Directed strongly regular graphs as elements of coherent algebras

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April 19, 1999

*Partially supported by the research grant No 6782–1–95 of the Israeli Ministry of Science.
1 Introduction

A systematical investigation of strongly regular graphs was started in the seminal paper [Bos-63] by R.C. Bose. During the last 30 years strongly regular graphs have become one of the most popular areas in algebraic combinatorics. A brief account of the modern state of the art can be found in [Bro-96]. It is well-known, see e.g. [God-93], that a regular simple graph is strongly regular if and only if it has at most three distinct eigenvalues.

It turns out that there are a few other classes of graphs with three distinct eigenvalues, among them the so-called doubly regular tournaments [BroR-72] and non-standard simple graphs (in the sense of [MuzK-98]). In the present paper we consider one more class of graphs with three distinct eigenvalues, namely directed strongly regular graphs (dsrg’s). This class was introduced in [Duv-88] by A.M. Duval who made a first significant input to the thorough investigation of dsrg’s. Further interest to dsrg’s was stimulated by [KliMMZ-97]. Our approach to dsrg’s is a continuation of the ideas which were introduced in [KliMMZ-97]. Namely, we are searching for dsrg’s whose adjacency matrices are elements of prescribed matrix algebras. As a rule, these are coherent (cellular) algebras, as they were introduced by D.G. Higman [Hig-70], [Hig-87] and, independently, by B.Ju. Weisfeiler and A.A. Leman [WeiL-68], [Wei-76]. The use of various modern computer algebra systems allows to discover new interesting infinite classes and sporadic examples of dsrg’s.

All used terminology is introduced in Section 2. Our methodology of construction of dsrg’s as elements of suitable coherent algebras is described in Section 3. The results of a computer enumeration of all vertex-transitive dsrg’s with less or equal than 20 vertices are presented in Section 4. Finally, in Section 5 we introduce a few infinite classes of dsrg’s as elements of coherent flag algebras (special class of coherent algebras with two natural generators).

The main goal of this paper is just to present new dsrg’s and to outline the methods which were used for their discovery. With more details these results will be submitted in further publications.

2 Preliminaries

For us a graph $\Gamma$ is a pair $(V, E)$ consisting of the finite vertex set $V$ and binary relation $E \subseteq V \times V$ which is called adjacency relation. Two vertices $v$ and $w$ are called adjacent if and only if $(v, w) \in E$. A pair $(v, w)$ of adjacent vertices is called arc of $\Gamma$. If also $(w, v)$ is an arc of $\Gamma$ then the set $\{v, w\}$ is called edge of $\Gamma$. A graph is called undirected if its adjacency relation is symmetric, otherwise
it is called directed graph. For an undirected graph the adjacency relation $E$ can be identified with the set $\hat{E}$ of all edges of $\Gamma$. In general we are only interested in graphs without loops (i.e. we assume that $\hat{E}$ is antireflexive). If $\Gamma$ does not have isolated vertices then it can be identified with its adjacency relation $\hat{E}$. Following, if $|V|=v$, then we will identify $V$ with the set of natural numbers $\{1, 2, \ldots, v\}$. In this case $\Gamma$ is called a graph of order $v$. An undirected graph without loops is called simple. For a good introduction into the classical notions and notations from graph theory we refer to [Big-93].

The adjacency matrix of a graph $\Gamma = (V, E)$ is defined as

$$A(\Gamma) = (a_{i,j}) \text{ where } a_{i,j} = \begin{cases} 1 & : \ (i,j) \in E \\ 0 & : \text{ otherwise.} \end{cases}$$

Oppositely, to a square $0/1$ matrix $A = (a_{i,j})$ of order $n$ we associate the graph $\Gamma(A) = (V, E)$ where $V = \{1, 2, \ldots, n\}$ and $E = \text{supp}(A) := \{(i,j) \mid a_{i,j} \neq 0\}$.

As usual we will denote the identity matrix and the all $1$ matrix by $I$ and $J$, respectively.

Let $\Gamma$ be a graph with adjacency matrix $A = A(\Gamma)$. Then we define $\bar{A} := J - I - A$ and $\bar{\Gamma} := \Gamma(\bar{A})$.

**Definition 2.1** A graph $\Gamma$ with adjacency matrix $A = A(\Gamma)$ is called regular if there exists a natural number $k$ such that

$$AJ = JA = kJ. \quad (1)$$

The number $k$ is called valency of $\Gamma$.

In [Bos-63] the notion of a strongly regular graph (srg) was introduced. There a simple regular graph of valency $k$ is said to be strongly regular if there exist integers $\lambda$ and $\mu$ such that

- for each edge $\{v, w\}$ the number of common neighbors of $v$ and $w$ is equal to $\lambda$,
- for each non-edge $\{v, w\}$ the number of common neighbors of $v$ and $w$ is equal to $\mu$.

Below we give the modern definition of srg’s in matrix notation.

