LOWER BOUNDS FOR INTERIOR NODAL SETS OF STEKLOV EIGENFUNCTIONS

CHRISTOPHER D. SOGGE, XING WANG, AND JIUYI ZHU

Abstract. We study the interior nodal sets, Z_λ of Steklov eigenfunctions in an n-dimensional relatively compact manifolds M with boundary and show that one has the lower bounds $|Z_\lambda| \geq c\lambda^{\frac{2-n}{2}}$ for the size of its $(n-1)$-dimensional Hausdorff measure. The proof is based on a Dong-type identity and estimates for the gradient of Steklov eigenfunctions, similar to those in [18] and [19], respectively.

1. **Introduction.**

This article is concerned with lower bounds for the size of nodal sets,

$$Z_\lambda = \{x \in M : e_\lambda(x) = 0\}, \tag{1.1}$$

of real Steklov eigenfunctions in a smooth relatively compact manifold (M, g) of dimension $n \geq 2$ with boundary ∂M. These eigenfunctions are solutions of the equation

$$\begin{cases} \Delta_g e_\lambda = 0, & \text{in } M \\ \partial_\nu e_\lambda = \lambda e_\lambda, & \text{on } \partial M, \end{cases} \tag{1.2}$$

where ν is the unit outward normal on ∂M.

The Steklov eigenfunctions were introduced by Steklov [17] in 1902. They describe the vibration of a free membrane with uniformly distributed mass on the boundary. The equation (1.2) was studied by Calderón [3] as its solutions can be regarded as eigenfunctions of the Dirichlet to Neumann map.

More specifically, the e_λ in (1.2) satisfy the eigenvalue problem

$$Pe_\lambda = \lambda e_\lambda,$$

if the Dirichlet to Neumann operator P is defined as

$$Pf = \partial_\nu Hf|_{\partial M},$$

where for $f \in C^\infty(\partial M)$, $Hf = u$ is the harmonic extension of f into M, i.e., the solution of

$$\begin{cases} \Delta_g u(x) = 0, & x \in M \\ u(x) = f(x), & x \in \partial M. \end{cases}$$

It is well known that P is a self-adjoint classical pseudodifferential operator of order one whose principal symbol agrees with that of the square root of minus the boundary

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Laplacian on ∂M coming from the metric. Furthermore, there is an orthonormal basis of real eigenfunctions $\{e_{\lambda_j}\}$ such that

$$Pe_{\lambda_j} = \lambda_j e_{\lambda_j}, \quad \text{and} \quad \int_{\partial M} e_{\lambda_j} e_{\lambda_k} dV_g = \delta_j^k.$$

The spectrum λ_j is discrete, with

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots, \quad \text{and } \lambda_j \to \infty.$$

Recently there has been much work on the study of nodal sets of Steklov eigenfunctions. It has largely been focused on the size of the nodal set

$$\mathcal{N}_\lambda = \{x \in \partial M : e_\lambda(x) = 0\}$$

on the boundary ∂M of M. Bellova and Lin [1] proved that $|\mathcal{N}_\lambda| \le C\lambda^6$, if $|\mathcal{N}_\lambda|$ denotes $d-1$ dimensional Hausdorff measure with here $d = n-1$ denoting the dimension of ∂M. Later, Zelditch [23] improved these results and gave the optimal upper bound $|\mathcal{N}_\lambda| \le C\lambda$ for analytic manifolds using microlocal analysis. In the smooth case, the last two authors showed in [20] showed that

$$|\mathcal{N}_\lambda| \ge c\lambda^{\frac{3-d}{2}}, \tag{1.3}$$

assuming that 0 is a regular value for e_λ. This agrees with the best known general lower bounds for the boundaryless case (see below), but in both [23] and [20] the nonlocal nature of the operators defining the eigenfunctions presented an obstacle which had to be overcome.

