Abstract—Evidence theory allows to build a large family of combination operators, based mostly on intersections and unions between the focal elements expressed by the experts, and multiplications and additions on the masses affected to these focal elements.

We explore algebraic structures where these operators behave differently, masses being linguistic labels, or focal elements being more, or less, than an union of singletons of a discernment space. In some cases, we have to forget this space and the notion of singleton to place ourselves in the space of possible focal elements.

We propose three new structures for linguistic labels, replacing masses. We show how to adapt the theory of evidence to six constrained spaces for the focal elements expressed by the experts.

Keywords: belief functions theory, algebra, discrete sets, linguistic labels.

I. INTRODUCTION

The belief functions theory, also called theory of evidence or Dempster-Shafer theory [3], [14], can be seen as an extension of the probability theory. It relies on the definition of basic belief assignments, and a large family of combination operators provide information fusion capabilities. An overview of their properties, based on the decisions they induce, has been made with some analysis of similarity tools [13].

However, an automatic processus is likely to express a mass between 0 and 1, but a human expert may find it most difficult. Interpreting the result can even be a bit more difficult. The operations applied on the focal elements, particularly the union, may lead to elements that can’t be interpreted, nor expressed in the expert’s syntax.

Here we browse the algebraic structures needed by the decision-aid functions and the combination operators of the theory of evidence. We determine how these functions and operators can be adapted to contexts where intersection and union differ of their usual behaviour (sections II-IV).

We propose three new types of linguistic labels – auto-indenting labels, unfinite auto-indenting labels and soft auto-indenting labels –, test their algebraic properties, and illustrate their differences on an example, including a max – min algebra reference (section V).

We show how to adapt the belief theory to six situations where the properties of the space containing the focal elements of the basic belief assignment are more constrained than a powerset or a hyper-powerset. We list the compatible operators, and keep an eye to the combinatorial complexity they require (section VI).

II. BASIC BELief ASSIGNMENTS

On a finite or discrete set $\Theta$, called the discernment space, it allows to provide mass on any subset of $\Theta$ instead of it singletons. Such a mass repartition is called a basic belief assignment (bba) $m$:

$$\sum_{X \in 2^\Theta} m(X) = 1$$

$$\forall X \in 2^\Theta, m(X) \geq 0$$

The hypothesis of closed world [14] can be added to this definition:

$$m(\emptyset) = 0$$

It is equivalent to allow an open world, or add a special element to $\Theta$, receiving the mass $\emptyset$ should receive, and use the properties of a closed world.

If $A$ is an element of $2^\Theta$ with a non-zero mass, it is called a focal element. As a possible bearer of mass, $\Theta$ is the ignorance. We will call $F(m)$ the set of the focal elements of $m$, the focal set of $m$. The notation $n$ will be reserved for the cardinal of $\Theta$.

A. Decision-aid functions

Belief ($\text{bel}$), Plausibility ($\text{pl}$) and pignistic probability ($\text{BetP}$) can be used to build monotonic functions on $2^\Theta$: $A \subset B$ implies $f(A) \leq f(B)$. As $\text{bel}(A) \leq \text{BetP}(A) \leq \text{pl}(A)$, the pignistic probability is often considered as an interesting compromise.

$$\text{bel}(X) = \sum_{Y \subset X, Y \neq \emptyset} m(Y)$$

$$\text{pl}(X) = \sum_{Y \cap X \neq \emptyset} m(Y)$$

$$\text{BetP}(X) = \sum_{Y \in 2^\Theta, Y \neq \emptyset} \frac{|X \cap Y|}{|Y|} \frac{m(Y)}{1 - m(\emptyset)}$$

To take a decision, one can choose the maximum of mass, the maximum of belief, the maximum of plausibility or the maximum of pignistic probability. As the three last functions are monotonic, their maximum is reached for the ignorance $\Theta$. They will be used for decision-making after selecting a
subset of $2^\Theta$, where all elements are pairwise incomparable, by example by fixing the cardinal of a possible decision, generally limiting it to the singletons. It is also possible to use a discounting method to deceive the larger elements of $2^\Theta$.

### B. Usual combination operators

A combination operator takes two or more bbas to build another bba. It is an inner operation (bel, pl or BetP are not).

