Receding Horizon Disturbance Attenuation for Takagi-Sugeno Fuzzy Switched Dynamic Neural Networks

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Abstract

In this paper, we propose a new receding horizon disturbance attenuator (RHDA) for Takagi-Sugeno (T-S) fuzzy switched Hopfield neural networks with external disturbance. First, a new set of linear matrix inequality (LMI) conditions is proposed for the finite terminal weighting matrix of the receding horizon cost function with a cross term. Second, under this condition, we show that the proposed RHDA attenuates the effect of external disturbance on T-S fuzzy switched Hopfield neural networks with a guaranteed infinite horizon $H_{\infty}$ performance. In addition, we prove that the proposed RHDA guarantees internal stability in closed-loop systems. A numerical example is presented to describe the effectiveness of the proposed RHDA scheme.

Keywords: neuro-fuzzy system; fuzzy system model; switched neural network; receding horizon disturbance attenuator (RHDA); linear matrix inequality (LMI)
1 Introduction

Hopfield neural networks [20] have been extensively investigated in recent years because of their widespread use in modeling many phenomena associated with signal processing, pattern recognition, associative memory, static image processing, and particularly in solving difficult optimization problems [18]. Therefore, the various stability properties (e.g., asymptotic, exponential, and stochastic stability) need to be studied for different types of Hopfield neural networks.

Switched systems form an important class of hybrid systems consisting of a finite number of subsystems described by dynamic systems and a switching signal that specifies the switching among them. Switched systems are formed when dynamic systems undergo abrupt changes due to parameter changes, component failures, or element switching. Experiments using various techniques have shown many important and interesting results for switched systems, owing to their theoretical and practical significance [17, 41, 42, 28]. Recently, the use of switched Hopfield neural networks, whose subsystems constitute a set of Hopfield neural networks, has been widely applied in the field of high-speed signal processing and in gene selection in DNA microarray analyses [34, 16, 27]. Some stability conditions for switched Hopfield neural networks were investigated in [22, 24, 1]. New results on learning, filtering, and estimation in switched Hopfield neural networks were presented in [3, 4, 9, 7, 10].

Recently, the Takagi-Sugeno (T-S) fuzzy model approach was used to describe neural networks and the problem of stability analysis for T-S fuzzy Hopfield neural networks was extensively studied in [21, 14, 23]. Among the different types of fuzzy methods, the T-S fuzzy models [32, 33] have attracted particular attention from researchers because these models can effectively approximate a wide class of complex nonlinear systems by using some local linear subsystems. The T-S fuzzy model approach is a multi-model approach in which some linear models are combined to form an overall single model through nonlinear membership functions to represent nonlinear system dynamics. The nonlinear system dynamics are captured by a set of fuzzy rules that characterize local correlation in the state space. Some new results on learning, identification, and filtering in T-S fuzzy Hopfield neural networks were presented in [2, 5, 6, 8, 11].
The receding horizon approach is now accepted as an important feedback strategy in many industry fields, especially in process industries [35, 36, 25, 19, 37]. This approach has many advantages, such as guaranteed robustness and adaptation to switched parameters. Since control methods based on the receding horizon approach are computed repeatedly under a cost function, the approach can adapt to unanticipated changes in system parameters. In [26], the receding horizon approach was applied to the nonlinear $H_\infty$ control problem. However, the inverse optimality based result was obtained simply via a Fake Hamilton-Jacobi-Isaacs equation. To the best of our knowledge, the problem of receding horizon disturbance attenuation for T-S fuzzy switched Hopfield neural networks with external disturbance has not been investigated thus far and remains an open and challenging research topic.

In this paper, we propose a new receding horizon disturbance attenuator (RHDA) for T-S fuzzy switched Hopfield neural networks with external disturbance. A new set of sufficient linear matrix inequality (LMI) conditions is proposed for the finite terminal weighting matrix of the receding horizon cost function with a cross term, under which the proposed RHDA reduces the effect of external disturbance in T-S fuzzy switched Hopfield neural networks. The proposed RHDA guarantees asymptotic stability in T-S fuzzy switched Hopfield neural networks without external disturbance. The finite terminal weighting matrix in the receding horizon cost function can be determined by solving a set of LMI conditions. This LMI problem can be solved efficiently by using standard convex optimization software [15].

This paper is organized as follows. In Section 2, we formulate the problem. In Section 3, a new set of sufficient LMI conditions is proposed for the receding horizon disturbance attenuation of T-S fuzzy switched Hopfield neural networks with external disturbance. In Section 4, a numerical example is given, and finally, conclusions are presented in Section 5.

