UNIVERSITY FOR ENSEMBLES OF MATRICES WITH POTENTIAL THEORETIC WEIGHTS ON DOMAINS WITH SMOOTH BOUNDARY

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Abstract. We investigate a two-dimensional statistical model of \( N \) charged particles interacting via logarithmic repulsion in the presence of an oppositely charged compact region \( K \) whose charge density is determined by its equilibrium potential at an inverse temperature corresponding to \( \beta = 2 \). When the charge on the region, \( s \), is greater than \( N \), the particles accumulate in a neighborhood of the boundary of \( K \), and form a determinantal point process on the complex plane. We investigate the scaling limit, as \( N \to \infty \), of the associated kernel in the neighborhood of a point on the boundary under the assumption that the boundary is sufficiently smooth. We find that the limiting kernel depends on the limiting value of \( N/s \), and prove universality for these kernels. That is, we show that, the scaled kernel in a neighborhood of a point \( \zeta \in \partial K \) can be succinctly expressed in terms of the scaled kernel for the closed unit disk, and the exterior conformal map which carries the complement \( K \) to the complement of the closed unit disk. When \( N/s \to 0 \) we recover the universal kernel discovered by Lubinsky in [13].

1. Introduction

1.1. Potential Theoretic Weights. Let \( K \subseteq \mathbb{C} \) be a compact subset whose boundary \( T = \partial K \) is a Jordan curve. We will assume that \( T \) is sufficiently nice in a way that will be made precise in the sequel. For such \( K \), there exists a unique measure \( \omega_K \), the equilibrium measure on \( K \), that minimizes the energy functional \( I[\sigma] := -\int \int \log |z - u|d\sigma(u)d\sigma(z) \) among all positive probability measures \( \sigma \) supported on \( K \) [17]. We define, \( P_K : \mathbb{C} \to (0, \infty) \), by

\[
P_K(z) := \exp \left\{ I[\omega_K] + \int \log |z - u|d\omega_K(u) \right\}.
\]

This function is simply the rescaled exponentiated equilibrium potential of \( K \).

It can be verified that \( \omega_K \) is supported on \( T \), \( P_K \) is identically one on \( K \) and, as \( z \to \infty \), \( P_K(z)/|z| \to \gamma_K^{-1} \), where \( \gamma_K := \exp\{-I[\omega_K]\} \) is the logarithmic capacity of \( K \).

In this paper we will be interested in random vectors whose joint density is given by

\[
\Omega_N(\lambda) := \frac{1}{Z_N} \prod_{n=1}^{N} w(\lambda_n) \prod_{m<n} |\lambda_n - \lambda_m|^2; \quad \lambda \in \mathbb{C}^N,
\]

where

\[
w(\lambda) = P_K(\lambda)^{-2s}, \quad Z_N := \int_{\mathbb{C}^N} \left\{ \prod_{n=1}^{N} w(\lambda_n) \right\} \prod_{m<n} |\lambda_n - \lambda_m|^2 dA^N(\lambda),
\]

and \( s > N \) (that is \( s \) is sufficiently large to guarantee that \( Z_N \) is finite). Here and throughout, \( A \) and \( A^N \) are Lebesgue measure on \( \mathbb{C} \) and \( \mathbb{C}^N \) respectively.

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We will often refer to the components of such random vectors as eigenvalues, since the joint density (2) can be thought of as a modification of the joint eigenvalue density of the ensemble of matrices with i.i.d. complex Gaussian entries. The eigenvalues of this latter matrix ensemble, originally introduced by Ginibre [11], have joint density given by (2) where \( w(\lambda) = e^{-|\lambda|^2} \).

In Section 3 we will give (i) a matrix model whose joint density of eigenvalues is given by (2) with weight given as in (3) for \( K \) equal to the closed unit disk, as well as (ii) models for more general \( K \) where the components of \( \lambda \) represent the positions of electrostatic particles confined to the plane and in the presence of a field determined by \( K \), and (iii) an ensemble of random polynomials chosen with respect to a height function determined by \( K \) whose roots are distributed as in (2).

Our primary goal is to demonstrate that, in the double scaling limit as \( s \) and \( N \) approach infinity, the local statistics of the eigenvalues near a point on the boundary of \( K \) depend only on the limiting ratio of \( s \) and \( N \), but are essentially independent of the specifics of \( K \). This will follow from the asymptotic behavior of the reproducing kernel of \( L^2(w) \), which in turn follow from the asymptotics of the leading coefficient of the related orthonormal polynomials. When \( s = \infty \), \( w \) is simply the characteristic function of \( K \) and our results collapse to those given by Lubinsky [13] for the universality of reproducing kernels formed with respect to Bergman polynomials for \( K \).

1.2. Eigenvalue Statistics. We briefly review some basic concepts for solvable ensembles of random matrices and how they relate to eigenvalue statistics. In this section we will assume that the joint density of eigenvalues is given by (2) where, for the purposes of this section, \( w : \mathbb{C} \rightarrow [0, \infty) \) is any non-negative function such that \( Z_N < \infty \).

We will suppose that \( \Xi = \{\xi_1, \xi_2, \ldots, \xi_N\} \subset \mathbb{C} \) is a random set corresponding to the eigenvalues of a random matrix from our ensemble. (Or, what amounts to the same thing, \( \Xi \) is the set corresponding to a random vector sampled from the density \( \Omega_N \)). Given a set \( E \subset \mathbb{C} \) we may construct a random variable \( X \) given by the cardinality of \( \Xi \cap E \). Given disjoint subsets \( E_1, E_2, \ldots, E_n \) we will let \( X_1, X_2, \ldots, X_n \) be the corresponding random variables. The \( n \)-th correlation function of our ensemble is defined to be \( R_n : \mathbb{C}^n \rightarrow [0, \infty) \), where

\[
E[X_1X_2\cdots X_n] := \int_{E_1} \int_{E_2} \cdots \int_{E_n} R_n(\lambda) \, dA_n(\lambda).
\]

It is straightforward to see that \( R_N = N! \Omega_N \). A less obvious exercise is to show that for \( 0 \leq n \leq N \),

\[
R_n(\lambda) = \frac{1}{(N-n)!} \int_{C^{N-n}} R_N(\lambda \lor x) \, dA^{N-n}(x),
\]

where \( \lambda \lor x = (\lambda_1, \ldots, \lambda_n, x_1, \ldots, x_{N-n}) \). Many probabilities of interest can be expressed in terms correlation functions. One particularly important example is the gap probability that there are no eigenvalues in \( E \),

\[
\text{Prob}\{X = 0\} = \sum_{n=0}^N \frac{(-1)^n}{n!} \int_{E^n} R_n(\lambda) \, dA^n(\lambda).
\]

Equations (4) and (5) are valid for a wide variety of symmetric measures on \( \mathbb{C}^N \). However, the presence of the square of the Vandermonde determinant which appears in (2) leads to additional structure which may be exploited.
Suppose $\pi_0, \pi_1, \ldots, \pi_{N-1}$ are the orthonormal polynomials with respect to the weight $w$. That is,
$$
\int_{\mathbb{C}} \pi_n(z)\pi_m(z)w(z)\,dA = \delta_{n,m},
$$
where, as usual, $\delta_{n,m}$ is 1 or 0 depending on whether or not $n = m$. The kernel of the ensemble is defined by
$$
\tilde{K}_N(z,u) := \sqrt{w(z)w(u)} \sum_{n=0}^{N-1} \pi_n(z)\pi_n(u).
$$
(Following Lubinsky’s notation, we will reserve the symbol $K_N$ for the unweighted analog of this kernel). In a celebrated result, Mehta and Gaudin [14], were able to express the correlation functions of ensembles with eigenvalue density $(2)$ in terms of determinants of matrices formed from this kernel,
$$
R_n(\lambda) = \det \left[ \tilde{K}_N(\lambda_j, \lambda_k) \right]_{j,k=1}^n.
$$
(See also [22] for a more modern derivation).

1.3. Universality. When $N$ is large we expect that, with high probability, the eigenvalues will accumulate in a neighborhood of $\partial K$. Slightly more precisely, if $\zeta \in \partial K$, then the number of eigenvalues in a disk of (small) radius $\epsilon$ about $\zeta$ is proportional to $N$ with the constant of proportionality is given by the integral of the equilibrium measure over the arc of $\partial K$ contained in the disk. The exact details of this phenomenon will be explored in a subsequent paper, for now we use this only as intuition to guess the proper scale on which we expect $\tilde{K}_N$ to converge.

From (5) and (6), the probability that there are no eigenvalues in a disk of radius $\epsilon$ centered at $\zeta$ is given by
$$
\sum_{n=0}^{N} \frac{(-1)^n}{n!} \int_{D^n} \det \left[ \epsilon^2 \tilde{K}_{N,s}(\zeta + \epsilon\lambda_j, \zeta + \epsilon\lambda_k) \right]_{j,k=1}^n \,dA^n(\lambda),
$$
where $D$ is the disk of radius 1 centered at the origin. Here we have made explicit that the kernel is dependent on $s$ as well as $N$.

Under the assumption that there are $O(N)$ eigenvalues in a neighborhood of $\zeta$, then we should scale $\epsilon$ like $1/N$ in order for (7) to approach a non-trivial limit. That is, the limiting gap probability of there being no eigenvalues in a shrinking neighborhood with radius $\epsilon = 1/N$ is given by
$$
\lim_{N \to \infty} \sum_{n=0}^{N} \frac{(-1)^n}{n!} \int_{D^n} \det \left[ \frac{1}{N^2} \tilde{K}_{N,s}(\zeta + \frac{\lambda_j}{N}, \zeta + \frac{\lambda_k}{N}) \right]_{j,k=1}^n \,dA^n(\lambda).
$$
Since $s > N$, this limit also depends on how $s$ scales with $N$ and we will assume that $N/s$ converges to some $\ell \in [0,1]$.