The mean operator is the simpler one. Its focal set is the union of the focal sets of the input bbas.

$$\text{Mean}(m_1, \ldots, m_N)(X) = \frac{1}{N} \sum_{i=1}^{N} m_i(X) \quad (7)$$

The conjunctive operator, proposed by Smets [19] for two input bbas $m_1$ and $m_2$, is given by the equation (8). It puts the mass $m_1(A)m_2(B)$ on the set $A \cap B$. It is an associative operator, so it is useless to write its expression for $N$ input bbas.\footnote{Such an expression would decline into a $O(p^N)$ algorithm, where the associativity can lead to an algorithm in $O(npN)$ operations, if the number of possible focal elements is limited to $n = |\Theta|$, the cardinal of $\Theta$.} Dempster [3] prefers a normalized version, multiplying all terms by $\frac{1}{1-m(\emptyset)}$: it has the same associativity property. Yager [21] puts the conflictual mass $m(\emptyset)$ on ignorance $m(\Theta)$, loosing associativity.

$$m_{\text{Conj}}(X) = \sum_{A \cap B = X} m_1(A)m_2(B) \quad (8)$$

The disjunctive operator puts the mass $m_1(A)m_2(B)$ on the set $A \cup B$. It is usually seen as insufficiently informative, as it puts mass on a local ignorance in case of distinct focal elements; it preserves the closed world hypothesis. Like the conjunctive operator, it is associative.

$$m_{\text{Disj}}(X) = \sum_{A \cup B = X} m_1(A)m_2(B) \quad (9)$$

The Dubois & Prade combination operator [4] is an interesting compromise between the conjunctive and the disjunctive ones. It puts the mass $m_1(A)m_2(B)$ on $A \cap B$ if $A \cap B$ is not empty, and on $A \cup B$ if it is. It respects the closed world hypothesis, adds information like the conjunctive rule (and even better, as local ignorance should be preferred to conflict), but is not associative.

$$m_{\text{DP}}(X) = \sum_{A \cap B = X} m_1(A)m_2(B) + \sum_{A \cap B = \emptyset} m_1(A)m_2(B) \quad (10)$$

$$m_{\text{DP}}(\emptyset) = 0 \quad (11)$$

A panel of conflict redistributing rules have been proposed [8], [9], [16], [17]: none is associative. The most used is the PCR5/6 combination operator (PCR5 and PCR6 are identical when combining two bbas, and differ for more).

$$m_{\text{PCR5/6}}(X) = m_{\text{Conj}}(X) + \sum_{Y \subset \Theta, \ X \cap Y = \emptyset} \left( \frac{m_1(X)^{2m_2(Y)}}{m_1(X)^{2m_2(Y)}} + \frac{m_2(X)^{2m_1(Y)}}{m_2(X)^{2m_1(Y)}} \right) \quad (12)$$

### III. Algebras

To define a basic belief assignment, to compute its belief, plausibility and pignistic probability, to apply combination operators, we need to have access to many operators on the masses and the focal elements:

- **Masses**: For most operators, they are multiplied and added. For the Mean combination, they are multiplied by a real number. For the normalization procedure of Dempster, the PCR5/6 operator or the pignistic probability, it is necessary to divide by a mass.

- **Elements**: They pass through intersection and union operators. They are also compared with $\emptyset$ and with a given element of $2^\Theta$.

The methods exposed in section II need that the masses are expressed in a $\text{R}$-algebra (addition, inner inversible multiplication, external multiplication). They need that the focal elements are expressed in a lattice where $\Theta$ and $\emptyset$ are extremum.

What happens if our masses and focal elements live in poorer algebraic structures?

#### A. Orders and partial orders

$(\Theta, \leq)$ is a partially ordered set (poset), if for any $a$, $b$, and $c$ in $\Theta$, we have that:

- reflexivity: $a \leq a$,
- antisymmetry: $a \leq b$ and $b \leq a$ implies $a = b$,
- transitivity: $a \leq b$ and $b \leq c$ implies $a \leq c$.

If $x \leq y$ and $x \neq y$, we will note $x < y$.

If for any $a$ and $b$ of $\Theta$ we have either $a \leq b$ or $b \leq a$ then $\Theta$ is called a totally ordered set or simply an ordered set.

Expressing a label in a ordered or partially ordered set is easier for an human expert than expressing a significant mass in $[0, 1]$.

