When considering the asymptotic availability is not a safe bet

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When evaluating the availability of an repairable equipment, it is customary to use its asymptotic (steady-state) value. Recent developments in the field of Information Technology and Telecommunications have shown that the transient regime should not be overlooked, since the availability can decrease *below* that limit. In a previous work presented in ESREL 2019, a repairable system described by a gamma failure distribution and an exponential repair distribution has been studied. The exact solution for the availability was obtained, exhibiting two regimes in which the minimum availability is indeed smaller than the asymptotic value. We show here that this phenomenon also happens for more general pairs of failure/repair distributions. We provide simple criteria for the occurrence of such a behavior, and apply them to various well-known distributions (lognormal, Weibull, Birnbaum-Saunders, and Inverse Gaussian). We show that replacing the availability by its steady-state value MTTF/(MTTF + MTTR) is not always a safe bet, and that more care should be exercised for the definition of a robust lower bound of the availability, applicable for all times. We conclude by providing methods to assess the true minimum of the availability.

Keywords: Availability, MTTF, MTTR, transient behavior, Laplace transform, failure/repair distributions, gamma distribution, lognormal distribution, Weibull distribution, Birnbaum-Saunders distribution, Inverse Gaussian distribution.

1. Introduction and context

It is well known that for the textbook model exponential distributions with failure rate λ and repair rates μ, the availability $A(t)$ is given by (Rausand and Høyland (2004))

$$A(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)\,t} . \qquad (1)$$

In most operational situations, $\lambda \ll \mu$. The availability is therefore barely distinguishable from its asymptotic value $A_\infty = \dfrac{\mu}{\lambda+\mu}$, which is a lower bound to $A(t)$. When the distributions are not exponential, the steady-state availability has to be replaced by another well-known, more general result, namely MTTF/(MTTF + MTTR), where MTTF is the Mean Time To Failure and MTTR the Mean Time To Repair.

Recently, a few papers have shown that the transient availability matters in operations in the field of Information Technology and Telecommunications (see for instance Distefano et al. (2010); Carnevali et al. (2015)), and brought proof that the assumption of exponential distributions is questionable when a realistic description of the behavior of equipments or systems is needed (Albert and Dorra (2018)). They have shown that $A(t)$ can exhibit different behaviors: (i) the asymptotic limit A_∞ may only be reached after a few MTTF (ii) $A(t)$ may undergo oscillations, and the asymptotic value is no longer a lower bound of the availability. Consequently, it is no longer safe to assume that the availability is given by MTTF/(MTTF + MTTR). Current research in telecommunications is devoted to 5G systems (Mauro et al. (2017)) and on Network Function Virtualization (NFV) (Mauro et al. (2018)). Realistic calculations of the availability of IP multimedia subsystems *at all times* imply using non-Markovian frameworks, that are sometimes numerically challenging. A robust lower bound of the availability is also required for Service Level Agreements (SLAs). Apart from the abovementioned references, to the best of our knowledge, this issue does not seem to have been addressed in the literature.

In this context and for these reasons, Tanguy (2019) and Tanguy et al. (2019)) investigated how such transient behaviors can occur for a system, for which the lifetime and repair distribution are not exponential. The exact availability $A(t)$ was calculated for a lifetime gamma distribution with a rational shape parameter α, and an exponential repair time distribution. Two regimes could be observed — for $0 < \alpha < 1$ and $\alpha > 2$ — in which $A(t)$ goes below A_∞. In these regimes, A_∞ overestimates the true availability for extended periods of time. Replacing the availability with A_∞ could therefore be misleading, when a robust assessment of an equipment's availability is needed.

In the present paper, we address the role of the lifetime and repair distributions, in order to see whether such a behavior only occurs for exponential and gamma distributions. Since the exact calculation of $A(t)$ is difficult in the general case, we

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have developed simple criteria that can ensure that the availability may be lower than its asymptotic limit. Our aim is thus to determine when caution should be exercised by reliability practitioners.

This paper is organized as follows: Notations and definitions are given in Section 2. Section 3 is devoted to the definition of our criteria, which we apply to the previously studied configuration (gamma failure distribution, and exponential repair distribution) in order to compare them with exact results. In Section 4, these criteria are applied to various lifetime distributions. We show in Section 5 that the roles of the failure and repair distributions are not symmetric. We then conclude and provide possible directions for the calculation of the true minimum, which could be very helpful for the robust determination of a lower bound to the availability.

