Abstract

We consider the impact of bundling products on retail merchandising. We consider two broad classes of retail products: basic and fashion. For these product classes, we develop models to calculate the optimal bundle prices, order quantities, and profits under bundling. We use this analysis to establish conditions and insights under which bundling is profitable. Our analysis confirms that bundling profitability depends on individual product demands, bundling costs, and the nature of the relationship between the demands of the products to be bundled. We also provide detailed numerical examples.

Key Words: Retailing, Product Bundling, Pricing, Inventory Management, Basic Products, Fashion Products.
1 Introduction

An increasing number of separate retail products are being bundled together and sold. Examples can be found across a range of products from several industries including food (cans of chicken broth), apparel (tee shirts), cosmetics (shampoo and conditioner), entertainment (concert tickets and cds), and electronics (computers and printers). There are several reasons why retailers bundle products. These include reducing logistic, packaging, and transaction costs, increasing market share and sales, and improving customer service; all of which could eventually contribute to increased profitability. However to realize this potential, it is critical for retailers to determine optimal bundle prices and quantities, and to then decide whether bundling a given set of products would be profitable or not.

In this paper, we address the issues of determining the optimal bundle prices, order quantities, and profits, and of assessing whether or not it is profitable to bundle. We perform this analysis on two broad categories of retail products: basic and fashion. Basic products like men’s underwear and dress shirts typically have more stable, predictable demand and longer life cycles than fashion products like women’s dresses, jewelry, sportswear etc., which are typified by highly unpredictable demand and extremely short life cycles. We also determine how product demand, costs, and the relationship of demand between products affect optimal prices, profits, and the decision to bundle. Finally, we compare the results between basic and fashion products to provide insights into how product category affects bundling decisions.

There have been several streams of research on bundling in the economics and marketing literatures. Starting with Stigler (1960), economists have studied bundling from the perspective of the customer. In Stigler’s (1960) paper, a vector of reservation prices captures customer demand information, and it is assumed that customers choose products that maximize the difference between these reservation prices and product prices. Stigler uses this framework to show that motion picture distributors would prefer to pure bundle or lease movies in multiple movie packages rather than lease individual movies. Adams and Yellen (1976) utilize this framework and consider three different sales strategies: unbundled sales in which the two goods are priced and sold separately, pure bundling in which only a bundle consisting of one unit of each good is sold, and mixed bundling in which both the bundle and the two goods are offered. They discuss the possible implications of switching from one strategy to another, but do not determine the optimal bundle prices...
and conditions under which bundling increases profits. Schmalensee (1984) extends their model by assuming that the distribution of reservation prices follows a bivariate normal distribution. This assumption precludes closed form analytical solutions, and the results are based on a numerical analysis. Also extending the Stigler/Adams and Yellen model, McAfee, McMillan, and Whinston (1989) provide a general sufficient condition for when mixed bundling is optimal in a two-product case. Hanson and Martin (1990) construct and solve a mixed integer program to determine the optimal bundle and prices when the number of customers in each segment and their reservation prices are known.

Recent work in the marketing literature includes papers by Bakos and Brynjolfsson (1999, 2000) that focus on bundling information products. However, as noted in these papers, due to the unique characteristics of information products that have zero or very low marginal costs of production, this approach would not be applicable to retail products. Stremersch and Tellis (2002) provide a comprehensive summary of the marketing literature on bundling, present a framework to classify the various types of bundling, and also discuss the legality of bundling in each case.

Our paper differs from these papers in several aspects. First, we consider bundling from a retail operations-management perspective. We consider two broad classes of retail products: basic and fashion. For basic products, we capture the notion that aggregate demand is stable by assuming that the size of the market is known. We capture the variation in disaggregate demand by modeling consumers’ reservation prices as a uniform random variable. For fashion products, we capture the notion that aggregate demand is unknown by modeling the size of the market as a uniform random variable. Second, in both cases, we provide closed-form analytical solutions to the optimal prices, order quantities, and profits under bundling. Third, we use these results to establish conditions under which it would be profitable for retailers to bundle products. This provides precise guidance on when retailers should consider forming a product bundle. Fourth, we show how individual product costs, individual product demands, bundle costs, and the relationship between the demands of the individual products affect optimal prices, profits, and the decision of whether or not to bundle. Finally, we compare prices and profits for basic and fashion products under bundling. This provides insight to retailers on how the type of product affects the efficacy of bundling.

This paper is organized as follows. In the next section, we present our model for pure bundling
of basic products. We use this model to determine the optimal prices, order quantities, and profits under bundling. (Henceforth, when we use the term ”bundling” we mean ”pure bundling” in which the retailer offers only the bundle or the individual products, but not a combination of both.) We establish conditions under which bundling is profitable and also determine how product costs, demand and the demand correlation between bundled products affects optimal prices, order quantities and the decision to bundle. In Section 3, we develop our model for bundling of fashion products and repeat this analysis. In Section 4, we compare our results between basic and fashion products and provide insight on how product category affects bundling decisions. A detailed numerical example is presented in Section 5. In the concluding section, we summarize our key results and provide future research directions.

2 Basic Products

In this section we introduce the preliminary model and notation. The firm produces two products, A and B, at constant marginal costs $c_A$ and $c_B$, respectively. We assume 0 fixed costs. Demand for each of the products is specified by consumer reservation prices $r_A$ and $r_B$. There is a potential market of size $M$ for each of the products. In this section we consider basic products, such as men’s white dress shirts, black socks, undergarments, etc., and assume that $M$ is known. The assumption that $M$ is known for basic products seems reasonable as often basic products have long life cycles and relatively stable aggregate demand; uncertainty occurs at the disaggregate level due to variability in customer reservation prices. For tractability, the distribution of consumers for product $i$ in reservation price space is assumed to be uniform between $r_{li}^i$ and $r_{ui}^i$. That is, the distribution of reservation prices for product A is uniform between $r_{lA}^A$ and $r_{uA}^A$, and the distribution of reservation prices for product B is uniform between $r_{lB}^B$ and $r_{uB}^B$. We assume without loss of generality that $r_{lA}^A = 0$ and $r_{uA}^A = 1$. Of course, $r_{lB}^B \geq 0$, $c_A \leq 1$, and $c_B \leq r_{uB}^B$. Finally, we assume $r_{uB}^B \leq 1$.