**Definition 2.2** A simple graph $\Gamma$ of order $v$ with adjacency matrix $A = A(\Gamma)$ is called strongly regular graph (srg) if there exist natural numbers $k$, $\lambda$, $\mu$ such that

$$A^2 = kI + \lambda A + \mu \bar{A}. \quad (2)$$

The tuple $(v, k, \lambda, \mu)$ is called the parameter set of $\Gamma$. 
In [Duv-88] the notion of srg’s was generalized to directed graphs.

**Definition 2.3** A directed regular graph $\Gamma$ of order $v$ and of valency $k$ with adjacency matrix $A = A(\Gamma)$ is called *directed strongly regular graph (dsrg)* if there exist natural numbers $t$, $\lambda$, $\mu$ such that

$$A^2 = tI + \lambda A + \mu \bar{A}. \quad (3)$$

The tuple $(v, k, \mu, \lambda, t)$ is called the *parameter set* of $\Gamma$.

One of the main results of [Duv-88] was the following theorem:

**Theorem 2.4** Let $\Gamma$ be a dsrg with the parameters $(v, k, \mu, \lambda, t)$ and with adjacency matrix $A = A(\Gamma)$. Then one of the following is true

1) $\Gamma$ is an srg ($t = k$),

2) $\Gamma$ is complete ($A = J - I$),

3) $\Gamma$ is a doubly regular tournament ($t = 0$),

4) $0 < t < k$ and there exists a positive integer $d$ such that

$$k(k + (\mu - \lambda)) = t + (v - 1)\mu, \quad (4)$$

$$(\mu - \lambda)^2 + 4(t - \mu) = d^2, \quad (5)$$

$$d(2k - (\mu - \lambda)(n - 1)), \quad (6)$$

$$\frac{2k - (\mu - \lambda)(n - 1)}{d} \equiv n - 1 \pmod{2}, \quad (7)$$

$$\left|\frac{2k - (\mu - \lambda)(n - 1)}{d}\right| \leq n - 1. \quad (8)$$

We refer to [BroR-72] for the definition of doubly regular tournaments.

A parameter set $(v, k, \mu, \lambda, t)$ will be called *feasible* if it complies to all the above stated necessary conditions.

The following simple lemma turns out to be useful.

**Lemma 2.5** If $A$ is the adjacency matrix of a directed strongly regular graph then so are $\bar{A}$ and $A^t$.

Let $A_1$ and $A_2$ be adjacency matrices of dsrg’s $\Gamma_1$ and $\Gamma_2$, respectively. Then $\Gamma_1$ and $\Gamma_2$ will be called *equivalent* if at least one of the following is true (here the sign $\cong$ denotes isomorphism of graphs):
1) $\Gamma_1 \cong \Gamma_2$,
2) $\Gamma_1 \cong \Gamma(\bar{A})$,
3) $\Gamma_1 \cong \Gamma(A^t)$,
4) $\Gamma_1 \cong \Gamma(\bar{A}^t)$.

Generally, we will consider dsrg’s only up to equivalence.

3 Methodology for the construction of dsrg’s

A directed strongly regular graph will be identified in this paper with its adjacency matrix.

A new approach to the search for dsrg’s was suggested in [KliMMZ-97]. It is based on the systematical use of coherent (cellular) algebras (see definitions below). Roughly speaking this approach allows to look through all 0/1 matrices inside a “candidate”-matrix algebra and to find all those that are adjacency matrices of dsrg’s.

**Definition 3.1** A set $W$ of square complex matrices of order $n$ is called *complex selfadjoint unital matrix algebra* of degree $n$ and of rank $r$ if

1) $W$ forms a complex matrix algebra,
2) $W$, if regarded as complex vector space, has dimension $r$,
3) $I \in W$,
4) $W$ is closed with respect to Hermitian adjunction (as usual the adjoint of $A$ will be denoted as $A^*$).

**Definition 3.2** A complex selfadjoint unital matrix algebra $W$ of degree $n$ and of rank $r$ is called *coherent algebra* if and only if $J \in W$ and for any two elements $A = (a_{i,j})$ and $B = (b_{i,j})$ the Schur-Hadamard product $C = A \circ B = (a_{i,j}b_{i,j})$ is again contained in $W$.

For a given set $M$ of complex $n \times n$ matrices the smallest coherent algebra that contains $M$ is denoted as $\langle \langle M \rangle \rangle$. This closure operator is called Weisfeiler-Leman closure or WL-closure.
It is well known (see e.g. [Wei-76]) that each coherent algebra \( W \) contains a unique basis \( \{ A_0, A_1, \ldots, A_{r-1} \} \) of 0/1 matrices such that

\[
\sum_{j=0}^{r-1} A_i = J.
\]

This basis is called standard basis of \( W \). For any two elements \( A_i \) and \( A_j \) of the standard basis we have the relation

\[
A_i A_j = \sum_{k=0}^{r-1} p_{i,j}^k A_k.
\]

The non-negative integer numbers \( p_{i,j}^k \) are called structure constants of the coherent algebra \( W \).