By the maximum principle, we know that the nodal sets in M must always intersect the boundary ∂M. In other words, there can be no component of the nodal set which is closed in M. Thus, it is natural to study the size of the nodal set in the interior, M. This question was also raised by Girouard and Polterovich in [9].

Let us briefly review the literature concerning the study of nodal sets for compact boundaryless Riemannian manifolds. Let ψ_λ denote an L^2-normalized eigenfunction on of the Laplace-Beltrami operator on such a smooth n-dimensional manifold, i.e.,

$$-\Delta_g \psi_\lambda = \lambda^2 \psi_\lambda.$$

Yau conjectured in [22] that one should have

$$c\lambda \le |Z_\lambda| \le C\lambda,$$

if Z_λ denotes the nodal set of ψ_λ, and $|Z_\lambda|$ its $(n-1)$-dimensional Hausdorff measure. In the real analytic case both the upper and lower bounds were established by Donnelly and Fefferman [6]. The lower bound was established in the two-dimensional case by Brüning [2] and Yau (unpublished); however, in all other cases, the conjecture remains open in the smooth case. Recently there has been much work on establishing lower bounds in the smooth case when $n \ge 3$. Colding and Minicozzi [4] and then later the first author and Zelditch [18], [19] showed that

$$|Z_\lambda| \ge c\lambda^{\frac{3-n}{2}}, \tag{1.4}$$

which matches up with the lower bounds in (1.3) which were obtained later. Another proof of (1.4) was given by Hezari and the second author in [11].

The arguments in [18], [19] and [11] involved establishing a Dong-type identity, similar to the one in [5], and then using either lower bounds for the L^1-norms of ψ_λ or upper-bounds for its gradient. We shall use similar arguments to establish our main result concerning lower bounds for the $(n-1)$-dimensional Hausdorff measure of the interior nodal sets of Steklov eigenfunctions contained in the following result.

Theorem 1.1. *Let M be a smooth relatively compact n-dimensional manifold with smooth boundary ∂M. Then there is a constant $c > 0$ so that*

$$|Z_\lambda| \geq c\lambda^{\frac{2-n}{2}} \tag{1.5}$$

for the $(n-1)$-dimensional Hausdorff measure of the nodal sets given by (1.1) *of the Steklov eigenfunctions* (1.2).

We note that this lower bound is off by a half-power versus the best known lower bounds, (1.4), for the boundaryless case. We shall explain what accounts for this difference after we complete the proof of Theorem 1.1. Also, it seems clear that in the two-dimensional case the lower bound (1.5) is far from optimal since the arguments of Brüning [2] and Yau (see also [12]) seem to give the optimal lower bound $|Z_\lambda| \geq c\lambda$ using the fact that the nodal set must intersect any $C\lambda^{-1}$ ball in M if C is large enough (see e.g. [9]).

2. An interior Dong-type identity for Steklov eigenfunctions.

As in [18] we shall want to use the Gauss-Green formula to establish a Dong-type identity which we can use to prove our lower bound (1.5). We shall be able to do this since the singular set

$$S_\lambda = \{x \in \overline{M} : \, e_\lambda(x) = 0 \text{ and } \nabla e_\lambda(x) = 0\}$$

is of Hausdorff codimension 2 or more, i.e., dim $S_\lambda \leq n-2$. This is true for $S_\lambda \cap M$ since e_λ is harmonic in M (see e.g. [10, Chapter 4]), while one can, for instance see that the same is true for $S_\lambda \cap \partial M$ using the doubling lemma in [24]. In addition, for each λ, there are only finitely many nodal domains (see e.g. [9]). Consequently, we may write $\overline{M}$ as the (essentially) disjoint union

$$\overline{M} = \bigcup_{i=1}^{k_\lambda} (D_i^+ \cup Z_i^+ \cup Y_i^+) \cup \bigcup_{j=1}^{m_\lambda} (D_j^- \cup Z_j^- \cup Y_j^-), \tag{2.1}$$