2 Problem Formulation

Consider the following T-S fuzzy switched Hopfield neural network:

**Fuzzy Rule** $R_i^\alpha$:

**IF** $\omega_1$ is $\mu_{\alpha_{1}}$ and $\cdots$ $\omega_s$ is $\mu_{\alpha_{s}}$ **THEN**
\[ \dot{x}(t) = A(i, \alpha)x(t) + W(i, \alpha)\phi(x(t)) + u(t) + w(t), \tag{1} \]

where \( \omega_j \) \((j = 1, 2, \cdots, s)\) is the premise variable, \( \mu_{ij}^\alpha \) \((i = 1, 2, \cdots, r, j = 1, 2, \cdots, s)\) is the fuzzy set that is characterized by a membership function, \( r \) is the number of IF-THEN rules, \( s \) is the number of premise variables, \( x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \mathbb{R}^n \) is the state vector, \( A(i, \alpha) \in \mathbb{R}^{n \times n} \) is the negative diagonal matrix representing the self-feedback term, \( W(i, \alpha) \in \mathbb{R}^{n \times n} \) is the connection weight matrix, \( \phi(x(t)) = [\phi_1(x(t)), \phi_2(x(t)), \cdots, \phi_n(x(t))]^T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the nonlinear function vector satisfying the global Lipschitz condition with Lipschitz constant \( L_\phi > 0 \), \( u(t) \in \mathbb{R}^n \) is the control input vector, and \( w(t) \in \mathbb{R}^n \) is the external disturbance vector. Here, \( \alpha \) is a switching signal that can have any value from the finite set \{1, 2, \cdots, N\}. The matrices \((A(i, \alpha), W(i, \alpha))\) are allowed to take values in the finite set \{(A(i, 1), W(i, 1)), \cdots, (A(i, N), W(i, N))\} at an arbitrary time for \( i = 1, 2, \cdots, r \). This study assumes that the switching rule \( \alpha \) is not known a priori and its instantaneous value is available in real time. A standard fuzzy inference method is used, and system (1) is inferred as follows:

\[ \dot{x}(t) = \sum_{i=1}^{r} h_{(i, \alpha)}(\omega)[A(i, \alpha)x(t) + W(i, \alpha)\phi(x(t)) + u(t) + w(t)], \tag{2} \]

where \( \omega = [\omega_1, \omega_2, \cdots, \omega_s]^T \), \( h_{(i, \alpha)}(\omega) = \psi_{(i, \alpha)}(\omega)/\sum_{j=1}^{r} \psi_{(j, \alpha)}(\omega) \), \( \psi_{(i, \alpha)} \) is the membership function of the system with respect to the fuzzy rule \( \mathbb{R}_{\alpha}^i \) \((i = 1, 2, \cdots, r)\). \( h_{(i, \alpha)}(\omega) \) can be regarded as the normalized weight of each IF-THEN rule and it satisfies \( h_{(i, \alpha)}(\omega) \geq 0 \) and \( \sum_{i=1}^{r} h_{(i, \alpha)}(\omega) = 1 \). The indicator function is defined as \( \xi(t) = [\xi_1(t), \xi_2(t), \cdots, \xi_N(t)]^T \), where \( \xi_k(t) = 1 \) when the neural network is described by the \( k \)-th mode \((A(i, k), W(i, k))\), and \( \xi_k(t) = 0 \) otherwise \((k = 1, 2, \cdots, N)\). This indicator function can be used to derive the model of the T-S fuzzy switched Hopfield neural networks (2) as

\[ \dot{x}(t) = \sum_{k=1}^{N} \sum_{i=1}^{r} \xi_k(t)h_{(i, k)}(\omega)[A(i, k)x(t) + W(i, k)\phi(x(t)) + u(t) + w(t)], \tag{3} \]
where $\sum_{k=1}^{N} \xi_k(t) = 1$ is satisfied under all switching rules. The following finite horizon cost with a cross term is associated with the T-S fuzzy switched Hopfield neural network (3):

$$J(x(t_0), t_0, t_1) = \int_{t_0}^{t_1} \left\{ \begin{bmatrix} x^T(t) & u^T(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} - \gamma^2 w(t)^T w(t) \right\} dt + x^T(t_1)Q_f x(t_1),$$

(4)

where

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0, \quad Q \geq 0, \quad R > 0, \quad Q_f = Q_f^T > 0,$$

(5)

where $\gamma > 0$ is the disturbance attenuation level, $t_0 > 0$ is the initial time, and $t_1$ is the final time. The finite horizon optimal differential game involves minimization with respect to $u(t)$ ($t_0 \leq t \leq t_1$), and maximization with respect to $w(t)$ ($t_0 \leq t \leq t_1$), of the receding horizon cost function (4). If a feedback saddle-point solution for the finite horizon optimal differential game exists, we denote the solution as $u^*(t)$ and $w^*(t)$ ($t_0 \leq t \leq t_1$). The saddle-point value function of the finite horizon optimal differential game will be denoted by $J^*(x(t_0), t_0, t_1)$.

The RHDA is then obtained by solving the finite horizon optimal differential game of cost function (4) with initial time $t_0$ and terminal time $t_1$ replaced by current time $t$ and future time $t + T$, respectively, where $T$ is a positive constant horizon size. The disturbance attenuation performance of the proposed attenuator depends on the choice of the finite terminal weighting matrix $Q_f$ in the cost function (4). In this study, we find a new set of LMI conditions for the finite weighting matrix $Q_f$ such that the T-S fuzzy switched Hopfield neural network (3) becomes asymptotically stable with a guaranteed infinite horizon $\mathcal{H}_\infty$ performance.