2.1 No Bundle

For a baseline comparison, we first determine the optimal prices for the case with no bundling. If the firm sets prices $p_A$ and $p_B$, then all customers with reservation prices above $p_A$ and $p_B$,
respectively, purchase the products. Thus, the demands \( D_A \) and \( D_B \) when the market for the products has size \( M \) are given by:

\[
D_A = M \int_{p_A}^{1} dx = M(1 - p_A), \quad \text{and}
\]

\[
D_B = M \int_{p_B}^{r^u_B} \frac{1}{r^u_B - r^l_B} dx = M \left( \frac{r^u_B - p_B}{r^u_B - r^l_B} \right). \tag{2}
\]

The firm’s objective is to choose prices \( p_A \) and \( p_B \) in order to maximize the profit function \( \pi_{(A,B)}(p_A, p_B) \):

\[
\pi^*_A(p_A, p_B) = \max_{p_A, p_B} M \left[ (p_A - c_A)(1 - p_A) + (p_B - c_B) \left( \frac{r^u_B - p_B}{r^u_B - r^l_B} \right) \right]. \tag{3}
\]

Solving for the optimal prices, quantities, and profit yields

\[
p^*_A = \frac{1 + c_A}{2} \geq c_A, \tag{4}
\]

\[
p^*_B = \frac{r^u_B + c_B}{2} \geq c_B, \tag{5}
\]

\[
Q^*_A = M \left( \frac{1 - c_A}{2} \right), \tag{6}
\]

\[
Q^*_B = M \left( \frac{r^u_B - c_B}{2(r^u_B - r^l_B)} \right), \quad \text{and}
\]

\[
\pi^*_A(p_A, p_B) = M \left[ \left( \frac{1 - c_A}{2} \right)^2 + \left( \frac{1}{r^u_B - r^l_B} \right) \left( \frac{r^u_B - c_B}{2} \right)^2 \right]. \tag{8}
\]

The optimal price of an unbundled good is increasing in the associated cost, and the profit function decreases in the costs.

### 2.2 Pure Bundle – Independent Demands

If the firm offers only the bundled product \( AB \) at price \( p \), a consumer will wish to purchase the bundle only if \( r_A + r_B \geq p \); that is, if the sum of his reservation prices exceeds the bundle price. Let \( D_{AB} \) be the demand of consumers in the market of size \( M \) whose sum of reservation prices exceeds the bundle price \( p \). Then, by considering the fact that \( r_A \sim U[0, 1] \) and \( r_B \sim U[r^l_B, r^u_B] \), we have:

\[
D_{AB} = M \int_{\max(p - 1, x_B)}^{r^u_B} \left( \int_{\max(p - x_B, 0)}^{1} dx_A \right) \frac{1}{r^u_B - r^l_B} dx_B, \tag{9}
\]
\[
\begin{align*}
E &= M \left\{ \begin{array}{ll}
1 - \left( \frac{1}{r_B - r_H} \right) \frac{(p - r_H)}{2} & \text{if } p < r_B^u \\
1 + \left( \frac{r_B^u + r_H}{2} \right) - p & \text{if } r_B^u \leq p \leq 1 + r_B^l \\
\frac{1}{r_B - r_H} \frac{(1 - p + r_H)}{2} & \text{if } p > 1 + r_B^l.
\end{array} \right.
\end{align*}
\]

As can be seen from the above equation, the changes in the formula representing the demand for the bundled good are generated by the changes in the limits of integration. Let \( c_{AB} \) be the cost to the firm of the bundled product. The firm chooses the bundle price \( p \) to maximize profits \( \pi_{AB}(p) \):

\[
\pi_{AB}^* = \max_p (p - c_{AB}) D_{AB}. \quad (11)
\]

For simplicity in what follows, we will restrict the bundle cost \( c_{AB} \) so that the optimal bundle price falls in the middle of the above three regions. That is, we want \( r_B^u \leq p \leq 1 + r_B^l \), though similar results hold when \( p \) falls in the other regions. To achieve the condition on \( p \), we assume \( 1.5r_B^u - 0.5r_B^l - 1 \leq c_{AB} \leq 1 + 1.5r_B^l - 0.5r_B^u \). Solving for the optimal bundle price \( p_{AB}^* \), the optimal quantity \( Q_{AB}^* \), and substituting in the profit function yields

\[
\begin{align*}
p_{AB}^* &= \left[ \left( 1 + \frac{r_B^u + r_B^l}{2} \right) + c_{AB} \right] / 2, \quad (12) \\
Q_{AB}^* &= M \left\{ \left( 1 + \frac{r_B^u + r_B^l}{2} \right) - c_{AB} \right\} / 2, \quad \text{and} \quad (13) \\
\pi_{AB}^* &= M \left\{ \left( 1 + \frac{r_B^u + r_B^l}{2} \right) - c_{AB} \right\} ^2. \quad (14)
\end{align*}
\]

Note that our assumption on \( c_{AB} \) results in \( Q_{AB}^* \geq 0 \), and there are positive profits. As in the case with no bundling, an increase in the cost leads to an increase in the optimal price and a decrease in the optimal profit. Our interest, though, is in a comparison of the prices and profits with and without bundling. Ordering the quantities is problematic, however. Recall that \( Q_A^* \) is units of \( A \) produced and \( Q_B^* \) is units of \( B \) produced in the unbundled state. On the other hand, \( Q_{AB}^* \) is the bundled units, each including one unit of \( A \) and one unit of \( B \). Therefore, unlike with the prices, it makes no sense to compare the bundle quantity with the sum of the unbundled quantities. We should (and will) analyze the ordering of the profits rather than the (intermediate) quantities.

**Proposition 1** Assume demands for the two products are independent and there are no diseconomies of bundling; i.e., \( c_{AB} \leq c_A + c_B \). Then

\[ I : \quad p_{AB}^* < p_A^* + p_B^*. \]
and

$$II: \pi_{AB}^* > \pi_{(A,B)}^* \iff (1 - c_A)^2 + \frac{(r_u^B - c_B)^2}{r_u^B - r_l^B} < \left(1 + \frac{r_u^B + r_l^B}{2} - c_{AB}\right)^2.$$  

Furthermore, if the two products have identically distributed reservation prices, $r_A \sim U[0, 1]$ and $r_B \sim U[0, 1]$, and $c_{AB} = c_A + c_B$, then

$$III: \pi_{AB}^* < \pi_{(A,B)}^*.$$  

All proofs are provided in the Appendix.

The claim in Part (I) of Proposition 1 is to be expected when there are economies to bundling as a reduction in the aggregate cost allows for a concomitant reduction in the bundle price while maintaining profit margins. Part (I) shows that even with no economies of bundling, the optimal bundle price is lower than the sum of the optimal prices of the individual goods without bundling. As noted below, this is partly explained by an increase in quantity sold. Because the optimal bundle price $p_{AB}^*$ is continuous in the cost $c_{AB}$, there must also be cases with small dis-economies of bundling, i.e., when $c_{AB} > c_A + c_B$, where the optimal bundle price is less than the sum of the prices of the unbundled products.