where D_i^+ and D_j^- are the connected components of $\{x \in M : \, e_\lambda(x) > 0\}$ and $\{x \in M : e_\lambda(x) < 0\}$, respectively, while $Z_k^\pm = \partial D_k^\pm \cap M$ and $Y_k^\pm = \overline{D_k^\pm} \cap \partial M$. Thus,

$$Z_\lambda = \bigcup_{i=1}^{k_\lambda} Z_i^+ \cup \bigcup_{j=1}^{m_\lambda} Z_j^-,$$

and

$$\partial M = \bigcup_{i=1}^{k_\lambda} Y_i^+ \cup \bigcup_{j=1}^{m_\lambda} Y_j^-.$$

The boundary of $D_k^\pm$ in $\overline{M}$ is $Z_k^\pm \cup Y_k^\pm$. Since S_λ has codimension 2 or more and ∂M is smooth, we may use the Gauss-Green formula (see e.g. Theorem 1 on p. 209 of [7]) for any $f \in C^\infty(\overline{M})$ to get

$$\begin{aligned}\int_{D_k^+} \Delta_g f e_\lambda \, dV &= \int_{D_k^+} f \Delta_g e_\lambda \, dV - \int_{\partial D_k^+} f \partial_\nu e_\lambda \, dS \\ &= -\lambda \int_{Y_k^+} f e_\lambda dS + \int_{Z_k^+} f |\nabla e_\lambda| \, dS.\end{aligned}$$

Here ∂_ν denotes the outward Riemann derivative on ∂D_k^+, and we used the equation (1.2) to get the last equality. Rearranging, we see from above that

$$\lambda \int_{Y_k^+} f e_\lambda \, dS + \int_{D_k^+} \Delta_g f \, e_\lambda dV = \int_{Z_k^+} f |\nabla e_\lambda| \, dS. \tag{2.2}$$

Similarly for each negative nodal domain we have

$$\begin{aligned}\int_{D_k^-} \Delta_g f e_\lambda \, dV &= \int_{D_k^-} f \Delta_g e_\lambda \, dV - \int_{\partial D_k^-} f \partial_\nu e_\lambda \, dS \\ &= -\lambda \int_{Y_k^-} f e_\lambda dS - \int_{Z_k^-} f |\nabla e_\lambda| \, dS,\end{aligned}$$

using in the last step that on each Z_k^-, unlike on each Z_k^+, $\partial_\nu e_\lambda = |\nabla e_\lambda|$ since e_λ increases as it crosses Z_k^- from D_k^-. Rearranging this time leads to

$$\lambda \int_{Y_k^-} f e_\lambda \, dS + \int_{D_k^-} \Delta_g f \, e_\lambda \, dV = - \int_{Z_k^-} f |\nabla e_\lambda| \, dS. \tag{2.3}$$

Since $e_\lambda > 0$ in D_k^+ and $e_\lambda < 0$ in D_k^-, we can combine (2.2) and (2.3) into

$$\lambda \int_{Y_k^\pm} f \, |e_\lambda| \, dS + \int_{D_k^\pm} \Delta_g f \, |e_\lambda| \, dV = \int_{Z_k^\pm} f |\nabla e_\lambda| \, dS. \tag{2.4}$$

Since almost every point in Z_λ belongs to exaclty one Z_i^+ and one Z_j^- and almost every point in ∂M belongs to just one of the sets $Y_k^\pm$, if we sum up the identity (2.4), we conclude that we have the Dong-type identity

$$\lambda \int_{\partial M} f \, |e_\lambda| \, dS + \int_M \Delta_g f \, |e_\lambda| \, dV = 2 \int_{Z_\lambda} f \, |\nabla e_\lambda| \, dS. \tag{2.5}$$

Of course if $f \equiv 1$ this simplifies to

$$\lambda \int_{\partial M} |e_\lambda| \, dS = 2 \int_{Z_\lambda} |\nabla e_\lambda| \, dS, \tag{2.6}$$

which is what we shall use in our proof of Theorem 1.1.

