ASYMPTOTIC ANALYSIS OF CHARGE CONSERVING POISSON-BOLTZMANN EQUATIONS WITH VARIABLE DIELECTRIC COEFFICIENTS

CHIUN-CHANG LEE∗

Department of Applied Mathematics
National Hsinchu University of Education
Hsinchu City 30014, Taiwan

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Abstract. Concerning the influence of the dielectric constant on the electrostatic potential in the bulk of electrolyte solutions, we investigate a charge conserving Poisson-Boltzmann (CCPB) equation [31, 32] with a variable dielectric coefficient and a small parameter ϵ (related to the Debye screening length) in a bounded connected domain with smooth boundary. Under the Robin boundary condition with a given applied potential, the limiting behavior (as ϵ ↓ 0) of the solution (the electrostatic potential) has been rigorously studied. In particular, under the charge neutrality constraint, our result exactly shows the effects of the dielectric coefficient and the applied potential on the limiting value of the solution in the interior domain. The main approach is the Pohozaev’s identity of this model. On the other hand, under the charge non-neutrality constraint, we show that the maximum difference between the boundary and interior values of the solution has a lower bound log 1/ε as ϵ goes to zero. Such an asymptotic blow-up behavior describes an unstable phenomenon which is totally different from the behavior of the solution under the charge neutrality constraint.

1. Introduction. Let Ω be a bounded connected domain with smooth boundary ∂Ω in ℝN (N ≥ 1). We are interested in the asymptotic behavior of solutions φε (as ϵ ↓ 0) to the following model that arises in the study of ion transport in electrolyte solutions:

\[ \epsilon^2 \nabla \cdot (g^2(x)\nabla \phi_\epsilon) = \frac{\alpha a(x)e^{\phi_\epsilon}}{\int_\Omega a(y)e^{\phi_\epsilon(y)}dy} - \frac{\beta b(x)e^{-\phi_\epsilon}}{\int_\Omega b(y)e^{-\phi_\epsilon(y)}dy} \quad \text{in} \quad \Omega, \]

\[ \phi_\epsilon + \eta \frac{\partial \phi_\epsilon}{\partial \nu} = \phi_{bd} \quad \text{on} \quad \partial \Omega, \]

where the unknown variable φε = \phi_\epsilon(x) ∈ ℝ represents the electrostatic potential, ϵ > 0 is a small parameter tending to zero, α and β are positive constants, g(x), a(x) and b(x) are functions keeping the positive sign on Ω̅, and the operator

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When N = 1, Ω becomes a bounded open interval.
\[ \nabla \cdot \left( g^2(x) \nabla \phi \right) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( g^2(x) \frac{\partial \phi}{\partial x_i} \right). \]

Besides, \( \eta_e \) is a non-negative parameter depending on \( \epsilon, \frac{\partial}{\partial x} = \sum_{i=1}^{N} \nu_i(x) \frac{\partial}{\partial x_i} \) and \( \phi_{\text{bd}} \equiv \phi_{\text{bd}}(x) \) is a function defined on \( \overline{\Omega} \), where \( \overrightarrow{\nu}(x) = (\nu_1(x), ..., \nu_N(x)) \) is the outward unit normal vector at \( x \in \partial \Omega \).

The equation (1.1)-(1.2), in addition to being natural extensions of Poisson-Boltzmann (PB) type equations describing the behavior of electrostatic potential in electrolyte solutions with monovalent anion and cation species (e.g., NaCl solutions), plays a crucial role in many electrochemical and physical problems. Physically, \( \Omega \) describes the bulk region of electrolyte solutions, its boundary \( \partial \Omega \) can be regarded as a charged surface where the electrical double layer (EDL) develops, and the EDL behaves as a capacitor [3, 14, 19, 29, 30]. The parameter \( \epsilon \) is a dimensionless quantity related to the ratio of the Debye screening length (the characteristic thickness of the EDL) to the diameter of the physical domain \( \Omega \) [7, 11]. The boundary condition (1.2) of the electrostatic potential \( \phi_{\epsilon} \) is related to the capacitance effect of the EDL [4, 16, 32, 33], where \( \phi_{\text{bd}} \) is the applied potential and \( \eta_e \) measures the surface capacitance [21].

The main motivation of studying this model is based on two different aspects of view. One view is that (1.1)-(1.2) is a generalization of the standard charge conserving Poisson-Boltzmann (CCPB) equation [31, 32, 34] with variable coefficients \( g^2(x), a(x) \) and \( b(x) \); the other view is that (1.1)-(1.2) is the steady-state of the classical Poisson-Nernst-Planck (cPNP) system [25] provided \( a(x) = b(x) = g^2(x) \). When \( g(x), a(x) \) and \( b(x) \) are non-zero constant-valued functions, equation (1.1) becomes the standard CCPB equation which describes the ion transport of monovalent ion species in electrolyte solutions. Thus, equation (1.1) can also be viewed as a CCPB equation with a variable dielectric coefficient \( g^2(x) \) and variable coefficients \( a(x) \) and \( b(x) \) which are related to bulk concentrations at position \( x \), where

\[ \mathcal{N}_e(x) = \frac{\alpha}{\int_{\Omega} a(y) e^{\phi(y)} dy} a(x) e^{\phi(x)} \quad \text{and} \quad \mathcal{P}_e(x) = \frac{\beta}{\int_{\Omega} b(y) e^{-\phi(y)} dy} b(x) e^{-\phi(x)} \]

represent the density of anion and cation species, respectively. The standard CCPB equation with constant-valued dielectric coefficient has been intensively studied in mathematics literatures [11, 21, 22, 23, 29]. To see the effect of the dielectric coefficient \( g^2(x) \) on the asymptotic behavior of solutions \( \phi_{\epsilon} \) of (1.1)-(1.2) as \( \epsilon \) tends to zero, we shall focus mainly on a general case that \( g \) is a positive valued function.