The existence of the standard basis and the knowledge of the structure constants make it particularly simple to construct all 0/1 matrices in \( W \), and to compute their squares. Namely, each 0/1 matrix \( X \in W \) can be expressed as sum

\[
X = \sum_{i \in \mathcal{I}} A_i
\]

where \( \mathcal{I} \subseteq \{0, 1, \ldots, r-1\} \) and

\[
X^2 = \left( \sum_{i \in \mathcal{I}} A_i \right)^2 = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \sum_{k=0}^{r-1} p_{i,j}^k A_k.
\]

Now it is easy to check whether \( X \) satisfies equations (1) and (3) for a suitable parameter set.

A simple but important necessary condition for a coherent algebra \( W \) to contain dsrg’s was given in [KliMMZ-97].

**Proposition 3.3** Let \( A \) be the adjacency matrix of some dsrg. Then \( \langle \langle A \rangle \rangle \) is non-commutative and has rank \( \geq 6 \).

Thus, in our search we may in advance restrict our efforts just to the consideration of non-commutative coherent algebras of rank \( \geq 6 \). We refer to [KliMMZ-97] for a number of constructions of dsrg’s that were accomplished with the use of coherent algebras.

One of the main sources for “candidate”-algebras is given by the centralizer algebras. Let \( \mathbb{C}^{n \times n} \) be the set of all complex \( n \times n \) matrices and let \( S_n \) be the
symmetric group of degree $n$. Between these sets a Galois-correspondence can be introduced: Namely for $X \subseteq \mathbb{C}^{n \times n}$ define

$$X' = \text{Aut}(X) := \{ g \in S_n \mid \forall A \in X : A \cdot M(g) = M(g) \cdot A \}$$

and for $Y \subseteq S_n$ define

$$Y' = \mathcal{V}(Y) := \{ A \in \mathbb{C}^{n \times n} \mid \forall g \in Y : A \cdot M(g) = M(g) \cdot A \}.$$

In both cases $M(g)$ denotes the permutation matrix of the permutation $g$.

The Galois-closed subsets are on one hand the so-called 2-closed subgroups of $S_n$ and on the other hand the so-called centralizer algebras. Obviously, each centralizer algebra is a coherent algebra. However, the opposite is not true in general. Centralizer algebras are also called Schurian coherent algebras.

The standard basis of a Schurian coherent algebra $W$ can be described very naturally. It consists of the adjacency matrices of the 2-orbits of Aut($W$). For more details see e.g. [FarKM-94].

**Example 3.4** Let $\Gamma := L_2(3)$ be the lattice graph on 9 points (see Figure 1). Then it can be easily checked that $\Gamma$ is invariant under the following permutation group:

$$G = \left\langle (1, 5, 9)(2, 6, 7)(3, 4, 8), (1, 5)(3, 8)(6, 7), (1, 6, 8)(2, 4, 9)(3, 5, 7), (1, 8)(3, 5)(4, 9) \right\rangle.$$

This group has order 36 and is isomorphic to the direct product $S_3 \times S_3$ (here $S_3$ denotes the symmetric group of order 6).

$G$ acts naturally on the 18 edges of $\Gamma$ (see Table 1).

Let us call this action $\tilde{G}$. Then

$$\tilde{G} = \left\langle (1, 13, 17)(2, 10, 18)(3, 14, 9)(4, 6, 15)(5, 11, 16)(7, 8, 12), (1, 6)(2, 14)(3, 10)(4, 13)(5, 7)(8, 16)(9, 18)(11, 12)(15, 17), (1, 11, 18)(2, 13, 16)(3, 15, 7)(4, 8, 14)(5, 10, 17)(6, 12, 9), (1, 7)(2, 14)(3, 18)(4, 16)(5, 6)(8, 13)(9, 10)(11, 15)(12, 17) \right\rangle.$$
Clearly, this action is transitive. Now we would like to compute the centralizer algebra of $\tilde{G}$. As was noted above, this is equivalent to the computation of the 2-orbits of $\tilde{G}$. As $\tilde{G}$ is transitive, each 2-orbit contains elements of the form $(1, x)$ (note that the number 1 refers to the edge $\{1, 2\}$ of $\Gamma$). Following, for each 2-orbit $O_i$, we give the set of all $x$ such that $(1, x) \in O_i$ together with a graphical representation in terms of a subgraph of $\Gamma$ (these sets are called suborbits of $\tilde{G}$ with respect to 1). Obviously this information describes the 2-orbits of $\tilde{G}$ completely. Except $O_6$ and $O_9$, all 2-orbits are symmetric.

$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
\text{Edge No.} & \{1, 2\} & \{1, 3\} & \{1, 4\} & \{1, 7\} & \{2, 3\} & \{2, 5\} & \{2, 8\} & \{3, 6\} & \{3, 9\} \\
\hline
\{4, 5\} & \{4, 6\} & \{4, 7\} & \{5, 6\} & \{5, 8\} & \{6, 9\} & \{7, 8\} & \{7, 9\} & \{8, 9\} \\
\text{Edge No.} & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
\hline
\end{array}$

Table 1: The edges of $\Gamma$ in lexicographical order

Figure 2: The suborbits of $\tilde{G}$ with respect to 1

adjacency matrices $A_i = A(O_i)$ ($1 \leq i \leq 10$) form the standard basis of the centralizer algebra $W := \mathcal{V}(\tilde{G})$.