2. Notations and definitions

2.1. *General definitions*

For a failure density of probability $f(t)$, the reliability $F(t)$ is given by

$$F(t) = 1 - \int_0^t f(\tau)\, d\tau\,; \tag{2}$$

it follows that $F(0) = 1$. The Laplace transform of f is

$$\widetilde{f}(s) = \int_0^\infty f(t)\, e^{-s\,t}\, dt\,. \tag{3}$$

An integration by parts shows that

$$\widetilde{F}(s) = \frac{1 - \widetilde{f}(s)}{s}\,. \tag{4}$$

$\widetilde{F}(0)$ is equal to the MTTF. Similar definitions and results are obtained for the repair density of probability $g(t)$.

Assuming that after each repair, the system is as good as new, it is possible to compute the Laplace transform $\widetilde{A}(s)$ of $A(t)$. Taking into account all the contributions of the possible scenarios, we obtain

$$\begin{aligned}\widetilde{A}(s) &= \frac{\widetilde{F}(s)}{1 - \widetilde{f}(s)\,\widetilde{g}(s)} \\ &= \frac{\widetilde{F}(s)}{s\,(\widetilde{F}(s) + \widetilde{G}(s) - s\,\widetilde{F}(s)\,\widetilde{G}(s))}\,. \end{aligned}\tag{5}$$

It is then straightforward to compute A_∞ and recover the well-known result

$$\begin{aligned}A_\infty = \lim_{s\to 0} s\,\widetilde{A}(s) &= \frac{\widetilde{F}(0)}{\widetilde{F}(0) + \widetilde{G}(0)} \\ &= \frac{\mathrm{MTTF}}{\mathrm{MTTF} + \mathrm{MTTR}}\,. \end{aligned}\tag{6}$$

Going back to the time domain requires performing an *inverse* Laplace transform, which is very difficult, except in special cases (Tanguy (2019); Tanguy et al. (2019)).

2.2. *Exponential distributions*

In the case of exponential distributions with rates λ and μ,

$$F(t) = e^{-\lambda\, t}\,, \tag{7}$$

$$G(t) = e^{-\mu\, t}\,. \tag{8}$$

This leads to

$$\widetilde{F}(s) = \frac{1}{s+\lambda}\,, \tag{9}$$

$$\widetilde{G}(s) = \frac{1}{s+\mu}\,, \tag{10}$$

and to the familiar $\mathrm{MTTF} = 1/\lambda$ and $\mathrm{MTTR} = 1/\mu$. Using Eqs. (9) and (10) in Eq. (5) gives

$$\begin{aligned}\widetilde{A}(s) &= \frac{s+\mu}{s\,(s+\lambda+\mu)}\,, \\ &= \frac{\mu}{\lambda+\mu}\,\frac{1}{s} + \frac{\lambda}{\lambda+\mu}\,\frac{1}{s+\lambda+\mu}\,, \end{aligned}\tag{11}$$

from which we deduce Eq. (1).

A consequence of this result is that A_∞ is reached after a few $1/(\lambda+\mu)$, which means that $A(t)$ is barely distinguishable from its asymptotic value after a few MTTR. For all purposes, the availability can therefore be safely replaced by $\dfrac{\mu}{\lambda+\mu}$.

2.3. *Gamma distributions*

Gamma distributions (Rausand and Høyland (2004)) are, after exponentials, among the most used distributions in Reliability Theory. In the following, the gamma distribution is defined by

$$f(t) = \frac{(\alpha\,\lambda)^\alpha\, t^{\alpha-1}}{\Gamma(\alpha)}\, e^{-\alpha\,\lambda\,t}\,, \tag{12}$$

where α is the so-called shape parameter, and Γ is the Euler gamma function. The definition given in Eq. (12) ensures that MTTF is equal to $1/\lambda$. The Laplace transform $\widetilde{f}(s)$ is

$$\widetilde{f}(s) = \frac{(\alpha\,\lambda)^\alpha}{(s+\alpha\,\lambda)^\alpha}\,. \tag{13}$$

The exponential case is recovered when $\alpha = 1$.