On the other hand, when \( a(x) = b(x) = g^2(x) \) and \( \Omega \equiv (-1, 1) \) is a one-dimensional domain (i.e., \( N = 1 \), equation (1.1) can be rewritten as

\[ \frac{\epsilon^2}{g^2(x)} \frac{d}{dx} \left( g^2(x) \frac{d \phi_{\epsilon}}{dx} \right) = n_{\epsilon} - p_{\epsilon}, \quad x \in (-1, 1), \]

\[ n_{\epsilon} = \frac{\alpha}{\int_{-1}^{1} g^2(y) e^{\phi_{\epsilon}(y)} dy} e^{\phi_{\epsilon}}, \quad p_{\epsilon} = \frac{\beta}{\int_{-1}^{1} g^2(y) e^{-\phi_{\epsilon}(y)} dy} e^{-\phi_{\epsilon}} \]

which is a steady-state of the following classical Poisson-Nernst-Planck system proposed in [24, 25] (for notation convenience, we still use \( (n_{\epsilon}, p_{\epsilon}, \phi_{\epsilon}) \) in the following system):

\[ \frac{\partial n_{\epsilon}}{\partial t} = - \frac{1}{g^2} \frac{\partial}{\partial x} \left( g^2 \frac{\partial \phi_{\epsilon}}{\partial x} \right), \quad \frac{\partial p_{\epsilon}}{\partial t} = \frac{1}{g^2} \frac{\partial}{\partial x} \left( g^2 \frac{\partial \phi_{\epsilon}}{\partial x} \right), \]

\[ J_n = -D_n g^2 \left( \frac{\partial n_{\epsilon}}{\partial x} - n_{\epsilon} \frac{\partial \phi_{\epsilon}}{\partial x} \right), \quad J_p = -D_p g^2 \left( \frac{\partial p_{\epsilon}}{\partial x} + p_{\epsilon} \frac{\partial \phi_{\epsilon}}{\partial x} \right). \]
\[
\frac{\epsilon^2}{g^2} \frac{\partial}{\partial x} \left( g^2 \frac{\partial \phi_e}{\partial x} \right) = n_e - p_e, \quad x \in (-1, 1), \quad t > 0, \tag{1.8}
\]

where the parameter \( \epsilon \) is related to the Debye screening length, \( \phi_e \) is the electrostatic potential, \( n_e \) is the density of anions, \( p_e \) is the density of cations, \( J_n \) and \( J_p \) are the ionic flux densities, \( D_n \) and \( D_p \) are their diffusion coefficients, and \( q \equiv g(x) \) represents the radius of the varying cross-section of the ion channel at position \( x \in [-1, 1] \) (cf. \([25, 26]\)). Indeed, under no-flux boundary conditions \( J_n = J_p = 0 \) at the boundary points \( x = \pm 1 \), we may use (1.6)-(1.7) to derive the conservation law of total charges of the individual ions, i.e.,

\[
\frac{d}{dt} \int_{-1}^{1} g^2 n_e \, dx = \frac{d}{dt} \int_{-1}^{1} g^2 p_e \, dx = 0 \quad \text{for } t > 0.
\]

As a consequence, there holds

\[
\begin{align*}
\int_{-1}^{1} g^2 n_e \, dx &= \int_{-1}^{1} g^2 n_e \, dx \bigg|_{t=0} \equiv \alpha > 0, \\
\int_{-1}^{1} g^2 p_e \, dx &= \int_{-1}^{1} g^2 p_e \, dx \bigg|_{t=0} \equiv \beta > 0.
\end{align*}
\tag{1.9}
\]

Moreover, from no-flux boundary conditions of \( J_n \) and \( J_p \), the steady-state, i.e., time-independent solutions of the PNP system (1.6)-(1.8) satisfy (1.11)-(1.12) in the domain \((-1, 1)\), which solves \( n_e = \hat{\alpha} e^{\phi_e} \) and \( p_e = \hat{\beta} e^{-\phi_e} \), where \( \hat{\alpha} \) and \( \hat{\beta} \) are positive constants. Along with (1.9) yields \( \hat{\alpha} = \frac{\alpha}{\int_{-1}^{1} g^2 e^{\phi_e} \, dy} \) and \( \hat{\beta} = \frac{\beta}{\int_{-1}^{1} g^2 e^{-\phi_e} \, dy} \) having the non-local dependence on \( \phi_e \), which gives (1.5). One can see that under various variable \( g(x) \), \( a(x) \) and \( b(x) \), equation (1.1) is broader than (1.4)-(1.5).