The condition in Part (II) of Proposition 1 warrants further discussion. Assume that there are no economies to bundling, i.e., $c_{AB} = c_A + c_B$, fix $c_{AB} = c$, and consider the condition as a function of $x = c_B$. Then bundling is optimal if and only if

$$(1 - (c - x))^2 + \frac{(r_u^B - x)^2}{r_u^B - r_l^B} < \left(1 + \frac{r_u^B + r_l^B}{2} - c\right)^2. \tag{15}$$

The left-hand side of this condition is quadratic in $x$, while the right-hand side is a constant. If the minimum of the quadratic function on the left (which occurs when $x = [r_u^B + (c - 1)(r_u^B - r_l^B)]/[1 + (r_u^B - r_l^B)]$) is less than the value on the right, then there is a range of values for $c_A$ and $c_B$ where bundling is optimal. If the minimum of the quadratic function on the left is greater than the value on the right, then bundling is not optimal for any costs. We provide a numerical example in a later section. For the special case considered in Part (III), with no economies to bundling, $r_l^B = 0$, $r_u^B = 1$, and $c_{AB} = 0.5$, the minimum of the quadratic function is attained at $x = 0.25$. It is then easy to check that the left-hand side is always greater than the right; hence, bundling is not optimal.
2.3 Pure Bundle – Perfectly Correlated Demands

While in the previous section we analyzed the case where the distributions of reservation prices for the two products were independent, in this section we analyze the case where the distributions of reservation prices are perfectly correlated. The analysis of the unbundled case is exactly as in section 2.1, as that section did not use the independence of the demands. We begin with the positively correlated case:

\[ r_A \sim U[0,1], \quad \text{and} \quad r_B = [r_A^u + (r_A^u - r_A^l)r_A^l] \sim U[r_B^l, r_B^u]. \]  \hspace{1cm} (16)

The reservation prices are perfectly positively correlated if, for example, the products are complements and the preference for the second item is determined by the preference for the first (shampoo and conditioner); or the products are identical (bars of soap or sets of sport socks), and if an individual wants one unit, they would have the same preference for a second unit. For the case where the firm offers only the bundled product at price \( p \), consumers whose sum of reservation prices for the individual products exceeds \( p \) will purchase the product. That is, if \( r_A + r_B \geq p \) the consumer will choose to purchase the bundled good. Substituting from (16) above, the purchase condition reduces to one on \( r_A \) only: the consumer will choose to purchase the bundled good if and only if

\[ r_A \geq \max \left( 0, \frac{p - r_B^l}{1 + r_B^u - r_B^l} \right). \]  \hspace{1cm} (17)

Let \( \hat{D}_{AB} \) be the demand of consumers in the market who will choose to purchase. When \( p \leq r_B^l \), \( D_{AB} = M \) and the optimal \( p^* = r_B^l \). Hence, it suffices to consider the case with \( p \geq r_B^l \), whence

\[ \hat{D}_{AB} = M \left[ \left( 1 + \frac{r_B^l}{1 + r_B^u - r_B^l} \right) - \left( \frac{1}{1 + r_B^u - r_B^l} \right) p \right]. \]

The first order condition yields the optimal price \( \hat{p}^* \), quantity \( \hat{Q}_{AB}^* \), and profits \( \hat{\pi}_{AB}^* \) as follows:

\[ \hat{p}_{AB}^* = \frac{(1 + r_B^u + c_{AB})}{2}, \]  \hspace{1cm} (18)
\[ \hat{Q}_{AB}^* = M \left( \frac{1 + r_B^u - c_{AB}}{2(1 + r_B^u - r_B^l)} \right) \quad \text{and} \]  \hspace{1cm} (19)
\[ \hat{\pi}_{AB}^* = \frac{M}{1 + r_B^u - r_B^l} \left[ \frac{1 + r_B^u - c_{AB}}{2} \right]^2. \]  \hspace{1cm} (20)

Note that the optimal bundle price for the positively dependent case (18) is strictly greater than the optimal bundle price for the independent case (12).
Proposition 2 Assume demands for the two products are perfectly positively correlated. If there are no economies or dis-economies to bundling, $c_{AB} = c_A + c_B$, then

$$I : \hat{p}_{AB}^* = p_A^* + p_B^*$$

and

$$II : \hat{\pi}_{AB}^* \leq \pi_{(A,B)}^*.$$ 

Unlike as in the uncorrelated demands case, when reservation prices are perfectly positively correlated, Proposition 2 part (I) shows that the bundled good is neither discounted nor marked up relative to the prices of the unbundled goods. It follows then that $\hat{p}_{AB}^* > p_{AB}^*$, that is, the price of the bundled good when demands are perfectly positively correlated is greater than the price of the bundled good when demands are uncorrelated. When the demands are perfectly positively correlated, if a consumer purchases one good, they will certainly purchase the other; and if they do not purchase one good, they will not purchase the other. Hence, a discounting of the bundle price yields no benefit. Furthermore, part (II) of Proposition 2 proves that bundle profits in the positively correlated case (20) are strictly less than unbundled profits (8). To finish the comparisons, $\hat{\pi}_{AB}^* < \pi_{AB}^*$ if and only if $c_{AB} < 1 - \sqrt{0.5}$; that is, the bundle profits in the positively correlated case are less than the bundle profits in the independent case as long as aggregate costs are small.

The reservation price $r_A + r_B$ for the bundled good in the independent case has lower variance than in the positively correlated case and the means are the same. When costs are low, prices are low, but the independent demand case makes up for it in quantity sold.

Now consider the case of perfectly negatively correlated reservation prices:

$$r_A \sim U[0,1], \text{ and } r_B = [r_B^u - (r_B^u - r_B^l) r_A] \sim U[r_B^l, r_B^u]. \tag{21}$$

For instance, suppose that the goods are close substitutes, for example, a DVD and a VHS tape of the same movie, and once a consumer has one of the products, the demand for the other goes down. Again, the analysis of the unbundled case remains unchanged. But for the bundle, the sum of the reservation prices exceeds the bundle price if

$$r_A \geq \max \left\{0, \frac{p - r_B^u}{1 - (r_B^u - r_B^l)} \right\}.$$ 

When $p \leq r_B^u$, $D_{AB} = M$, and the optimal price is $p^* = r_B^u$. As previously, we are interested in the case when $p$ is not at the boundary, i.e., $p \geq r_B^u$. Solving for the optimal price, quantity, and
profits as before:

$$
\tilde{p}_{AB}^* = \frac{1 + r_B^l + c_{AB}}{2},
$$

(22)

$$
\tilde{Q}_{AB}^* = M \left( \frac{1 + r_B^l - c_{AB}}{2[1 - (r_B^u - r_B^l)]} \right)
$$

and

(23)

$$
\tilde{\pi}_{AB}^* = \frac{M}{1 - (r_B^u - r_B^l)} \left[ \frac{1 + r_B^l - c_{AB}}{2} \right]^2.
$$

(24)

**Proposition 3** Assume demands for the two products are perfectly negatively correlated. If there are no economies or dis-economies to bundling, i.e., $c_{AB} = c_A + c_B$, then

$I : \tilde{p}_{AB}^* < p_A^* + p_B^*$.