3. Criteria for the determination of minima below the asymptotic availability

3.1. *First criterion*

We define our first criterion by

$$\Delta_0 = \int_0^\infty \left(A(t) - A_\infty\right)\,dt\,. \tag{14}$$

Obviously, $A(0) = 1 > A_\infty$. Consequently, if Δ_0 is negative, there must be at least one time interval during which $A(t) - A_\infty$ is negative. If that happens, the availability would be overestimated, were one to use A_∞. Our first criterion is therefore $\Delta_0 < 0$.

We can compute Δ_0 without needing the exact knowledge of $A(t)$, because we can still use $\widetilde{A}(s)$ with profit. Indeed,

$$\begin{aligned}\Delta_0 &= \lim_{s\to 0}\int_0^\infty e^{-s\,t}\,\left(A(t) - A_\infty\right)\,dt\,,\\ &= \lim_{s\to 0}\left(\widetilde{A}(s) - \frac{A_\infty}{s}\right)\,.\end{aligned} \tag{15}$$

From Eq. (5), we find

$$\Delta_0 = \frac{\widetilde{F}^2(0)\,\widetilde{G}(0) + \widetilde{F}'(0)\,\widetilde{G}(0) - \widetilde{F}(0)\,\widetilde{G}'(0)}{(\widetilde{F}(0)+\widetilde{G}(0))^2}\,. \tag{16}$$

$\widetilde{F}'(0)$ and $\widetilde{G}'(0)$ must now be linked to properties of the distributions F and G. For instance,

$$\begin{aligned}\widetilde{F}'(0) &= \lim_{s\to 0}\frac{d}{ds}\int_0^\infty F(t)\,e^{-s\,t}\,dt\,,\\ &= \lim_{s\to 0}\int_0^\infty (-t)\,F(t)\,e^{-s\,t}\,dt\,,\\ &= -\int_0^\infty t\,F(t)\,dt\,,\\ &= -\frac{1}{2}\int_0^\infty t^2\,(-F'(t))\,dt\,,\\ &= -\frac{1}{2}\,\langle\, t^2\,\rangle_f\,.\end{aligned} \tag{17}$$

$\widetilde{F}'(0)$ is proportional to the average of t^2 for the distribution density f. More generally,

$$\widetilde{F}(s) = \sum_{n=0}^{+\infty}(-1)^n\,\frac{s^n}{(n+1)!}\,\langle\, t^{n+1}\,\rangle_f\,. \tag{18}$$

Going back to the expression of Δ_0 in Eq. (16), we find

$$\begin{aligned}\Delta_0 = &\frac{1}{\left(\langle\, t\,\rangle_f + \langle\, t\,\rangle_g\right)^2}\\ &\times\Bigg\{\langle\, t\,\rangle_g\,\left(\langle\, t\,\rangle_f^2 - \frac{1}{2}\,\langle\, t^2\,\rangle_f\right)\\ &-\langle\, t\,\rangle_f\,\left(\langle\, t\,\rangle_g^2 - \frac{1}{2}\,\langle\, t^2\,\rangle_g\right)\\ &+\langle\, t\,\rangle_f\,\langle\, t\,\rangle_g^2\Bigg\}\,.\end{aligned} \tag{19}$$

In the following, we will keep $\widetilde{F}(0) = 1/\lambda$ and $\widetilde{G}(0) = 1/\mu$ to better compare the influence of failure and repair distributions. Because of the presence of $\langle\, t^2\,\rangle_f$ and $\langle\, t^2\,\rangle_g$, the variances of the distributions play an important role too. Let us recall that

$$\mathrm{var}_f = \langle\, t^2\,\rangle_f - \langle\, t\,\rangle_f^2 \;\; > 0\,. \tag{20}$$

3.2. Δ_0 *for exponential and gamma distributions*

When failures and repairs are described by exponential distributions, the variance is equal to the square of MTTF, and the first two lines between the curly brackets of Eq. (19) vanish, leaving only

$$\Delta_0(\exp,\exp) = \frac{\langle\, t\,\rangle_f\,\langle\, t\,\rangle_g^2}{\left(\langle\, t\,\rangle_f + \langle\, t\,\rangle_g\right)^2} = \frac{\lambda}{(\lambda+\mu)^2}\,. \tag{21}$$