#### B. Lattice

$(\Theta, \lor, \land)$ is a lattice if the supremum operator $\lor$ and the infimum operator $\land$ satisfy for any $a$, $b$ and $c$ of $\Theta$:

- commutativity: $a \lor b = b \lor a$, $a \land b = b \land a$,
- associativity: $(a \lor b) \lor c = a \lor (b \lor c)$, $(a \land b) \land c = a \land (b \land c)$,
- idempotence: $a \lor a = a$, $a \land a = a$,
- absorption: $a \lor (a \land b) = a$, $a \land (a \lor b) = a$.

The lattice is closed if it has a smallest element $\bot$ (its infimum) and a greatest element $\top$ (its supremum): for any $x \in \Theta$, $\bot \land x = x$ and $\top \lor x = x$.\footnote{If we define a supremum operator, we get a semi-lattice. For any set $\Theta$, $(\Theta, \cap, \cup)$ is a lattice. If we define the relation $\leq$ by $a \leq b$ iff $a \lor bb$, then $(\Theta, \leq)$ is a poset.}

If we just define a supremum operator, we get a semi-lattice.
C. Ring

\((\Theta, +, \times)\) is a ring if

- the addition operator is inversible, associative and commutative, and has a neutral element, called 0;
- the multiplication operator is associative, commutative, and has a neutral element, called 1;
- the multiplication distributes over the addition:

\[ a \times (b + c) = (a \times b) + (a \times c) \]

For most of the applications, the addition does not need to be inversible. So a semi-ring should be sufficient.

D. Field

A field \((\Theta, +, \times)\) is a ring where the multiplication is inversible on \(\Theta \backslash \{0\}\). The set of real numbers \(\mathbb{R}\) with the usual addition and multiplication is a field. It is because the real interval \([0, 1]\) is a part of a field that we can use all of the operators and functions of the section II.

E. Vector space

\((\Theta, +, \bullet)\) is a vector space over a field \(F\) if + is an inner operator which is inversible, associative and commutative, and has a neutral element 0. The operator \(\bullet\) is the external multiplication, also called scalar multiplication, by an element of \(F\), satisfying, for any \(a\) and \(b\) in \(F\) and \(x\) and \(y\) in \(\Theta\):

- Distributivity of \(\bullet\) over +: \((a \bullet (x + y)) = a \bullet x + a \bullet y,\)
- Distributivity of addition over \(\bullet\): \((a + b) \bullet x = a \bullet x + b \bullet x,\)
- Associativity of multiplications: \((a \times b) \times x = a \times (b \times x)\).

F. Algebra over a field

\((\Theta, +, \times, \bullet)\) is an algebra over a field \(F\) if \((\Theta, +, \times)\) is a ring, \((\Theta, +, \bullet)\) is vector space over \(F\), and for any \(a\) and \(b\) in \(F\) and \(x\) and \(y\) in \(\Theta\) we have \((a \bullet x) \times (b \bullet y) = (ab) \bullet (x \times y)\).

A field can be seen as an algebra over itself, identifying the inner addition and the scalar multiplication: \((\mathbb{R}, +, \bullet, \bullet)\).

We will not consider any richer algebraic structure for our fusion applications.

IV. Basic Belief Assignments

As \(2^\Theta\) is built by taking all the possible unions of elements of \(\Theta\), it is a semi-lattice. If we don’t have an union operator, but a supremum \(\lor\), we speak of the closure of \(\Theta\) by \(\lor\).

So we obtain the lattice \((2^\Theta, \cap, \cup)\) (powerset of \(\Theta\)) by closing \(\Theta\) by the operator \(\cup\). Its infimum is \(\emptyset\), its supremum is \(\Theta\).

We obtain the lattice \((D^\Theta, \cap, \cup)\) (hyper-powerset of \(\Theta\)), basis of the Dezert-Smarandache theory (DSmT) [15] by closing \(\Theta\) by the operators \(\cap\) and \(\cup\). If \(\Theta = \{\theta_1, \ldots, \theta_n\}\), its infimum is \(\cap_{i=1}^n \theta_i\), its supremum is \(\Theta\). Adding constraints on intersections and unions to build an equivalence class for \(\emptyset\) corresponds to an anti-chain in the more general lattice. The anti-chain cuts the lower part of the lattice, and its infimum becomes \(\emptyset\), as an efficient element of the equivalence class.

The section VI explores some of the lattices we can use to express focal elements. The set of the possible focal elements will be called the extension of \(\Theta\), noted \(E(\Theta)\). It can be \(2^\Theta\), \(D^\Theta\), or another closure of \(\Theta\).