3. **Interior estimates for Steklov eigenfunctions.**

We shall prove interior estimates for the e_λ which are natural analogs of the ones obtained earlier in the boundaryless case by Sogge and Zelditch [18], [19]. We shall use arguments which are similar to those of Shi and Xu [14] and [21] and H. Smith (unpublished).

Specifically, we have the following:

Proposition 3.1. *If e_λ is as above and if $d = d(x)$ denotes the distance from $x \in M$ to ∂M,*

$$\|(\lambda^{-1}+d)\,\nabla_g e_\lambda\|_{L^\infty(M)} + \|e_\lambda\|_{L^\infty(M)} \le C\lambda^{\frac{n-2}{2}}\|e_\lambda\|_{L^1(\partial M)}. \tag{3.1}$$

Let us first argue that on the boundary, we have these estimates. Indeed,

$$\lambda^{-\alpha}\|D^\alpha e_\lambda\|_{L^\infty(\partial M)} \le C_\alpha \lambda^{\frac{n-2}{2}}\|e_\lambda\|_{L^1(\partial M)}, \tag{3.2}$$

with D^α here referring to α boundary derivatives. This inequality follows from arguments in [13] and [18]–[19], since $Pe_\lambda = \lambda e_\lambda$ where P is a classical self-adjoint pseudodifferential of order one operator whose principal symbol agrees with that of the square root of minus the boundary Laplacian. As a result we can use Lemma 5.1.3 in [15] to write $e_\lambda = T_\lambda e_\lambda$, where T_λ is an integral operator on the $(n-1)$-dimensional boundary of M whose kernel $K_\lambda(x,y)$ satisfies $D^\alpha K = O(\lambda^{\alpha+\frac{n-2}{2}})$ for each α, which immediately gives us (3.2).

For the next step, we use that by the maximum principle, the bounds in (3.2) for e_λ yield

$$\|e_\lambda\|_{L^\infty(M)} \le C\lambda^{\frac{n-2}{2}}\|e_\lambda\|_{L^1(\partial M)}, \tag{3.3}$$

as desired. Thus, we only need to prove the bounds in (3.1) for $\nabla_g e_\lambda$.

As a first step we realize that we can obtain this estimate in the region of M which is of distance $\delta\lambda^{-1}$ from the boundary just by using standard Schauder estimates for a given $\delta > 0$. Indeed, since e_λ is harmonic in M and (3.2) is valid, it follows from Corollary 6.3 in [8] applied to balls centered at points $x \in M$ or radius $r \le d(x)/2$ that we have

$$\|d\,\nabla_g e_\lambda\|_{L^\infty(\{x\in M:\mathrm{dist}(x,\partial M)\ge\delta\lambda^{-1})} \le C_\delta \lambda^{\frac{n-2}{2}}\|e_\lambda\|_{L^1(\partial M)}. \tag{3.4}$$

Here, the constant C_δ depends on δ and (M,g), but not on λ.

To finish the proof of (3.1), it suffices to show that if $\delta > 0$ is sufficiently small we also have the uniform bounds

$$\lambda^{-1}\|\nabla_g e_\lambda\|_{L^\infty(M\cap B(x_0,\delta\lambda^{-1}))} \le C_\delta \lambda^{\frac{n-2}{2}}\|e_\lambda\|_{L^1(\partial M)}, \quad x_0 \in \partial M, \tag{3.5}$$

with $B(x_0,\delta\lambda^{-1})$ denoting the geodesic ball of radius $\delta\lambda^{-1}$ about the boundary point x_0.