For a gamma distribution with shape parameter α and MTTF equal to $1/\lambda$, noted in short gamma(λ,α),

$$\langle\, t^2\,\rangle_{\mathrm{gamma}} = \frac{1+\alpha}{\alpha\,\lambda^2}\,. \tag{22}$$

In the case of a failure distribution gamma(λ,α) and an exponential repair distribution characterized by μ,

$$\Delta_0(\mathrm{gamma},\exp) = \frac{2\,\alpha\,\lambda + (\alpha-1)\,\mu}{2\,\alpha\,(\lambda+\mu)^2}\,. \tag{23}$$

In most cases of practical interest, $\lambda \ll \mu$. We deduce from Eq. (23) that if $\alpha < 1$, Δ_0 will be negative, implying that $A(t)$ exhibits a minimum $A_{\min}$ which is smaller than $\frac{\mu}{\mu+\lambda}$. We thereby recover very easily a previously obtained result, namely the first of the two regimes.

3.3. *Interpretation of* $\boldsymbol{\Delta_0}$

In our previous study of the gamma(λ,α)/exp(μ) case, we found that for $1 \le \alpha \le 2$, the asymptotic limit A_∞ was indeed the minimum of $A(t)$. We shall show how Δ_0 can be linked to the average time needed to reach the asymptotic value.

Let us first define

$$B(t) = A(t) - A_\infty \,, \tag{24}$$

and

$$\langle\, t \,\rangle_{\text{asympt}} = \frac{\displaystyle\int_0^\infty t\,(-B'(t))\,dt}{\displaystyle\int_0^\infty (-B'(t))\,dt} \,. \tag{25}$$

An integration by parts leads to

$$\begin{aligned} \langle\, t \,\rangle_{\text{asympt}} &= \frac{\displaystyle\int_0^\infty B(t)\,dt}{B(0)} \\ &= \frac{\Delta_0}{1 - A_\infty} \\ &= \frac{\mu+\lambda}{\lambda}\,\Delta_0 \,. \end{aligned} \tag{26}$$

Inserting Eq. (23) in the above expression gives

$$\langle\, t \,\rangle_{\text{asympt}}(\text{gamma}, \exp) = \frac{2\,\alpha\,\lambda + (\alpha-1)\,\mu}{2\,\alpha\,\lambda\,(\lambda+\mu)} \,. \tag{27}$$

For $\alpha = 1$, we recover the expected result $1/(\lambda+\mu)$, whereas for $\alpha = 2$, we obtain $\dfrac{\mu + 4\,\lambda}{4\,\lambda\,(\lambda+\mu)} \approx \dfrac{1}{4\,\lambda}$, which is nothing but the MTTF/4 already identified in Tanguy et al. (2019).

3.4. *Higher-order criteria*

In our previous work (Tanguy (2019); Tanguy et al. (2019)) we found that there are also minima — and possibly large oscillations — of $A(t)$ when $\alpha > 2$. This second regime was not detected by Δ_0. This shows that the criterion $\Delta_0 < 0$, while interesting, is not sufficient. We can however define other quantities, namely

$$\Delta_n = \int_0^\infty t^n\,\left(A(t) - A_\infty\right)\,dt \,. \tag{28}$$

They generalize Δ_0, and ensure that the contribution for t close to zero does not weigh too much. Like before, a negative Δ_n implies that there is a time domain during which the availability is below its asymptotic limit; this quantity can also be calculated by using the Taylor expansion of $\widetilde{A}(s)$ at the origin

$$\Delta_n = (-1)^n\,\frac{d^n}{ds^n}\left(\widetilde{A}(s) - \frac{A_\infty}{s}\right)\Bigg|_{s=0} \,. \tag{29}$$

Alternatively, we have

$$\widetilde{B}(s) = \widetilde{A}(s) - \frac{A_\infty}{s} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\,\Delta_n\,s^n \,. \tag{30}$$

We expect therefore Δ_1's expression to include values such as $\widetilde{F}''(0)$ and $\widetilde{G}''(0)$, which correspond to $\langle\, t^3 \,\rangle_f$ and $\langle\, t^3 \,\rangle_g$, and therefore to the skewness of the respective distributions.