Knowing the standard basis of $\mathcal{V}(\tilde{G})$ it is easy, using a computer, to find its structure constants and to test each of its nonsymmetric 0/1 matrices, whether it belongs to a dsrg. The result is that (up to equivalence) the following is a
complete list of dsrg’s in $W$:

<table>
<thead>
<tr>
<th>matrix</th>
<th>$v$</th>
<th>$k$</th>
<th>$\mu$</th>
<th>$\lambda$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_2 + A_3 + A_6$</td>
<td>18</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$A_2 + A_3 + A_6 + A_8$</td>
<td>18</td>
<td>7</td>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

These two graphs were discovered in [FieKM]. In the next section more results from [FieKM] are sketched.

4 Small vertex-transitive dsrg’s

Most examples of dsrg’s that were presented in [Duv-88] and [KliMMZ-97] have a transitive automorphism group. Hence a rather natural problem that arises is the description of vertex-transitive dsrg’s. Using the Galois correspondence between coherent algebras of degree $v$ and permutation groups of degree $v$ it is possible to compute all automorphism groups of graphs with $v$ vertices. Since we are interested in vertex-transitive dsrg’s we consider centralizer algebras of transitive permutation groups and the corresponding 2-closed groups. Given such a lattice of 2-closed permutation groups all graphs having a vertex-transitive automorphism group can be constructed with the aid of the principle of inclusion and exclusion, cf. [FarKM-94].

In [Duv-88] A.M. Duval presented a list of feasible parameter sets of dsrg’s on up to 20 vertices. This list served as challenge to construct vertex-transitive dsrg’s on up to 20 vertices and to find examples for all feasible parameter sets (or to prove the non-existence). An essential background of the constructive enumeration of small vertex-transitive dsrg’s is the knowledge of all minimal transitive permutation groups of degree $\leq 20$. In fact, all transitive permutation groups up to degree 31 are known [Hul-96]. A catalogue of these groups is available in GAP [Sch+95], release 3.4.4 or later.

Starting from these groups a GAP-catalogue of vertex-transitive directed graphs has been built [Fie-98], [FieK], [FieKP]. Basing on this catalogue and using computer all small dsrg’s with a transitive automorphism group are constructed and examined thoroughly in [FieKM]. This includes graphical representation as well as a theoretical description. In most cases the careful investigation and interpretation of a dsrg revealed the graph to belong to some series. Besides the already known methods presented in [Duv-88] and [KliMMZ-97] some new series have been described.

Lemma 4.1 ¹ Let $n$ be even, $c \in C_n$ where $c \neq e$ is an involution and $X,Y \subseteq C_n$ such that

¹This lemma was firstly presented (in a slightly different formulation) in [HobS-98]
Let \( p \) yields a dsrg.

Together with [KliMMZ-97], Lemma 6.1, this implies that each dihedral group yields a dsrg.

Let \( d \in D_n \setminus C_n \). Then the Cayley graph \( \Gamma \) which corresponds to \( \Gamma = X + dY + d \) is a dsrg with parameters \((2n, n - 1, \frac{n}{2} - 1, \frac{n}{2} - 1, \frac{n}{2})\).

### Proposition 4.2

The graphs \( \Gamma_i = (B, \mathcal{R}_i) \) with vertex set \( B \) and arc set \( \mathcal{R}_i \) are dsrg’s with parameter \((2q, q - 1, \frac{q-1}{2}, \frac{q-3}{2}, \frac{q-1}{2})\) where

- \( \mathcal{R}_1 = \{(S, S + x) \mid x \in S\}^{(G, B)} \cup \{(S, N + x) \mid x \in S\}^{(G, B)} \);
- \( \mathcal{R}_2 = \{(S, S + x) \mid x \in S\}^{(G, B)} \cup \{(S, N + x) \mid x \in N\}^{(G, B)} \);
- \( \mathcal{R}_3 = \{(S, S + x) \mid x \in N\}^{(G, B)} \cup \{(S, N + x) \mid x \in S\}^{(G, B)} \);
- \( \mathcal{R}_4 = \{(S, S + x) \mid x \in N\}^{(G, B)} \cup \{(S, N + x) \mid x \in N\}^{(G, B)} \).

Each relation \( \mathcal{R}_i \) is a union of two of the six 2-orbits of the transitive action of \((G, \mathcal{B})\). Hence the graphs \( \Gamma_i \) are invariant with respect to \((G, \mathcal{B})\). Moreover, they are equivalent in the sense of Lemma 2.5.