It is worth noting that under the transformation \( \Phi_e(x) = \phi_e(x) + \frac{1}{2} \log \frac{a(x)}{\hat{a}} \), equation (1.1) is equivalent to the following CCPB equation with a permanent charge density \( \epsilon^2 Q(x) \):

\[
\epsilon^2 \nabla \cdot \left( g^2(x) \nabla \Phi_e \right) = \frac{\alpha}{\int_{-1}^{1} h(y) e^{\Phi_e(y)} \, dy} - \frac{\beta}{\int_{-1}^{1} h(y) e^{-\Phi_e(y)} \, dy} + \epsilon^2 Q(x),
\tag{1.10}
\]

where \( h(x) = \sqrt{a(x)b(x)} > 0 \) and \( Q(x) = \frac{1}{2} \nabla \cdot \left( g^2(x) \nabla \log \frac{a(x)}{\hat{a}} \right) \). From this point of view, for any smooth permanent charge density \( \epsilon^2 Q(x) \), we can study the behavior of solutions of the CCPB equation (1.10) via solutions of (1.1) under suitable variable coefficients \( a(x) \), \( b(x) \) and boundary conditions.

In this paper, we are interested in investigating the asymptotic behavior (as \( \epsilon \downarrow 0 \)) of solutions \( \phi_e \) of (1.1)-(1.2) under a wide variety of coefficients \( g(x) \), \( \alpha \), \( \beta \), \( a(x) \), \( b(x) \) and \( \eta_e \). For direct connection to results in the present work, we shall mention the previous works in \([22, 23, 29]\). The authors in \([22, 23, 29]\) have studied the asymptotic behavior (as \( \epsilon \downarrow 0 \)) of one-dimensional solutions of (1.1)-(1.2) in the case when \( g^2(x) = a(x) = b(x) \equiv 1 \), i.e.,

\[
\epsilon^2 \phi_e'' = \frac{\alpha}{\int_{-1}^{1} e^{\phi_e(y)} \, dy} - \frac{\beta}{\int_{-1}^{1} e^{-\phi_e(y)} \, dy} \quad \text{in } (-1, 1), \tag{1.11}
\]

\[
\phi_e(-1) - \eta_e \phi'_e(-1) = \phi_{bd}^-, \quad \phi_e(1) + \eta_e \phi'_e(1) = \phi_{bd}^+,
\tag{1.12}
\]

where \( \phi_{bd}^\pm \equiv \phi_{bd}(\pm 1) \) are constants independent of \( \epsilon \). Physically, the constraint \( \alpha = \beta \) is called the global charge neutrality which means that the total charges of anions and cations are equal (cf. (1.9) for \( g \equiv 1 \)). The other situation under \( \alpha \neq \beta \) is called the charge non-neutrality which means that the total charges of anions and cations are not equal to each other (cf. \([22, 32]\)). As \( \epsilon \) goes to zero, the asymptotic behavior of solutions \( \phi_e \) of (1.11)-(1.12) in the domain \((-1, 1)\) can be clearly classified into two situations \( \alpha = \beta \) (global charge neutrality) and \( \alpha \neq \beta \) (charge non-neutrality):
Roughly speaking, we show that for any compact subset \( K \subset \epsilon \) in the interior domain as \( \epsilon \to 0 \) (cf. Theorem 1.7 of [22]),

- When \( \alpha = \beta \), \( \max_{[-1,1]} |\phi_\epsilon| \) is uniformly bounded for all \( \epsilon > 0 \) (cf. Theorem 2.1(i) of [23] and Theorem 3.1 of [21]);
- When \( \alpha \neq \beta \), for each \( x \in (-1,1) \), \( \phi_\epsilon(x) \) asymptotically blows up as \( \epsilon \to 0 \) (cf. Theorem 1.7 of [22]).

We refer the reader to [21, 22, 23, 29] and references therein for more detail results corresponding to the asymptotic behavior for solutions \( \phi_\epsilon \) of (1.11)-(1.12).

However, in cases that \( g(x) \), \( a(x) \) and \( b(x) \) are not constant-valued functions, situation becomes more complicated when we study the asymptotic behavior of solutions \( \phi_\epsilon \) to equation (1.1)-(1.2) in any dimensional domains \((N \geq 1)\). Based on the study of the asymptotic behavior of solutions of (1.11)-(1.12) under different situations \( \alpha = \beta \) and \( \alpha \neq \beta \), we shall investigate (1.1)-(1.2) with broader constraints on coefficients \( \alpha, \beta, g(x), a(x) \) and \( b(x) \). We are also interested in the effect of the dielectric constant \( g^2(x) \) on the limiting behavior of the electrostatic potential \( \phi_\epsilon \) as \( \epsilon \) tends to zero, which is of importance issue in electrolyte solutions. To provide a guideline for readers, we now setup the problems and summary the main results as follows:

**Problem I. The effect of the dielectric coefficient on the electrostatic potential.**

Almost electrolyte solutions possess the pointwise charge neutrality in the bulk, and creates the EDL at the charged surface. This motivates us to seek solutions \( \phi_\epsilon \) of (1.1)-(1.2) such that when \( \epsilon \) tends to zero, \( \phi_\epsilon \) forms a boundary layer (related to the EDL) at the boundary and becomes flat in the interior domain, as was studied in [22, 23, 29]. Hence, it suffices to assume that in the interior domain, \( \phi_\epsilon \) approaches a constant value and \( \mathcal{N}_\epsilon - \mathcal{P}_\epsilon \) (the difference between the total concentrations of cation and anion species) approaches zero as \( \epsilon \) goes to zero. Thus by (1.1) and (1.3), we find that the ratio \( \frac{b(x)}{a(x)} \) is a constant due to the following formal approximation as \( \epsilon \) tends to zero:

\[
\frac{b(x)}{a(x)} \approx \frac{\int_{\Omega} b(y) e^{-\phi_\epsilon(y)} dy}{\int_{\Omega} a(y) e^{\phi_\epsilon(y)} dy} \approx \frac{\int_{\Omega} b(y) dy}{\int_{\Omega} a(y) dy}.
\]

As a consequence, we make the following assumption:

(GN) \( \alpha = \beta \) (global charge neutrality), \( g(x) \), \( a(x) \) and \( b(x) \) are strictly positive and the ratio \( \frac{b(x)}{a(x)} \) is a constant on \( \Omega \).