As a mass is usually a real number, we will use the term label when the values assigned to focal elements are not necessarily in a field.

To define a basic belief assignment, the normalization condition (1) implies the labels can be added, and we have to choose a constant value to take the role of “1”. The fact 1 is the neutral element for the multiplication operator eliminates the normalization step for the Conj, Dis, DP or PCR combination operators.

This normalization condition may have to be relaxed in an other label algebra, if the “addition” operator cannot have these comfortable properties.

To calculate the plausibility \(\text{pl}(A)\) (5), we need an inner addition for the labels (semi-group structure) and to determine if an intersection between \(A\) and another element of \(E(\Theta)\) is empty.

To calculate the belief \(\text{bel}(A)\) (4), we need an inner addition for the labels and to determine if an element \(X\) of \(E(\Theta)\) is included in \(A\). This can be extended to any partial order \(\leq\) on \(E(\Theta)\):

\[
\text{bel}(A) = \sum_{X \leq A} m(X)
\]

To calculate the pignistic probability \(\text{bel}(A)\) (6), we need an inner addition, inner multiplication and scalar multiplication – an algebra over \(\mathbb{R}\) or \(\mathbb{Q}\) – for the labels. Without the closed world hypothesis \((m(\emptyset)\) can be nonzero) the inner multiplication operator must be inversible.

Non-numeric labels will hardly support a pignistic transformation. On \(E(\Theta)\), we need to define an intersection and a cardinal; the later can lead to some time-consuming calculus on \(D^\Theta\), with or without constraints.

For the Mean operator (7), we only need an inner addition and a scalar multiplication. Labels can be elements of a vector space.

For the Conjunction (8), we need an intersection operator, distinct of the one used to extend \(\Theta\) to \(E(\Theta)\); we get \(2^\Theta\). Then \(E(\Theta)\) only needs to be a semi-lattice \((\Theta, \lor)\). Labels live in a ring.

For the Disjunction (9), we need an intersection operator, distinct of the one used to extend \(\Theta\) to \(E(\Theta)\). So we need a complete lattice structure on \(\Theta\). If \(E(\Theta)\) exists without any reference to \(\Theta\), a semi-lattice \((E(\Theta), \lor)\) can be sufficient. Labels are in a ring too. The DP combination operator (10) and the Yager rule have the same constraints.

The normalized Dempster rule needs the multiplication and the addition on the labels to be inversible, because of the multiplication by \(\frac{1}{1-m(\emptyset)}\).

The PCR5/6 operator, like the pignistic transformation, needs the labels to be expressed in a \(\mathbb{R}\)-algebra, with an
inversible inner multiplication. An intersection operator is needed, but not the cardinal. That makes the hyper-powerset $D^\Theta$ a convenient lattice for this operator.

V. Discrète Labels

Smarandache and Dezert proposed a field structure for linguistic labels [18], allowing all the combination operators and functions of the sections II-A and II-B. However, the normalization condition (1) is hard to satisfy, and they have to make a strong hypothesis of equirepartition of the linguistic labels.

The three algebraic structures we propose here are not concerned by an equirepartition hypothesis, as the labels are plunged in a total order rather than a field. The finite linguistic set $L$ is predetermined, but can evolve for the soft auto-indenting labels.

The $\text{Conj}$, $\text{Dis}$ and DP combination operators are based on a ring structure over the labels: $(L, +, \times)$. We can replace these operators to get some other rings: $(L, +, \max)$ or $(L, \max, \min)$.

In the first case, we have a structure equivalent to $\mathbb{N}$: we can take a positive non-zero element of $L$, and define the successor of an element $\ell$ of $L$ by $\ell + x$. So $L$ either is not finite, and therefore inadequate for linguistic labels, as there is an element whose successor is 0, and we can’t define an order on $L$ (that’s why $\mathbb{Z}/n\mathbb{Z}$ is not useful for semantic labels).

As $L$ is a finite ordered set, we will denote by $s(x)$ the successor of an element $x$: $x \leq s(x)$, $x \neq s(x)$, and if $x \neq y$ and $x \leq y$, then $s(x) \leq y$. We call $0_L$ the minimum of $L$, and $M_L$ its maximum. An element of $E(\Theta)$ with a label $0_L$ isn’t a focal element.

