Value of the golden ratio (number Φ) knowing the the side length of a square

Christophe Jouis[1], Mercedes Orús–Lacort[2]
[1]*Centre d'Analyse et de Mathématique Sociales – CAMS,* [2]*Online teacher of College Mathematics UOC–UNED*

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Abstract

This paper explains how to obtain the number Φ, using a square with side length equal to a, the right triangle with sides $\frac{a}{2}$ and a, and a circle with radius equal to the hypotenuse of this right triangle. In particular, from a square whose side length is equal to a, we will show how to obtain a segment b in such a way that the value of $\frac{a}{b}$ is the number Φ. It is well known that this ratio is also calculated from equating the ratios obtained by dividing a segment of length $a+b$ by a (being a always the largest segment) and a by b, that is, $\frac{a+b}{a} = \frac{a}{b}$. This equality is a consequence of the ratio of proportionality in triangles applying Thales's Theorem. And, we must mention also how this golden ratio it is obtained as a consequence of the Fibonacci sequence. However, the golden ratio as a consequence of the limit of Fibonacci sequence was found later than many masterpieces, as for instance the ones of Leonardo da Vinci. This is the main reason because we analyzed how to find the proportionality golden ratio using the most common geometric figures and its symmetries. This paper aims to show how the golden ratio can be obtained knowing the side length a of a square.

Key Words: Number Φ, Golden ratio, Fibonacci.

1 Introduction

The Φ number, the golden ratio, has always been and is a number that has amazed and surprised us.

Transcendent number, we usually call it. He and someone else received that name because they simply have characteristics that make them different from others.

This number leaves us thoughtful. It seems that the beauty of the most harmonic forms resides in it, and it makes us wonder: how can this happen?.

Many times, as a mathematicians, we have felt a very small human being when we observe certain results, because they seem to be not by chance, but the work of someone whose conjunction between mind and soul is supreme, divine.

It is well known that this ratio is also calculated from equating the ratios obtained by dividing a segment of length $a+b$ by a (being a always the largest segment) and a by b, that is, $\frac{a+b}{a} = \frac{a}{b}$. This equality is a consequence of the ratio of proportionality in triangles applying Thales's Theorem, see Fig.(1).

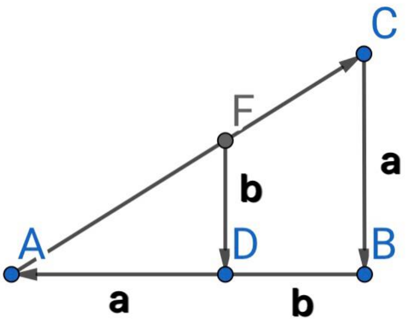

Figure 1: Right triangle where $\frac{a+b}{a} = \frac{a}{b}$.

And, we must also mention how this golden ratio it is obtained as a consequence of the Fibonacci sequence. This sequence, which terms are $\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$ has numerous applications in computer science, mathematics, game theory, in topological quantum computing with a system of Fibonacci anyons described by the Yang–Lee model the SU(2) special case of the Chern-Simons theory and Wess-Zumino-Witten models [25], as well as in Linguistics in the syntactic derivation of sentence structures [26]. It also appears in biological configurations, such as in the branches of trees, in the arrangement of leaves on the stem, in the flowers of artichokes and sunflowers, in the inflorescences of Romanesco broccoli, in the configuration of coniferous conifers, in the reproduction of rabbits and in how DNA encodes the growth of complex organic forms. And allow us to draw a spiral using squares with side length equal to each term of the sequence, as it shows Fig. (2). The growth ratio is Φ, that is, the golden ratio.

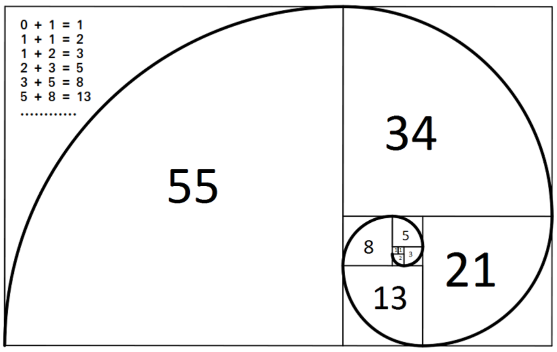

Figure 2: Fibonacci spiral.