(GN) is a sufficient condition which makes the solution approach the charge neutrality at each interior point, i.e., \( \mathcal{N}_\epsilon - \mathcal{P}_\epsilon \to 0 \) as \( \epsilon \) goes to zero. Under constraint of (GN), we are interested in investigating the effect of the dielectric coefficient \( g^2 \) and the applied potential \( \phi_{bd} \) on solution values (electrostatic potential) \( \phi_\epsilon \) of (1.1)-(1.2) in the interior domain as \( 0 < \epsilon \ll 1 \). Such a result is stated in Theorem 1.2. Roughly speaking, we show that for any compact subset \( K \subset \Omega \),

\[
\lim_{\epsilon \to 0} \phi_\epsilon(x) = \frac{\int_{\partial \Omega} g^2 \phi_{bd} dS}{\int_{\partial \Omega} g^2 dS} \quad \text{uniformly in } K.
\]

(1.13) exactly shows the effect of the dielectric coefficient \( g^2 \) and the applied potential \( \phi_{bd} \) on the limiting value of \( \phi_\epsilon \) in the interior domain. In particular, when \( N = 1 \) and \( \Omega = (-1,1) \), (1.13) implies \( \lim_{\epsilon \to 0} \phi_\epsilon(x) = \frac{g^2(1)\phi_{bd}(1)+g^2(-1)\phi_{bd}(-1)}{g^2(1)+g^2(-1)} = \frac{\phi_{bd}(1)+\phi_{bd}(-1)}{2} \) for \( x \in (-1,1) \) provided that \( \frac{g^2(1)}{g^2(-1)} = \varpi \), which formally gives
• If $\varpi \sim \infty$ (resp., $\varpi \sim 0$), i.e., $g(1)$ is very large (resp., small) compared with $g(-1)$, then for $x \in (-1, 1)$, $\phi(x)$ is close to the value $\phi_{bd}(1)$ (resp. $\phi_{bd}(-1)$).

• If $\varpi \sim 1$, then for $x \in (-1, 1)$ $\phi(x)$ is close to $\frac{1}{2} (\phi_{bd}(-1) + \phi_{bd}(1))$, i.e., the mean value of $\phi_{bd}$ on the boundary.

Hence, various dielectric coefficients result in distinct solution behaviors in the interior domain. Such results seem difficult to observe from the equation (1.1)-(1.2) in a direct way.

We also want to point out that from the framework of (1.4)-(1.5), $g(\pm 1)$’s are the radius of the cross-section of the ion channel at boundary points. Hence, (1.13) also demonstrates the effect of $g(\pm 1)$’s on the intra-channel electrostatic potential (note that in this case, $a(x) = b(x) = g^2(x)$ satisfies the condition (GN)).

**Problem II. Estimate of a lower bound of the asymptotic blow-up solution.**

In electrolyte solutions, charge non-neutrality phenomena can be found near the electrodes when the Faradaic current is driven by redox reactions occurring at the electrodes [3]. To describe such an unstable phenomenon of the electrostatic potential under charge non-neutrality constraints, we also focus our attention on the asymptotic blow-up behavior (as $\epsilon \downarrow 0$) of solutions $\phi_\epsilon$ of (1.1)-(1.2) under some specific conditions on $\alpha$, $\beta$, $a(x)$ and $b(x)$, which excludes the condition (GN).

Hereafter, $\phi_\epsilon$ is said to be an asymptotic blow-up solution of (1.1)-(1.2) with $\epsilon \downarrow 0$ if for each $\epsilon > 0$, $\phi_\epsilon$ is smooth in the whole domain $\Omega$, and the maximum difference between two values of $\phi_\epsilon$ in $\Omega$ diverges to infinity as $\epsilon$ tends to zero. Namely, there exist $x_1^\epsilon$, $x_2^\epsilon \in \Omega$ such that $\lim_{\epsilon \downarrow 0} |\phi_\epsilon(x_1^\epsilon) - \phi_\epsilon(x_2^\epsilon)| = \infty$. For such a phenomenon, we are interested in estimating a lower bound of the value $\max_{\Omega} \phi_\epsilon - \min_{\Omega} \phi_\epsilon$ for $0 < \epsilon \ll 1$.

To obtain an asymptotic blow-up phenomenon for solutions $\phi_\epsilon$ of (1.1)-(1.2) as $\epsilon$ goes to zero, we assume that $g(x) \alpha, \beta, a(x)$ and $b(x)$ are strictly positive and satisfy one of the following assumptions:

(A1) $\max_{x \in \Omega} a(x) \max_{x \in \Omega} b(x) < \beta \min_{x \in \Omega} a(x) \min_{x \in \Omega} b(x)$; or

(A2) $\beta \max_{x \in \Omega} a(x) \max_{x \in \Omega} b(x) < \alpha \min_{x \in \Omega} a(x) \min_{x \in \Omega} b(x)$.