If, in addition, $r_B \sim U[d, 1]$ for $d > 0$, then

$$IIa : \tilde{\pi}_{AB}^* > \pi_{(A,B)}^* \iff \frac{1}{d} \left( \frac{1 + d - c_A - c_B}{2} \right)^2 > \left( \frac{1 - c_A}{2} \right)^2 + \frac{1}{1 - c} \left( \frac{1 - c_B}{2} \right)^2;$$

Alternatively, if $r_B \sim U[0,d]$ for $d < 1$, then

$$IIb : \tilde{\pi}_{AB}^* > \pi_{(A,B)}^* \iff \frac{1}{1 - d} \left( \frac{1 - c_A - c_B}{2} \right)^2 > \left( \frac{1 - c_A}{2} \right)^2 + \frac{1}{d} \left( \frac{d - c_B}{2} \right)^2.$$ 

Proposition 3 part (I) shows that for the perfectly negatively correlated demands, the bundle price offers a discount over the unbundled prices: $\tilde{p}_{AB}^* < p_A^* + p_B^*$. Note the difference with the positively correlated case, highlighting the dictum that the firm must have an understanding of the relationship between the demands for the two products.

Because the bundle profits in the perfectly positive correlated case with no economies to bundling are strictly less than the unbundled profits, a first intuition might lead one to believe that the opposite would be true for perfect negative correlation; that is, $\tilde{\pi}_{AB}^* \geq \pi_{(A,B)}^*$. Parts (IIa) and (IIb) of Proposition 3 shows that this intuition is wrong: unlike the positively correlated case, there is not an unambiguous ordering of profits. Because the bundle price is lower with negative correlation, the firm must have a more-than-offsetting increase in volume if profits are to increase.

To summarize, with no economies to bundling and perfectly positive correlated reservation prices, the optimal bundle price is equal to the sum of the unbundled prices, and bundle profits are lower than unbundled profits. The firm would not bundle. With no economies to bundling and
perfect negative correlation, the optimal bundle price is lower than the sum of the unbundled prices, and there are cases where the bundle profits exceed the unbundled profits and alternative cases where the opposite holds. The firm may, but need not, find it optimal to offer the bundle. With no economies to bundling and independent reservation prices, again the bundle price is less than the sum of the unbundled prices, and again, there are cases where the bundle profits exceed the unbundled profits and alternative cases where the opposite holds. Similar results for bi-variate normally distributed reservation prices have been derived by Schmalensee (1984). But the intractability of the resulting model does not permit analytical comparisons with economies of scale.

The economy to bundling is given by $c_A + c_B - c_{AB}$. Greater economies to bundling leads to greater bundle profits, increasing the range wherein bundling is optimal. Similarly, greater diseconomies to bundling leads to lower bundle profits, shrinking the range wherein bundling is optimal. The above results highlight the importance of economies to bundling to the bundling decision, that is, the bundling decision is as much an operational decision as it is a marketing one. Next, we consider the case of fashion products.

### 3 Fashion Products

In this section we generalize from the previous assumption that the market size $M$ is known, substituting instead an assumption about the distribution of the possible (random) market size $\mathcal{M}$. This can be expected for fashion products for which aggregate demand is often unknown and highly variable. If the market size (i.e., demand) is random but decreasing in price, and the firm makes its bundling (i.e., production) decision before demand is known, we have, in essence, a newsboy model with pricing, a model well known to be nearly intractable analytically. To overcome this near intractability, we make several simplifying assumptions. Rather than introduce a new set of notation for this section, we reuse earlier notation: the meaning should be clear from the context. For example, $\pi^*_{AB}$ will represent the optimal profit from bundling with independent demands, in the current section with the additional assumption that the market size is unknown, while in the previous section with the assumption that market size is known.

As per a newsboy model, let $Q_i$ be the order quantity and $D_i$ the demand distribution of product
\( i = A, B, \) and \( AB. \) (While it is straightforward to also include a salvage value \( s_i, \) for ease of notation, we assume the salvage value is 0 for all products.) If the firm produces \( Q \) units of a good at a unit cost of \( c \) for sale at price \( p \) when realized demand is \( x, \) then the firm’s profits \( \pi(Q, x) \) are given by

\[
\pi(Q, x) = (p - c) \min\{Q, x\} - c(Q - x)^+.
\]

Expected profits \( \pi(Q) = E_X\pi(Q, X) \) when demand \( X \) has distribution \( F \) are maximized by the familiar

\[
Q^* = F^{-1}\left(\frac{p - c}{p}\right),
\]

\[
\pi^* = p \int_0^{Q^*} x dF(x).
\]

If demand is uniformly distributed in the interval \([0, 2d(p)]\) where \( d(p) \) is some downward sloping function of price, equations (25) and (26) become

\[
Q^*(p) = 2d(p)\left(\frac{p - c}{p}\right),
\]

\[
\pi^*(p) = d(p)\left(\frac{(p - c)^2}{p}\right).
\]

That is, if price is given, the optimal order quantity is \( Q^*(p) \) and the optimal profits are \( \pi^*(p) \) as given in (27) and (28). Recall, however, we allow the firm to optimize over price. For example, assume demand is linear in price, i.e.,

\[
d(p) = \delta - \alpha p,
\]

for some scalars \( \delta \) and \( \alpha, \) where \( \delta \) represents the size of the potential market and \( \alpha \) the price sensitivity.

**Proposition 4** Assume demand is linear in price. Then the profit function (28) is unimodal in price.

In light of Proposition 4, there is a unique price \( p^* \) that maximizes profits. It can be found by substituting (29) in (28) and solving for the first order condition to give

\[
p^* = \frac{\delta + \sqrt{\delta^2 + 8\alpha \delta c}}{4\alpha}.
\]
Note that the optimal price is increasing in the market size $\delta$, decreasing in the price sensitivity $\alpha$, and increasing in the cost $c$. Substituting the optimal price (30) into the quantity (27) and profit (28) equations yields:

\begin{align*}
Q^* &= \left(\frac{3\delta - \sqrt{\delta^2 + 8\alpha\delta c}}{2}\right) \left(1 - \frac{4\alpha c}{\delta + \sqrt{\delta^2 + 8\alpha\delta c}}\right) \\
\pi^* &= \frac{1}{4} \left(3\delta - \sqrt{\delta^2 + 8\alpha\delta c}\right) \left(\frac{\delta + \sqrt{\delta^2 + 8\alpha\delta c} - 4\alpha c}{4\alpha(\delta + \sqrt{\delta^2 + 8\alpha\delta c})}\right).
\end{align*}

(31) (32)

Notice that both the optimal order quantity $Q^*$ and the optimal profit $\pi^*$ are decreasing in the cost $c$. A numerical example in an upcoming section will confirm this.

