Reconstructing Graphs from Their k-Edge Deleted Subgraphs

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Let G be a graph with m edges and n vertices. We show that if $2^{m-k} > n!$ or if $2m > \binom{n}{2} + k$ then G is determined by its collection of k-edge deleted subgraphs.

1. INTRODUCTION

A graph H is an edge reconstruction of the graph G if there is a bijection $\beta$ from $E(G)$ to $E(H)$ such that for each edge $e$ in $E(G)$, $G \setminus e$ is isomorphic to $H \setminus \beta(e)$. (Here $G \setminus e$ denotes the graph obtained from G by deleting the edge e.) We call G edge reconstructible if any edge reconstruction of G is isomorphic to G. The well-known edge-reconstruction conjecture, due to Harary [4], asserts that any graph with at least four edges is edge reconstructible. If G has m edges and n vertices then it is known that it is edge-reconstructible if either $2m > \binom{n}{2}$ or $2^{m-1} > n!$ These results are due, respectively, to Lovász [6] and Müller [7].

In this paper we provide analogues of these results for k-edge reconstruction. A graph $H$ is a k-edge reconstruction of $G$ if there is a bijection $\beta$ from $\binom{E(G)}{m-k}$ to $\binom{E(H)}{m-k}$ such that for each k-subset $S$ of $E(G)$, $G \setminus S$ is isomorphic to $H \setminus \beta(S)$. As might be expected G is k-edge reconstructible if any k-edge reconstruction of G is isomorphic to it. Our result is the following:
1.1. Theorem. Let $G$ be a graph with $m$ edges and $n$ vertices. If $2^{m-k} > n!$ or if $2m \geq \binom{n}{2} + k$ then $G$ is $k$-edge reconstructible.

If $k \geq 2$ and all $k$-edge reconstructions of the graph $G$ are isomorphic to $G$ then so are all $(k-1)$-edge reconstructions. Thus this theorem is a very natural extension of the results of Lovász and Müller. Our proof of the first part is similar in spirit to Stanley's proof [8] of Lovász's result, but our proof of the second is based on a new approach. Finally we note that a very fine survey of the reconstruction problem can be found in [1].

Remark. This note is based on an earlier manuscript by the second and third authors, where a different approach was used. This yielded a weaker version of Theorem 1.1, but could also be used (with some effort) to prove Stanley's result on reconstruction from vertex switching [9]. For more details see [5].

2. Proof of Theorem 1.1

Let $n$ be a fixed integer. Let $\mathcal{G}_m$ denote the set of all graphs on $n$ vertices with $m$ edges. We view a graph on $n$ vertices as a subset of the $\binom{n}{2}$ edges of a fixed copy of $K_n$. Let $H(m, k)$ be the $0$-$1$ matrix with rows indexed by the elements of $\mathcal{G}_m$, columns indexed by the elements of $\mathcal{G}_{m-k}$ and with $(H(m, k))_{ij} = 1$ iff the $j$th element of $\mathcal{G}_{m-k}$ is a subgraph of the $i$th member of $\mathcal{G}_m$. If $G \in \mathcal{G}_m$ let $\chi_G$ be the function from $\mathcal{G}_m$ to $\{0, 1\}$ defined by

$$
\chi_G(F) = \begin{cases} 
1 & \text{if } F \cong G \\
0 & \text{otherwise}
\end{cases}
$$

We can, and will, regard $\chi_G$ as a row vector. Hence the product $\chi_G H(m, k)$ is defined.

2.1 Lemma. If $N = \binom{n}{2}$ then

$$
\text{rank}(H(m, k)) = \min \left\{ \binom{N}{m}, \binom{N}{m-k} \right\}.
$$

If $\binom{N}{m} > \binom{N}{m-k}$ and $xH(m, k) = 0$ for some nonzero vector $x$ then $x$ has at least $2^{m-k+1}$ non-zero entries.

Proof. There is nothing new here. Our matrix $H(m, k)$ is just the obvious incidence matrix for $m$-subsets of an $N$-set versus $(m-k)$-subsets. The rank of this matrix is well known (See, e.g., [3]. In their notation our $H(m, k)$ is the transpose of their matrix $H_{N, (m-k,m)}$.) For the second claim we note that a vector $x$ such that $xH(m, k) = 0$ is the same thing as a
m-uniform null \((m-k)\)-design in [2]. From their Corollary 1 it follows immediately that \(x\) must have at least \(2^{m-k+1}\) non-zero elements.

2.2. \textsc{Lemma.} Two graphs \(F\) and \(G\) in \(\mathcal{G}_m\) have the same collection of \(k\)-edge deleted subgraphs iff

\[
(\lvert \text{Aut}(F) \rvert \chi_F - \lvert \text{Aut}(G) \rvert \chi_G) H(m, k) = 0.
\]

\textit{Proof.} Let \(A\) be an element of \(\mathcal{G}_{m-k}\). The columns of \(H(m, k)\) correspond to the elements of \(\mathcal{G}_{m-k}\). The entry of \(\chi_G \cdot H(m, k)\) corresponding to \(A\) is just the number of graphs isomorphic to \(G\) containing \(A\). So

\[
\lvert \text{Aut}(G) \rvert \chi_G \cdot H(m, k)
\]

is equal to

\[
\lvert \{ x \in \text{Sym}(n) \mid A \leq G^x \} \rvert = \lvert \{ x \in \text{Sym}(n) \mid A^{x^{-1}} \leq G \} \rvert.
\]

Here the r.h.s. is \(\lvert \text{Aut}(A) \rvert\) times the number of subgraphs of \(G\) isomorphic to \(A\). Our claim now follows immediately.

2.3. \textsc{Corollary.} If \(2m > \binom{n}{2}\) and \(k \leq 2m - \binom{n}{2}\), each element of \(\mathcal{G}_m\) is \(k\)-edge reconstructible.

\textit{Proof.} If \(m > \frac{1}{2} \binom{n}{2}\) and \(k \leq 2m - \binom{n}{2}\) then the rows of \(H(m, k)\) are linearly independent. So if \(xH(m, k) = 0\) then \(x = 0\). The claim now follows from the lemma above.

2.4. \textsc{Corollary.} If \(2^{m-k} > n!\) Then each graph with \(m\) edges is \(k\)-edge reconstructible.

\textit{Proof.} Let \(F\) and \(G\) be two elements of \(\mathcal{G}_m\) and set

\[
x = \lvert \text{Aut}(F) \rvert \chi_F - \lvert \text{Aut}(G) \rvert \chi_G.
\]

The number of non-zero entries in \(\chi_F\) is \(n!/\lvert \text{Aut}(F) \rvert\), similarly \(\chi_G\) has \(n!/\lvert \text{Aut}(G) \rvert\) non-zero entries. Hence \(x\) has at most \(2n!\) non-zero entries. If \(xH(m, k) = 0\) then, by Lemma 2.2, we must have \(2n! > 2^{m-k+1}\).

\textbf{REFERENCES}


7. V. Müller, The edge reconstruction conjecture is true for graphs with more than $n \log_2 n$ edges, J. Combinatorial Theory Ser. B 22 (1977), 281–283.