Note that if both $a(x)$ and $b(x)$ are constant-valued functions, then assumption (A1) (respectively, (A2)) becomes $\alpha < \beta$ (respectively, $\alpha > \beta$), which has been studied in [22, 29] for the one-dimensional case. Thus, it is expected that under assumptions (A1) and (A2), solutions $\phi_\epsilon$ of (1.1)-(1.2) may asymptotically blow up as $\epsilon$ goes to zero. In this paper, we show that under assumption (A1) or (A2), the maximum difference between two values of the solution $\phi_\epsilon$ of (1.1)-(1.2) in the whole domain $\Omega$ has a lower bound $\log \frac{1}{\epsilon}$ as $\epsilon$ goes to zero (cf. (1.15)). Such an asymptotic blow-up phenomenon of $\phi_\epsilon$ is totally different from the asymptotic behavior of $\phi_\epsilon$ in the case of (GN).

To the author’s best knowledge, the current work is the first rigorous mathematical analysis of the CCPB equation with the variable dielectric coefficient. We want to emphasize that the results established in this paper seems simple but their proofs are highly nontrivial. Since the method used in [21, 22, 23] cannot be applied to the model (1.1)-(1.2), there are a number of difficulties that one has to overcome.

1.1. **Statement of the main results.** For analysis convenience, we may assume that $g$, $a$, $b$, $\phi_{bd} \in C^\infty(\Omega)$ are smooth functions. As mentioned previously, $\eta_\epsilon$ is a non-negative parameter depending on $\epsilon$; $\alpha$, $\beta$, $g$, $a$, $b$ and $\phi_{bd}$ are independent of $\epsilon$, ...
and \( g, a, b > 0 \) on \( \Omega \). Equation (1.1)-(1.2) can be regarded as the Euler-Lagrange equation of the energy functional

\[
E[u] = \frac{\epsilon^2}{2} \int_\Omega g^2(x)|\nabla u|^2 dx + \alpha \log \int_\Omega a(x)e^ux dx + \beta \log \int_\Omega b(x)e^{-ux} dx + B_\eta[u], \quad u \in \mathbb{H},
\]

where \( B_\eta[u] = 0 \) and \( \mathbb{H} = \{ u \in H^1(\Omega) : u - \phi_{\text{bd}} \in H^1_0(\Omega) \} \) if \( \eta_\epsilon = 0 \); \( B_\eta[u] = \frac{\epsilon^2}{2m} \int_{\partial \Omega}(u - \phi_{\text{bd}})^2 dS \) and \( \mathbb{H} = H^1(\Omega) \) if \( \eta_\epsilon > 0 \). Applying the Jensen's inequality and the Hölder's inequality to both the second and third terms of the right-hand side of (1.14), and using the Friedrichs' inequality (cf. Theorem 6.1 of [20]) to the first term of the right-hand side of (1.14), one obtains that (cf. Lemma 5.1 in the Appendix):

- \( E[u] \) is bounded below, i.e., \( \inf_{u \in \mathbb{H}} E[u] > -\infty \), and
- \( E[u] \) is strictly convex on \( \mathbb{H} \), i.e., \( E[tu_1 + (1 - t)u_2] < tE[u_1] + (1 - t)E[u_2] \) for any \( u_1, u_2 \in \mathbb{H} \) with \( u_1 \neq u_2 \) and \( t \in (0, 1) \).

Hence, we may apply standard arguments of direct methods in the calculus of variation to the energy functional \( E[u] \) to get the uniqueness of the minimizer of \( E[u] \) over \( \mathbb{H} \). Then, by using the similar argument of Theorem 1.1 of [21], we establish the existence and uniqueness of classical solutions to (1.1)-(1.2) as follows:

**Proposition 1.** Let \( \epsilon > 0 \) and \( \Omega \) be a bounded connected domain in \( \mathbb{R}^N \) with smooth boundary, where \( N \geq 1 \). Assume that \( \alpha, \beta > 0, \eta_\epsilon \geq 0, g, a, b \in C^\infty(\overline{\Omega}) \) are positive-valued functions, and \( \phi_{\text{bd}} \in C^\infty(\overline{\Omega}) \). Then the equation (1.1) with the boundary condition (1.2) has a unique solution \( \phi_\epsilon \in C^\infty(\overline{\Omega}) \).

The proof of Proposition 1 is similar to Theorem 1.1 of [21] so we omit the detailed proof here.

For the charge non-neutrality, we focus on cases (A1) and (A2). As \( \epsilon \) tends to zero, the solution \( \phi_\epsilon \) of (1.1)-(1.2) has asymptotic blow-up phenomenon which can be demonstrated as follows:

**Theorem 1.1.** Under the same hypotheses as in Proposition 1, in addition, we assume that \( g(x), \alpha, \beta, a(x), b(x) \) and \( \phi_{\text{bd}}(x) \) are independent of \( \epsilon \). Then we have

- (i) If (A1) holds true, then \( \phi_\epsilon \) attains its maximum value at an interior point of \( \Omega \) and minimum value on the boundary \( \partial \Omega \) as \( 0 < \epsilon \ll 1 \).
- (ii) If (A2) holds true, then \( \phi_\epsilon \) attains its minimum value at an interior point of \( \Omega \) and maximum value on the boundary \( \partial \Omega \) as \( 0 < \epsilon \ll 1 \).