3 Receding Horizon Disturbance Attenuation for T-S Fuzzy Switched Neural Networks

In this section, we derive a new set of LMI conditions on the finite terminal weighting matrix $Q_f$ for the receding horizon disturbance attenuation for T-S fuzzy switched neural networks.
Theorem 1. Given a disturbance attenuation level $\gamma > 0$, assume that there exist $X = X^T \in R_{n \times n}^n > 0$ and $Y(i,k) \in R_{n \times n}$ such that

$$
\begin{bmatrix}
\Gamma(i,k) & W(i,k) & I & X & Y(i,k)^T & XS & X \\
W(i,k)^T & -I & 0 & 0 & 0 & 0 & 0 \\
I & 0 & -\gamma^2 I & 0 & 0 & 0 & 0 \\
X & 0 & 0 & -Q^{-1} & 0 & 0 & 0 \\
Y(i,k) & 0 & 0 & -(R+I)^{-1} & 0 & 0 & 0 \\
S^TX & 0 & 0 & 0 & 0 & -I & 0 \\
X & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\gamma^2} I \\
\end{bmatrix} \leq 0,
$$

(6)

where

$$
\Gamma(i,k) = A(i,k)X + Y(i,k) + [A(i,k)X + Y(i,k)]^T,
$$

(7)

for $i = 1, 2, \cdots, r$ and $k = 1, 2, \cdots, N$. Then, the saddle-point value function satisfies

$$
\frac{\partial J^*(x(\tau), \tau, \sigma)}{\partial \sigma} \leq 0,
$$

(8)

where $\tau$ and $\sigma$ are the initial and final times of the saddle-point value function $J^*(x(\tau), \tau, \sigma)$ ($\tau \leq \sigma$), respectively, and the finite terminal weighting matrix is given by $Q_f = X^{-1}$.

Proof:

$$
\frac{\partial J^*(x(\tau), \tau, \sigma)}{\partial \sigma} = \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ J^*(x(\tau), \tau, \sigma + \Delta) - J^*(x(\tau), \tau, \sigma) \right\}
$$

$$
= \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \int_{\tau}^{\sigma} \begin{bmatrix} x^{(1)}(t) \\ u^{(1)}(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x^{(1)}(t) \\ u^{(1)}(t) \end{bmatrix} - \gamma^2 w^{(1)}(t)w^{(1)}(t) \right\} dt
$$

$$
+ J^*(x^{(1)}(\sigma), \sigma, \sigma + \Delta)
$$

$$
- \int_{\tau}^{\sigma} \begin{bmatrix} x^{(2)}(t) \\ u^{(2)}(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x^{(2)}(t) \\ u^{(2)}(t) \end{bmatrix} - \gamma^2 w^{(2)}(t)w^{(2)}(t) \right\} dt
$$
\[
- x^{(2)T}(\sigma)Q f x^{(2)}(\sigma)
\]

where \((u^{(1)}(t), w^{(1)}(t))\) and \((w^{(2)}(t), w^{(2)}(t))\) are the saddle-point solutions for \(J(x(\tau), \tau, \sigma + \Delta)\) and \(J(x(\tau), \tau, \sigma)\), respectively. If \(u^{(1)}(\cdot)\) and \(w^{(2)}(\cdot)\) are replaced by \(u^{(2)}(\cdot)\) and \(w^{(1)}(\cdot)\) up to \(\sigma\), respectively, and the following switched fuzzy controller for \(t \geq \sigma\):

**Fuzzy Rule \( R^i_{\alpha} \):**

**IF** \( \omega_1 \) is \( \mu_{\alpha_1}^i \) and \( \cdots \omega_s \) is \( \mu_{\alpha_s}^i \)** **THEN**

\[
u^{(1)}(t) = K(i, \alpha)x(t),
\]

which can be represented by

\[
u^{(1)}(t) = \sum_{k=1}^{N} \sum_{i=1}^{r} \xi_k(t) h_{(i,k)}(\omega) K(i,k)x(t),
\]

then

\[
\frac{\partial J^*(x(\tau), \tau, \sigma)}{\partial \sigma} \leq \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ J^*(x(\sigma), \sigma, \sigma + \Delta) - x^T(\sigma)Q f x(\sigma) \right\}
\]

\[
= \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \int_{\sigma}^{\sigma+\Delta} \sum_{k=1}^{N} \sum_{i=1}^{r} \xi_k(t) h_{(i,k)}(\omega) \begin{bmatrix} x(t) \\ K(i,k)x(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ ST & R \end{bmatrix} \begin{bmatrix} x(t) \\ K(i,k)x(t) \end{bmatrix} dt + x^T(\sigma + \Delta)Q f x(\sigma + \Delta) \\
- x^T(\sigma)Q f x(\sigma) \right\}
\]

\[
= \sum_{k=1}^{N} \sum_{i=1}^{r} \xi_k(\sigma) h_{(i,k)}(\omega) \begin{bmatrix} x(\sigma) \\ K(i,k)x(\sigma) \end{bmatrix}^T \begin{bmatrix} Q & S \\ ST & R \end{bmatrix} \begin{bmatrix} x(\sigma) \\ K(i,k)x(\sigma) \end{bmatrix} \\
- \gamma^2 w^T(\sigma)w(\sigma) + \frac{d}{d\sigma} \{ x^T(\sigma)Q f x(\sigma) \}
\]
Using (13), we have