To prove this we shall use local coordinates and a scaling argument. We shall work in such coordinates and scale and normalize e_λ by replacing it by

$$u_\lambda(x) = \lambda^{-\frac{n-2}{2}} e_\lambda(x/\lambda). \tag{3.6}$$

Similarly, we shall scale the $\delta\lambda^{-1}$ ball so that it becomes a δ ball $\tilde{B}(x_0,\delta)$ and use the "stretched" Laplacian with principal part $\sum g^{jk}(x/\lambda)\partial_j\partial_k$ (coming from the "stretched" metric $g_{jk}(x/\lambda)$), which denote by L. It follows from (3.2) that we have the uniform bounds

$$\|D^\alpha u_\lambda\|_{L^\infty(\partial\tilde{M})} \le C_\alpha\|e_\lambda\|_{L^1(\partial M)}, \tag{3.7}$$

where $\tilde{M}$ denotes the stretched version of M in our local coordinates. Additionally, the coefficients of our "stretched" Laplacian L belong to a bounded subset of C^∞ as $\lambda \ge 1$ and $x_0 \in \partial M$ vary. Also, because of (3.6) we can find a function φ_λ in our local coordinate system which agrees with u_λ on $\partial\tilde{M}$ and has bounded $C^{2,\alpha}(\tilde{B}(x_0,2\delta)\cap\tilde{M})$ norm independent of $\lambda \ge 1$ and $x_0 \in \partial M$ for a given $0 < \alpha < 1$. Therefore, if we apply

Corollary 8.36 in [8] to $u = u_\lambda - \varphi_\lambda$ and $f = -L\varphi_\lambda$, we conclude that the $C^{1,\alpha}(\tilde{B}(x_0, \delta))$ norm u_λ is bounded uniformly with respect to these parameters if α is fixed. Thus, we in particular have the uniform bounds

$$\|Du_\lambda\|_{L^\infty(\tilde{B}(x,\delta)\cap\tilde{M})} \le C.$$

If we go back to the original local coordinates and recall (3.6), we obtain (3.5), which completes the proof of Proposition 3.1.

4. **Conclusion.** It is now very easy to prove Theorem 1.1. If we use (2.6) and (3.1), we conclude that

$$\lambda\|e_\lambda\|_{L^1(\partial M)} = 2\int_{Z_\lambda} |\nabla e_\lambda|\, dS \le C\lambda^{\frac{n-2}{2}} \|e_\lambda\|_{L^1(\partial M)} \int_{Z_\lambda} (\lambda^{-1} + d(x))^{-1}\, dS,$$

where, as before, $d(x)$ denotes the distance from $x \in M$ to ∂M. From this, we deduce that

$$\lambda^{2-\frac{n}{2}} \le C \int_{Z_\lambda} (\lambda^{-1} + d(x))^{-1}\, dS. \tag{4.1}$$

Clearly this inequality yields (1.5), establishing Theorem 1.1.

Remarks: There is a simple explanation of why the lower bounds (1.5) are off by a half power versus the corresponding best lower bounds (1.4) for the boundaryless case. This is because the Dong-type identity in [18] involved λ^2 in the left side instead of λ, which accounts for a relative loss of a full power of λ, but, on the other hand, the estimates for the gradient here are one half power better due to the fact that the boundary of M is of one less dimension, accounting for a relative gain of a half power.

In some cases one can use (4.1) to get improved lower bounds. For instance if we let

$$Z_{\lambda,k} = \{x \in Z_\lambda :\, d(x) \in [2^{-k}, 2^{-k+1})\}$$

and if $|Z_{\lambda,k}| \le C2^{-k}|Z_\lambda|$ for $C \le k \le \log_2 \lambda$ and if $|\{x \in Z_\lambda : d(x) \le \lambda^{-1}\}| \le C\lambda^{-1}|Z_\lambda|$, with C fixed, we then get the lower bound $|Z_\lambda| \ge c\lambda^{2-\frac{n}{2}}/\log\lambda$, which is essentially optimal when $n = 2$. The subsets $Z_{\lambda,k}$ of Z_λ have this property, for instance, for the Steklov eigenfunctions $r^m \sin m\theta$ on the disk in $\mathbb{R}^2$ (written in polar coordinates).