Another series of dsrg’s can be described using certain antiflags of a vector space. The method outlined here is very similar to the method developed in [Pec-98], see next section. Let \( q \) be a prime number, \( q > 2 \). \( \mathbb{F} = \mathbb{F}_q \) be the finite field with \( q \) elements and \( V \) be the 2-dimensional vector space over \( \mathbb{F} \). \( \overline{\mathbf{0}} \) be the zero vector of \( V \). Let \( L \) denote all affine lines in \( V \), and \( V^* = V \setminus \{\overline{0}\} \). \( L^* \) be the set of all affine lines in \( V \) which do not include \( \overline{0} \). We consider the incidence structure \( \mathcal{D} = (V^*, L^*) \) with natural incidence relation, namely inclusion. Let \( \mathcal{A} \) be the set of such antiflags \((l, P)\) of \( \mathcal{D} \) that \( l \) is an affine line not including the zero vector, and \( P \neq \overline{0} \) is a point on the line parallel to \( l \) which includes the zero vector, that is

\[
\mathcal{A} = \{(l, P) \mid l \in L^* \land P \neq \overline{0} \land \exists l' \in L(\overline{0} \in l' \land l' \parallel l \land P \in l')\}.
\]
Evidently, \(|A| = (q - 1)(q^2 - 1)\). On \(A\) we introduce eight natural relations. We emphasize that all relations \(R_i\) are regular. Let \(f = (p, P)\) and \(g = (q, Q)\) be antiflags in \(A\). \(PQ\) may denote the affine line determined by \(P\) and \(Q\), \(P \neq Q\):

- \(R_0 = \{(f, g) \mid P = Q \land p = q\}\);
- \(R_1 = \{(f, g) \mid P = Q \land p \parallel q\}\);
- \(R_2 = \{(f, g) \mid P \neq Q \land \vec{0} \in PQ \land p = q\}\);
- \(R_3 = \{(f, g) \mid P \neq Q \land \vec{0} \in \overline{PQ} \land p \parallel q\}\);
- \(R_4 = \{(f, g) \mid P \neq Q \land \vec{0} \notin \overline{PQ} \land P \notin q \land Q \notin p\}\);
- \(R_5 = \{(f, g) \mid P \neq Q \land \vec{0} \notin \overline{PQ} \land P \notin q \land Q \in p\}\);
- \(R_6 = \{(f, g) \mid P \neq Q \land \vec{0} \notin \overline{PQ} \land P \in q \land Q \notin p\}\);
- \(R_7 = \{(f, g) \mid P \neq Q \land \vec{0} \notin \overline{PQ} \land P \in q \land Q \in p\}\).

**Proposition 4.3** The graphs \(\Gamma_i = (A, R_i)\) are dsrg’s with the parameter set \(((q - 1)(q^2 - 1), q^2 - 2, q, 2q - 3, 2q - 2)\) where

- \(\mathcal{R}_1 = R_1 \cup R_5 \cup R_7\);
- \(\mathcal{R}_2 = R_1 \cup R_6 \cup R_7\);
- \(\mathcal{R}_3 = R_2 \cup R_5 \cup R_7\);
- \(\mathcal{R}_4 = R_2 \cup R_6 \cup R_7\).

In particular, they are equivalent in the sense of Lemma 2.5.

Finally, we can now claim that for all feasible parameter sets on up to 20 vertices either an example was found, or the non-existence of a dsrg was proved ([Duv-88], [KliMMZ-97], [FieKM], [Jor-97]), see Table 2.

<table>
<thead>
<tr>
<th>#</th>
<th>(v)</th>
<th>(k)</th>
<th>(\mu)</th>
<th>(\lambda)</th>
<th>(t)</th>
<th>Existence</th>
<th>Remarks</th>
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Table 2: List of feasible parameters and number of non-equivalent realizations

<table>
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<tr>
<th>#</th>
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<th>µ</th>
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<td>18</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>+ 1 1</td>
</tr>
<tr>
<td>17</td>
<td>18</td>
<td>7</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>? ? 2</td>
</tr>
<tr>
<td>18</td>
<td>18</td>
<td>8</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>+ 1 1</td>
</tr>
<tr>
<td>19</td>
<td>18</td>
<td>8</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>+ 2 5</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>+ 1 1</td>
</tr>
<tr>
<td>21</td>
<td>20</td>
<td>7</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>? 2 1</td>
</tr>
<tr>
<td>22</td>
<td>20</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>+ 2 3</td>
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<td>23</td>
<td>20</td>
<td>9</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>+ 2 5</td>
</tr>
</tbody>
</table>

Table 2: List of feasible parameters and number of non-equivalent realizations

5 Dsrg’s inside of flag algebras

There is an interesting class of coherent algebras with two generators—the flag algebras of block designs, see, e.g., [KilS-73], [Smi-88]. It follows from [KliMMZ-97] that such algebras are “candidates” to contain dsrg’s. This is why we decide to make a systematical search inside of flag algebras.

**Definition 5.1** An incidence structure is a triple $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{F})$ where $\mathcal{P}$ is a set of $v$ points, $\mathcal{B}$ is a set of $b$ blocks and $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{B}$ is a set of flags (the incidence relation).