If it can be shown that there is some limiting kernel $\tilde{H}_{\zeta,\ell}$ so that
$$
\frac{1}{N^2} \tilde{K}_{N,s}(\zeta + \frac{z}{N}, \zeta + \frac{u}{N}) \to \tilde{H}_{\zeta,\ell}(z,u)
$$
uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}$, then (8) converges to
$$
\sum_{n=0}^{N} \frac{(-1)^n}{n!} \int_{D^n} \det \left[ \tilde{H}_{\zeta,\ell}(\lambda_j, \lambda_k) \right]_{j,k=1}^n \,dA^n(\lambda).$$
(See for instance [1, §3.4]). Our primary result here is that \( \tilde{H}_{\zeta,\ell} \) exists, and is dependent on \( \zeta \) and \( K \) in only the most trivial manner. More specifically, we will express \( \tilde{H}_{\zeta,\ell} \) in terms of the limiting kernel for the ensemble formed from the closed unit disk and the value of a conformal map from \( \mathbb{C} \setminus K \) to \( \mathbb{C} \setminus \mathbb{D} \) evaluated at \( \zeta \). This is what is called universality for potential theoretic ensembles.

We will also demonstrate that \( \tilde{H}_{\zeta,\ell} \) is a convex combination of \( \tilde{H}_{\zeta,0} \) (which is Lubinsky’s limiting kernel) and \( \tilde{H}_{\zeta,1} \).

### 1.4. Potential Theoretic Orthogonal Polynomials

We denote the orthonormal polynomials for the weight \( P_K^{-2s} \) by \( \{\pi_{n,s}\}_{n=0}^{\lfloor s-2 \rfloor} \). That is,

\[
\int_{\mathbb{C}} \pi_{n,s}(z) \overline{\pi_{m,s}(z)} P_K^{-2s}(z) \, dA = \delta_{nm}.
\]

The reproducing kernel for this system of polynomials is given by

\[
K_{N,s}(z,u) := \sum_{n=0}^{N-1} \pi_{n,s}(z) \overline{\pi_{n,s}(u)}, \quad N \leq \lfloor s - 1 \rfloor,
\]

with the weighted kernel given by

\[
\tilde{K}_{N,s}(z,u) := P_K^{-s}(z) P_K^{-s}(u) K_{N,s}(z,u).
\]

Our derivation of \( \tilde{H}_{\zeta,\ell} \) will follow from the asymptotics of \( K_{N,s} \), which in turn will follow from the asymptotics of the orthogonal polynomials. These latter asymptotics are of independent interest, and they provide the other primary results of the paper.

### 2. Statement of Results

In what follows, we assume that \( T := \partial K \) is a rectifiable Jordan curve which is either analytic or of class \( C^{p+1,\alpha} \), where \( p \) is a nonnegative integer and \( \alpha \in (0,1) \). That is, the arclength function of \( T \) is \( p \) times continuously differentiable as a periodic function on the real line and its \( p \)-th derivative is \( \alpha \)-Hölder continuous. Denote by \( \Phi \) the conformal map of \( O := \overline{\mathbb{C}} \setminus K \) onto \( \Omega := \overline{\mathbb{C}} \setminus \mathbb{D} \) such that \( \Phi(\infty) = \infty \) and \( \Phi'(<\infty) > 0 \). In the case where \( T \) is an analytic Jordan curve we denote by \( \rho(T) < 1 \) a number such that \( \Phi^{-1} \) has a univalent extension into \( |w| > \rho(T) \). Moreover, we put \( O_\rho := \Phi^{-1}(\{|w| > \rho\}) \) for each \( \rho > \rho(T) \).

It is known that \( |\Phi| \) is identically equal to \( P_K \) in \( O \) and therefore \( \Phi'(\infty) = \gamma_K^{-1} \). Hence, orthogonality relations (9) can be rewritten as

\[
\int_D \pi_{n,s}(z) \overline{\pi_{m,s}(z)} \, dA + \int_O \pi_{n,s}(z) \overline{\pi_{m,s}(z)} |\Phi(z)|^{-2s} \, dA = \delta_{nm},
\]

\( n, m \leq \lfloor s - 2 \rfloor \), where \( D \) is the interior domain of \( T \). Since \( |\Phi| > 1 \) in \( O \), we can formally set \( \pi_{n,\infty} \) to be polynomials satisfying

\[
\int_D \pi_{n,\infty}(z) \overline{\pi_{m,\infty}(z)} \, dA = \delta_{nm}.
\]

In a sense, potential theoretic polynomials \( \pi_{n,s} \) can be considered as perturbations of \( \pi_{n,\infty} \). The latter were initially studied by Carleman [2] who derived their exterior asymptotics (asymptotics in \( O \)) for the case \( T \) being an analytic Jordan curve. The results in [2] were subsequently extended by Suetin [21] to include \( C^{p+1,\alpha} \) Jordan curves. Other aspects of the behavior of \( \pi_{n,\infty} \), such as zero distribution and interior asymptotics, were investigated in [15, 6, 7]. The following theorem provides an analog of [21, Theorem 1.2] for potential theoretic polynomials \( \pi_{n,s} \).
Figure 1. In this figure, $K$ (respectively $\overline{\mathbb{D}}$) is the region enclosed by the black contour. Here, $K$ has analytic boundary, and the dashed contour on the left is the inner-most contour outside of which we can find a univalent extension of $\Phi^{-1}$. $O_\rho$ is represented by the region outside of the contour corresponding to radius $\rho$. The curves outside of $T$ are level lines of $P_K$.

**Theorem 1.** Let $T = \partial K$ be a Jordan curve of class $C^{p+1,\alpha}$, $p + \alpha > 1/2$, and \{\(\pi_{n,s}\)\}_{n=0}^{\lfloor s-2 \rfloor}$ be a sequence of polynomials satisfying orthogonality relations (9). The leading coefficient $\kappa_{n,s}$ of $\pi_{n,s}$ satisfies

$$
\kappa_{n,s} = \frac{1}{\gamma_{n+1}} \sqrt{\frac{n+1}{\pi}} \left(1 - \frac{n+1}{s}\right) \left[1 + \mathcal{O}\left(\frac{1}{n^{2(p+\beta)}}\right)\right]
$$

as $n, s \to \infty$, where $\beta$ is a real number such that $1/2 < p + \beta < p + \alpha$. Moreover, if $T$ is an analytic Jordan curve, then the error terms in (13) can be replaced by $\mathcal{O}(\rho^{2n})$ for any $\rho(T) < \rho < 1$.

It also holds that

$$
\pi_{n,s} = \sqrt{\frac{n+1}{\pi}} \left(1 - \frac{n+1}{s}\right) \Phi^n \Phi' \left[1 + \mathcal{O}(\Sigma_n)\right]
$$

uniformly on $\overline{O}$ as $n, s \to \infty$, where $\Sigma_n$ is given by Table 1.

In general, the location of the zeros of $\pi_{n,s}$ depends on $s$ (as well as, obviously, $K$ and $n$). However, as the following proposition shows, this is not the case for a family of ellipses which interpolate between the unit circle and the interval $[-2, 2]$.

**Proposition 2.** Let $T$ be such that $\phi(w) = w + \frac{q}{w}$, $q \in [0, 1]$, where $\phi$ is the inverse of $\Phi$. Then

$$
\pi_{n,s} = \sqrt{\frac{n+1}{\pi}} \left(1 - \frac{n+1}{s}\right) / \left(1 - q^{2n+2} \frac{s-n-1}{s+n+1}\right) \Phi^n \Phi' \left(1 - \frac{q^{n+1}}{\Phi^{2n+2}}\right)
$$
Table 1. The error term $\Sigma_n$ depending on the smoothness of $T$

<table>
<thead>
<tr>
<th>$T$</th>
<th>Analytic</th>
<th>$C_{p+1,\alpha}$</th>
<th>$\limsup_{n,s \to \infty} \frac{n}{s} &lt; 1$</th>
<th>$\limsup_{n,s \to \infty} \frac{n}{s} = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho(T) &lt; \rho &lt; 1$</td>
<td>$p \geq 2$</td>
<td>$p = 1$</td>
<td>$p = 0$</td>
<td>$p = 1$</td>
</tr>
<tr>
<td>$\Sigma_n$</td>
<td>$\rho^n$</td>
<td>$\log \frac{n}{n^p + \beta}$</td>
<td>$n^{1-2\beta}$</td>
<td>$n^{-2\beta}$</td>
</tr>
</tbody>
</table>

for all $n \leq |s-2|$ and all $s$ including $s = \infty$ except for $q = 1$. That is, polynomials $\pi_{n,s}$ are the renormalized Chebyshev polynomials of the second kind for the interval $[-2\sqrt{q}, 2\sqrt{q}]$, where $\pm 2\sqrt{q}$ are the foci of $T$.

Notice that in the proposition above all the ellipses have unit logarithmic capacity (i.e., $\gamma_K = 1$). Observe also that, if $q = 0$, $T = T(K = \overline{D})$ and

$$\pi_{n,s}(z) = \sqrt{\frac{n+1}{\pi}} \left( \frac{1}{s} - \frac{n+1}{s} \right) z^n.$$  

On the other hand, if $q = 1$, $T = K = [-2, 2]$ and

$$\pi_{n,s}(z) = \sqrt{\frac{s^2-(n+1)^2}{2\pi s}} \frac{1}{\sqrt{s^2-4}} \left[ \left( \frac{z + \sqrt{s^2-4}}{2} \right)^{n+1} - \left( \frac{z - \sqrt{s^2-4}}{2} \right)^{n+1} \right].$$

(15) It is easy to see that the asymptotic behavior of the normalizing constant in (15) is different from the one in (13). However, the case $K = [-2, 2]$ is not covered by Theorem 1.