This Fibonacci spiral, it is found in the spiral structure of the shell of some mollusks, such as the nautilus as it shows Fig. (3), and also in Leonardo da Vinci's masterpieces, as for instance it is shown in Fig. (4).

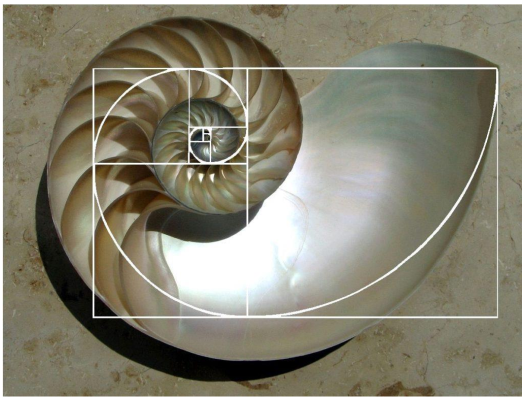

Figure 3: Nautilus and Fibonacci spiral.

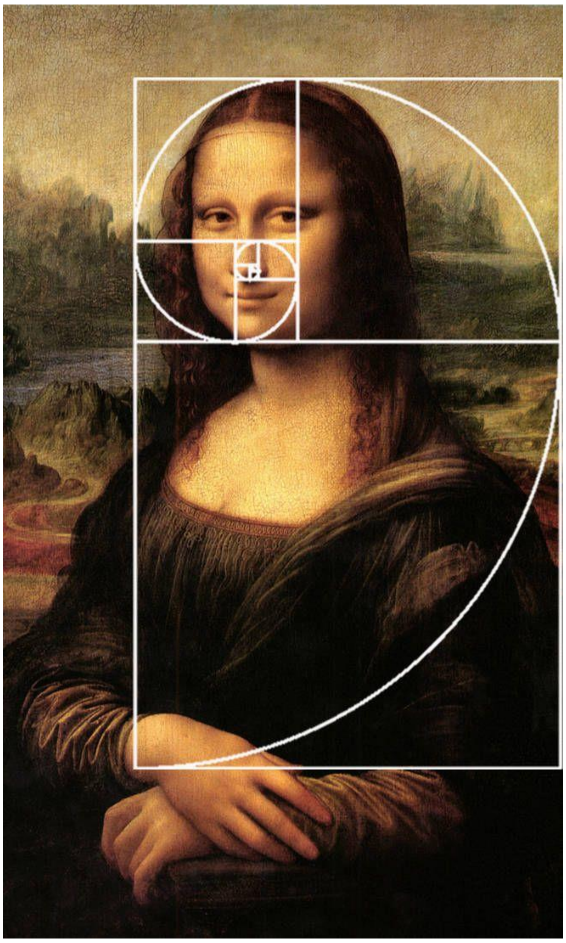

Figure 4: Mona Lisa and Fibonacci spiral.

Mathematicians like Edouard Lucas [24] and Kepler studied this sequence, and the Scottish mathematician Robert Simson found in 1753 (later than Leonardo da Vinci's masterpieces) that the relationship between two successive Fibonacci numbers approaches the golden ratio Φ when n tends to infinity.

And like them, many others worked and wrote about it [1] to [24].

In our case, one day observing Leonardo da Vinci's masterpieces, we wondered how it was possible that the Fibonacci spiral and consequently the number phi were present in his masterpieces, if Fibonacci was born centuries later than Leonardo da Vinci.

The Vetruvian Man caught our attention too. Analyzing this masterpiece, anyone can observe that he was using squares and circles circumscribing a human body to obtain certain body proportions.

It was at that moment when we begin to build the idea shown in this paper, playing with the most basic geometric figures and concepts.

This paper only aims to show how we can obtain the Φ number using the most basic geometric shapes: a square, a right triangle and a circle.

In particular, from a square whose side length is equal to a, we will show how to obtain a segment b in such a way that the value of $\frac{a}{b}$ is the number Φ.

The paper is organized as follows: in Sec.2 we explain how we can obtain Φ using these basic geometric figures and concepts and in Sec.3 we wrap up our conclusions.

2 How to obtain Φ number

First, as it shows the Fig. (5), we draw a square with side length equal to a.