In order to exclude degenerate incidence structures we always assume that each point is incident with at least two blocks and each block is incident with at least two points.
If an incidence structure contains points that are incident with exactly the same set of blocks, then it is called incidence structure with repeated points (otherwise incidence structure without repeated points). Dually, the notion incidence structure with (without) repeated blocks can be introduced.

In the sequel we will only consider incidence structures with additional regularities, namely the block designs.

**Definition 5.2** A nontrivial incidence structure \( D = (\mathcal{P}, \mathcal{B}, \mathcal{F}) \) without repeated points or blocks is called block design with parameters \((v, b, r, k)\) if \( v = |\mathcal{P}|, b = |\mathcal{B}| \) and if there exist integers \( r \) and \( k \) such that each point is incident with exactly \( r \) blocks and each block is incident with exactly \( k \) points.

On the set of flags \( \mathcal{F} \) of a block design we can introduce two natural relations:

- two flags \( f = (P, p) \) and \( g = (Q, q) \) are called collinear if and only if \( f \neq g \) and \( p = q \).
- two flags \( f = (P, p) \) and \( g = (Q, q) \) are called concurrent if and only if \( f \neq g \) and \( P = Q \).

The collinearity and the concurrency relation over \( \mathcal{F} \) will be denoted as \( R_L \) and \( R_N \) respectively. The corresponding adjacency matrices will be called \( L \) and \( N \).

**Definition 5.3** Let \( D \) be a block design and let \( L \) and \( N \) be as defined above. Then \( W_\mathcal{F}(D) = \langle \langle L, N \rangle \rangle \) is called the flag coherent algebra or simply the flag algebra of \( D \).

The class of block designs is very wide and there is no general way for the description of their flag algebras. Therefore we restrict ourselves to some special, more regular subclasses of block designs:

**Definition 5.4** An incidence structure is called a balanced incomplete block design (BIBD) with parameters \((v, b, r, k, \lambda)\) if it is a block design with parameters \((v, b, r, k)\) and if each pair of points is incident with exactly \( \lambda \) blocks.

The following lemma will help us to introduce a new class of dsrg’s.

**Lemma 5.5** Let \( D = (\mathcal{P}, \mathcal{B}, \mathcal{F}) \) be a \( 2-(v, b, r, k, \lambda) \) design and let \( L, N \) be its collinearity and concurrency relation, respectively. Then

1) \[ L^2 = (k - 1)I + (k - 2)L, \]
2) \( N^2 = (r - 1)I + (r - 2)N \),

3) \( NLN = \lambda J - L - LN - NL - \lambda I - \lambda N \).

\( \square \): 1) and 2) are evident.

**About 3)** First we introduce 5 additional binary relations on \( F \), namely \( R_{(0,0)} \), \( R_{(0,1)} \), \( R_{(1,0)} \), \( R_{(1,1)} \) and \( R_\emptyset \) with adjacency matrices \( A_{(0,0)} \), \( A_{(0,1)} \), \( A_{(1,0)} \), \( A_{(1,1)} \) and \( A_\emptyset \), respectively: (following assume that \( f = (P,p) \) and \( g = (Q,q) \) are flags of \( D \))

\[
\begin{align*}
R_{(0,0)} &= \{(f,g) \mid p \cap q \neq \emptyset, P \notin q, Q \notin p \}, \\
R_{(0,1)} &= \{(f,g) \mid p \cap q \neq \emptyset, P \notin q, Q \in p \}, \\
R_{(1,0)} &= \{(f,g) \mid p \cap q \neq \emptyset, P \in q, Q \notin p \}, \\
R_{(1,1)} &= \{(f,g) \mid p \cap q \neq \emptyset, P \in q, Q \in p \}, \\
R_\emptyset &= \{(f,g) \mid p \cap q = \emptyset \}.
\end{align*}
\]

Note that

\[ I + L + N + A_{(0,0)} + A_{(0,1)} + A_{(1,0)} + A_{(1,1)} + A_\emptyset = J. \]

By counting configurations of flags (see [Pec-98]) it can be shown that

\[
\begin{align*}
LN &= A_{(1,1)} + A_{(0,1)}, \\
NL &= A_{(1,1)} + A_{(1,0)}, \\
NLN &= (\lambda - 1)L + \lambda A_{(0,0)} + (\lambda - 1)(A_{(0,1)} + A_{(1,0)}) + (\lambda - 2)A_{(1,1)} + \lambda A_\emptyset.
\end{align*}
\]

Simple algebraical reformulations lead to the desired result.

\[ \square \]

**Theorem 5.6** Let \( D \) be any 2-(\( v, b, r, k, \lambda \)) design. Let \( L \) and \( N \) be its collinearity and concurrency matrix respectively. Then

1) \( L + LN \) is the adjacency matrix of a directed strongly regular graph with parameters

\[ (\tilde{v}, \tilde{k}, \tilde{\mu}, \tilde{\lambda}, \tilde{t}) = (vr, r(k - 1), \lambda(k - 1), \lambda(k - 2), \lambda(k - 1)). \]

2) \( L + N + LN \) is the adjacency matrix of a directed strongly regular graph with parameters

\[ (\tilde{v}, \tilde{k}, \tilde{\mu}, \tilde{\lambda}, \tilde{t}) = (vr, rk - 1, \lambda k, \lambda(k - 1) + r - 2, \lambda(k - 1) + r - 1). \]
This Theorem was formulated in [Pec-98]. Here the main ideas of the its proof are repeated.