Moreover, as \( \epsilon \) goes to zero, the value \( \max_{x \in \Omega} \phi_\epsilon(x) - \min_{x \in \Omega} \phi_\epsilon(x) \) diverges to infinity and

\[
\liminf_{\epsilon \downarrow 0} \left( \log \frac{1}{\epsilon} \right)^{-1} \left( \max_{x \in \Omega} \phi_\epsilon(x) - \min_{x \in \Omega} \phi_\epsilon(x) \right) \geq 1. \tag{1.15}
\]

**Remark 1.** Assume that one of (A1) and (A2) holds true. Theorem 1.1 shows that as \( \epsilon \) tends to zero, the maximum difference between the boundary and interior values of the electrostatic potential \( \phi_\epsilon \) in the whole domain \( \Omega \) asymptotically blows up. Moreover, as \( \epsilon \) tends to zero, the leading order term of the asymptotic expansion of \( \max_{x \in \Omega} \phi_\epsilon(x) - \min_{x \in \Omega} \phi_\epsilon(x) \) with respect to \( \epsilon \) has a lower bound \( \log \frac{1}{\epsilon} \). This indicates an unstable phenomenon of the electrostatic potential under charge non-neutrality constraints.
Now we are in a position to point out the main difficulty for proving (1.15). When either (A1) or (A2) holds, \( \max |\phi_\epsilon| \) may not be uniformly bounded as \( \epsilon \to 0 \). This is caused by the non-local coefficients \( \frac{a(x)}{\int_{\Omega} a(y)e^{\phi_\epsilon(y)}dy} \) and \( \frac{b(x)}{\int_{\Omega} b(y)e^{\phi_\epsilon(y)}dy} \) which may diverge to infinity or tend to zero, which becomes an extra challenge for estimating its solutions. In order to overcome this difficulty, we should deal with its non-local coefficients before estimating the solution \( \phi_\epsilon \). To achieve the idea, we construct an auxiliary function

\[
B_\epsilon(x) = \frac{A e^{\phi_\epsilon(x)}}{\int_{\Omega} e^{\phi_\epsilon(y)}dy} - \frac{B e^{-\phi_\epsilon(x)}}{\int_{\Omega} e^{-\phi_\epsilon(y)}dy}, \quad x \in \Omega, 
\]

where

\[
A = \frac{\alpha \max_{\Omega} a(x)}{\min_{\Omega} a(x)}, \quad B = \frac{\beta \min_{\Omega} b(x)}{\max_{\Omega} b(x)} \quad \text{if (A1) holds,} \\
A = \frac{\alpha \min_{\Omega} a(x)}{\max_{\Omega} a(x)}, \quad B = \frac{\beta \max_{\Omega} b(x)}{\min_{\Omega} b(x)} \quad \text{if (A2) holds.}
\]

We will establish the second-order differential equation of \( B_\epsilon \). Along with the comparison principle, we can prove (1.15).

To see the effect of the dielectric coefficient \( g^2 \) and the applied potential \( \phi_{bd} \) under a charge neutrality constraint (GN), we establish the following result:

**Theorem 1.2.** Under the same hypotheses as in Proposition 1, in addition, we assume that \( g(x), \alpha, \beta, a(x), b(x) \) and \( \phi_{bd}(x) \) are independent of \( \epsilon \), and that (GN) holds true. Then

\[
\text{-i) For any } \epsilon > 0, \phi_\epsilon \text{ attains both its maximum and minimum values on the boundary } \partial \Omega, \text{ and}
\]

\[
\min_{\partial \Omega} \phi_{bd} \leq \phi_\epsilon(x) \leq \max_{\partial \Omega} \phi_{bd}, \quad \forall x \in \Omega. \tag{1.19}
\]

Moreover, for any compact subset \( K \) of \( \Omega \), \( \max_{x_1, x_2 \in K} |\phi_\epsilon(x_1) - \phi_\epsilon(x_2)| \) exponentially converges to zero as \( \epsilon \to 0 \).

\[
\text{-ii) In addition, assume that } \lim_{\epsilon \to 0} \frac{\Omega}{\epsilon} = \infty \text{ and that } 0 \in \Omega \text{ and } \Omega \text{ is a strictly star-shaped domain with respect to } 0. \text{ Then for any compact subset } K \text{ of } \Omega, \text{ we have}
\]

\[
\lim_{\epsilon \to 0} \phi_\epsilon(x) = \theta, \quad \text{uniformly in } K, \tag{1.20}
\]

and \( \lim_{\epsilon \to 0} \int_{\partial \Omega} (\phi_\epsilon - \theta)^2 dS = 0 \), where \( \theta = \frac{\int_{\partial \Omega} g^2 \phi_{bd} dS}{\int_{\partial \Omega} g^2 dS} \) is a constant depending on \( g \) and \( \phi_{bd} \).

It is worth mentioning that the Pohozaev’s identity of (1.1)-(1.2) (cf. (4.12)) plays an important role in the proof of Theorem 1.2.