Theorem 1 is the essential building block in proving results on asymptotic behavior of kernels $K_{N,s}$ and $\tilde{K}_{N,s}$ defined in (10) and (11), respectively. To continue, denote by $A_D^2$ the Hilbert space of holomorphic functions on $D$ whose moduli are square-integrable with respect to the area measure. We equip $A_D^2$ with the norm induced by the inner product

$$\langle f, g \rangle := \int_D f(z)\overline{g(z)} \, dA.$$

Denote by $K_D(z, w)$, $z, w \in D$, the reproducing kernel for $A_D^2$. That is,

$$f(z) = \int_D f(w)K_D(z, w) \, dA, \quad K_D(z, w) = \frac{1}{\pi} \frac{\psi'(z)\overline{\psi'(w)}}{\left(1 - \psi(z)\overline{\psi(w)}\right)^2},$$

(17) for any $f \in A_D^2$, where $\psi$ is any conformal map from $D$ onto $\mathbb{D}$ [9, §1.5]. It is known [9, Theorem 1.5.2] that $K_{N,\infty}$ is the reproducing kernel for the set of polynomials of degree at most $N-1$ in the sense of (17) and that $K_{N,\infty}(\cdot, w) \to K_D(\cdot, w)$ as $N \to \infty$ locally uniformly in $D$ for each $w \in D$.  


Theorem 3. Let $N \leq [s - 1]$. Under the conditions of Theorem 1, it holds that

$$K_{N,s}(z, w) = \frac{\Phi'(z)\Phi'(w)}{\pi} \left[ \left(1 - \frac{N + 1}{s}\right) \left(-(N + 1)\left[\Phi(z)\Phi(w)\right]^N + \frac{1 - \left[\Phi(z)\Phi(w)\right]^{N+1}}{1 - \Phi(z)\Phi(w)}\right) \right] + O(\max\{1, N^2\Sigma_N\})$$

uniformly for $z, w \notin K$, $z \neq w$, dist$(z, \partial K) \leq \text{const.}/N$ and dist$(w, \partial K) \leq \text{const.}/N$. Moreover, it holds that

$$K_{N,s}(z, z) = \frac{\Phi'(z)^2}{\pi} \left[ \frac{N(N + 1)}{2} \left(1 - \frac{N + 1}{s}\right) + \frac{N(N + 1)(N + 2)}{6s} \right] + O(\max\{1, N^2\Sigma_N\})$$

uniformly for $z \in \partial K$. Finally, we have that

$$\lim_{N,s \to \infty} K_{N,s}(\cdot, w) = K_D(\cdot, w)$$

locally uniformly in $D$ for each $w \in D$.

To describe the asymptotic behavior of kernels (10) near the boundary $\partial K$, it is convenient to introduce the following notation. Set

$$H_0(\tau) := \frac{2}{\pi} \frac{e^\tau(\tau - 1) + 1}{\tau^2} \quad \text{and} \quad H_1(\tau) := \frac{6}{\pi} \frac{e^\tau(\tau - 2) + \tau + 2}{\tau^3},$$

and define $H_\ell$ to be the convex combination,

$$H_\ell(\tau) := \frac{3 - 3\ell}{3 - 2\ell} H_0(\tau) + \frac{\ell}{3 - 2\ell} H_1(\tau), \quad \ell \in (0, 1).$$

Note that the value at the origin for each of these functions is determined by taking a limit; that is, $H_\ell(0) = 1$ for all $\ell \in [0, 1]$. The following theorem is an analog of [13, Theorem 2.1].

Theorem 4. Let $\tau(a, z) := a\Phi'(z)\Phi(z)^{-1}, \ell := \lim_{N,s \to \infty} NS^{-1} \in [0, 1]$ and for $\ell > 0$, set

$$\omega(a, z) := \begin{cases} \exp\left\{ -\text{Re} \left( \tau(a, z) / \ell \right) \right\}, & \text{Re} \left( \tau(a, z) \right) > 0, \\ 1, & \text{otherwise.} \end{cases}$$

Under the conditions of Theorem 1, and assuming $z \in \partial K$,

$$\lim_{N,s \to \infty} \frac{K_{N,s}(z + \frac{\alpha}{N}, z + \frac{b}{N})}{K_{N,s}(z, z)} = H_\ell \left( \tau(a, z) + \tau(b, z) \right)$$

where we assume that $s$ and $N$ go to $\infty$ in such a way that $N \leq [s - 1]$. Moreover, if $\ell > 0$, it holds that

$$\lim_{N,s \to \infty} \frac{K_{N,s}(z + \frac{\alpha}{N}, z + \frac{b}{N})}{K_{N,s}(z, z)} = \omega(a, z)\omega(b, z)H_\ell \left( \tau(a, z) + \tau(b, z) \right)$$

\(^3\)The argument of $\tau(a, z)$ is equal to the angle between $a$ and $\Phi(z)/\Phi'(z)$, the normal to $T$ at $z$. 
If $\ell = 0^2$,
\begin{equation}
\lim_{N,s \to \infty} \frac{K_{N,s}(z + \frac{a}{N}, z + \frac{b}{N})}{K_{N,s}(z, z)} = \begin{cases} H_0 \left( \tau(a, z) + \tau(b, z) \right), & \text{Re}(\tau(a, z)) < 0, \\
0, & \text{Re}(\tau(a, z)) > 0 \text{ or } \text{Re}(\tau(b, z)) > 0.
\end{cases}
\end{equation}

The convergence in (24) and (25) is uniform for $a, b$ in compact subsets of $C$.

Observe that by putting $s = \infty$ (that is $\ell = 0$), formulae (18)–(23) specialize to the asymptotic formulae obtained in [13] for Carleman polynomials. Notice also that when $s = N + 1$ ($\ell = 1$), the first summands in (18)–(23) disappear and only the second ones remain. For general $\ell$, formulae (18)–(23) turn out to be convex combination of these two extreme cases.

3. THREE MODELS OF POTENTIAL THEORETIC ENSEMBLES

Before proceeding to the proofs of our main results, we will present three models, a matrix model, an electrostatic model and a polynomial model, whose joint density of eigenvalues, particles and roots coincide with the potential theoretic ensembles we are considering.

3.1. Entropic Normal Matrix Ensembles. The entropy of a self-map $T$ on a metric space $X$ is a measure of how the distance between nearby points is stretched under iteration of $T$. In the case where $Z$ is an $N \times N$ complex matrix acting on $C^N$, the entropy of $Z$ is given by
\[ h(Z) = \sum_{n=1}^{N} \log \max \{1, |\lambda_n|\}, \]
where $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues of $Z$ [23]. We may use this to create a probability measure on normal $N \times N$ complex matrices, which we will denote by $\mathcal{N}_N(C)$.

There exists a canonical measure on $\mathcal{N}_N(C)$ induced by the standard metric on $C^{N \times N}$ and we may define a probability density with respect to this measure by writing
\[ P_N(Z) = \frac{1}{Z_N} e^{-2s h(Z)}, \]
where $Z_N$ is a normalization constant and $s > N$ is a real number necessary so that the probability measure is actually finite.

This probability measure on normal matrices induces a symmetric probability measure on $C^N$ as identified with vectors of eigenvalues. This measure is absolutely continuous with respect to Lebesgue measure and its density is given as in (2) with $w(\lambda) = \max \{1, |\lambda|\}^{-2s}$ [4, 16]. Normal matrix ensembles, and in particular the statistics of their eigenvalues, were first considered in [3] and [4].

The function $\lambda \mapsto \log \max \{1, |\lambda|\}$ is the logarithmic (equilibrium) potential of the closed unit disk, and the weight for the entropic ensemble is formed from this in the obvious manner. We therefore see the eigenvalue statistics of the entropic normal matrix ensemble coincides with the potential theoretic ensemble with $K = \overline{D}$.

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2As is clear from (11), the function $\omega(a, z)$ is designed to describe the limit of $P_{K}^{-s}(z + a/N)$ as $N, s \to \infty$. This limit depends on whether or not the points $z + a/N$ belong to $O$ for $N$ large enough. The case $\text{Re}(\tau(a, z)) = 0$ corresponds to the situation when the sequence $\{z + a/N\}$ approaches $z \in \partial K$ tangentially to the boundary. This does not cause a problem in (22) as this function is continuous with respect to $a$. However, when $\ell = 0$ formula (22) cannot be used as the limit is described by a discontinuous function of $a$ and the convexity of the boundary $\partial K$ at $z$ starts to play a role.
3.2. **Two-Dimensional Electrostatics.** In two-dimensional electrostatics, charged particles are identified with points in the extended complex plane. The potential energy of a system of two like charged particles located at \( z, w \in \mathbb{C} \) is proportional to \(-\log|z-w|\). More generally, if \( z_1, z_2, \ldots, z_N \) are the locations of \( N \) identically charged particles, then \( \mathbf{z} \) determines the state of the system and the potential energy of this state is given by

\[
-E(\mathbf{z}) = \sum_{m<n} \log|z_n - z_m|.
\]

The energy is minimized when the particles are all at \( \infty \). In order for the system to be found in a state where the particles are at finite positions, there needs to be a potential (or other obstructions) which repels the particles from \( \infty \). We represent this field by \( V \) so that the interaction energy between a particle located at \( z \) and the field is given by \( V(z) \). The total potential energy of the system comprised of the \( N \) particles in the field is given by

\[
E(\mathbf{z}) = \sum_{n=1}^{N} V(z_n) - \sum_{m<n} \log|z_n - z_m|.
\]

The system is assumed to be in contact with a heat reservoir so that the energy of the system is variable, but the temperature is fixed. In this setting, \( \beta \) denotes the reciprocal of the temperature, and the Boltzmann factor for the state \( \mathbf{z} \) is given by

\[
e^{-\beta E(\mathbf{z})} = \left\{ \prod_{n=1}^{N} e^{-\beta V(z_n)} \right\} \prod_{m<n} |z_n - z_m|^\beta.
\]

This quantity gives the relative density of states, so that the probability (density) of finding the system in state \( \mathbf{z} \) is given by

\[
\frac{1}{Z_N} e^{-\beta E(\mathbf{z})} \quad \text{where} \quad Z_N = \int_{\mathbb{C}^N} e^{-\beta E(\mathbf{z})} dA^N(\mathbf{z}).
\]

Comparing with (2) we see that, when \( \beta = 2 \) the density of states is identical with the density of eigenvalues of the normal matrix ensemble with weight and \( w(z) = e^{-2V(z)} \).