Multiplying both sides of equation 3) in the formulation of Lemma 5.5 from left with \( L \) gives:

\[
LNLN = \lambda(k-1)J - L^2 - L^2N - LNL - \lambda L - \lambda LN.
\]

**About 1)**

With \( A = (L + LN) \) we get

\[
A^2 = L^2 + L^2N + LNL + LNLN
\]
\[
= L^2 + L^2N + LNL + \lambda(k-1)J - L^2 - L^2N - LNL - \lambda(L + LN)
\]
\[
= \lambda(k-1)J - \lambda A.
\]

From the other hand we can compute

\[
\tilde{t}I + \tilde{\lambda}A + \tilde{\mu}A = \lambda(k-1)I + \lambda(k-2)A + \lambda(k-1)\tilde{A}
\]
\[
= \lambda(k-2)A + \lambda(k-1)(J - A)
\]
\[
= \lambda(k-2)A - \lambda(k-1)A + \lambda(k-1)J
\]
\[
= \lambda(k-1)J - \lambda A.
\]

Finally we see that

\[
AJ = LJ + LJN = (k-1)J + (r-1)LJ = (k-1)J + (k-1)(r-1)J = r(k-1)J
\]

and

\[
JA = JL + JLN = r(k-1)J.
\]

**About 2)**

With \( A = (L + N + LN) \) we get

\[
A^2 = L^2 + LN + L^2N + NL + N^2 + NNL + LNL + LN^2 + LNLN
\]
\[
= LN + NL + N^2 + NNL + LN^2 + \lambda(k-1)J - \lambda L - \lambda LN
\]
\[
= (r-1)I + \lambda(k-1)J + (r-1 - \lambda)L + (r-2)N + (r - \lambda - 1)LN
\]
\[
+NL + NNL
\]
\[
= (r - \lambda - 1)I + \lambda kJ + (r - \lambda - 2)A.
\]

On the other hand we can compute

\[
\tilde{t}I + \tilde{\lambda}A + \tilde{\mu}A = (r - \lambda - 1)I + \lambda kJ + (r - \lambda - 2)A
\]
Finally we check easily that
\[
AJ = JA = JN + JL + JLN = ((r - 1) + (k - 1) + (k - 1)(r - 1))J = (r + k - 2 + kr - k - r + 1)J = (rk - 1)J.
\]
This completes the proof.

In [Pec-98] it was also shown that for BIBD \(D\) where any two blocks are either disjoint or intersect in exactly \(\mu\) points, the following equality holds:
\[
A_\emptyset = \frac{1 - \lambda}{\lambda} L + \frac{\mu - 1}{\mu} N + \frac{\mu - \lambda}{\lambda \mu} (LN + NL) - \frac{1}{\mu} LNL + \frac{1}{\lambda} NLN.
\]
where \(A_\emptyset\) is defined as above. Using this equality, the following theorem was proved:

**Theorem 5.7** Let \(D\) be a BIBD with parameters \((v, b, r, k, \lambda)\) such that any two blocks are either disjoint or intersect in exactly \(\mu\) of points (for a fixed positive integer \(\mu\)). Then \((N + NL + A_\emptyset)\) is the adjacency matrix of a directed strongly regular graph if and only if either \(\lambda = 1\) or \(k = 2\mu\). In the case \(k = 2\mu\) the dsrg has parameters
\[
\begin{align*}
\bar{v} &= vr \\
\bar{k} &= \frac{r(2r\mu + \lambda - r - 2\mu\lambda)}{\lambda} \\
\bar{\mu} &= \frac{2r^2\mu + 2\mu\lambda^2 + 2r\lambda - r^2 - 4r\mu\lambda - \lambda^2}{\lambda} \\
\bar{\lambda} &= \frac{2r^2\mu + r\lambda - 4r\lambda\mu + 2\lambda^2\mu - r^2}{\lambda} \\
\bar{t} &= \bar{\mu}.
\end{align*}
\]
The proof of this theorem is very tedious. Therefore we omit it here. It can be found in [Pec-98]. The case \(\lambda = 1\) was treated in [KliMMZ-97].

Note that Theorem 5.7 applies in particular for the Hadamard 3-designs.

**Remark 5.8** The previous theorems hold for classes of BIBDs for which the flag algebras have not been described yet in a unified way. In fact the proofs do not require the knowledge of the flag algebra itself but only of the smallest selfadjoint unital matrix algebra that is generated by \(L\) and \(N\). However, in many cases this algebra can be described more easily in terms of defining relations over the alphabet \(\{L, N\}\) (cf. [Pec-98]).
One interesting class of block designs for which the flag algebras are known completely is that of the finite generalized quadrangles.