In order to draw a parallel with the analysis of the previous section on basic products, we assume that the market size $\mathcal{M}$ is uniformly distributed in the range $[0, 2M]$ where $M$ is defined as before. Thus, $E(\mathcal{M}) = M$. When market size is known, demand for product $A$ is characterized in (1), for product $B$ in (2), and for the bundle $AB$ in (10). With uncertain market size, we extend the demand characterizations in (1), (2), and (10), and define new demands via:

\begin{align*}
D_A &= \mathcal{M}(1 - p_A) \\
D_B &= \mathcal{M}(r_B^u - p_B)/(r_B^u - r_B^l) \quad \text{and} \\
D_{AB} &= \mathcal{M}\left(1 + \frac{r_B^l + r_B^u}{2} - p_{AB}\right).
\end{align*}

(33) (34) (35)

As in (10), the demand $D_{AB}$ for the bundled good given by (35) assumes the demands for the individual goods are independent of each other, and the cost of the bundled good is in a middle range. Note that (33), (34), and (35), all have the linear form of (29). That is, for product $A$, $\delta_A = M$ and $\alpha_A = M$; for product $B$, $\delta_B = Mr_B^u/(r_B^u - r_B^l)$ and $\alpha_B = M/(r_B^u - r_B^l)$; and for the bundled product $AB$, $\delta_{AB} = M[1 + ((r_B^l + r_B^u)/2)]$ and $\alpha_{AB} = M$.

### 3.1 No Bundle

If the firm only offers the individual products, $A$ and $B$, with random demand for the goods, $D_A$ and $D_B$, given by (33) and (34), respectively, substitution in the optimal price, quantity, and profit equations (30) – (32) yields the following values for product $A$:

\[ p_A^* = \frac{1 + \sqrt{1 + 8c_A}}{4} \] 

(36)
\[
Q_A^* = \frac{M}{2} \left( 3 - \sqrt{1 + 8c_A} \right) \left( 1 - \frac{4c_A}{1 + \sqrt{1 + 8c_A}} \right) \quad (37)
\]
\[
\pi_A^* = \frac{M}{4} \left( 3 - \sqrt{1 + 8c_A} \right) \left[ \frac{(1 + \sqrt{1 + 8c_A} - 4c_A)^2}{4(1 + \sqrt{1 + 8c_A})} \right]. \quad (38)
\]

Similarly for product B:
\[
p_B^* = \frac{r_u^B + \sqrt{(r_u^B)^2 + 8r_u^B c_B}}{4} \quad (39)
\]
\[
Q_B^* = \frac{M}{2(r_u^B - r_l^B)} \left( 3r_u^B - \sqrt{(r_u^B)^2 + 8r_u^B c_B} \right) \left( 1 - \frac{4c_B}{r_u^B + \sqrt{(r_u^B)^2 + 8r_u^B c_B}} \right) \quad (40)
\]
\[
\pi_B^* = \frac{M}{4(r_u^B - r_l^B)} \left( 3r_u^B - \sqrt{(r_u^B)^2 + 8r_u^B c_B} \right) \left[ \frac{(r_u^B + \sqrt{(r_u^B)^2 + 8r_u^B c_B} - 4c_B)^2}{4\left( r_u^B + \sqrt{(r_u^B)^2 + 8r_u^B c_B} \right)} \right]. \quad (41)
\]

At this point all that can be said is that, as would be expected, the optimal prices \( p_A^* \) and \( p_B^* \) are increasing, and the optimal quantities \( Q_A^* \) and \( Q_B^* \) and optimal profits \( \pi_A^* \) and \( \pi_B^* \) are decreasing, in the costs \( c_A \) and \( c_B \), respectively. The unbundled prices, quantities, and profits are important, however, for a comparison with the bundled values.

### 3.2 Pure Bundle – Independent Demands

If the firm only offers the bundled good \( AB \) produced at cost \( c_{AB} \), with \( 1.5r_u^B - 0.5r_l^B - 1 \leq c_{AB} \leq 1 + 1.5r_l^B - 0.5r_u^B \), and with random demand \( D_{AB} \) given in (35), substitution in (30) – (32) yields the following for the optimal bundle price,
\[
p_{AB}^* = \frac{1}{4} \left[ \left( 1 + \frac{r_l^B + r_u^B}{2} \right) + \sqrt{\left( 1 + \frac{r_l^B + r_u^B}{2} \right)^2 + 8 \left( 1 + \frac{r_l^B + r_u^B}{2} \right) c_{AB}} \right], \quad (42)
\]
the optimal order quantity,
\[
Q_{AB}^* = \frac{M}{2} \left[ 3 \left( 1 + \frac{r_l^B + r_u^B}{2} \right) - \left( 1 + \frac{r_l^B + r_u^B}{2} \right)^2 + 8 \left( 1 + \frac{r_l^B + r_u^B}{2} \right) c_{AB} \right] \left[ 1 - \frac{4c_{AB}}{\left( 1 + \frac{r_l^B + r_u^B}{2} \right) + \sqrt{\left( 1 + \frac{r_l^B + r_u^B}{2} \right)^2 + 8 \left( 1 + \frac{r_l^B + r_u^B}{2} \right) c_{AB}}} \right], \quad (43)
\]
and the optimal profits,
\[
\pi_{AB}^* = \frac{M}{4} \left[ 3 \left( 1 + \frac{r^l_B + r^u_B}{2} \right) - \sqrt{\left( 1 + \frac{r^l_B + r^u_B}{2} \right)^2 + 8 \left( 1 + \frac{r^l_B + r^u_B}{2} \right) c_{AB}} \right]
\]
\[
\left( 1 + \frac{r^l_B + r^u_B}{2} \right) + \sqrt{\left( 1 + \frac{r^l_B + r^u_B}{2} \right)^2 + 8 \left( 1 + \frac{r^l_B + r^u_B}{2} \right) c_{AB}} \right)^2 \right)
\]
\[
(44)
\]
As in the unbundled case, the optimal price is increasing and the optimal profit is decreasing in the cost of production.

### 3.3 Comparing No Bundling with Pure Bundling

So far, we have derived price, order quantities, and profits for the separate cases of no bundling and pure bundling. We next compare prices and profits between no bundling and pure bundling. As in the case of known demand, comparing quantities without additional assumptions is problematic, and so we focus on profits.

In the previous section on basic products with known market size we proved that when reservation prices for the individual products are independent, the optimal bundle price is less than the sum of the unbundled prices. A similar result holds for fashion products when the market size is unknown.

**Proposition 5** Assume market size is random and demands for the two products are independent. If there are no economies to bundling, i.e., \(c_{AB} = c_A + c_B\), and the range of reservation prices for product B is \(U[0, 1]\). Then
\[
I : \quad p_{(A,B)}^* < p_A^* + p_B^*.
\]
If, in addition, the costs of the individual products are the same, i.e., \(c_A = c_B\), then
\[
II : \quad \pi_{AB}^* < \pi_{(A,B)}^*.
\]
As the selling price for the bundle is lower than the sum of the unbundled prices when there are no economies to bundling, in order for bundling to be a profitable undertaking, it must be that
the firm “increases” the quantity sold. Proposition 5 shows that it cannot increase the quantity sold enough to compensate for the lower price. Moreover, Proposition 5 mirrors the results of Proposition 1.

