These efficiency gains are clearly lower compared to those reported earlier for manufacturing layouts, but not negligible. And if the manager is reluctant to accept that some of his forwards should shift positions with other midfielders, his a priori conditions regarding the position of his forwards will deteriorate the unrestricted optimal layouts by 10´´ to 30´´. Thus, there are still efficiency gains in the restricted layouts of about 14´´ to 74´´, compared to the initial formation of players. Given the fact that some matches have been decided on the last seconds of the game, it is not wise to throw away 14-30´´ by not assigning the appropriate players to the correct position. And it is not clever either sometimes to accomplish the functions of the players 45-74´´ earlier, instead of delaying it, in order to hold the victory until the last second.

There is still room for more research on this area. One can follow various directions, such as: (i) use more functions (such as "fouls committed" and "Shots on goal") to find out more robust layouts; (ii) make the graph asymmetric and the T-matrices as well, by disaggregating the flows of passes and markings into "inflows" and "outflows and perform a kind of sensitivity analysis in the values of all functions to examine how robust these layouts are, in case of measurement errors; (iii) use more observations from both home and away matches, with victories, draws and even defeats; (iv) apply it to the whole team formation and to other formations as well, with or without the manager´s a priori conditions.

REFERENCES

Drira, A., Pierreval, H. & Hajri-Gabouj, S. (2007), Facility layout problems: A survey, *Annual Reviews in Control*, 31, 255-267.

Elshafei, A. (1977), Hospital Lay-out as a Quadratic Assignment Problem, *Operational Research Quartrely*, 28, 167-169.

Francis, R.L., McGinnis L.F.and White J.A. (1992), *Facility Layout and Location: An Analytical Approach*, Prentice-Hall, Englewood Cliffs, NJ.

Hahn, P.M., Kim, B-J., Stützle, T., Kanthak, S., Hightower, W.L., Samra, H., Ding, Z., Guignard, M. (2008), The quadratic three-dimensional assignment problem: Exact and approximate solution methods, *European Journal of Operational Research*, 184, 416-428.

Knowles, J.D., & Corne, D.W. (2002), Towards landscape analyses to inform the design of a hybrid local search for the multiobjective quadratic assignment problem; in: Abraham, A., Ruiz-del-Solar, J., Koppen, M. (Eds.), *Soft Computing Systems:Design, Management and Applications,* IOS Press, Amsterdam, 271-279.

Loiola, E.M., Abreu, N.M.M., Boaventura-Netto, P.O., Querido, T.M. & Hahn, P. (2007), A survey for the quadratic assignment problem, *European Journal of Operational Research*, 176, 657-690.

Nahmias, S. (1997), *Production and Operations Analysis,* 3rd edition, Irwin/McGraw-Hill, *Singapore.*

Papahristodoulou, C. (2008), An analysis of UEFA Champions League match statistics, *International Journal of Applied Sports Sciences*, 20 (1), 67-93.

Papahristodoulou, C. (2003), A simple binary LP model to the facility layout problem, *The Empirical Economics Letters*, 2 (4).

Raman, D., Nagalingam, SV. & Lin, Grier CI. (2009), Towards measuring the effectiveness of a facilities layout, *Robotics and Computer-Integrated Manufacturing*, 25 (1), 191-203.

Schrage, L. (2002), *Optimization Modeling with Lingo,* Lindo Systems, Chicago Illinois.

Sherali, H. & Adams, W. (1999), *A Reformulation-Linearization Technique for Solving Discrete and Continuous NonConvex Problems*, Kluwer Academic Publisher, Dordrecht.

Skiena, S. (1990), *Implementing Discrete Mathematics*, Addison-Wesley, Redwood City, CA.

Tompkins, J.A., White J.A., Bozer, Y.A., Frazelle, E.H., Tanchoco, J.M., & Trevino, J. (1996), *Facilities planning*, Wiley, New York.

Yang, T., & Kuo, C. (2003), A hierarchical AHP/DEA methodology for the facilities layout
design proble design pro, *European Journal of Operational Research*, 147, 128-136.

Internet sources

http://www.fifa.com/
http://www.gazzetta.it/
http://sport.virgilio.it/calcio/serie-a/statistiche/index.html
http://www.uefa.com
http://en.wikipedia.org/wiki/Formation_%28association_football%29
http://en.wikipedia.org/wiki/Association_football_pitch