Applying the calculation of Δ_1 to the configuration gamma(λ,α)/exp(μ) gives

$$\Delta_1(\text{gamma}, \exp) = \frac{(\alpha^2-1)\mu\,(\mu+4\,\lambda) + 6\,\alpha\,(\alpha+1)\,\lambda^2}{12\,\alpha^2\,\lambda\,(\lambda+\mu)^3} \,. \tag{31}$$

Assuming $\lambda \ll \mu$, we see that the sign of Δ_1 is basically given by the sign of $(\alpha^2-1)\mu\,(\mu+4\,\lambda)$. It is negative for $\alpha < 1$. So far, no new information has been gained we did not obtain from Δ_0. Still, we can consider the limit $\lambda \to 0$ to the numerator of Δ_1 to determine the sign.

Using MATHEMATICA[a], we have computed the Taylor expansion of $\widetilde{B}(s)$ and obtained the following results in the limit $\lambda \to 0$

$$\Delta_2 \propto \alpha^2 - 1 \,, \tag{32}$$

$$\Delta_3 \propto (\alpha^2-1)\,(19-\alpha^2) \,, \tag{33}$$

$$\Delta_4 \propto (\alpha^2-1)\,(9-\alpha^2) \,. \tag{34}$$

Theirs signs are negative for $\alpha < 1$. The novelty is here that for $\alpha > \sqrt{19}$ (for Δ_3) and $\alpha > 3$ (for Δ_4), the sign is also negative. We have therefore identified at least part of the second region for α in which the minimum $A_{\min}$ is less than A_∞. Increasing the value of n up to 50 indicates that the lower bound of the domain where $A_{\min} < A_\infty$ seems to be 2^+, as expected.

In conclusion, we have found a simple way to identify whether the minimum value of the availability lies below the asymptotic limit, and checked the previous results obtained in the case

[a] https://www.wolfram.com/mathematica/

of the gamma(λ,α)/exp(μ) configuration. Let us now turn to the study of other configurations.

4. Application to various lifetime distributions

As a first step towards a generalization of our previous results, we have kept an exponential repair time distribution, but changed the lifetime distribution, using other well-known examples from the safety and reliability literature (Rausand and Høyland (2004)). By doing so, we can assess the influence of the lifetime distribution, in cases where the exact calculation of the availability is not possible, since the Laplace inverse transform is very difficult, even numerically.

In the following, the distribution parameters will be rescaled with respect to the standard parameters (Rausand and Høyland (2004)) so that the MTTF is always $1/\lambda$. The second parameter will be called α. When it corresponds to a traditional shape factor, we shall not change its meaning; otherwise, the "new" α will be defined by $\alpha = \lambda^2 \,\mathrm{var}_f$. The corresponding Laplace transforms are listed in Appendix A.

4.1. *Lognormal (LN) distribution*

Setting $\alpha = \gamma - 1$, and following the same procedure as in the preceding Section, we found that in the limit $\lambda \to 0$,

$$\Delta_3(\mathrm{LN}, \exp) \propto 45 - 90\,\gamma + 20\,\gamma^2 + 30\,\gamma^3 - 6\,\gamma^6\,, \tag{35}$$

so that the condition $\Delta_3 < 0$ is satisfied if $\gamma > 0.770221938$. This means that for all positive values of α, Δ_3 is negative. Conqequently, there is always a minimum of $A(t)$ below A_∞ for a lognormal lifetime distribution, in sharp contrast with the previous exponential and gamma distributions.

4.2. *Weibull (W) distribution*

The standard definition of α (the exponent) is kept for the Weibull distribution. A first result is

$$\Delta_0(\mathrm{W}, \exp) \propto 2\,\Gamma\left(1 + \frac{1}{\alpha}\right)^2 - \Gamma\left(1 + \frac{2}{\alpha}\right)\,, \tag{36}$$

which is negative for $\alpha < 1$ (recall that when $\alpha = 1$ we have an exponential distribution, so no negative sign should be found). As in the gamma distribution configuration, only for $n \geq 3$ can we infer the existence of another regime in which the minimal availability is smaller than the asymptotic value. Δ_3 is negative when $\alpha > \alpha_3 = 1.97290629$. The study of higher-order terms shows that the lower bound α_n decreases with n; in particular, $\alpha_{18} = 1.0763248975$. The variation of α_n with n seems to indicate that the lower limit is likely 1^+. That would mean that unless $\alpha = 1$, there is always a minimum of $A(t)$ below the asymptotic limit.