In this model, a compact set \( K \) is identified with a conducting region. A charge supported on \( K \) will distribute itself to minimize its potential energy, and this distribution, suitably normalized, leads to the equilibrium measure on \( K \). In this way, we can think of the function \(-s \log P_K(z)\) as the potential energy felt by an oppositely charged particle at \( z \) when placed in the field given by the minimal energy configuration formed by placing a total charge of \( s \) on \( K \). In this situation where our system consists of \( N \) charged particles, the condition that \( s > N \) is required to make \( \infty \) repulsive (or rather to make \( K \) sufficiently attractive so that the particles do not flee to \( \infty \)). It follows that the statistics of particles in this model agree with those of the potential theoretic ensemble for \( K \).

3.3. **Roots of Random Polynomials.** The Mahler measure of a polynomial \( f(x) \in \mathbb{C}[x] \) is given by

\[
M(f) = \exp \left\{ \int_{0}^{1} \log|f(e^{2\pi i \theta})| d\theta \right\},
\]

is an example of a height function; that is a function which measures the complexity of arithmetic objects, in this case polynomials. One type of problem of interest to number theorists is to provide asymptotic estimates for the number of arithmetic objects whose height is bounded by \( C \) as \( C \to \infty \). For instance, for the Mahler measure, such estimates for the
number of integer polynomials of fixed degree and Mahler measure bounded by $C$ as $C \to \infty$ was given by Chern and Vaaler in [5]. They also gave a similar estimate for the number of polynomials with Gaussian integer ($\mathbb{Z}[i]$) coefficients.

In the latter case, the main term in their estimate came from the calculation of the Lebesgue measure of the set of polynomials of degree $N$ with complex coefficients whose Mahler measure is at most 1. A key aspect of their proof is to show that this volume is equal to

$$
\frac{\pi}{N+1} \int_{\mathbb{C}^N} \left\{ M \left( x^N + \sum_{n=1}^{N} a_n z^{N-n} \right) \right\}^{-2N-2} dA^N(a),
$$

That is the volume is proportional to an integral of a (negative) power of the Mahler measure of monic polynomials with respect to Lebesgue measure on the non-leading coefficients of such polynomials. Moreover, after the change of variables from coefficients to roots of polynomials, this volume reduces to

$$
\frac{\pi}{N+1} \int_{\mathbb{C}^N} \left\{ \prod_{n=1}^{N} \exp \left\{ \int_0^1 \log |\alpha_n - e^{2\pi i \theta}| \, d\theta \right\}^{-2s} \prod_{m<n} |\alpha_n - \alpha_m|^2 dA^N(\alpha); \quad s = N + 1.
$$

That is, this volume, up to the factor of $\pi/(N + 1)$ is equal to the normalization constant $Z_N$ for the potential theoretic ensemble for the unit circle for the value $s = N + 1$. In fact, Chern and Vaaler were able to show that this normalization constant as a function of $s$ is a rational function in $s$ with poles at positive integers $\leq N$. This striking result can be seen as a consequence of determinantal nature of the correlation functions.

The derivation of (26) shows that the roots of a polynomial chosen randomly from the volume of complex polynomials of degree $N$ and Mahler measure at most 1 obey the same statistics as those of the potential theoretic ensemble for the disk. This gives a polynomial model for these statistics.

The computation of the normalization constant of this polynomial model for potentials for certain other compact regions (in particular the ellipses considered in Proposition 2) is given in [19], while a more general treatment for more general potentials is given in [20]. The special case where the family of ellipses degenerates to the interval $[-2, 2]$ on the real axis, and its application to the estimation of counting reciprocal polynomials with bounded Mahler measure is given in [18] and [19].

### 4. Proofs

Let us state two formulæ that shall be useful later. The first one is the Cauchy-Green identity for the domain $D$, which says that

$$
\int_D g(z) h'(z) \, dA = \frac{1}{2i} \oint_T g(z) \overline{h(z)} \, dz
$$

whenever $g$ and $h'$ are holomorphic functions in $D$ that continuously extend to $T$, where $\oint_T$ always means integration in the counter-clockwise direction unless specified otherwise. Now, assume that $g$ and $h$ are holomorphic functions in $O$ such that $h$ vanishes at infinity, $g$ has at least a double zero there, and $g, h$ and $h'$ continuously extend to $T$. Then by using the transformation $z \mapsto 1/z$ and (27), one can show that the Cauchy-Green identity for $O$ assumes the form

$$
\int_O g(z) \overline{h'(z)} \, dA = -\frac{1}{2i} \oint_T g(z) \overline{h(z)} \, dz.
$$
To prove Theorem 1 we use the method of normal moments in which we rely on the results in [21, Ch. I]. We must therefore discuss Faber polynomials before proceeding to the proof of Theorem 1.

4.1. Faber Polynomials. Denote by \( F_n \) the \( n \)-th Faber polynomial for \( D \) associated with \( \Phi \). That is,

\[
F_n(z) = \oint_T \frac{\Phi^n(t)\Phi'(t)}{t-z} \, dt \, 2\pi i, \quad z \in D.
\]

In other words, \( F_n \) is the polynomial part of \( \Phi^n \Phi' \). Then it follows from Plemelj-Sokhotski formulæ [10] that

\[
(29) \quad F_n = \Phi^n \Phi' + E_n,
\]

where \( E_n \) is a holomorphic function in \( O \) vanishing at infinity with integral representation

\[
(30) \quad E_n(z) := \oint_T \frac{\Phi^n(t)\Phi'(t)}{t-z} \, dt \, 2\pi i, \quad z \in O.
\]

We are interested in the asymptotic behavior of

\[
(31) \quad m_{j,k}^s := \int_C F_j F_k P_k^2s \, dA = \int_D F_j F_k \, dA + \int_O F_j F_k |\Phi|^{-2s} \, dA, \quad j, k \leq |s-2|,
\]

where we used (12) for the second representation. It was shown in [21, Equation (1.32) combined with (28) above] that the first integral on the right-hand side of (31) can be written as

\[
(32) \quad \frac{\pi}{k+1} \left( \delta_{jk} - \frac{k+1}{\pi} \int_O E_j E_k \, dA \right) =: \frac{\pi}{k+1} (\delta_{jk} + I_D).
\]

Moreover, it was also obtained there, see [21, Equation (1.45) and Lemma 1.5], that

\[
(33) \quad |I_D| \leq \frac{\text{const.}}{(j+1)^{p+\alpha} (k+1)^{p+\alpha}} \quad \text{or} \quad |I_D| \leq \text{const.} \rho^{j+k},
\]

where both constants are independent of \( j \) and \( k \), but depend on \( T \) and \( \rho \) (in the analytic case). Hereafter, by stating a double estimate of the form (33), we always assume that the first bound is given for \( T \) of class \( C^{p+1,\alpha} \) and the second one for an analytic \( T \) with \( \rho(T) < \rho < 1 \).

Similar to (32), we shall show that the second integral on the right-hand side of (31) can be written as

\[
(34) \quad \frac{\pi}{s-(k+1)} \left( \delta_{jk} + \frac{s-(k+1)}{\pi} \int_O E_j E_k |\Phi|^{-2s} \, dA \right) =: \frac{\pi}{s-(k+1)} (\delta_{jk} + I_O),
\]

where the modulus of the integral \( I_O \) satisfies a similar bound to (32), namely,

\[
(35) \quad |I_O| \leq \frac{\text{const.}}{(j+1)^{p+\beta} (k+1)^{p+\beta}} \quad \text{or} \quad |I_O| \leq \text{const.} \rho^{j+k}
\]

for any \( 0 < \beta < \alpha \) (from here forward, we assume \( \beta \) satisfies \( p + \beta > 1/2 \)). Indeed, the integral on the left-hand side of (34) can be written with the help of (29) as

\[
(36) \quad \int_O \Phi^j \Phi^k |\Phi|^2 |\Phi|^{-2s} \, dA + \int_O \Phi^j \Phi E_k |\Phi|^{-2s} \, dA + \int_O E_j \Phi \Phi |\Phi|^{-2s} \, dA + \int_O E_j E_k |\Phi|^{-2s} \, dA.
\]

It can be immediately computed by conformity of \( \Phi \) that

\[
(37) \quad \int_O \Phi^j \Phi^k |\Phi|^2 |\Phi|^{-2s} \, dA = \int_O w^j \bar{w}^k |w|^{-2s} \, dA = \frac{\pi}{s-(k+1)} \delta_{jk}.
\]
To evaluate the second and the third integrals in (36), set

$$E_k(z) := \Phi^s(z) \int_z^\infty \frac{E_k(t)}{\Phi^s(t)} \, dt,$$

which is clearly a holomorphic function in $O$. Using this function as well as Cauchy-Green identity (28) applied to $g = \Phi^{j-s} \Phi'$ and $h' = E_k \Phi^{-s}$, the second integral in (36) can be written as

$$\frac{1}{2i} \oint_T \Phi_j(z) \Phi'(z) E_k^*(z) \, dz = \frac{1}{2i} \oint_T w^j (E_k^* \circ \phi)(w) \, dw = \frac{1}{2i} \oint_T w^j E_k^*(1/w) \, dw = 0,$$

where $T := \{ |w| = 1 \}$, $\phi$ is the inverse of $\Phi$, and the last equality follows from the holomorphy of $(E_k^* \circ \phi)(1/w)$ in the unit disk. Analogously, we obtain that

$$\int_O E_j \Phi_k \Phi' |\Phi|^{-2s} \, dA = \frac{1}{2i} \oint_T \Phi(1/z) E_j^*(z) \, dz = 0.$$