**Remark 2.** Theorem 1.2 presents that as (GN) holds true, the solution remains bounded for \( \epsilon > 0 \) and the solution profile becomes flat in the interior domain as \( \epsilon \) goes to zero. Moreover, as \( \epsilon \) goes to zero, the solution value uniformly converges to a constant \( \frac{\int_{\partial \Omega} g^2 \phi_{bd} dS}{\int_{\partial \Omega} g^2 dS} \) in any compact subset of \( \Omega \). Such an asymptotic behavior under the assumption (GN) is totally different from that under assumptions (A1) and (A2).
The rest of this paper is organized into five sections. In the following section we will give a proposition (see Proposition 2) which is useful for proving Theorems 1.1 and 1.2. The proofs of Theorems 1.1 and 1.2 are stated in Sections 3 and 4, respectively. In Section 3, (1.16) plays an important role on establishing a lower bound of the asymptotic blow-up solution of (1.1)-(1.2) as $\epsilon \downarrow 0$. We will use (1.16) and Proposition 2 to give the proof of Theorem 1.1. In Section 4, we shall divide it into two subsections. In the first subsection, we give the proof of Theorem 1.2(i). In the second subsection, we establish the Pohozaev’s identity and estimate the each term of this identity in the limit of $\epsilon \downarrow 0$ (cf. Lemmas 4.2 and 4.3). Under these estimates, we then give the proof of Theorem 1.2(ii). In Section 5, we state a brief discussion on the present work and introduce our next project in the near future. Finally, we will prove the convexness of the energy functional (1.14) and a stability property in $H^1$-norm for solutions of equation (1.1)-(1.2) with respect to $\phi_{bd}$ in the Appendix (Section 5).

2. Preliminaries. The following estimate plays a crucial role on the proof of Theorems 1.1 and 1.2.

**Proposition 2.** Let $\epsilon > 0$, $g \in C^\infty(\Omega)$, and $V_\epsilon$ be the unique solution of

$$
\epsilon^2 \nabla \cdot (g^2(x)\nabla V_\epsilon) = CV_\epsilon \quad \text{in } \Omega, \quad V_\epsilon = 1 \quad \text{on } \partial \Omega,
$$

where $C$ is a positive constant independent of $\epsilon$. Then $V_\epsilon \geq 0$ satisfies the following estimates:

- i) For any compact subset $K$ of $\Omega$, there exist positive constants $C(K, g)$ and $M(K, g)$ independent of $\epsilon$ such that

$$
\max_K V_\epsilon \leq C(K, g)e^{-\frac{M(K, g)}{\epsilon}}.
$$

- ii) For any $p > 0$, there exists a positive constant $\tilde{C}(\Omega, g, p)$ independent of $\epsilon$ such that

$$
\frac{1}{\epsilon} \int_{\Omega} V_\epsilon^p dx \leq \tilde{C}(\Omega, g, p).
$$

2.1. Proof of Proposition 2. Applying the maximum principle to (2.1), we get $0 \leq V_\epsilon \leq 1$ in $\Omega$. Let

$$
W_\epsilon = gV_\epsilon.
$$

One may check that (2.1) is equivalent to the following problem:

$$
\epsilon^2 \Delta W_\epsilon = \left(\frac{\epsilon^2 \Delta g}{g} + \frac{C}{g^2}\right)W_\epsilon \quad \text{in } \Omega, \\
W_\epsilon = g \quad \text{on } \partial \Omega.
$$

Note that

$$
\frac{\epsilon^2 \Delta g}{g} + \frac{C}{g^2} \geq -\frac{\epsilon^2 \max_{\Omega} |\Delta g|}{\min_{\Omega} g} + \frac{C}{\min_{\Omega} g} > \frac{C}{2 \min_{\Omega} g^2} > 0 \quad \text{as } 0 < \epsilon < \delta(g),
$$

where

$$
\delta(g) = \begin{cases} 
\infty, & \text{if } \Delta g = 0, \\
\frac{c}{\sqrt{2(\min_{\Omega} g)(\max_{\Omega} |\Delta g|)}}, & \text{if } \Delta g \neq 0.
\end{cases}
$$
Thus, by the maximum principle, we have $0 \leq W_\varepsilon \leq \max_{\overline{\Omega}} g$ on $\overline{\Omega}$ for $0 < \varepsilon < \delta(g)$. In order to estimate $W_\varepsilon$ (and $V_\varepsilon$), we let $U_\varepsilon$ satisfy

$$
e^2 \Delta U_\varepsilon = C^* U_\varepsilon \quad \text{in} \quad \Omega,$$

$$U_\varepsilon = 1 \quad \text{on} \quad \partial \Omega,$$

where $C^* = \frac{c}{2 \min_{\overline{\Omega}} g^2} > 0$ is independent of $\varepsilon$. Applying the comparison theorem to both equations (2.5)-(2.6) and (2.8)-(2.9), we immediately get $0 \leq W_\varepsilon \leq (\max_{\overline{\Omega}} g) U_\varepsilon$ on $\overline{\Omega}$. Along with (2.4), we immediately find

$$0 \leq V_\varepsilon \leq \left( \frac{\max_{\overline{\Omega}} g}{\min_{\overline{\Omega}} g} \right) U_\varepsilon \quad \text{on} \quad \overline{\Omega}.$$  

(2.10)

Hence, it is sufficient to establish the following estimate:

**Lemma 2.1.** Let $U_\varepsilon \in C^\infty(\overline{\Omega})$ be the unique solution of (2.8)-(2.9), where $C^*$ is a positive constant independent of $\varepsilon$. Then there exists $\hat{C}^* > 0$ independent of $\varepsilon$ such that