4 Comparison of Known with Uncertain Market Size

Our specification of the demand for the products with random market size is such that the expected market size \( E(M) \) equals the market size \( M \) in the non-random case. It is reasonable to ask, then, what effect uncertainty has on optimal prices and profits. First consider the case of the individual products.

**Proposition 6**

I: The prices of the individual basic products with known market size are lower than that of the individual fashion products with unknown market size.

II: The profits of the individual basic products with known market size are greater than the individual fashion products with unknown market size.

Part I can be expected as the variability associated with unknown market size does not have to be covered, thus allowing the possibility of lower prices. This is consistent with practice, where basic products are usually priced lower than comparable fashion products. This lower price is offset by an increase in demand, in turn, increasing overall profits.

Now consider what happens to the bundle.

**Proposition 7**

I: The price of the bundled basic product with known market size is lower than the price of the bundled fashion product with unknown market size.

II: The profit of the bundled basic product with known market size is greater than the profit of the bundled fashion product with unknown market size.

Proposition 7 shows that the results of Proposition 6 carries over to the bundled product. This is because, here again, the variability associated with the unknown market size does not have to be covered. This in turn leads to lower prices, which is offset by an increase in demand. This eventually increases overall profits.
5 Numerical Examples

In this section, we construct numerical examples to better illustrate the ideas developed in this paper. The first example relates to basic products. Here, we let $M = 100$, $r_A \sim U[0, 1]$ and $r_B \sim U[0, 1]$, $c_A = c_B = 0.25$ and $c_{AB} = c_A + c_B = 0.5 = c$. Substituting these values in (4) to (8), (12) to (14) and (17) to (19), we get the optimal prices, order quantities and profits under no bundling and bundling with independent demand and perfectly positive correlated demand. These results are summarized in Table 1. As expected, by Proposition 1, these results confirm that $p^*_A = 1.05 < (p^*_A + p^*_B) = 0.625 + 0.625$. In addition, since $c = 0.5$, $\pi^*_A = 25 < \pi^*_A, B = 28.125$. Finally, as implied by Proposition 2, we have $\tilde{p}^*_A = 1.25 = p^*_A + p^*_B$ and $\tilde{\pi}^*_A = 28.125 \leq \pi^*_A, B = 28.125$.

INSERT TABLE 1 HERE

To consider the impact of perfectly negatively correlated demand, we set $r_A \sim U[0, 1]$ and $r_B \sim U[0.2, 1]$ and recomputed the optimal price, order quantities and profits under no bundling and the various cases of bundling (i.e., independent, positive and negative dependent demand). These results are summarized in Table 2. Consistent with Proposition 3, these results show that $\tilde{p}^*_A = 0.85 < p^*_A + p^*_B = 1.25$ and $\tilde{\pi}^*_A = 61.25 > \pi^*_A, B = 31.64$. Finally, in this example, observe that $\tilde{\pi}^*_A = 61.25 > \tilde{\pi}^*_A = 31.25 > \pi^*_A = 30.25$.

INSERT TABLE 2 HERE

To illustrate the impact of costs on optimal prices and profits, we set $r_A \sim U[0, 1]$ and $r_B \sim U[0.2, 1]$, $x = c_A = c_B$ and $c_{AB} = c_A + c_B$ and vary $x$ from 0.2 to 0.4 in increments of 0.02. We consider 4 scenarios, namely, no bundling, bundling with independent demands, positively correlated demands and negatively correlated demands. The impact of these changes on optimal prices are summarized in Figure 1, while the impacts on optimal profits are summarized in Figure 2. Figure 1 shows that, as expected, prices are increasing in costs. In addition, this figure shows that prices under bundling with negative correlated and independent demands are lower than prices of the sum of the individual products, results that are consistent with Propositions 1 and 3, respectively. In addition, as expected from Proposition 2, prices under positively correlated demand are always equal to the prices of the unbundled quantities. Figure 2 shows the impact of costs on profits for the various scenarios: no bundling, bundling under independent, perfectly
positive and negatively correlated demands. This figure illustrates that when costs are relatively low (i.e., \(0.2 < c < 0.4\)) the optimal strategy is to bundle perfectly negatively correlated products. From this analysis, we also confirmed that profit from bundling under perfectly positive correlated demands is dominated by not bundling.

INSERT FIGURES 1 AND 2 HERE

The second example deals with fashion products. Here again, we let \(M = 100\), \(r_A \sim U[0,1]\) and \(r_B \sim U[0,1]\), \(c_A = c_B = 0.25\) and \(c_{AB} = c_A + c_B = 0.5\). Substituting these values in (35) to (43), we get the optimal prices, order quantities and profits under no bundling and bundling with independent demand. These results are consistent with Proposition 5: \(p^*_{AB} = 1.09 < (p^*_A + p^*_B) = (0.68 + 0.68)\) and \(\pi^*_{AB} = 13.09 < \pi^*_{(A,B)} = 17.4\).

To analyze the impact of reservation price variability on optimal prices and profits, we set \(r_A \sim U[0,1]\) and \(r_B \sim U[0,r^*_B]\), \(c = c_A = c_B = 0.15\) and \(c_{AB} = c_A + c_B\) and vary \(r^*_B\) from 0 to 0.8 in increments of 0.1. These results for optimal prices and profits are summarized in Figures 3 and 4, respectively, showing that \(p^*_{AB} < (p^*_A + p^*_B)\) and \(\pi^*_{AB} < \pi^*_{(A,B)}\) across the entire range of \(r^*_B\). This is expected from Proposition 5. In addition, note that prices are increasing in \(r^*_B\). This is because it is optimal for the retailer to increase prices to extract the surplus from customers who have higher reservation prices. However, Figure 3 also shows that the rate of increase of optimal price varies with and without bundling. It is higher for no bundling as the retailer can increase the price on one product without affecting the chances that the customer would buy the other product. The rate of increase is smaller for the bundled product, as the increase in price for one product, affects the chance the customer would buy the other product due to the higher bundled price. Similarly, observe from Figure 4 that profits are increasing in \(r^*_B\). This is because the gain in prices offsets the loss in sales due to higher prices, which in turn increases overall profits. Since the rate of increase in prices differ with and without bundling, so does the rate of increase in profits as shown in Figure 4.

INSERT FIGURES 3 AND 4 HERE

Finally, by considering the case in which the demands are independent, we compared prices and profits between basic and fashion products when \(M = 100\), \(r_A \sim U[0,1]\) and \(r_B \sim U[0,1]\), \(c = c_A = c_B = 0.25\) and \(c_{AB} = c_A + c_B = 0.5\). These results are summarized in Table 3 and
confirm Proposition 6. For instance, prices of individual basic products are lower than prices of the corresponding fashion products, while profits from the individual basic products are higher. In addition, as expected from Proposition 7, prices of the bundled basic good are lower than the bundled fashion good, while profits of the basic bundle are higher.