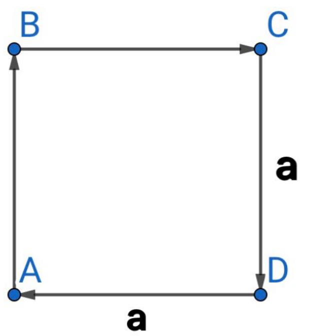

Figure 5: Square with side length equal to a.

Next, we divide this square in two equal rectangles, see Fig. (6), and we draw the rectangle diagonal OC (hypotenuse h of the right triangle OCD) as it shows Fig. (7).

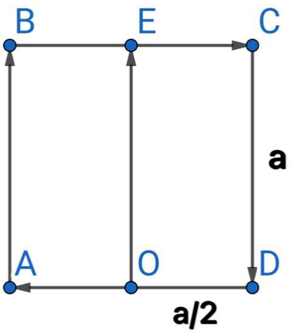

Figure 6: Square divided in two equal rectangles.

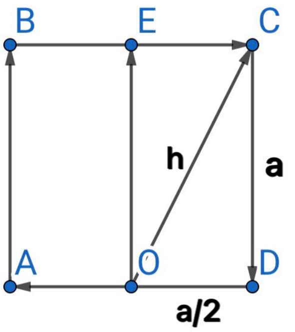

Figure 7: Right triangle with sides $\frac{a}{2}$ and a and hypotenuse h.

Now, considering the point O as the center and the diagonal OC as the radius, we draw a circle as it shows Fig. (8), and we call b to the distance between D and F.

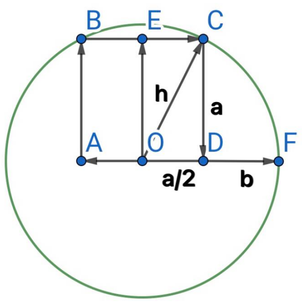

Figure 8: Circle with center O and radius OC.

Note that the distance OF is the radius OC, that is, the hypotenuse of the right triangle OCD, and at the same time it is equal to $\frac{a}{2} + b$.

The number Φ is the proportion between a and b, that is, $\frac{a}{b}$.

Hence, to obtain $\frac{a}{b}$ value, we calculate it as follows.

From the right triangle OCD, the radius OC, that is, the hypotenuse is equal to:

$$OC = \sqrt{(\frac{a}{2})^2 + a^2} = \frac{a\sqrt{5}}{2} \tag{1}$$

And since $OC = OF$, we have:

$$\begin{aligned}
&\frac{a\sqrt{5}}{2} = \frac{a}{2} + b \\
&\frac{a\sqrt{5}}{2} - \frac{a}{2} = b \\
&a(\frac{\sqrt{5}}{2} - \frac{1}{2}) = b \\
&\frac{a}{b} = \frac{2}{\sqrt{5}-1} = \frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{2(\sqrt{5}+1)}{4} \\
&\frac{a}{b} = \frac{\sqrt{5}+1}{2}
\end{aligned} \tag{2}$$

Therefore, the golden ratio $\frac{a}{b}$ also known as the number Φ is equal to:

$$\Phi = \frac{a}{b} = \frac{1+\sqrt{5}}{2} \tag{3}$$

3 Conclusions

In this paper we have shown how to calculate two segments a and b, so that $\frac{a}{b}$ is the number Φ.

In particular, we have shown how from a square whose side measures a, this length being arbitrary, the appropriate segment of length b is the one that allows us to build a circle of radius $\frac{a}{2} + b$, centered on the base of the square at point $\frac{a}{2}$, and so that the semicircle of this circle circumscribes the upper part of the square, as it is shown in Fig. (9).

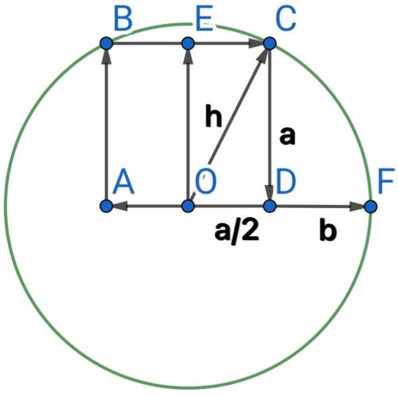

Figure 9: Circle with radius $OC = OF$ and segment $b = DF$.

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