References

[1] K. Bellova and F.H. Lin:*Nodal sets of Steklov eigenfunctions*, arXiv:1402.4323.

[2] J. Brüning: *Über Knoten von Eigenfunktionen des Laplace-Beltrami-Operators*, (German) Math. Z. **158** (1978), 15–21.

[3] A.P. Calderón: *On a inverse boundary value problem*, in "Seminar in Numerical Analysis and Its Applications to Continuum Physics", 1980, 65–73, Soc. Brasileira de Matemática, Rio de Janeiro.

[4] T.H. Colding and W. P. Minicozzi II:*Lower bounds for nodal sets of eigenfunctions*, Comm. Math. Phys. **306** (2011), 777–784.

[5] R-T Dong: *Nodal sets of eigenfunctions on Riemann surfaces*, J. Differential Geom. **36** (1992), 493–506.

[6] H. Donnelly and C. Fefferman: *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math. **93** (1988), 161–183.

[7] L.C. Evans and R. Gariepy: *Measure theory and fine properties of functions*, CRC Press, Boca Raton, Ann Arbor, and London, 1992.

[8] D. Gilbarg and N. S. Trudinger:*Elliptic partial differential equations of second order*, Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[9] A. Girouard and I. Polterovich: *Spectral geometry of the Steklov problem*, arXiv:1411.6567.
[10] Q. Han and F.H. Lin: *Nodal sets of solutions of elliptic differential equations*, book in preparation (online at http://www.nd.edu/qhan/nodal.pdf).
[11] H. Hezari and C.D. Sogge: *A natural lower bound for the size of nodal sets*, Anal. PDE. **5** (2012) 1133–1137.
[12] A. Savo: *Lower bounds for the nodal length of eigenfunctions of the Laplacian*, Ann. Global Anal. Geom. **19** (2001), 133–151.
[13] A. Seeger and C.D. Sogge: *Bounds for eigenfunctions of differential operators*, Indiana Math. J. **38** (1989), 669–682.
[14] Y. Shi, and Xu, Bin: *Gradient estimate of an eigenfunction on a compact Riemannian manifold without boundary*, Ann. Global Anal. Geom. **38** (2010), 21–26.
[15] C.D. Sogge: *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics, **105** Cambridge University Press, Cambridge, 1993.
[16] C.D. Sogge: *Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds*, J. Funct. Anal. **77** (1988), 123–138.
[17] W. Stekloff: *Sur les problèmes fondamenbetax de la physique mathematique, Ann. Sci. École Norm. Sup.* **19** (1902), 191-259.
[18] C.D. Sogge and S. Zelditch: *Lower bounds on the Hausdorff measure of nodal sets*, Math. Res. Lett. **18** (2011), 25–37.
[19] C.D. Sogge and S. Zelditch: *Lower bounds on the Hausdorff measure of nodal sets II*, Math. Res. Lett. **19** (2012), 1361–1364.
[20] X. Wang and J. Zhu: *A lower bound for the nodal sets of Steklov eigenfunctions*, arXiv:1411.0708.
[21] X. Xu: *Gradient estimates for the eigenfunctions on compact manifolds with boundary and Hörmander multiplier theorem*, Forum Math. **21** (2009), 455–476.
[22] S. T. Yau: *Survey on partial differential equations in differential geometry*, Seminar on Differential Geometry, pp. 371, Ann. of Math. Stud., **102**, Princeton Univ. Press, Princeton, N.J., 1982.
[23] S. Zelditch: *Measure of nodal sets of analytic Steklov eigenfunctions*, arXiv:1403.0647.
[24] J. Zhu: *Doubling property and vanishing order of Steklov eigenfunctions*, Comm. Partial Differential Equations.

E-mail address: sogge@jhu.edu

E-mail address: xwang@math.jhu.edu

E-mail address: jzhu43@math.jhu.edu

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218