4.3. *Inverse Gaussian (IG) distribution*

In the limit $\lambda \to 0$,

$$\Delta_0(\mathrm{IG}, \exp) \propto 1 - \alpha\,, \tag{37}$$

meaning that there is a minimum below A_∞ when $\alpha > 1$. Then,

$$\Delta_3(\mathrm{IG}, \exp) \propto -(1 - 15\,\alpha^2)^2\,. \tag{38}$$

This is always negative unless $\alpha = 1/\sqrt{15}$. This particular case does not cause any trouble for Δ_4, since

$$\Delta_4(\mathrm{IG}, \exp) \propto -(1 - 25\,\alpha^2 + 210\,\alpha^4)\,, \tag{39}$$

which is always negative. There will therefore always be a minimum of $A(t)$ lying below A_∞.

4.4. *Birnbaum-Saunders (BS) distribution*

When $\lambda \to 0$,

$$\Delta_0(\mathrm{BS}, \exp) \propto 1 - \alpha^4\,; \tag{40}$$

there is a minimum below A_∞ when $\alpha > 1$. Increasing n leads to

$$\begin{aligned}\Delta_4(\mathrm{BS}, \exp) \propto -(&16 + 84\,\alpha^2 - 208\,\alpha^4 \\ &-1878\,\alpha^6 - 720\,\alpha^8 + 11170\,\alpha^{10} + 12080\,\alpha^{12} \\ &+5475\,\alpha^{14} + 1200\,\alpha^{16} + 105\,\alpha^{18})\,.\end{aligned} \tag{41}$$

Since the polynomial between parentheses is positive for $\alpha > 0$, Δ_4 is always negative. Here again, $A(t)$ will dip below its asymptotic limit.

4.5. *Summary and further remarks*

We have summarized in Table 1 the results of the previous paragraphs.

We observe that, contrary to what happens for the exponential and gamma distributions, the existence of a minimum A_{min} below A_∞ is the rule, not the exception. We would also like to point out that even for moderate values of n, the domain for which $\Delta_n < 0$ is actually a set of intervals. It is therefore quite frequent to have, for a given α, $\Delta_n < 0$ while for some $m > n$, $\Delta_m > 0$. This should not be seen as a contradiction, but merely an indication of the existence of oscillations of $A(t) - A_\infty$, the positive or negative parts of which are not sampled identically when using n or m.

Table 1. Summary of the results for an exponential repair distribution.

Lifetime distribution	$A_{\min} < A_\infty$
Exponential	Never
Gamma	$0 < \alpha < 1$ and $\alpha > 2$
Lognormal	Always
Weibull	$\alpha \neq 1^*$
Birnbaum-Saunders	Always
Inverse Gaussian	Always

Note: * likely, but not proven result.

5. Influence of the repair time distribution: First results

Following a remark made by Pierre Dersin at the ESREL 2019 conference in Hannover on the asymmetry between failures and repairs with respect to the availability, we have looked at the possible occurrence of minima, when we swap the two distributions. A good starting point is to take an exponential lifetime distribution — a standard assumption — and various repair time distribution, starting with the gamma(μ,β) distribution. For the sake of clarity, we keep α for the second parameter of the lifetime distribution, whereas β will always be associated with repairs.

5.1. *Gamma(μ,β) repair time distribution*

5.1.1. *Case $\beta = 2$*

When $\beta = 2$, the Laplace transform of the availability reads

$$\widetilde{A}(s) = \frac{(s+2\,\mu)^2}{s\,(s^2 + (\lambda + 4\,\mu)\,s + 4\,\mu\,(\mu+\lambda))}\,. \tag{42}$$

This leads to the inverse Laplace transform:

$$A(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}\, e^{-\frac{1}{2}\,(\lambda+4\,\mu)\,t}\, H(\lambda,\mu,t)\,, \tag{43}$$

where

$$H(\lambda,\mu,t) = \cos\phi + \frac{2\,\mu - \lambda}{\sqrt{\lambda\,(8\,\mu-\lambda)}}\,\sin\phi\,, \tag{44}$$

and

$$\phi = \frac{\sqrt{\lambda\,(8\,\mu-\lambda)}}{2}\,t\,. \tag{45}$$

Obviously, $H(\lambda,\mu,t)$ may be negative. This implies that $A(t)$ undergoes oscillations and repeatedly dips below $\frac{\mu}{\mu+\lambda}$. When we considered the configuration described by a gamma(λ,2) failure distribution and an exponential repair distribution (Tanguy et al. (2019)), the availability exhibited a smooth, monotonous decrease toward the asymptotic limit. This is a first example of the asymmetry between the two distributions, even if the same MTTF and MTTR were kept.