In the last inequality in (12), the following inequality is obtained:

\[ SK_k \leq K(i, k) \]

\[ \partial J^*(\tau) = \frac{\partial^2 J(T)}{\partial T^2} \Phi(T) \Omega(i, k) \Phi(T), \]
where

\[
\Phi(\sigma) = \begin{bmatrix}
    x(\sigma) \\
    \phi(x(\sigma)) \\
    w(\sigma)
\end{bmatrix},
\quad
\Omega(i,k) = \begin{bmatrix}
    \Theta(i,k) & Q_f W(i,k) & Q_f \\
    W(i,k)^T Q_f & -I & 0 \\
    Q_f & 0 & -\gamma^2 I
\end{bmatrix},
\]

\[
\Theta(i,k) = Q + K(i,k)^T (R + I) K(i,k) + SS^T + Q_f A(i,k) + A(i,k)^T Q_f
\]

\[
+ Q_f K(i,k) + K(i,k)^T Q_f + L_2^2 I. \tag{15}
\]

If \(\Omega(i,k) \leq 0\) for \(i = 1, 2, \cdots, r\) and \(k = 1, 2, \cdots, N\), we have \(\frac{\partial J^*(x,\tau,\sigma)}{\partial \sigma} \leq 0\). From the Schur complement, \(\Omega(i,k) \leq 0\) is equal to

\[
\begin{bmatrix}
    \Theta(i,k) & Q_f W(i,k) & Q_f & I & K(i,k)^T & S & I \\
    W(i,k)^T Q_f & -I & 0 & 0 & 0 & 0 & 0 \\
    Q_f & 0 & -\gamma^2 I & 0 & 0 & 0 & 0 \\
    I & 0 & 0 & -Q^{-1} & 0 & 0 & 0 \\
    K(i,k) & 0 & 0 & 0 & -(R + I)^{-1} & 0 & 0 \\
    S^T & 0 & 0 & 0 & 0 & -I & 0 \\
    I & 0 & 0 & 0 & 0 & 0 & -\frac{1}{L_2^2 I}
\end{bmatrix} \leq 0, \tag{16}
\]

for \(i = 1, 2, \cdots, r\) and \(k = 1, 2, \cdots, N\), where \(\Theta(i,k) = Q_f A(i,k) + A(i,k)^T Q_f + Q_f K(i,k) + K(i,k)^T Q_f\). By pre-multiplying and post-multiplying (16) by diag\(Q_f^{-1}, I, I, I, I, I\) and introducing a change of variables such as \(X = Q_f^{-1}\) and \(Y(i,k) = K(i,k)Q_f^{-1}\), (16) becomes equal to a set of LMIs (6). This completes the proof. 

The following three results will be used to show the infinite horizon \(H_\infty\) performance of the RHDA:

**Lemma 1.** \(\frac{\partial J^*(x(\tau'),\tau',\sigma)}{\partial \sigma} \leq 0\) implies \(\frac{\partial J^*(x(\tau''),\tau'',\sigma)}{\partial \sigma} \leq 0\), where \(\tau' \leq \tau'' \leq \sigma\).

**Proof:**

\[
\frac{\partial J^*(x(\tau'),\tau',\sigma)}{\partial \sigma} = \lim_{\Delta \to 0} \frac{1}{\Delta} \{ J^*(x(\tau'), \tau', \sigma + \Delta) - J^*(x(\tau'), \tau', \sigma) \} = 0.
\]
\[
\lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \int_{\tau''}^{\tau'''} \left\{ \begin{bmatrix} x^{(3)}(t) \\ u^{(3)}(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ ST & R \end{bmatrix} \begin{bmatrix} x^{(3)}(t) \\ u^{(3)}(t) \end{bmatrix} - \gamma^2 w^{(3)T}(t)w^{(3)}(t) \right\} dt \\
+ J^*(x^{(3)}(\tau'''), \tau'', \sigma + \Delta) - \int_{\tau''}^{\tau'''} \left\{ \begin{bmatrix} x^{(4)}(t) \\ u^{(4)}(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ ST & R \end{bmatrix} \begin{bmatrix} x^{(4)}(t) \\ u^{(4)}(t) \end{bmatrix} \\
- \gamma^2 w^{(4)T}(t)w^{(4)}(t) \right\} dt - J^*(x^{(4)}(\tau'''), \tau'', \sigma) \right\}, \tag{17}
\]

where \((u^{(3)}(t), w^{(3)}(t)))\) and \((u^{(4)}(t), w^{(4)}(t)))\) are the saddle-point solutions for \(J(e(\tau'), \tau', \sigma + \Delta)\) and \(J(e(\tau'), \tau', \sigma)\), respectively. If \(u^{(3)}(\cdot)\) and \(w^{(4)}(\cdot)\) are replaced by \(u^{(4)}(\cdot)\) and \(w^{(3)}(\cdot)\) up to \(\sigma\), respectively, then we have

\[
\frac{\partial J^*(x(\tau'), \tau', \sigma)}{\partial \sigma} \leq \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ J^*(x(\tau'''), \tau'', \sigma + \Delta) - J^*(x(\tau'''), \tau'', \sigma) \right\} = \frac{\partial J^*(x(\tau'''), \tau'', \sigma)}{\partial \sigma}. \tag{18}
\]

Thus, \(\frac{\partial J^*(x(\tau'), \tau', \sigma)}{\partial \sigma} \leq 0\) implies \(\frac{\partial J^*(x(\tau'''), \tau'', \sigma)}{\partial \sigma} \leq 0\). This completes the proof. \(\blacksquare\)

**Lemma 2.** Under condition (6), the following relations are satisfied:

\[
J^*(x(t), t, t + T) \geq 0, \quad J^*(0, t, t + T) = 0, \tag{19}
\]

for a positive constant \(T\).