A. Max-Min algebra

The $\max$ and $\min$ operators effectively fulfills the distribution property, and define a ring on $L$:

$$\min(a, \max(b, c)) = \max(\min(a, b), \min(a, c))$$

Note that this ring

For any elements $a$ and $b$ of an ordered set $L$, we have $\min(a, b) \in L$ and $\max(a, b) \in L$. So, the result of any expression involving elements of $L$, $\min$ and $\max$ is still in $L$. If such an expression involves elements $a_1, \ldots, a_k$, then

$$f(a_1, \ldots, a_k) \leq \max(a_1, \ldots, a_k)$$

Therefore, if $\top$ is the supremum of the lattice or semi-lattice $E(\Theta)$,

$$\pi(\top) = \max_{A \in F(m)} m(A)$$

A normalization condition can be that at least one label of $m$ is $M_L$. This condition is robust to the $\text{Conj}$, $\text{Dis}$ and DP combination operators.

However, this algebra lacks a fine property of the usual combination operators of the theory of evidence: many small amounts of evidence can’t make a high amount of evidence.

In the following cases, we keep $\min$ as replacement for the inner multiplication, but we slightly transform the $\max$ operator.

B. Auto-indenting labels

The operators for the auto-indenting labels (AIL) treat differently the case of equality for the $\max$.

$$x \oplus_{\text{AIL}} y = \begin{cases} \max(x, y) & \text{if } x \neq y \\ s(x) & \text{if } x = y, x \neq 0_L, x \neq M_L \\ 0_L & \text{if } x = y = 0_L \\ M_L & \text{if } x = y = M_L \end{cases}$$

(16)

The second condition allows to enforce a focal element receiving many similar labels.

The third condition guarantees that $O_L$ is a neutral element for $\oplus$.

The fourth condition guarantees that $M_L$ remains the higher possible label. If it is removed, new labels over $M_L$ are allowed. This defines another operator, who creates unfinite auto-indenting labels (AIL$_\infty$):

$$x \oplus_{\text{AIL}_\infty} y = \begin{cases} \max(x, y) & \text{if } x \neq y \\ s(x) & \text{if } x = y, x \neq 0_L, x \neq M_L \\ 0_L & \text{if } x = y = 0_L \end{cases}$$

(17)

Note that, for AIL and AIL$_\infty$, $x \oplus x \oplus x = s(x)$. The later example, in the section V-D, corresponds to $x \oplus x \oplus x \oplus s(x) = x(s(x))$.

These operators are not distributive over min:

$$\min(1, 2 \oplus 2) = \min(1, 3) = 1$$

$$\min(1, 2) \oplus \min(1, 2) = 1 \oplus 1 = 2$$

(18) (19)

So using AIL or AIL$_\infty$ suppresses the associativity property of the $\text{Conj}$ and $\text{Dis}$ combination operators.

AIL respects the normalization property (1) through the $\text{Conj}$, $\text{Dis}$ and DP combination operators, but AIL$_\infty$ does not.

C. Soft auto-indenting labels

To distinguish between a label reached by the bbas’ information and a label reached by accumulation of lower labels, we should prefer to create intermediary labels than jump to the next one\(^3\). This new label, taking place between $x$ and $s(x)$, is noted $x^+$ and called “a bit more than $x$”.

$$x \oplus_{\text{SAIL}} y = \begin{cases} \max(x, y) & \text{if } x \neq y \\ x^+ & \text{if } x = y, x = y^+ \text{ or } x^+ = y \\ 0_L & \text{if } x = y = 0_L \\ M_L & \text{if } x = y = M_L \end{cases}$$

(20)

The following table gives the value of $x \oplus_{\text{SAIL}} y$ for $x$ and $y$ taking their values in a label set extended from $\{0, 1, 2, M\}$.

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\(^2\)As this ring is compatible with the multiplication by a positive number, it is usually called an algebra. Here, of course, we just use its ring operators.

\(^3\)An African swallow is stronger than an European swallow, but how many European swallows are required to carry as much weight as an African swallow?
SAIL respects the normalization property (1) through the Conj, Dis and DP combination operators.

**D. Example**

Here we consider a set of linguistic labels \( L = \{ \text{no}, \text{low}, \text{mod}, \text{high} \} \). The label “no” is the non-focal label \( 0_L \), and the label “high” is the maximum of \( L, M_L \).