INSERT TABLE 3 HERE

6 Conclusion

In this paper, we have analyzed the impact of product bundling on retail merchandising. To perform this analysis, we considered two broad classes of retail products: basic and fashion. For basic products, demand is specified by uniformly distributed reservation prices and also assumes that the market size is known. For fashion products, we generalize from this assumption and treat the market size to be random. For both basic and fashion products, we consider pairs of products and develop models to calculate the optimal bundle prices, order quantities and profits under bundling. Our analysis confirms that bundling profitability depends on individual product demands, bundling costs and the nature of the relationship between demands of the products to be bundled. We also provide detailed numerical examples.

For basic products, we find that with no economies to bundling and when product demands are independent or perfectly negatively correlated, the optimal bundle price is lower than the sum of the unbundled prices. We also show that there are cases where bundled profits exceed unbundled profits and alternative cases where the opposite holds. Therefore, the firm might or might not bundle. For products with perfectly positive correlated demand, bundle prices are equal to the sum of unbundled prices and profits are always lower, and therefore, the firm should not bundle. Thus, the firm needs to carefully understand costs, demand and the relationship between demands between the products before deciding to bundle. Our paper provides a framework to determine if bundling is profitable, and also to determine optimal prices, order quantities and profits under bundling. Our numerical example shows how prices and profits are affected by costs.

For fashion products, when there are no economies to bundling and when product demands are independent, we find that the optimal bundle prices are lower than the sum of the unbundled prices. Our numerical example illustrates the impact of reservation price variability on optimal
prices and profits. Finally, we compare basic and fashion products and find that, as expected, prices and profits of individual basic products are higher. These results also hold when we compare bundled basic products with bundled fashion products.

This paper provides several new avenues for future research. First, it would be constructive to consider the case with mixed bundling, where the retailer offers the individual products and the bundle. While this is a realistic setting in many situations, the exact analysis of this problem is complex due to intractability of the joint profit function. Second, one could consider different distributions of reservation prices, such as the multinomial logit distribution that could also capture the impact of difference in market share among the bundled products. Third, it could be instructive to examine product bundling under retail competition. Finally, it could be important to identify and incorporate the additional financial, operational and marketing constraints that may be needed to implement bundling across various product lines. We hope the ideas developed in this paper will be used as building blocks to address these extensions.

In conclusion, we believe that the methods developed in this paper provide a simple and useful framework to understand and analyze bundling in the context of retail merchandising.
7 Appendix: Proofs of Propositions

Proof of Proposition 1. Part (I) follows from (4), (5), and (12). Compare the profits without bundling (8) and the profits with bundling (14) to establish parts (II) and (III) of the proposition.

Proof of Proposition 2, part (I) follows from (4), (5), and (17). Proof of Proposition 2, part (II): bundle profits (20) in the positively correlated case, known market size, are less than the unbundled profits (8).

\[
\pi^*_{(A,B)} : \hat{\pi}^*_{AB} \\
(1 - c_A)^2 + \frac{1}{r_B^u - r_B^l}(r_B^u - c_B)^2 = \frac{[(1 - c_A) + (r_B^u - c_B)]^2}{1 + r_B^u - r_B^l} \\
(1 + r_B^u - r_B^l)(1 - c_A)^2 + \frac{1}{r_B^u - r_B^l}(r_B^u - c_B)^2 = (1 - c_A)^2 + (r_B^u - c_B)^2 \\
+ 2(1 - c_A)(r_B^u - C_B) \\
(r_B^u - r_B^l)(1 - c_A)^2 + \frac{1}{r_B^u - r_B^l}(r_B^u - c_B)^2 = 2(1 - c_A)(r_B^u - c_B) \\
(r_B^u - r_B^l)(1 - c_A)^2 - 2(1 - c_A)(r_B^u - c_B) + \frac{1}{r_B^u - r_B^l}(r_B^u - c_B)^2 = 0 \\
\left[ \sqrt{r_B^u - r_B^l}(1 - c_A) - \frac{1}{\sqrt{r_B^u - r_B^l}}(r_B^u - c_B) \right]^2 > 0.
\]

Thus, the profits in the known market size unbundled case (the left column) dominate the known market size bundled case (the right column) when there are no economies to bundling \((c_{AB} = c_A + c_B)\) and reservation prices are perfectly positively correlated.

Proof of Proposition 3, part (I) follows from (4), (5), and (21). Proof of Proposition 3 part (IIb): Consider when the range of the reservation price for product \(B\) is \([0, d]\) where \(d < 1\). Thus, \(r_B = d(1 - r_A)\). The optimal bundle price given in (22) shows that the bundle price offers a discount over the unbundled prices. The bundle profits for the known market size, perfectly negatively correlated case, \(\hat{\pi}^*_{AB}\) given by (24), are larger than the known market size unbundled profits \(\pi^*_{(A,B)}\) given by (8) if and only if

\[
\frac{1}{1 - d} \left( \frac{1 - c_A - c_B}{2} \right)^2 > \left( \frac{1 - c_A}{2} \right)^2 + \frac{1}{d} \left( \frac{d - c_B}{2} \right)^2
\]
\[ \iff \quad d(1 - c_A - c_B)^2 > d(1 - d)(1 - c_A)^2 + (1 - d)(d - c_B)^2. \]

Rearranging terms, bundle profits exceed unbundle profits if and only if

\[ d^2(2 - 2(c_A + c_B) + c_A^2) + d(2c_B^2 + 2c_Ac_B - 1) - c_B^2 > 0. \]

For example, if \( c_A = c_B = c \), calculations show that this inequality is satisfied if \( d = 0.8 \) and \( c \leq 0.3 \), but that it is not satisfied for any value of \( c \) when \( d = 0.4 \).

********************************************************************************

Proof of Proposition 4: If demand is linear, then profits are unimodal in \( p \). The profit function is given by:

\[ \pi(p) = (\delta - \alpha p)(p - c)^2/p. \]

Note that at \( p = c \), profits are 0. Differentiating the profit function and simplifying yields

\[ \pi'(p) = (1/p)^2(p - c)[-2\alpha p^2 + \delta p + \delta c]. \]

When \( p = c \), \( \pi'(p) = 0 \), but the third term on the right-hand side is positive. Thus, when \( p \) is slightly above \( c \), \( \pi'(p) \) is positive. There are two other zeros of \( \pi' \), one negative and one positive. The positive zero is

\[ p = \frac{\delta + \sqrt{\delta^2 + 8\alpha \delta c}}{4\alpha}. \]

Thus, \( \pi(p) \) is unimodal for \( p \geq c \).