By combining (37), (39), and (40), and applying Cauchy-Green identity (28) once more, we deduce that

$$I_O = \frac{s - (k + 1)}{2\pi i} \oint_T E_j(z) E_k^*(z) \, dz.$$

Thus, to estimate $I_O$, we need to estimate $E_k$ and $E_k^*$ first. It was shown in [21, Lemma 1.3] that

$$\max_T |E_k| \leq \text{const.} \frac{\log(k + 2)}{(k + 1)^{p+\alpha}} \quad \text{or} \quad \max_T |E_k| \leq \text{const.} \rho^k.$$

This consequently implies for $z \in T$ that

$$|E_k(z)| \leq \left| \int_\infty^{\Phi(z)} \frac{w(E_k \circ \phi)(w)}{w^{s+1}} \phi'(w) \, dw \right| \leq \text{const.} |E_k| \Phi \int_1^{\infty} \frac{dx}{x^{s+1}} = \frac{1}{s} \left( \frac{1 - (k + 1)^{s}}{s} \right) |E_k|,$$

since $\phi'$ extends continuously onto $T$ whenever the latter is $C^{1,\delta}$, $\delta > 0$ [21, §2], and by the maximum modulus principle applied to $E_k \Phi$ (recall that $E_k$ vanishes at infinity). Clearly, (35) follows now from (41), (42), and (43).

Gathering together (32) and (34), we get that

$$m_{j,k}^s = \frac{s\pi}{(k + 1)(s - (k + 1))} \left( \delta_{jk} + \epsilon_{j,k}^s \right),$$

where

$$\epsilon_{j,k}^s := \frac{s - (k + 1)}{s} I_D + \frac{k + 1}{s} I_O = -\frac{k + 1}{\pi} \left( 1 - \frac{k + 1}{s} \right) \int_O E_j E_k (1 - |\Phi|^{-2s}) \, dA.$$

Moreover, it readily follows from (33) and (35) that

$$|\epsilon_{j,k}^s| \leq \frac{\text{const.}}{(j + 1)^{p+\beta}(k + 1)^{p+\beta}} \quad \text{or} \quad |\epsilon_{j,k}^s| \leq \text{const.} \rho^{j+k}.$$
4.2. The Von Koch-Riesz Algebra. Denote by $D$ the algebra of all operators defined on $\ell_2(\mathbb{N})$ by matrices $A = [a_{j,k}]_{j,k=0}^{\infty}$ with respect to the standard basis for which

$$\|A\|_D := \max \left\{ \sum_{k=0}^{\infty} |a_{k,k}|, \left( \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} |a_{j,k}|^2 \right)^{1/2} \right\} < \infty. $$

It is known [12, Theorem II.2.1], if $\{A_n\}$ is a sequence in $D$ converging to $A \in D$ (with respect to $\| \cdot \|_D$), then the determinant of $I + A_n$ ($I$ being the identity operator) converges to the determinant of $I + A$.

Let $\{s_n\}$ be an increasing sequence of positive reals such that $s_n \to \infty$ as $n \to \infty$. Set, for convenience, $\epsilon_{j,k}^n := 0$ when either $j$ or $k$ is greater than $[s_n - 2]$ and define $E_{s_n} := [\epsilon_{j,k}^n]_{j,k=0}^{\infty}$. We also set $E_\infty := [\epsilon_{j,k}^\infty]_{j,k=0}^{\infty}$, where we put

$$(47) \quad \epsilon_{j,k}^\infty := -\frac{k+1}{\pi} \int O E_k E_j dA. $$

Observe that the estimate in (46) is also valid for $s = \infty$. Using this bound, it is simple to verify that

$$\|E_{s_n}\|_D \leq \text{const.} \sum_{k=1}^{\infty} \frac{1}{k^{2(p+\beta)}} \quad \text{or} \quad \|E_{s_n}\|_D \leq \frac{\text{const.}}{1 - \rho^2} $$
for each $n$ including the case $n = \infty$, where the constant $\sum_{k=1}^{\infty} k^{-2(p+\beta)}$ is finite as $p + \beta > 1/2$. Thus, all the operators $E_{s_n}$ belong to the Von Koch-Riesz algebra $D$. Moreover, it holds that

$$(48) \quad \|E_{s_n} - E_\infty\|_D \to 0 \quad \text{as} \quad N \to \infty. $$

Indeed, let $\{k_n\}$ be a non-decreasing sequence of integers such that $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$. Then

$$(49) \quad \sum_{k=k_n}^{\infty} |\epsilon_{k,k}^n - \epsilon_{k,k}^\infty| \leq \sum_{k=k_n}^{\infty} \left( |\epsilon_{k,k}^n| + |\epsilon_{k,k}^\infty| \right) \leq \frac{\text{const.}}{(k_n + 1)^{p+\beta}} \quad \text{or} \quad \sum_{k=k_n}^{\infty} |\epsilon_{k,k}^n - \epsilon_{k,k}^\infty| \leq \text{const.} \rho^{2k_n} $$

by (46). Furthermore, we can readily deduce from (45) and (47) using the notation of (32) and (34) that

$$|\epsilon_{k,k}^n - \epsilon_{k,k}^\infty| = \frac{k+1}{n} |I_0 - I_D| \leq \frac{k_n}{n} \frac{\text{const.}}{(k + 1)^{2(p+\beta)}} \quad \text{or} \quad |\epsilon_{k,k}^n - \epsilon_{k,k}^\infty| \leq \frac{k_n}{n} \text{const.} \rho^{2k_n} $$

by (33) and (35) for all $k \in \{0, \ldots, k_n - 1\}$. Therefore, it holds that

$$(50) \quad \sum_{k=0}^{k_n-1} |\epsilon_{k,k}^n - \epsilon_{k,k}^\infty| \leq \frac{k_n}{n}. $$

Combining (49) and (50), we deduce that

$$(51) \quad \sum_{k=0}^{\infty} |\epsilon_{k,k}^n - \epsilon_{k,k}^\infty| \to 0 \quad \text{as} \quad n \to \infty $$

by the choice of the sequence $\{k_n\}$. Analogously, one can show that

$$(52) \quad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\epsilon_{j,k}^n - \epsilon_{j,k}^\infty|^2 \to 0 \quad \text{as} \quad n \to \infty, $$
which finishes the proof of (48).

Naturally [12, Section I.1], it holds that det(I + E_{s,n}) = det \left[ \delta_{jk} + \epsilon_{s,k} \right]_{j,k=0}^{s,n-2} and therefore we deduce from the remark made at the beginning of this section that

\begin{equation}
\det(I + E_{s,n}) \to \det(I + E_{\infty}) > 0 \quad \text{as} \quad n \to \infty,
\end{equation}

where the last inequality was shown in [21, Section I.4].

4.3. Proof of Theorem 1. Since \{F_n\} is a complete system of polynomials, each \pi_{n,s} can be expressed as a linear combination of \{F_0, \ldots, F_n\} with the coefficients determined via orthogonality relations (9). In fact, it holds that

\begin{equation}
\pi_{n,s}(z) = \frac{1}{\sqrt{D_{n-1,s}D_{n,s}}} \begin{bmatrix} m_{0,0} & m_{0,1} & \cdots & m_{0,n} \\ m_{1,0} & m_{1,1} & \cdots & m_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1,0} & m_{n-1,1} & \cdots & m_{n-1,n} \\ \frac{F_0(z)}{\gamma(s+1,n+1)} & \frac{F_1(z)}{\gamma(s+1,n+1)} & \cdots & \frac{F_n(z)}{\gamma(s+1,n+1)} \end{bmatrix},
\end{equation}

where the moments \(m_{j,k}\) are defined in (31) and \(D_{n,s} := \det[m_{j,k}]_{j,k=0}^{n}\).

Set \(\Delta_{n,s} := \det[\delta_{jk} + \epsilon_{s,k}]_{j,k=0}^{n}\) and observe that

\begin{equation}
D_{n,s} = \Delta_{n,s} \prod_{j=0}^{n} \frac{s!}{(j+1)(s-(j+1))} \quad \text{and} \quad D_{n,s}(k) = \Delta_{n,s}(k) \prod_{j=0,j\neq k}^{n} \frac{s!}{(j+1)(s-(j+1))}
\end{equation}

by (44), where the determinants \(D_{n,s}(k)\) and \(\Delta_{n,s}(k)\) are obtained from the same matrices as \(D_{n,s}\) and \(\Delta_{n,s}\) only with the last row and the \((k+1)\)-st column removed. Given (46), it is a straightforward algebraic computation using Hadamard’s inequality, see [21, Lemma 1.7], to derive that

\begin{equation}
\Delta_{n,s}(k) \leq \frac{\text{const.}}{(k+1)^{p+b}(n+1)^{p+b}} \quad \text{or} \quad \Delta_{n,s}(k) \leq \text{const.} \rho^{k+n}
\end{equation}

for any \(k \in \{0, 1, \ldots, n-1\} \).