In the following table, we consider the labels in a ring or a pseudo-ring \((L, \oplus, \odot)\). We eventually transform the \( \odot \) operator in a more usual multiplication symbol on the labels \((x, y) \) or \( x^2 \). The Conj combination operator gives:

\[
\begin{align*}
m(A) &= m_1(A) \odot m_2(A) \oplus m_1(A) \odot m_2(A \cap B) \oplus m_1(A) \odot m_2(A \cup C) \oplus m_1(A \cup B) \odot m_2(A) \oplus m_1(A \cup B) \odot m_2(A \cup C) \oplus m_1(\Theta) \odot m_2(A) \\
&= \text{low}^2 \odot \text{low}^2 \odot \text{low} \odot \text{high} \odot \text{high} \odot \text{low} \oplus \text{high}^2 \oplus \text{mod} \odot \text{low} \\
m(A \cup B) &= m_1(A \cup B) \odot m_2(A \cup B) \oplus m_1(\Theta) \odot m_2(A \cup B) \\
&= \text{high} \odot \text{low} \odot \text{mod} \\
m(A \cup C) &= m_1(\Theta) \odot m_2(A \cup C) = \text{mod} \odot \text{high}
\end{align*}
\]

All these label systems are purely discrete and semantic; they allow a certain form of normalization (at least, the bias they produce respect a normalization constraint); they allow a decision step by maximizing belief, plausibility or mass; they can’t produce a pignistic probability.

**VI. CONSTRAINTS OVER THE FOCAL ELEMENTS**

What happens if the lattice \( E(\Theta) \) uses operators different of \( \cap \) and \( \cup \)? These operators may create focal elements incompatible with the model elements that should appear in the bba produced by the experts (human or artificial).

The following examples browse some of these situations, from an order set to a formalism near the natural language, through some classification models. It is possible that the “singletons of \( \Theta \)” are difficult to exhibit; in this case, \( E(\Theta) \) should be considered as our set of interest, as browsing the singletons is interesting only if they are privileged by the experts, or to calculate a pignistic probability.

**A. Ordered set**

If \( \Theta \) is an ordered set, a subset \( A \) of \( \Theta \) is connected if, for any \( x \) and \( y \) in \( A \), \( x \leq y \) brings \( x \leq z \leq y \) implies \( z \in A \) (resp. \( y \leq x \) brings \( y \leq z \leq x \) implies \( z \in A \)). Disconnected subsets don’t have any signification is the context of an ordered set \( \Theta \) can be a discretization of some real value: \( \{36, 39, 42, 45, 48, 51\} \). Therefore, the elements of \( E(\Theta) \) should be the intervals of \( \leq \), noted \([x, y]\). If \( x = y \) the interval is a singleton. If \( y < x \), the interval is \( \emptyset \).

To remain within \( E(\Theta) \), the \( \cap \) operator is convenient, as it preserves the connectedness of its operands, but \( \cup \) is not: \( \{36\} \cup \{42\} = \{36, 42\} \notin E(\Theta) \), but \( \{36, 39, 42\} \) is. We use the hull of the operands to obtain the smallest interval:

\[
[x_1, y_1] \cup_{\leq} [x_2, y_2] = [\min(x_1, x_2), \max(x_2, y_2)]
\]

The cardinal of an element of \( E(\Theta) \) holds as usual, so all the combination operators and decision-aid functions of section II can be used.

As the cardinal of \( F(m) \) is bounded by \( \binom{n+2}{n+1} \), where \( 2^n \) has a size of 2, the constraint of a total order on \( \Theta \) can limit the combinatorial explosion inherent to most combinatorial operators.

**B. Intervals of \( \mathbb{R}^N \)**

In a context of interval analysis \([6], [11]\), the objects manipulated and the results are intervals of \( \mathbb{R}^N \). If the theory guarantees some non-void intersections when manipulating the solutions of an equation, its application in an information fusion system with unpredicted events may lead to conflictual situations.