**Proof:**

\[
J^*(x(t), t, t + T) = \int_t^{t+T} \left\{ \begin{bmatrix} x^{(*)T}(\xi) \\ u^{(*)T}(\xi) \end{bmatrix} \begin{bmatrix} Q & S \\ ST & R \end{bmatrix} \begin{bmatrix} x^{(*)}(\xi) \\ u^{(*)}(\xi) \end{bmatrix} \\
- \gamma^2 w^{(*)T}(\xi)w^{(*)}(\xi) \right\} d\xi + x^{(*)T}(t + T)Q_f x^{(*)}(t + T). \tag{20}
\]
If $w^{(\epsilon)}(\cdot)$ is replaced by 0 from $t$ to $t + T$, then

$$J^*(x(t), t, t + T) \geq \int_t^{t + T} \left[ x^{(\epsilon)}(\xi) \quad u^{(\epsilon)}(\xi) \right] \left[ \begin{array}{cc} Q & S \\ S^T & R \end{array} \right] \left[ \begin{array}{c} x^{(\epsilon)}(\xi) \\ u^{(\epsilon)}(\xi) \end{array} \right] d\xi$$

$$+ x^{(\epsilon)}(t + T)Q_f x^{(\epsilon)}(t + T) \geq 0. \quad (21)$$

Thus, we have $J^*(x(t), t, t + T) \geq 0$. From Theorem 1, we have

$$0 \leq J^*(0, t, t + T) \leq J^*(0, t, t) = 0. \quad (22)$$

Thus, we have $J^*(0, t, t + T) = 0$. This completes the proof.

The RHDA is obtained by replacing $t_0$ and $t_1$ by $t$ and $t + T$, respectively, where $T$ denotes the horizon length. The infinite horizon $H_\infty$ performance of the proposed RHDA can be stated as the following theorem:

**Theorem 2.** Assume that the terminal weighting matrix $Q_f$ in (4) satisfies a set of LMI conditions (6). Then, the RHDA guarantees infinite horizon $H_\infty$ performance.

**Proof:** The saddle-point solutions for $J^*(x(t + \Delta), t + \Delta, t + \Delta + T)$ and $J^*(x(t), t, t + T)$ are denoted by $(u^{(5)}(t), w^{(5)}(t))$ and $(u^{(6)}(t), w^{(6)}(t))$, respectively. Then,

$$\frac{dJ^*(x(t), t, t + T)}{dt} = \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ J^*(x(t + \Delta), t + \Delta, t + \Delta + T) - J^*(x(t), t, t + T) \right\}$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \int_{t+\Delta}^{t+\Delta+T} \left[ \begin{array}{c} x^{(5)}(\xi) \\ u^{(5)}(\xi) \end{array} \right]^T \left[ \begin{array}{cc} Q & S \\ S^T & R \end{array} \right] \left[ \begin{array}{c} x^{(5)}(\xi) \\ u^{(5)}(\xi) \end{array} \right] - \gamma^2 w^{(5)T}(\xi)w^{(5)}(\xi) \right] d\xi$$

$$+ x^{(5)}(t + \Delta + T)Q_f x^{(5)}(t + \Delta + T) - \int_t^{t+T} \left[ \begin{array}{c} x^{(6)}(\xi) \\ u^{(6)}(\xi) \end{array} \right]^T \left[ \begin{array}{cc} Q & S \\ S^T & R \end{array} \right] \left[ \begin{array}{c} x^{(6)}(\xi) \\ u^{(6)}(\xi) \end{array} \right]$$

$$- \gamma^2 w^{(6)T}(\xi)w^{(6)}(\xi) \right\} d\xi - x^{(6)}(t + T)Q_f x^{(6)}(t + T). \quad (23)$$
If \( u^{(5)}(\cdot) \) and \( w^{(6)}(\cdot) \) are replaced by \( u^{(6)}(\cdot) \) and \( w^{(5)}(\cdot) \) from \( t + \Delta \) up to \( t + T \), respectively, and \( u^{(5)}(\cdot) = \sum_{k=1}^{N} \sum_{i=1}^{r} \xi_k(t)h_{(i,k)}(\omega)K(i,k)x(\cdot) \) from \( t + T \) up to \( t + \Delta + T \), then

\[
\frac{dJ^*(x(t),t,t+T)}{dt} \\
\leq - \left[ \begin{array}{c} x(t) \\ u(t) \end{array} \right]^T \left[ \begin{array}{cc} Q & S \\ S^T & R \end{array} \right] \left[ \begin{array}{c} x(t) \\ u(t) \end{array} \right] + \gamma^2 w^T(t)w(t) \\
+ \left[ \begin{array}{c} x(t+T) \\ u(t+T) \end{array} \right]^T \left[ \begin{array}{cc} Q & S \\ S^T & R \end{array} \right] \left[ \begin{array}{c} x(t+T) \\ u(t+T) \end{array} \right] - \gamma^2 w^T(t+T)w(t+T) \\
+ \frac{d}{d\xi} \{x^T(\xi)Q_f x(\xi)\} \bigg|_{\xi=t+T}.
\]

(24)