On the other hand, the family \(\{\Delta_{n,s}\}\) is bounded away from zero. Indeed, as mentioned just before (12), \(\Phi'(\infty) = \gamma_{K}^{-1}\) and therefore the leading coefficient of \(F_n\) is equal to \(\gamma_{K}^{n-1}\). Hence, we get from (54) and (55) that

\begin{equation}
x_{n,s}^{-n+1} = \sqrt{\frac{D_{n-1,s}}{D_{n,s}}} = \sqrt{\frac{k+1}{\pi} \left( 1 - \frac{k+1}{s} \right) \frac{\Delta_{n-1,s}}{\Delta_{n,s}}}.
\end{equation}

Recall that any monic orthogonal polynomial has the smallest \(L^2\)-norm with respect to the weight of orthogonality among all monic polynomials of the same degree. In particular,

\begin{equation}
\frac{1}{x_{n,s}^2} = \int_C |\pi_{n,s}/x_{n,s}|^2 |P_K|^{-2s} dA \leq \int_C |\gamma_{K}^{n+1}F_n|^2 |P_K|^{-2s} dA = \gamma_{K}^{2n+2}m_{n,n}^s.
\end{equation}

Therefore, it follows from (57), (44), and (45) that

\begin{equation}
\Delta_{n,s} = \frac{k+1}{\pi} \left( 1 - \frac{k+1}{s} \right) \frac{\Delta_{n-1,s}}{\gamma_{K}^{2n+2}x_{n,s}^2} \leq \frac{k+1}{\pi} \left( 1 - \frac{k+1}{s} \right) m_{n,n}^s \Delta_{n-1,s}
\end{equation}

\begin{equation}
= (1 + \epsilon_{n,n}^s) \Delta_{n-1,s} < \Delta_{n-1,s}.
\end{equation}

Hence, it holds that

\begin{equation}
\inf_{s} \min_{1 \leq n \leq [s-2]} \Delta_{n,s} = \inf_{s} \Delta_{\lfloor s-2 \rfloor,s} > 0
\end{equation}
by (53) since $\Delta_{[s-2],s} = \det(I + E_s)$, which proves the claim.

Thus, expanding the determinant $\Delta_{n,s}$ by the last row, we get that

$$
\Delta_{n,s} = (1 + \epsilon_{n,s}) \Delta_{n-1,s} + \sum_{k=0}^{n-1} (-1)^{n+k} \epsilon_{n,k} \Delta_{n,s}(k).
$$

Dividing both sides of the equality above by $\Delta_{n-1,s}$ and using (46), (56), and (58) yields

$$
\frac{\Delta_{n,s}}{\Delta_{n-1,s}} = 1 + O\left(\frac{1}{n^{2(p+\beta)}}\right) \quad \text{or} \quad \frac{\Delta_{n,s}}{\Delta_{n-1,s}} = 1 + O\left(\rho^{2n}\right).
$$

Clearly, we get (13) by taking the reciprocal of (60) and substituting it into (57).

Now, expanding the determinant in (54) by the last row as in (59) yields

$$
\pi_{n,s} = \sqrt{n + 1} \left(1 - \frac{n + 1}{s}\right) \sqrt{\frac{\Delta_{n-1,s}}{\Delta_{n,s}}} \left(F_n + \sum_{k=0}^{n-1} (-1)^{n+k} \frac{(k+1)(s-k-1)}{(n+1)(s-n-1)} \frac{\Delta_{n,s}(k)}{\Delta_{n-1,s}} F_k\right).
$$

Hence, by factoring out $\Phi^n\Phi'$ and using (29), the error term in (14) can be written as

$$
\sqrt{\frac{\Delta_{n-1,s}}{\Delta_{n,s}}} - 1 + \sqrt{\frac{\Delta_{n-1,s}}{\Delta_{n,s}}} \left[\frac{E_n}{\Phi^n\Phi'} + \sum_{k=0}^{n-1} (-1)^{n+k} \frac{(k+1)(1-k+\frac{1}{s})}{(n+1)(1-n+\frac{1}{s})} \frac{\Delta_{n,s}(k)}{\Delta_{n-1,s}} \left(1 + \frac{E_k}{\Phi^n_{k}\Phi'}\right) \frac{1}{\Phi^{n-k}}\right].
$$

Since $|\Phi| > 1$ and $|\Phi'|$ is bounded away from zero in $O$, we get from (42), (56), (58), and (60) that the error term in (14) is of order

$$
\frac{\log(n+1)}{(n+1)^{p+\alpha}} + \frac{1}{(n+1)^{2(p+\beta)}} \sum_{k=0}^{n-1} \frac{(k+1)(1-k+\frac{1}{s})}{(n+1)(1-n+\frac{1}{s})} \frac{1}{(n+1)^{p+\beta}},
$$

If $\limsup_{n,s \to \infty} n/s < 1$, then the fractions $(1 - k+1)/(1 - n+1)$ are uniformly bounded above and it easily follows from (61) that the error term in (14) is of order

$$
\frac{\log(n+1)}{(n+1)^{p+\alpha}} + \frac{1}{(n+1)^{2(1-\beta)}},
$$

where the first summand is larger for all $p \geq 1$ and and the second one is larger when $p = 0$.

Clearly, the estimate for the error term in the case of analytic curve, can be derived in a similar fashion. On the other hand, if $\limsup_{n,s \to \infty} n/s = 1$, then we use the estimate

$$
1 - \frac{n+1}{s} \geq 1 - \frac{n+1}{n+2} = \frac{1}{n+2},
$$

which is valid since $n \leq [s - 2]$. In this case (62) gets replaced by

$$
\frac{\log(n+1)}{(n+1)^{p+\alpha}} + \frac{1}{(n+1)^{2(1-\beta)}},
$$

where the first summand is larger for all $p \geq 2$ and and the second one is larger when $p = 1$ (we exclude $p = 0$ as in this case the above bound grows as $n^{2(1-\beta)}$). Analogous estimate shows that the error term in (14) is of order $\rho^{-n}$ when $T$ is an analytic curve. This finishes the proof of Theorem 1. \qed
4.4. **Proof of Proposition 2.** Let $U_n$ be the monic Chebyshev polynomial of the second kind for the interval $[-2\sqrt{q}, 2\sqrt{q}]$. That is,

$$U_n = \Phi^n \Phi' \left( 1 - \frac{q^{n+1}}{\Phi^{2n+2}} \right), \quad \Phi(z) = \frac{z + \sqrt{z^2 - 4q}}{2}, \quad z \in O.$$

It can be readily checked that the inverse of $\Phi$ is indeed $\phi(w) = w + q/w$, $\Phi$ is the conformal map of the complement of $[2\sqrt{q}, 2\sqrt{q}]$ onto $\{ w : |w| > \sqrt{q} \}$ with positive derivative at infinity, and the level lines of $\Phi$ are ellipses with foci $\pm 2\sqrt{q}$.

Let us show that polynomials $U_n$ are orthogonal on $D$ with respect to the area measure. It follows from the Cauchy-Green identity (27) that

$$2i(k + 1) \int_D U_n(z) \bar{z}^kdA = \oint_T (\Phi^n(z) - \frac{q^{n+1}}{\Phi^{n+2}(z)}) \Phi'(z) \bar{z}^{k+1}dz$$

(63)

where we used the identity $w = 1/\bar{w}$ on $T$ and the last equality is a consequence of the facts $\int_T w^j w^l dw = 0$ for all $j \geq -n$ and $\int_T w^{-n-2} w^j dw = 0$ for all $j \leq n$.

In another connection, the Cauchy-Green identity (28) yields

$$2i \int_O U_n(z) \bar{z}^k|\Phi(z)|^{-2s} dA = \oint_T U_n(z)Q_{s,k}(z)dz = \oint_T (w^n - q^{n+1}/w^{n+2})(Q_{s,k} \circ \phi)(w)dw,$$

where

$$(Q_{s,k} \circ \phi)(w) = \Phi(\phi(w))^s \int_{\phi(w)}^{\infty} \frac{t^k dt}{\Phi^s(t)} = w^n \int_{w}^{\infty} \frac{\phi^k(t)}{t^s} \phi'(t) dt.$$

It is easy to check using the expression $\phi(w) = w + q/w$ that $Q_{s,k} \circ \phi$ is a Laurent polynomial in $w$ with exponents ranging from $(k + 1)$ to $-(k + 1)$. Hence, as in the case for $D$, we get that polynomials $U_n$ are orthogonal on $O$ with the weight $|\Phi|^{-2s}$.

Altogether, polynomials $U_n$ are orthogonal over $\mathbb{C}$ with respect to the measure $P_K^{-2s} dA$. In fact, it can be easily shown that they are also Faber polynomials for this $K$. So, it only remains to compute the normalizing factor. Evaluating as in (63) and (64), we get that

$$\int_D U_n(z) \bar{z}^kdA = \frac{1}{2i(n + 1)} \oint_T \left( w^n - \frac{q^{n+1}}{w^{n+2}} \right) \left( (qw)^{n+1} + \cdots + \frac{1}{w^{n+1}} \right) dw$$

$$= \frac{\pi}{n + 1} (1 - q^{2n+2})$$

and

$$\int_O U_n(z) \bar{z}^k|\Phi(z)|^{-2s} dA = \frac{1}{2i} \oint_T \left( w^n - \frac{q^{n+1}}{w^{n+2}} \right) \left( \frac{-(qw)^{n+1}}{s + n + 1} + \cdots + \frac{1}{s - (n + 1) w^{n+1}} \right) dw$$

$$= \pi \left( \frac{1}{s - (n + 1)} + \frac{q^{2n+2}}{s + n + 1} \right).$$

Thus, we deduce that

$$K_{n,s}^{-2} = \int_D |U_n(z)|^2 P_K^{-2s}(z) dA = \int_D U_n(z) \bar{z}^kdA + \int_O U_n(z) \bar{z}^k|\Phi(z)|^{-2s} dA$$

$$= \frac{\pi s}{(n+1)(s-n-1)} \left( 1 - q^{2n+2} \frac{s - n - 1}{s + n + 1} \right). \quad \Box$$
4.5. Weak Convergence in $A^2_D$. The authors have no doubt that the following claim is well-known. However, we were unable to find a particular reference.