Here an interval \([x, y] \) corresponds to a cartesian product \([x_1, y_1] \times \ldots \times [x_N, y_N] \). The intersection works as usual, giving us an infimum operator \( \wedge_I \) for the lattice \( E(\Theta) \):

\[
[x^1, y^1] \wedge_I [x^2, y^2] = \prod_{i=1}^{N} \max(y_i^1, y_i^2)
\]

If for some dimension \( i \), \( \max(x_i^1, x_i^2) > \min(y_i^1, y_i^2) \), then

\[
[x^1, y^1] \wedge_I [x^2, y^2] = \emptyset
\]

For the \( \vee_I \) operator, we take the smallest interval of \( \mathbb{R}^N \) containing the operands:

\[
[x^1, y^1] \vee_I [x^2, y^2] = \prod_{i=1}^{N} \max(y_i^1, y_i^2)
\]

We can use the measure of the Lebesgue as the cardinal of interval:

\[
\mu([x, y]) = \prod_{i=1}^{N} (y_i - x_i)
\]

Therefore, all the combination operators and decision-aid functions of section II can be used in a context of interval calculus.
Figure 1. Non-distributivity

Unlike the usual \((∪, ∩)\) lattice or the \((∨_I, ∧_I)\) lattice on an ordered set, the \((∨_I, ∧_I)\) lattice is just a lattice, not a ring: \(∨_I\) does not distribute over \(∧_I\). On figure 1 we take \(A = [0, 1] × [2, 3], B = [2, 3] × [4, 5], C = [2, 3] × [0, 1], D = [4, 5] × [2, 3].\) We find:

\[
(A ∨_I B) ∧_I (C ∨_I D) = [2, 3] × [2, 3]
\]

\[
(A ∧_I C) ∨_I (A ∧_I D) ∨_I (B ∧_I C) ∨_I (B ∧_I D) = \emptyset
\]

C. Partitions

A set \(P\) of subsets of \(Θ\) is a partition if for any pair \(A, B\) of elements of \(P\), either \(A = B\) or \(A \cap B = \emptyset\), and \(\bigcup\{A \in P\} = Θ\). This structure is popular for unsupervised classification problems; a vast family algorithms, around \(K\)-means [10], produce results within this model.

The intersection between two partitions, \(P_1\) and \(P_2\), can be easily defined:

\[
P_1 ∧_P P_2 = \{A \cap B \mid A \in P_1, B \in P_2\} \tag{26}
\]

However, replacing \(∩\) by \(∪\) in (26) does not produce a partition. An operator \(∨_P\) should be constructed by considering the connected parts of the hypergraph \(P_1 ∪ P_2\), but this tends to give us a degenerated partition even if \(P_1\) and \(P_2\) differs only slightly. See GUÉNOCHÉ and Garreta [5] for robust methods of comparison between partitions.

We have to limit the operators to a closed semi-lattice, whose infimum \(⊥\) is a partition in \(n\) singletons and supremum \(⊤\) is a partition containing only \(Θ\). The cardinal of a partition \(n = |P|\) where \(n\) is the cardinal of \(Θ\) and \(|P|\) the number of subsets of \(Θ\) in \(P\). So

\[
\begin{align*}
\text{Card (⊥)} &= 0 \\
\text{Card (⊤)} &= n - 1 \\
\text{Card (}P_1 \land_\ P_2\text{)} &\leq \min(\text{Card (}P_1\text{)}, \text{Card (}P_1\text{))}
\end{align*}
\]

We can use the partitions on \(Θ\) as focal elements for bbas, and use them for all the decision-aid functions, including BetP, and for the Conj and PCR5/6 combination operators.

D. Hierarchies

A hierarchy on \(Θ\) is a set \(H\) of subsets of \(Θ\) such that:

- for any \(x \in Θ\), \(\{x\} \in H\);
- \(Θ \in H\);
- for any \(A\) and \(B\) in \(H\), \(A \cap B \in \{A, B, \emptyset\}\).

This structure is very popular for unsupervised classification. It is produced by Ward’s algorithm, single linkage, complete linkage and many others [7].

We can define a merging operation between two hierarchies \(H_1\) and \(H_2\) by \(A \in (H_1 ⊔_H 2)\) if

\[
A = \bigcap \{X \in H_1 \mid X \in H_2 \mid A \cap X \neq X\}
\]

This operation is not associative, but it is idempotent, and admits \(H_0\), the hierarchy containing only the singletons and \(Θ\), as a neutral element. The structure we define is only a pseudo-semi-lattice using this operator \(⊙\). Its “infimum” is \(H_0\). It has no unique supremum, but the complete hierarchies (containing exactly \(2^n\) subsets of \(Θ\)) have no dominating hierarchy: if \(H\) and \(H’\) are complete, \(H’ ⊔_H H = H\) implies that \(H’ = H\).