Using the arguments in the proof of Theorem 1 at a future time \( t + T \), we have

\[
\frac{dJ^*(x(t),t,t+T)}{dt} \\
\leq - \left[ \begin{array}{c} x(t) \\ u(t) \end{array} \right]^T \left[ \begin{array}{cc} Q & S \\ S^T & R \end{array} \right] \left[ \begin{array}{c} x(t) \\ u(t) \end{array} \right] + \gamma^2 w^T(t)w(t) + \sum_{k=1}^{N} \sum_{i=1}^{r} \xi_k(t+T)h_{(i,k)}(t+T) \\
\times \left\{ x^T(t+T)Q + K(i,k)^T R K(i,k) + K(i,k)^T K(i,k) + SS^T + Q_f A(i,k) + A(i,k)^T Q_f \\
+ Q_f K(i,k) + K(i,k)^T Q_f + L^2_{\phi} f(x(t+T)) + x^T(t+T)Q_f W(i,k)\phi(x(t+T)) \\
+ \phi^T(x(t+T))W(i,k)^T Q_f x(t+T) - \gamma^2 w^T(t+T)w(t+T) + x^T(t+T)Q_f w(t+T) \\
+ w^T(t+T)Q_f x(t+T) - \phi^T(x(t+T))\phi(x(t+T)) \right\} \\
= - \left[ \begin{array}{c} x(t) \\ u(t) \end{array} \right]^T \left[ \begin{array}{cc} Q & S \\ S^T & R \end{array} \right] \left[ \begin{array}{c} x(t) \\ u(t) \end{array} \right] + \gamma^2 w^T(t)w(t) + \sum_{k=1}^{N} \sum_{i=1}^{r} \xi_k(t+T)h_{(i,k)}(\omega)\Phi(t+T)^T \\
\times \Omega(i,k)\Phi(t+T) - \phi^T(x(t+T))\phi(x(t+T)).
\]

(25)
If $Q_f$ satisfies (6) for $i = 1, 2, \cdots, r$ and $k = 1, 2, \cdots, N$, then

$$
\frac{dJ^*(x(t), t, t + T)}{dt} \leq - \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + \gamma^2 w^T(t)w(t). \quad (26)
$$

Integrating both sides from 0 to $\infty$ yields

$$
J^*(x(t), t, t + T)|_{t=\infty} - J^*(x(0), 0, T) \leq - \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt + \gamma^2 \int_0^\infty w^T(t)w(t)dt. \quad (27)
$$

From Lemma 2, when $x(0) = 0$, we have

$$
0 \leq - \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt + \gamma^2 \int_0^\infty w^T(t)w(t)dt, \quad (28)
$$

which implies

$$
\frac{\int_0^\infty z^T(t)\Xi z(t)dt}{\int_0^\infty w^T(t)w(t)dt} \leq \gamma^2, \quad (29)
$$

where

$$
z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad \Xi = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}. \quad (30)
$$

This completes the proof.

Next, we show that the RHDA ensures asymptotic stability without external disturbance under relation (6).

**Corollary 1.** Assume that the terminal weighting matrix $Q_f$ in (4) satisfies a set of LMI conditions (6). Then, the RHDA guarantees asymptotic stability without external disturbance.
Proof:

\[ J^*(x(t), t, t + T) = \int_t^{t+\mu} \left\{ \begin{bmatrix} x^{(s)}(\xi) & u^{(s)}(\xi) \end{bmatrix} \begin{bmatrix} Q & S \\ ST & R \end{bmatrix} \begin{bmatrix} x^{(s)}(\xi) \\ u^{(s)}(\xi) \end{bmatrix} \right\} d\xi + \int_{t+\mu}^{t+T} \left\{ \begin{bmatrix} x^{(s)}(\xi) & u^{(s)}(\xi) \end{bmatrix} \begin{bmatrix} Q & S \\ ST & R \end{bmatrix} \begin{bmatrix} x^{(s)}(\xi) \\ u^{(s)}(\xi) \end{bmatrix} \right\} d\xi \]

\[ - \gamma^2 w^{(s)}(\xi) w^{(s)}(\xi) d\xi \]

where \(0 < \mu < T\). If \(w^{(s)}(\cdot)\) is replaced by 0 from \(t\) to \(t + \mu\), then

\[ J^*(x(t), t, t + T) \geq \int_t^{t+\mu} \left\{ \begin{bmatrix} x^{(s)}(\xi) & u^{(s)}(\xi) \end{bmatrix} \begin{bmatrix} Q & S \\ ST & R \end{bmatrix} \begin{bmatrix} x^{(s)}(\xi) \\ u^{(s)}(\xi) \end{bmatrix} \right\} d\xi \]

\[ + J^*(x(t + \mu), t + \mu, t + T). \] (32)

Let \(\tau = t\) and \(\sigma = t + T\) in (8). Then, we have

\[ \frac{\partial J^*(x(t), t, \sigma)}{\partial \sigma} \bigg|_{\sigma = t+T} \leq 0. \] (33)