Since there is a minimum $A_{\min}$ for the availability, it might be worthwhile to look at its actual value, in this exactly solvable configuration. Setting $\delta = A_{\min} - A_\infty$, we obtain

$$\delta = -\frac{\lambda}{2\,(\mu+\lambda)}\,\exp\left(-\frac{\lambda+4\,\mu}{\sqrt{\lambda\,(8\,\mu-\lambda)}}\,\theta\right)\,, \tag{46}$$

with

$$\theta = \pi - 2\arcsin\sqrt{\frac{\lambda}{8\,\mu}}\,. \tag{47}$$

When $\lambda \ll \mu$,

$$\delta \approx -\frac{\lambda}{2\,\mu}\, e^{1-\pi\,\sqrt{\frac{2\,\mu}{\lambda}}}\,. \tag{48}$$

For $\lambda = 10^{-2}\,\mu$, we get $\delta \approx -5.95\,10^{-22}$, while for $\lambda = 10^{-3}\,\mu$, $\delta \approx -1.25\,10^{-64}$. Even though there is a minimum below A_∞, it is inconsequential. Keeping A_∞ as a lower bound to the availability would therefore not be such a big mistake.

5.1.2. *Other cases*

Another configuration can be solved exactly, that of an exponential failure distribution and a gamma repair distribution with shape parameter $\frac{1}{2}$. We already considered the "symmetrical" configuration gamma(λ,$\frac{1}{2}$)/exp(μ) (Tanguy (2019)). Whereas we observed a pronounced dip in the former study, no such feature appears here, only a smooth, monotonous decrease towards the asymptotic limit. This can be further demonstrated by the different value of Δ_0 (compare with Eq. (23)):

$$\Delta_0(\exp,\text{gamma}) = \frac{(1+\beta)\,\lambda}{2\,\beta\,(\lambda+\mu)^2}\,, \tag{49}$$

leading to, for $0 < \beta \leq 1$,

$$\langle\, t\,\rangle_{\text{asympt}}(\exp,\text{gamma}) = \frac{(1+\beta)}{2\,\beta\,(\lambda+\mu)}\,. \tag{50}$$

The expected result for $\beta = 1$ is recovered, while for $\beta = \frac{1}{2}$, $\frac{3}{2\,(\lambda+\mu)} \approx \frac{3}{2}\,\text{MTTR}$. This shows that MTTR is now the relevant time scale, in contrast to Eq. (23).

Numerical experiments have shown that the possibility of $A_{\min} < A_\infty$ depends on the ratio λ/μ. For instance, when $\lambda = 10^{-2}\,\mu$, the calculation of Δ_{50} gives a negative sign for $\beta > 1.82929$, while for $\beta = \frac{3}{2}$ there is no minimum except at infinity.

Further work is clearly needed to complete a more accurate assessment of the behavior of $A(t)$ for such parameters. This also applies to other configurations, when the gamma distribution is replaced by the distributions considered in Section 4.

6. Conclusion and ongoing work

In a previous work on a system described by a gamma failure distribution and an exponential repair distribution, we showed that the time-dependent availability $A(t)$ may reach values smaller than the asymptotic, well-known limit $A_\infty = \mathrm{MTTF}/(\mathrm{MTTF} + \mathrm{MTTR})$. We have considered here various distributions, not only exponential and gamma distributions, and developed a way to identify when such a minimum is likely to occur, without having to solve exactly for $A(t)$. This method is based on the Taylor expansion of the Laplace transforms of the lifetime and repair distributions.

When the repair distribution is exponential, we have shown that the existence of a minimum below A_∞ is the rule, not the exception. However, that does not necessarily means that all the studies using A_∞ badly overestimate the true availability. In many cases, since the MTTR is much smaller than the MTTF, the error is nevertheless likely to be negligible (an example is provided in Section 5).