![Figure 2. Dealing with hierarchies](image-url)
hierarchies.

\[ A \in (\mathcal{H}_1 \cap \mathcal{H}_2) \text{ iff } A \in \mathcal{H}_1 \text{ and } A \in \mathcal{H}_2 \quad (29) \]

Taking the number of subsets of \( \Theta \) the hierarchy contains is efficient to identify the complete hierarchies, but it is not decreasing with \( \wedge_H \). Other definitions are hardly constant on the complete hierarchies.

The pseudo-semi-lattice defined by \( \odot \) can be used to model bbas in a hierarchy space, apply on them the decision-aid functions bel and pl, and combine them through the PCR5/6 and Conj operators. However, in this latter case, we loose the associativity of the operator.

**E. Binary clustering systems**

A binary clustering system [1] on \( \Theta \) is a set \( \mathcal{E} \) of subsets of \( \Theta \) (called clusters) such that, for any \( x \) and \( y \) in \( \Theta \), the set \( \mathcal{E}(x,y) = \{ A \in \mathcal{E} \mid x \in A \land y \in A \} \) admits an unique smallest element, called \( A(x,y) \). It is said proper if \( A(x,x) = \{ x \} \).

Hierarchies (section VI-D) are binary clustering systems; partitions (section VI-C) are non-proper binary clustering systems (add singletons and \( \Theta \), containing all the possible hyperedges with 1 or 2 vertices. Its cardinal function can be defined by:

\[ \text{Card} (\mathcal{E}) = \sum_{\{x,y\} \subseteq \Theta} (|A(x,y)| - 2) \quad (31) \]

It defines a semi-lattice whose infimum is \( \mathcal{E}_1 \), a hypergraph containing all the possible hyperedges with 1 or 2 vertices. Its non-proper version is the complete graph whose vertex set is \( \Theta \). Its supremum is the hypergraph \( \mathcal{E}_\top \) whose only hyperedge is \( \Theta \).

A cardinal function can be defined by:

\[ \text{Card} (\mathcal{E}) = \sum_{\{x,y\} \subseteq \Theta} (|A(x,y)| - 2) \quad (31) \]

So \( \text{Card} (\mathcal{E}_L) = 0 \), \( \text{Card} (\mathcal{E}_\top) = \frac{1}{2}n(n+1)(n-2) \), and \( \mathcal{E} \leq \mathcal{E}' \) (see section III-B for the definition of \( \leq_\ell \) in a lattice) brings \( \text{Card} (\mathcal{E}) \leq \text{Card} (\mathcal{E}') \).

So we can use the binary clustering systems on \( \Theta \) as focal elements for bbas, and use them for all the decision-aid functions, including BetP, and for the Conj and PCR5/6 combination operators.

**F. Semantic assertions**

Semantic assertions can be modeled by **conceptual graphs** [20] or ontologies. Infimum and supremum operators can be defined, but these operations can lead to NP-hard problems in a general case. However, some sub-classes of conceptual graphs avoid this combinatorial problem [12].

With the same restrictions on the shape of graphs than the other operations on them, to avoid combinatorial explosion, they can be combined by Conj and Dis. As obtaining \( \emptyset \) by conjunction of two conceptual graphs is very unlikely, the PCR5/6 and DP combination operators should not be used: they are nearly equivalent to Conj.

The usual ways to calculate the cardinal of a graph (number of edges or number or vertices) is not compatible with the infimum operator, and does not make sense with the specialization of labels. We should limit the decision-aid functions to pl and bel.

Then, the conjunctive combination operator, receiving bbas containing “Johann have seen a Leclerc” and “A man have seen a tank near the river” can put a mass (or a label) on the assertion “Johann have seen a Leclerc tank near the river”.

**VII. CONCLUSION**

The combination operators of the theory of evidence are often heavy to manipulate: cumbersome equations, data ill-adapted to matrix calculus under popular scientific software, real risks of combinatorial explosion, by example. However, they are more likely than Bayesian approaches or fuzzy sets to adapted to many forms of symbolic data.

We have shown that the link between the theory of evidence and the probabilities, the pignistic transformation, relies on “difficult” operations: scalar multiplication and cardinal calculus. Dropping only this link and keeping most of their properties allows bbas to explore many facets of experts’ opinions, and build a fused information from them, while other theories only deal with a projection of the experts’ assertions on a too small space.

**REFERENCES**


See the M-bbas version of PCR6 in [8].