Let $\{f_n\}$ be a sequence in $A^2_D$ with uniformly bounded norms (induced by the inner product (16)). Then $\{f_n\}$ converges weakly to zero if and only if this sequence converges to zero locally uniformly in $D$. Indeed, since point-evaluations are locally uniformly bounded functionals on $A^2_D$ \cite[Theorem 6.12]{8}, the “if” part follows. Analogously,

$$\left\langle f_n, K_D(z, \cdot) \right\rangle = \int_D f_n(w) K_D(z, w) \, dA = f(z) \to 0 \quad \text{as} \quad n \to \infty$$

for any $z \in D$ when $f_n$ converge to zero locally uniformly in $D$. Let now $g$ be an arbitrary function in $A^2_D$ and $\{\zeta_j\}_{j=1}^{\infty}$ be a sampling sequence in $D$ \cite[§ 6.1]{8}. Then $g$ can be written as

$$g(w) = \sum_{j=1}^{\infty} g_j K_D(z_j, w), \quad \psi(z_j) = \zeta_j,$$

where $\psi$ is a conformal map of $D$ onto $\mathbb{D}$, the series converges in $A^2_D$-norm, and the sequence $\{g_j\}$ is $l^2$-summable \cite[Theorem 6.12]{8}. Then for any $\epsilon > 0$, there exist $J \in \mathbb{N}$ and $N_J \in \mathbb{N}$ such that

$$\left\| f_n - \sum_{j=1}^{J} g_j K_D(z_j, \cdot) \right\|_{2,D} \leq \left\| f_n - \sum_{j=1}^{\infty} g_j K_D(z_j, \cdot) \right\|_{2,D} \leq \epsilon$$

for all $n$ as the norms $\|f_n\|_{2,D} := \langle f_n, f_n \rangle$ are uniformly bounded above, and such that

$$\left\| f_n - \sum_{j=1}^{J} g_j K_D(z_j, \cdot) \right\|_{2,D} \leq \sqrt{J} \sum_{j=1}^{\infty} |g_j| \max_{j \in \{1, \ldots, J\}} |f(z_j)| \leq \epsilon$$

for all $n \geq N_J$ since $J$ is a fixed number. That is, $|\langle f_n, g \rangle| \leq \epsilon$ for any $\epsilon > 0$. This shows that $\{f_n\}$ converges weakly to zero in $A^2_D$.

4.6. Proof of Theorem 3. Fix $N \leq [s - 1]$ and let $\zeta$ be a point such that

$$\zeta \notin K \quad \text{and} \quad \text{dist}(\zeta, K) \leq c/N$$

for some fixed constant $c$. Further, let $\zeta_0 \in T$ be such that $|\zeta - \zeta_0| = \text{dist}(\zeta, K)$. Since $|\Phi(\zeta_0)| = 1$ and $\Phi$ is continuously differentiable in $\mathcal{O}$ as $T$ is at least $C^{1,\alpha}$-smooth, it holds that

$$|\Phi(\zeta)| \leq 1 + |\Phi(\zeta) - \Phi(\zeta_0)| \leq 1 + \mathcal{O}(|\zeta - \zeta_0|) = 1 + \mathcal{O}(N^{-1}),$$

where the estimate $\mathcal{O}(\cdot)$ does not depend on the choice of $\zeta$ satisfying (65). Hence,

$$\max_{k \in \{1, \ldots, N\}} |\Phi^k(\zeta)| \leq \text{const.}$$

for some absolute constant.

Select $z, w$ satisfying (65) and assume that $z \neq w$. Put, for brevity, $u := \Phi(z)\Phi(w)$. Then we get from the definition of $K_{N,s}$, (14), and (66) that

$$K_{N,s}(z, w) = \frac{\Phi^2(z)\Phi^2(w)}{\pi} \sum_{n=0}^{N-1} \left( (n+1) - \frac{(n+1)^2}{s} \right) u^n \left[ 1 + \mathcal{O}(\Sigma_n) \right]$$

$$\leq \frac{\Phi^2(z)\Phi^2(w)}{\pi} \left( \sum_{n=0}^{N-1} (n+1)u^n - \frac{1}{s} \sum_{n=0}^{N-1} (n+1)^2u^n \right) + \mathcal{O} \left( \max \{1, N^2\Sigma_N\} \right).$$
Since,
\[
\sum_{n=0}^{N-1} (n+1)u^n = -(N+1) \frac{u^N}{1-u} + \frac{1-u^{N+1}}{(1-u)^2},
\]
and
\[
\sum_{n=0}^{N-1} (n+1)^2u^n = -(N+1)^2 \frac{u^N}{1-u} + (N+1) \frac{1-u^{N+1}}{(1-u)^2} - (N+2) \frac{1+u^{N+1}}{(1-u)^2} + 2 \frac{1-u^{N+2}}{(1-u)^3},
\]
the validity of (18) follows. In a similar but simpler fashion, we also get that (19).

Fix now \(w \in D\). To prove (20), we show that \(\{K_{N,s}(\cdot, w)\}\) is a normal family and the norms \(\|K_{N,s}(\cdot, w)\|_{2,D}\) are uniformly bounded above. Then, for an arbitrary limit point, say \(K_w\), of this family, it holds that \(K_w \in A^2_D\) and for each \(k \in \mathbb{Z}^+\),
\[
\langle K_{N,s}(\cdot, w), \pi_{k,\infty} \rangle \to \langle K_w, \pi_{k,\infty} \rangle \quad \text{as} \quad N, s \to \infty,
\]
within a proper subsequence by the claim made in Section 4.5. As \(\{\pi_{n,\infty}\}_{n=0}^{\infty}\) is an orthonormal basis for \(A^2_D\), any \(g \in A^2_D\) is uniquely determined by its Fourier coefficients \(\langle g, \pi_{k,\infty} \rangle\).

Then we finish the proof of (20) by showing that
\[
(68) \quad \langle K_w, \pi_{k,\infty} \rangle = \pi_{k,\infty}(w) = \langle K_D(\cdot, w), \pi_{k,\infty} \rangle.
\]

Recall the Christoffel variational principle:
\[
K_{N,s}(z, z) = \max_{\deg(p)<N} \frac{|p(z)|^2}{\int_{C} |p|^2 P_{K}^{-2s} dA}, \quad z \in \mathbb{C}.
\]
Hence,
\[
(69) \quad K_{N,s}(z, z) \leq K_{N,\infty}(z, z) \leq K_D(z, z),
\]
where the second inequality follows from the fact that \(K_D(z, w) = \sum_{n=0}^{\infty} \overline{\pi_{n,\infty}(w)} \pi_{n,\infty}(z) [9, \S\,1.5]\. This together with the Cauchy-Swartz inequality implies that
\[
|K_{N,s}(z, w)|^2 \leq K_D(z, z)K_D(w, w),
\]
from which we deduce that \(\{K_{N,s}(\cdot, w)\}\) is a normal family. Moreover,
\[
\|K_{N,s}(\cdot, w)\|_{2,D}^2 \leq \int_{C} |K_{N,s}(z, w)|^2 P_{K}^{-2s}(z) dA = K_{N,s}(w, w) \leq K_D(w, w).
\]
That is, the norms \(\|K_{N,s}(\cdot, w)\|_{2,D}\) are uniformly bounded above for all \(N\) and \(s\). Hence, to prove (20), it only remains to verify (68). Fix \(k \in \mathbb{Z}^+\). Then,
\[
\langle K_{M,s}(\cdot, w), \pi_{k,\infty} \rangle = \sum_{n=0}^{M-1} \overline{\pi_{n,s}(w)} \int_{D} \pi_{n,s} \pi_{k,\infty} dA
\]
\[
= \pi_{k,s}(w) \int_{D} \pi_{k,s} \pi_{k,\infty} dA + \sum_{n=k+1}^{M-1} \overline{\pi_{n,s}(w)} \int_{D} \pi_{n,s} \pi_{k,\infty} dA
\]
\[
= \sum_{n=k}^{M-1} \pi_{k,s}(w) - \sum_{n=k+1}^{M-1} \overline{\pi_{n,s}(w)} \int_{D} \pi_{n,s} \pi_{k,\infty} |\Phi|^{-2s} dA,
\]
where we used the orthogonality of \(\pi_{k,\infty}\) over \(D\) with respect to \(dA\) and the orthogonality of \(\pi_{n,s}\) over \(C\) with respect to \(P_{K}^{-2s} dA\). It is a straightforward estimate using (29) to show
that the moments (31) satisfy $m^s_{ij} \to m^\infty_{ij}$ as $s \to \infty$. Therefore, we get from (54), since it is valid for $s = \infty$ as well, that

\[(71) \quad \frac{\pi_{k,s} - \pi_{k,\infty}}{\pi_{k,\infty}}(w) \to \pi_{k,\infty}(w) \quad \text{as} \quad s \to \infty.\]

Moreover, it follows from (14) and (37) that, for sufficiently large $s$, by (19) and since $\Sigma$ is valid for $s < N$ for any $a,b$ to prove (23) only for those $z \in \Omega$ in this family are indexed by $\zeta$ in (14), we see that

\[(72) \quad |K_D(z,w) - K_{M,s}(z,w)| \to 0 \quad \text{whenever} \quad M/s \to 0 \quad \text{as} \quad M,s \to \infty\]

locally uniformly in $D$.