We have also evidenced the asymmetry between the two lifetime and repair time distributions: Even when the same MTTF and MTTR are kept, swapping the distributions provides quite different time variations of the availability.

An extension of the present work is naturally to find a way to determine the minimum availability $A_{\min}$ for various pairs of distributions, and MTTR/MTTF ratios. This is by no means easy, especially for distributions that are not exponential. We have begun to explore two paths:

- Instead of computing the various Δ_n studied in the present paper, we could consider sampling $A(t) - A_\infty$ by a normalized function that is sharply centered at a particular time $t_{\min}$. Such a function could be a gamma distribution of parameters Λ and k (we would consider an integer k) so that $\Lambda = \frac{k}{t_{\min}}$. A possible choice is to consider

$$\chi(t_{\min}) = \int_0^\infty \frac{\Lambda^k\, t^{k-1}}{\Gamma[k]}\, e^{-\Lambda\, t}\,(A(t) - A_\infty)\, dt\,. \tag{51}$$

  $\chi(t_{\min})$ can be calculated by using the Laplace transform $\widetilde{A}(s)$ again, and finally

$$\chi(t_{\min}) = (-1)^{k-1}\frac{\Lambda^k}{(k-1)!}\,\frac{d^{k-1}\widetilde{A}(\Lambda)}{d\Lambda^{k-1}} - A_\infty\,. \tag{52}$$

  For reasonably large enough values of k, $\chi(t_{\min})$ could provide a good assessment of $A_{\min} - A_\infty$, if the minimum is not too narrow.
- Another way to look for the minimum of $A(t)$ is to compute the optimal Padé approximant of $\widetilde{A}(s)$. Since it is merely a fraction of two polynomials in s, finding the inverse Laplace transform would then be easy after a partial fraction decomposition. This should provide a good approximation to $A(t)$, especially for large values of t. This approximation could extend to a time domain close to $t_{\min}$, so that a good estimate of $A_{\min} - A_\infty$ would be obtained.

In conclusion, we have shown that in many cases, the time-dependent availability may decrease below its asymptotic limit MTTF/(MTTF + MTTR). We have provided tools to ascertain if this is the case. The occurrence of such a situation depends on the pair of distributions, as well as on the ratio MTTR/MTTF. Determining a robust, lower bound for the availability should be made easier by the methods we propose.

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Appendix A. A few Taylor expansions for useful time distributions

We list here the Taylor expansions of $\widetilde{F}(s)$ used in this study. They have been obtained by calculating the moments of orders higher than 2 of the distributions. Note that for $\widetilde{G}(s)$, μ and β should replace λ and α, respectively.

- Gamma distribution

$$\widetilde{F}(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\,\frac{s^n}{(\alpha\,\lambda)^{n+1}}\,\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)}\,. \tag{A.1}$$

- Lognormal distribution

$$\widetilde{F}(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \frac{s^n}{\lambda^{n+1}} (1+\alpha)^{\frac{n\,(n+1)}{2}} . \tag{A.2}$$

- Weibull distribution

$$\widetilde{F}(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \frac{s^n}{\lambda^{n+1}} \frac{\Gamma\left(1+\frac{n+1}{\alpha}\right)}{\Gamma\left(1+\frac{1}{\alpha}\right)^{n+1}} . \tag{A.3}$$

- Birnbaum-Saunders distribution

$$\widetilde{F}(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \frac{s^n}{\lambda^{n+1}} \left(\frac{2}{2+\alpha^2}\right)^{n+1} \times \frac{K_{n+\frac{1}{2}}\left(\frac{1}{\alpha^2}\right) + K_{n+\frac{3}{2}}\left(\frac{1}{\alpha^2}\right)}{2\,K_{\frac{1}{2}}\left(\frac{1}{\alpha^2}\right)} , \tag{A.4}$$

where K_ν is the modified Bessel function of the second kind.

- Inverse Gaussian distribution

$$\widetilde{F}(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \frac{s^n}{\lambda^{n+1}} \times \sum_{m=0}^{n} \frac{\alpha^m}{2^m\, m!} \frac{(n+m)!}{(n-m)!} . \tag{A.5}$$

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