If we let \(\tau' = t\), \(\tau'' = t + \mu\), and \(\sigma = t + T\) in Lemma 1, (33) implies

\[ \frac{\partial J^*(x(t + \mu), t + \mu, \sigma)}{\partial \sigma} \bigg|_{\sigma = t+T} \leq 0 \] (34)

from Lemma 1. Thus, we have

\[ J^*(x(t), t, t + T) \geq \int_t^{t+\mu} \left\{ \begin{bmatrix} x^{(s)}(\xi) & u^{(s)}(\xi) \end{bmatrix} \begin{bmatrix} Q & S \\ ST & R \end{bmatrix} \begin{bmatrix} x^{(s)}(\xi) \\ u^{(s)}(\xi) \end{bmatrix} \right\} d\xi \]

\[ + J^*(x(t + \mu), t + \mu, t + T + \mu). \] (35)
When \( w(t) = 0 \),

\[
\frac{dJ^*(x(t), t, t + T)}{dt} \leq - \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \leq 0
\]

from (26). It is clear that \( J^*(x(t), t, t + T) \) is a decreasing function. Thus,

\[
\lim_{t \to \infty} J^*(x(t), t, t + T) = \text{const},
\]

where \( \text{const} \) is a positive constant. From relation (35), we have

\[
\lim_{t \to \infty} \int_{t}^{t+\mu} \begin{bmatrix} x^{(*)}(\xi) \\ u^{(*)}(\xi) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x^{(*)}(\xi) \\ u^{(*)}(\xi) \end{bmatrix} d\xi = 0,
\]

which guarantees \( \lim_{t \to \infty} x(t) = 0 \) and \( \lim_{t \to \infty} u(t) = 0 \). This completes the proof.

Remark 1. Disturbance or noise always exists in real secure communication systems based on chaotic neural networks under switching. In this case, the extension of the proposed RHDA to synchronization for the interconnected or master-slave structure of T-S fuzzy switched neural networks with disturbance has a practical importance. Once we obtain an RHDA for synchronization of this interconnected or master-slave structure, it can be used efficiently in the disturbance attenuation of real secure communication systems. In addition, the proposed RHDA can be also extended to studying the synchronization behavior of complex brain networks, whose nodes are represented by T-S fuzzy switched neural networks. In this case, several topological and spatial features in complex brain networks can be considered.

Remark 2. Until now, some results have been published that consider learning, filtering, and estimation either for fuzzy neural networks [2, 5, 6, 8, 11] or for switched neural networks [3, 4, 9, 7, 10]. However, very few studies have focused on fuzzy switched neural networks. This paper considers fuzzy switched neural networks and proposes a new disturbance attenuator for these neural networks based on the receding horizon control concept. This paper is a new contribution to the study of fuzzy switched neural networks.

Remark 3. Novel Takagi-Sugeno fuzzy controller and filter design methods were recently pro-
posed in [31] and [30], respectively. In addition, model reduction and sliding model control for switched systems were studied in [39] and [40], respectively. However, these studies do not deal with neural networks. The extension of the methods proposed in [31, 30, 39, 40] to fuzzy switched neural networks can be an interesting research topic. This extension remains to be evaluated in future research work.

Remark 4. The results proposed in this paper can be utilized in several control applications. For example, T-S fuzzy switched neural networks are applied to represent unknown nonlinear systems with external disturbance and the proposed RHDA can be then utilized to stabilize these nonlinear systems in the $\mathcal{H}_\infty$ sense. Thus, the proposed RHDA for T-S fuzzy switched neural networks is of significance for many practical applications from the point of view of control.

Remark 5. This paper deals with fuzzy switched neural networks. The results presented in this paper for fuzzy switched neural networks can be further researched by using existing results [29, 13, 38, 12]. In [29], an induced $l_2$ fuzzy filter was proposed for stochastic systems, but that study does not focus on neural networks. However, a new induced $l_2$ RHDA for fuzzy switched neural networks can be developed based on the results given in [29]. The authors in [13] studied exponential $L_2$-$L_\infty$ stability of Takagi-Sugeno fuzzy neural networks, but their work does not consider fuzzy switched neural networks. Using the results in [13], it is possible to extend the RHDA to a new exponentially convergent $L_2$-$L_\infty$ RHDA for fuzzy switched neural networks. In [38], a fuzzy filter was designed for Itô stochastic systems, and this result can also be applied to obtain a new receding horizon fuzzy filtering method for stochastic fuzzy switched neural networks. Finally, some results on stability margins of the RHDA for fuzzy switched neural networks can be obtained by using the results presented in [12]. Although these four studies do not deal with fuzzy switched neural networks, their findings can be effectively used to obtain some new and additional results for the RHDA of fuzzy switched neural networks.

4 Numerical Example

Consider the following T-S fuzzy switched Hopfield neural network:

$$\dot{x}(t) = \sum_{k=1}^{2} \sum_{i=1}^{2} \xi_k(t)h_{(i,k)}(\omega) [A(i,k)x(t) + W(i,k)\phi(x(t)) + u(t) + w(t)],$$  \hspace{1cm} (39)

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where

\[ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}, \quad \phi(x(t)) = \begin{bmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{bmatrix}, \]

\[ A(1,1) = \begin{bmatrix} -2 & 0 \\ 0 & -2.05 \end{bmatrix}, \quad A(2,1) = \begin{bmatrix} -2.6 & 0 \\ 0 & -2.13 \end{bmatrix}, \]

\[ A(1,2) = \begin{bmatrix} -3.4 & 0 \\ 0 & -3 \end{bmatrix}, \quad A(2,2) = \begin{bmatrix} -3.1 & 0 \\ 0 & -2.56 \end{bmatrix}, \]

\[ W(1,1) = \begin{bmatrix} -0.6 & 0.4 \\ 0 & -0.1 \end{bmatrix}, \quad W(2,1) = \begin{bmatrix} 0.1 & -0.8 \\ 0.4 & 0.5 \end{bmatrix}, \]

\[ W(1,2) = \begin{bmatrix} -0.5 & -1.11 \\ 0.35 & -0.15 \end{bmatrix}, \quad W(2,2) = \begin{bmatrix} 0.24 & 0.8 \\ -0.14 & 1.1 \end{bmatrix}. \]