********************************************************************************

Proof of Proposition 5: To validate part (I) of Proposition 5, compare the bundle price (42) with the sum of the unbundled prices, (36) plus (39). The bundle price is

\[ p_{AB}^* = \frac{1.5 + \sqrt{(1.5)^2 + 8(1.5)(c_A + c_B)}}{4} \]  \hspace{1cm} (45)

while the sum of the unbundled prices is

\[ p_A^* + p_B^* = \frac{2 + \sqrt{1 + 8c_A + \sqrt{1 + 8c_B}}}{4}. \]  \hspace{1cm} (46)

Because \( c_A < 1 \) and \( c_B < 1 \), it follows that the bundle price (45) is less than (46). This is easiest to see if \( c_A = c_B = c \), in which case \( c_{AB} = 2c \).
Assume the range of reservation prices for product B is $[0, 1]$, there are no economies to bundling, and the costs of the unbundled goods are the same, i.e. $c_A = c_B = c$, whence $c_{AB} = 2c$. The profit for the bundled good (44) is

$$\pi^*_A = \frac{M}{4} \left( 3(1.5) - \sqrt{1.5^2 + 8(1.5)2c} \right) \left[ \frac{(1.5 + \sqrt{1.5^2 + 8(1.5)2c} - 4(2c))^2}{4(1.5 + \sqrt{1.5^2 + 8(1.5)2c})} \right],$$

while the sum of the unbundled profits is

$$\pi^*_A + 2M \left( 3 - \sqrt{1 + 8c} \right) \left[ \frac{(1 + \sqrt{1 + 8c} - 4c)^2}{4(1 + \sqrt{1 + 8c})} \right].$$

Some algebraic manipulations show that the bundle profits (47) exceed the sum of the unbundled profits (48), i.e., bundling is optimal, if and only if

$$9 \left( 3 - \sqrt{1 + 8\left( \frac{4}{3} \right)c} \right) \left[ \frac{(1 + \sqrt{1 + 8\left( \frac{4}{3} \right)c} - 4\left( \frac{4}{3} \right)c)^2}{1 + \sqrt{1 + 8\left( \frac{4}{3} \right)c}} \right] > 8 \left( 3 - \sqrt{1 + 8c} \right) \left[ \frac{(1 + \sqrt{1 + 8c} - 4c)^2}{1 + \sqrt{1 + 8c}} \right].$$

Calculations show that this last condition is satisfied only when the cost $c < 0.09$, which violates the assumption regarding the value of $c_{AB}$, hence bundling is not optimal.

**************************

Proof of Proposition 6 I: The prices, $p^*_A$ and $p^*_B$, of the individual products are lower when the market size is known.

We begin by showing that the price for product B with known market size is less than the price for product B when the market size is random. The price of product B with known demand given in (5) is less than the price of product B with random demand given in (39) if and only if

$$\frac{(r^u_B - c_B)}{2} < \frac{r^u_B + \sqrt{(r^u_B)^2 + 8r^u_Bc_B}}{4}$$

$$\iff r^u_B + 2c_B < \sqrt{(r^u_B)^2 + 8r^u_Bc_B}$$

$$\iff (r^u_B)^2 + 4r^u_Bc_B + 4c^2_B < (r^u_B)^2 + 8r^u_Bc_B$$

$$\iff c_B < r^u_B.$$
(with \( r_A^u = 1 \)) leads to an equivalent result for product \( A \): the price is lower when the market size is known.

Proof of Proposition 6 II: The profit to the individual products \( \pi_A^* \) and \( \pi_B^* \) are greater when demand is known than when it is random.

We establish this proposition for product \( A \); the proof for \( B \) is similar. The profit to product \( A \) in the known demand case is greater than the profit to product \( A \) in the random demand case if and only if:

\[
M \cdot \left( \frac{1 - c_A}{2} \right)^2 > M \cdot \left( 3 - \sqrt{1 + 8c_A} \right) \left( \frac{1 + \sqrt{1 + 8c_A} - 4c_A}{4(1 + \sqrt{1 + 8c_A})} \right). 
\]

(54)

Several tedious algebraic steps and some simple calculations show that for all costs such that \( 0 < c_A < 1 \), this inequality is satisfied. Thus, the firm earns greater profits from product \( A \) when demand is known. A similar formulation shows the same is true for product \( B \). That is, the firm earns greater (expected) profits for the unbundled goods when demand is known.

Proof or Proposition 7 I: The price, \( p_{AB}^* \), of the bundled good is lower with known market size.

Let \( x = [1 + (r_B^u + r_B^l)/2] \). Follow the same path outlined following Proposition 6: the bundle price with known market size given in (12) is less than the bundle price with random market size given in (42) if and only if

\[
\frac{x + c_{AB}}{2} < \frac{1}{4} \left[ x + \sqrt{x^2 + 8xc_{AB}} \right] 
\]

\[
\iff x + 2c_{AB} < \sqrt{x^2 + 8xc_{AB}} 
\]

\[
\iff x^2 + 4xc_{AB} + 4c_{AB}^2 < x^2 + 8xc_{AB} 
\]

\[
\iff c_{AB} < x = [1 + (r_B^u + r_B^l)/2].
\]

(55)

The last inequality holds by assumption.

Proof of Proposition 7 II: The profit to the bundled good \( \pi_{AB}^* \) when demand is known is greater than when demand is random.
The bundle profit, known demand, given in (14) is greater than the bundle profit, random demand, given in (44) if and only if

\[(x - c_{AB})^2 > \left(3x - \sqrt{x^2 + 8xc_{AB}}\right) \left[\frac{x + \sqrt{x^2 + 8xc_{AB} - 4c_{AB}}}{4(x + \sqrt{x^2 + 8xc_{AB}})}\right]^2,\]  

(59)

where \(x = [1 + (r_B^u + r_B^l)/2]\). As long as \(x > c_{AB}\), this inequality holds. This is the same inequality which establishes when the price of the bundled good in the known demand case is less than the bundle price in the random demand case.

***********************
References


## Table 1: Results For Basic Products With Symmetric Demand

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<th>Bundling with Independent Demand</th>
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## Table 2: Results For Basic Products With Asymmetric Demand

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<td>46.875</td>
<td>55</td>
<td>175</td>
</tr>
<tr>
<td><strong>Optimal Profits</strong></td>
<td>14.06</td>
<td>17.58</td>
<td>30.25</td>
<td>61.25</td>
</tr>
</tbody>
</table>

## Table 3: Comparing Basic And Fashion Products

<table>
<thead>
<tr>
<th></th>
<th>Basic: No Bundling</th>
<th>Fashion: No Bundling</th>
<th>Bundled Basic Products with Independent Demand</th>
<th>Bundled Fashion Products with Independent Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Product A</td>
<td>Product B</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Optimal Prices</strong></td>
<td>0.625</td>
<td>0.625</td>
<td>0.68</td>
<td>1</td>
</tr>
<tr>
<td><strong>Optimal Order Quantities</strong></td>
<td>37.5</td>
<td>37.5</td>
<td>20.1</td>
<td>50</td>
</tr>
<tr>
<td><strong>Optimal Profits</strong></td>
<td>14.0625</td>
<td>14.0625</td>
<td>8.7</td>
<td>25</td>
</tr>
</tbody>
</table>
Figure 1: Prices Versus Costs For Basic Products

Figure 2: Profits Versus Costs For Basic Products
Figure 3: Price Vs. Reservation Price Variability for Fashion Products

Figure 4: Expected Profit Vs. Reservation Price Variability for Fashion Products