**Definition 5.9** A block design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{F})$ with parameters $(v, b, r, k)$ is called *finite generalized quadrangle* of order $(s, t)$ if

- $s = k - 1, t = r - 1,$
- each pair of blocks intersects in at most one point,
- for each block $b$ and each point $P$ not on $b$ there exists exactly one block $c$ that has a non-empty intersection with $b$ and that contains $P$.

The following is an elementary property of generalized quadrangles:

**Lemma 5.10** Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{F})$ is a finite generalized quadrangle of order $(s, t)$ and let $\mathcal{F}^* = \{(p, P) \mid (p, P) \in \mathcal{F}\}$. Then $\mathcal{D}^* := (\mathcal{B}, \mathcal{P}, \mathcal{F}^*)$ is a generalized quadrangle of order $(t, s)$. It is called the dual quadrangle of $\mathcal{D}$.

For this class of block designs the flag algebras were described completely, e.g. in [KilS-73], [Zie-95] or [Pec-98].

**Proposition 5.11** Let $\mathcal{D}$ be a generalized quadrangle of order $(s, t)$ and let $W_\mathcal{F}(\mathcal{D})$ be its flag algebra. Then

1) the following is a complete set of defining relations for $W_\mathcal{F}(\mathcal{D})$ over the alphabet $\{L, N\}$:

$$L^2 = sI + (s - 1)L$$
$$N^2 = tI + (t - 1)N$$
$$(LN)^2 = (NL)^2$$

2) the standard basis of $W_\mathcal{F}(\mathcal{D})$ is

$$\langle A_0, A_1, A_2, A_4, A_5, A_6, A_7 \rangle$$

where $A_0 = I, A_1 = L, A_2 = N, A_3 = LN, A_4 = NL, A_5 = LNL, A_6 = NLN, A_7 = LNLN = NNLN$.

With this knowledge it is possible to describe all structure constants in terms of polynomials in $s$ and $t$. This was done e.g. in [Pec-98] and [KliPZ-98]. Now the above described method may be used for the enumeration of all dsrg’s in $W_\mathcal{F}(\mathcal{D})$. 

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Theorem 5.12 Let $\mathcal{D}$ be a generalized quadrangle of order $(s,t)$ and let $W_\mathcal{F}$ be its flag algebra with standard basis as described in Proposition 5.11. Then, the following is a complete list of equivalence classes of dsrg’s in $W_\mathcal{F}(\mathcal{D})$ up to duality of $\mathcal{D}$: with $\tilde{v} = (s+1)(t+1)(ts+1)$

<table>
<thead>
<tr>
<th>Matrix</th>
<th>$\tilde{v}$</th>
<th>$\tilde{k}$</th>
<th>$\tilde{\mu}$</th>
<th>$\tilde{\lambda}$</th>
<th>$\tilde{t}$</th>
<th>$(s,t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1 + A_2 + A_3$</td>
<td>$\tilde{v}$</td>
<td>$ts + t + s$</td>
<td>$1$</td>
<td>$t + s - 1$</td>
<td>$t + s$</td>
<td>$(s,t)$</td>
</tr>
<tr>
<td>$A_1 + A_3 + A_5$</td>
<td>$2(t+1)^2$</td>
<td>$2t + 1$</td>
<td>$1$</td>
<td>$t$</td>
<td>$t + 1$</td>
<td>$(1,t)$</td>
</tr>
<tr>
<td>$A_3 + A_5 + A_6$</td>
<td>$8$</td>
<td>$3$</td>
<td>$1$</td>
<td>$1$</td>
<td>$2$</td>
<td>$(1,1)$</td>
</tr>
</tbody>
</table>

$\Box$:
See [Pec-98] or [KliPZ-98].

It turns out that a part of this result can be generalized to a slightly wider class of block designs — the partial geometries.

Definition 5.13 A block design $\mathcal{D}$ with parameters $(v, b, R, K)$ is called partial geometry if there exists a positive integer $T$ such that

- any pair of blocks intersects in at most one point,
- for any block $b$ and any point $P$ such that $P \notin b$ there exist exactly $T$ blocks that contain $P$ and intersect $b$.

The triple $(K, R, T)$ is called parameter set of the partial geometry.

Theorem 5.14 Let $\mathcal{D}$ be a partial geometry with parameters $(K, R, T)$. Let $L$ and $N$ be its collinearity and concurrency matrix, respectively. Then $L + N + LN$ is the adjacency matrix of a dsrg with the following parameters:

\[
\begin{align*}
 v &= RK \left(1 + \frac{(K - 1)(R - 1)}{T}\right) \\
 k &= RK - 1 \\
 \mu &= T \\
 \lambda &= K + R - 3 \\
 t &= K + R - 2.
\end{align*}
\]

6 Acknowledgments

We thank A. Munemasa, M. Muzychuk and P.-H. Zieschang for helpful discussions on various stages of the investigation of dsrg’s.
References