The fuzzy membership functions are taken as
\[ h_{(1,1)}(\omega) = \sin^2(x_1(t)), \quad h_{(2,1)}(\omega) = \cos^2(x_1(t)), \]
\[ h_{(1,2)}(\omega) = \exp(t)/(\exp(t) + \exp(100t)), \quad h_{(2,2)}(\omega) = \exp(100t)/(\exp(t) + \exp(100t)). \]
Further, the switching signal is given by

\[ \alpha = \begin{cases} 1, & 0.2k \leq t \leq 0.2k + 0.1, \ k \in \mathbb{Z} , \\ 2, & \text{otherwise}, \end{cases} \tag{40} \]

where \( \mathbb{Z} \) is the whole set of non-negative integers. Figure 1 shows the switching signal. Let \( Q = R = 0.1I \in R^{2 \times 2} \) and \( S = O \in R^{2 \times 2} \), where \( I \) is the identity matrix and \( O \) is the zero matrix. In this example, we have five LMI variables \( X, Y(1,1), Y(1,2), Y(2,1), \) and \( Y(2,2) \).

Solving a set of LMIs (6) with \( \gamma = 0.499 \) gives

\[ X = \begin{bmatrix} 1.8220 & 0.0099 \\ 0.0099 & 2.0340 \end{bmatrix}, \]

\[ Y(1,1) = \begin{bmatrix} -0.9088 & 0.0007 \\ 0.0007 & -0.8973 \end{bmatrix}, \quad Y(1,2) = \begin{bmatrix} -0.8392 & 0.0005 \\ 0.0005 & -0.8355 \end{bmatrix}, \]

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\[
Y(2,1) = \begin{bmatrix}
-0.8741 & 0.0061 \\
0.0061 & -0.8986
\end{bmatrix}, \quad Y(2,2) = \begin{bmatrix}
-0.8444 & -0.0138 \\
-0.0137 & -0.8829
\end{bmatrix}.
\]

Thus, the finite terminal weighting matrix is given by
\[
Q_f = \begin{bmatrix}
0.5488 & -0.0027 \\
-0.0027 & 0.4917
\end{bmatrix}.
\]

In this simulation, the initial condition is given by
\[
\begin{bmatrix}
x_1(0) \\
x_2(0)
\end{bmatrix} = \begin{bmatrix}
-0.1 \\
1.5
\end{bmatrix},
\]
and the external disturbance \(w_i(t)\ (i = 1, 2)\) is a Gaussian noise with mean 0 and variance 0.1. Figure 2 shows state trajectories for the T-S fuzzy switched Hopfield neural network (39) under the switching signal (40). Figure 3 shows state trajectories in the phase plane. These figures demonstrate that the proposed RHDA reduces the effect of external disturbance on the state variable of the T-S fuzzy switched Hopfield neural network (39) under the switching signal (40).

Next, we check the \(H_\infty\) performance of the RHDA. We use the same parameters in the above simulation. Let \(w_1(t) = \cos(10t)\) and \(w_2(t) = \sin(10t)\). Assume the zero initial state condition. Define the following function:
\[
H(t) = \frac{\int_0^t z^T(\nu)\Xi z(\nu) d\nu}{\int_0^t w^T(\nu)w(\nu) d\nu},
\]
where \(z(\nu)\) and \(\Xi\) are given by (30). The \(H_\infty\) performance (29) is then restated as \(H(\infty) < \gamma^2\). Figure 4 verifies \(H(\infty) < \gamma^2 = 0.2490\), which means that the proposed RHDA guarantees the \(H_\infty\) performance under the zero initial state condition.
5 Conclusion

In this paper, we proposed a new disturbance attenuator, called RHDA, for T-S fuzzy switched Hopfield neural networks with external disturbance. A new set of sufficient LMI conditions is proposed for the finite terminal weighting matrix in the receding horizon cost function with a cross term, under which T-S fuzzy switched Hopfield neural networks were shown to be asymptotically stable with a guaranteed infinite horizon $\mathcal{H}_\infty$ performance. The finite terminal weighting matrix can be easily determined by solving the LMI feasibility problem. A numerical example with simulation results was provided to illustrate the applicability and usefulness of the proposed RHDA. The proposed RHDA is expected to be useful in practical applications such as synchronization controller design of secure communication systems and complex brain networks. In addition, the extension of the RHDA to time-delayed fuzzy switched neural networks is a further and important problem. The analysis of computational complexity of the RHDA in terms of system matrices is also an interesting research topic. The exploration of these research topics remains as further work.

References


Figure captions

Figure 1. Switching signal
Figure 2. State trajectories of the T-S fuzzy switched Hopfield neural network (39)
Figure 3. State trajectories in the phase plane
Figure 4. $H(t)$ versus time