Now, employing (69) and the Cauchy-Schwarz inequality once more, we get that

\[(73) \quad |K_N(z,w) - K_{M,s}(z,w)| \leq \frac{1}{s}\left|\int |\Phi|^{-2s} dA\right| \leq const \sqrt{n} \leq const \frac{1}{s},\]

where the constants depend on $k$ but are independent of $n$. Using the above estimate and the Cauchy-Schwarz inequality, we find that

\[(74) \quad |K_N(z,w) - K_{M,s}(z,w)| \leq \frac{1}{s}\left|\int |\Phi|^{-2s} dA\right| \leq const \frac{M}{s} \sum_{n=k+1}^{M-1} \pi_{n,s}(w)^2 \leq const. K_D(w,w) M/s.\]

Substituting (71) and (72) into (70), we see that (68) holds whenever $M/s \to 0$ as $M,s \to \infty$. That is,

\[\lim_{N,s \to \infty} K_{N,s}(z,w) = K_{M,s}(z,w) \quad \text{whenever} \quad M/s \to 0 \quad \text{as} \quad M,s \to \infty\]

4.7. Proof of Theorem 4. Let now $z \in T$. Observe that

\[\lim_{N,s \to \infty} K_{N,s}(z,z) = \frac{1}{\pi} \left(1 - \frac{\ell}{2} + \frac{\ell}{6}\right) = \frac{1}{\pi} \left(\frac{\ell}{2} - \frac{2\ell}{6}\right)\]

by (19) and since $\Sigma_N \to 0$ as $N \to \infty$. Fix $c > 0$ and let $a,b$ be such that $|a|, |b| < c$. It follows from the Cauchy-Schwarz inequality that

\[\left|K_{N,s}(z,a) - K_{M,s}(z,a)\right| \leq \left|K_D(z,w) - K_{M,s}(z,w)\right|,\]

which together with (73) yields (20) by choosing $M$ so that $M/s \to 0$. \qed

where the maximum is taken over all $\zeta$ satisfying (65). Furthermore, we get from the Bernstein-Walsh inequality and (66) that

\[\left|K_{N,s}(z,a) - K_{M,s}(z,a)\right| \leq \frac{1}{\pi} \left(1 - \frac{\ell}{2} + \frac{\ell}{6}\right) = \frac{1}{\pi} \left(\frac{\ell}{2} - \frac{2\ell}{6}\right)\]

for any $\zeta$ satisfying (65) with some absolute constant. Combining (76) and (77) with (75), we see that $\{K_{N,s}(z,a) - K_{M,s}(z,a)\}$ is a normal family for $|a|, |b| < c$, where the functions in this family are indexed by $z \in T$, $N \in \mathbb{N}$, and $s \in [N + 1, \infty)$. This observation is enough to prove (23) only for those $a,b$ for which $z + \frac{a}{N}, z + \frac{b}{N} \notin K$ (since $T$ has a tangent at $z$ we can assume that this condition holds for all $N$ large enough).
As $|\Phi'|$ is bounded above in $\overline{\mathcal{O}}$, there exists a path $\gamma \subset \overline{\mathcal{O}}$ connecting $z$ and $z + \frac{a}{N}$ whose length is proportional to $1/N$. Hence,

$$
\Phi \left( z + \frac{a}{N} \right) - \Phi(z) - \frac{a}{N} \Phi'(z) = \int_{\gamma} (\Phi'(t) - \Phi'(z)) \, dt = o \left( \frac{1}{N} \right),
$$

where the estimate holds uniformly for $z \in T$ and $a, b$ on compact subsets of $\mathbb{C}$. As $|\Phi(z)| = 1$, this means that

$$
\Phi \left( z + \frac{a}{N} \right) \Phi \left( z + \frac{b}{N} \right) = 1 + \frac{\tau(a, z) + \tau(b, z)}{N} + o \left( \frac{1}{N} \right)
$$

and respectively

$$
\left[ \Phi \left( z + \frac{a}{N} \right) \Phi \left( z + \frac{b}{N} \right) \right]^N = \exp \left\{ \frac{\tau(a, z) + \tau(b, z) + o(1)}{N} \right\},
$$

where $\tau(\cdot, \cdot)$ was defined before (22). As before, we can assume without loss of generality that $\tau(a, z) + \tau(b, z) \neq 0$. Then we get from (18), (79), (80), and the continuity of $\Phi'$ that

$$
\lim_{N,s \to \infty} K_{N,s} \left( z + \frac{a}{N}, z + \frac{b}{N} \right) N^{-2} =
\frac{\Phi'(z)^2}{\pi} \left( \frac{1 - \ell}{2} H_0 \left( \tau(a, z) + \tau(b, z) \right) + \frac{\ell}{6} H_1 \left( \tau(a, z) + \tau(b, z) \right) \right).
$$

The limit in (23) now follows from (75), (81), and (21).

As obvious from (11), to prove (24), it suffices to show that

$$
\lim_{N,s \to \infty} P_{\mathcal{K}}^{-s}(z + a/N) = \omega(a, z)
$$

uniformly for $z \in T$ and locally uniformly for $a \in \mathbb{C}$. To this end observe that an outward normal to $T$ at $z$ is given by $\Phi(z)/\Phi'(z)$. Hence, the angle between the vectors $a/N$ and $\Phi(z)/\Phi'(z)$ is less than $\pi/2$ if and only if the vector $a \Phi'(z) / \Phi(z) = \tau(a, z)$ belongs to the right half-plane. That is, if $\text{Re} \tau(a, z) > 0$. Hence, the limit in (82) holds for $\text{Re} \tau(a, z) < 0$ as $P_{\mathcal{K}}(z + a/N) = 1$ for such $a$. Moreover, when $\text{Re} \tau(a, z) > 0$, (82) follows immediately from (80). The case $\text{Re} \tau(a, z) = 0$ can be deduced by continuity and the uniformity of the estimate follows from the uniform character of the estimate in (78). Finally, the same arguments yield (24) for $\ell = 0$. \hfill \square

### Appendix A. Plots of Correlation Functions

To provide intuition for the results reported here we consider the scaled limit of $R_1$ and $R_2$ of the entropic (potential theoretic, with $K = \overline{\mathbb{D}}$) ensemble in a neighborhood of a point on the unit circle. By the radial symmetry of the weight it suffices to restrict ourselves to a neighborhood of 1. In this case $\tau(a, 1)$ reduces to $a$ and $\omega(1, 1,a) = e^{-2\text{Re}(a)/\ell}$ for $\text{Re}(a) > 0$ and $\omega(1, 1,a) = 1$ otherwise. As before, if $\ell = 0$ and $\text{Re}(a) > 0$ then we take $\omega(1, 1,a) = 0$. For convenience we define the scaled kernel at $z = 1$ by

$$
\tilde{H}_\ell(a,b) = \omega(1,a)\omega(1,b)H_\ell \left( a + \overline{b} \right).
$$

The limiting density of scaled eigenvalues is then given by

$$
R_1^\ell(a) = H_\ell(a,a),
$$

$$
R_2^\ell(a) = \text{Re} H_\ell(a,a).
$$
and the scaling limit of the second correlation function is given by

\[ R_2^\ell(a, b) = \tilde{H}_\ell(a, a)\tilde{H}_\ell(b, b) - \tilde{H}_\ell(a, b)\tilde{H}_\ell(b, a) \]

The visualizations provided here are for the cases where \( a \) and \( b \) are either real or on the imaginary axis.

**A.1. Tangent to the curve.** The tangent line of the circle at \( z = 1 \) is parallel to the imaginary axis and the local density of eigenvalues in this direction is given by

\[ \tilde{H}_\ell(it, it) = 1 \]

This is expected since the spatial density of eigenvalues on the unit circle must be invariant under rotation (and locally, this rotation is given by translation up the imaginary axis).

Looking at the second correlation function, when \( a \) and \( b \) are on the imaginary axis, we see that \( \tilde{H}_\ell \) is a function of \( t = -i(a + b) \). Figures 2, 3 and 4 show plots of the second correlation function for various values of \( \ell \) in various regions as a function of \( t \).

![Figure 2](image1.png)  
**Figure 2.** Plots of \( R_2^1(a, b) \) as a function of \( t = -i(a + b) \). The second plot is an enlargement of the shaded region.

![Figure 3](image2.png)  
**Figure 3.** Plots of \( R_2^0(a, b) \) as a function of \( t = -i(a + b) \).

By way of comparison we also provide plots of the second scaled correlation function for ensembles with the sine kernel. Specifically,

\[ S(a, b) = 2 \frac{\sin ( (a - b)/2 )}{(a - b)} \quad \text{and} \quad R_2^{\sin}(a, b) = 1 - S(a, b)^2. \]
Figure 4. Plot of the interpolation between $R_0^2(a,b)$ and $R_1^2(a,b)$ as a function of $t = -i(a + b)$.

(The slightly unusual normalization given by the superfluous appearing factors of 2 in the first equation arises in the scaling limit when we take the expected distance between eigenvalues to be $2\pi$—this allows for the most accurate comparison with our other figures).

Figure 5. Plots of $R_2^{\sin}(a,b)$ as a function of $t = a - b$.

A.2. Normal to the curve. In the regime where $a$ and $b$ are real, we are looking in a neighborhood of $z = 1$ in a direction perpendicular to that where the density of eigenvalues becomes constant. That is, the first scaled correlation function should decay as $a$ moves away from 0. Negative $a$ corresponds to moving into $K$ (where the potential is constant) whereas positive $a$ corresponds to moving away from $K$ where the potential acts to make $\infty$ repulsive. As $\ell$ decreases to 0, the field increases in strength until at $\ell = 0$ there is no possibility that an eigenvalue can be outside $K$. That is, when $\ell = 0$ the first correlation function vanishes for $a > 0$. 
Figure 6. Plot of the interpolation between $R_0^\ell(a)$ and $R_1^\ell(a)$ as a function of $t = -\text{Re}(a)$. When $\ell = 0$ there is a sharp cutoff at $t = 0$.

When $a$ and $b$ are real, $R_2^\ell(a,b)$ is no longer a function of a linear combination of $a$ and $b$, and we plot this as a surface for $\ell = 0$ and $\ell = 1$.

Figure 7. Plot of $R_2^\ell(a,b)$ when $a$ and $b$ are real, with part of the surface removed to see the cross-section.

References


Figure 8. Plot of $R^0_a(b)$ when $a$ and $b$ are real. Note that $R^0_a(b)$ is identically zero if either $a$ or $b$ is greater than 0.


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