- i) For any compact subset $K$ of $\Omega$, there exist positive constants $C(K)$ and $M(K)$ independent of $\varepsilon$ such that

$$\max_K U_\varepsilon \leq C(K) e^{-\frac{M(K)}{\varepsilon}} \quad \text{for} \quad 0 < \varepsilon < \hat{C}^*.$$  

(2.11)

- ii) For any $p > 0$, there exists a positive constant $C_2(\Omega, p)$ independent of $\varepsilon$ such that

$$\frac{1}{\varepsilon} \int_{\Omega} U_\varepsilon^p dx \leq C_2(\Omega, p) \quad \text{for} \quad 0 < \varepsilon < \hat{C}^*.$$  

(2.12)

Before proving Lemma 2.1, we briefly introduce the main idea of the proof. We firstly establish the interior estimate of $U_\varepsilon$ in any compact subset $K$ of $\Omega$. Then, we shall deal with the estimate of $U_\varepsilon$ near the boundary $\partial \Omega$. Due to the compactness of $K$ in $\Omega$, one may choose finitely many open balls whose union covers $K$ and is contained in $\Omega$. We will construct a series of supersolutions on each open ball, which are radially symmetric so that we can establish the interior estimate of $U_\varepsilon$. On the other hand, due to $\partial \Omega \in C^\infty$, $\Omega$ satisfies the uniform interior ball condition. Hence, at each boundary point $x_0$ of $\partial \Omega$, we can use the same argument to estimate $U_\varepsilon$ near the boundary $\partial \Omega$ in the direction of the inward normal vector $-\nu$ at $x_0$.

Now we state the proof of Lemma 2.1 as follows.

**Proof.** When $N = 1$ and $\Omega \equiv (l_1, l_2) \subset \mathbb{R}$, it is easy to see that $0 \leq U_\varepsilon(x) \leq e^{-\frac{\sqrt{\sigma(x-1)}}{\varepsilon}} + e^{-\frac{\sqrt{\sigma(l_2-x)}}{\varepsilon}}$ for $x \in (l_1, l_2)$. Hence, Lemma 2.1 immediately follows.

Now we state the proof for $N \geq 2$. Since $K$ is a compact subset of $\Omega$, there exists finitely many open balls $B_{R_j}(x_j) \subset \Omega$ with $x_j \in K \ (j = 1, ..., m)$ such that $K \subseteq \bigcup_{j=1}^m B_{R_j}(x_j)$. We will construct a series of supersolutions $U_{\varepsilon,j}$’s of $U_\varepsilon$ on each $B_{R_j}(x_j)$. Note that $C^* > 0$. Thus the standard maximum principle implies $0 \leq U_\varepsilon \leq 1$ in $\overline{\Omega}$. For each $B_{R_j}(x_j)$, we let $U_{\varepsilon,j}$ be the unique solution of the equation

$$e^2 \Delta U_{\varepsilon,j} = C^* U_{\varepsilon,j} \quad \text{in} \quad B_{R_j}(x_j), \quad \text{and} \quad U_{\varepsilon,j} = 1 \quad \text{on} \quad \partial B_{R_j}(x_j).$$  

(2.13)

Note that $U_\varepsilon \leq U_{\varepsilon,j}$ on $\partial B_{R_j}(x_j)$. Thus by (2.1) and (2.13), we may apply the standard comparison theorem to get $U_\varepsilon \leq U_{\varepsilon,j}$ in $B_{R_j}(x_j)$.
Along with the elementary inequality
\[ \left| \frac{N-1}{r} u''_{c,j} - C_{c,j} u_{c,j} \right| = 0 \quad \text{in} \quad (0, R_j), \]  \[ u'_{c,j}(0) = 0, \quad u_{c,j}(R_j) = 1. \]  

The estimate of \( \tilde{u}_{c,j} \) is given as follows:

**Lemma 2.2.** For \( 0 < \epsilon < \frac{\pi^2}{2} \sqrt{C_{c,j}} R_j \), we have
\[ \tilde{u}_{c,j}(r) \leq \begin{cases} 2^{N-2} \sqrt{\pi} e^{-\frac{\pi}{2\epsilon} R_j} & \text{for} \quad r \in [0, R_j/2], \\ 2^{N-2} \sqrt{\pi} e^{-\frac{\pi}{2\epsilon} (R_j-r)} & \text{for} \quad r \in [R_j/2, R_j]. \end{cases} \]  

**Proof.** We state the proof of (2.16) in two cases. First, we consider the dimension \( N \geq 3 \) and define an auxiliary function
\[ g_j(r) = \left( \frac{R_j}{r} \right)^{\frac{N+1}{2}} e^{-\frac{\pi}{\epsilon} (R_j-r)} \quad \text{for} \quad r \in (0, R_j]. \]  

Then one may check that
\[ \epsilon^2 \left( g''(r) + \frac{N-1}{r} g'(r) \right) - C_{c,j} g_j = \frac{(N-1)(3-N)}{4r^2} \epsilon^2 g_j \leq 0, \]  
\[ \lim_{r \to 0} g_j(r) = \infty \quad \text{and} \quad g_j(R_j) = 1. \]  

On the other hand, by the maximum principle of (2.14)-(2.15), we have \( 0 \leq \tilde{u}_{c,j}(r) \leq 1 \). Thus by applying the standard comparison theorem to (2.14)-(2.15) and (2.17)-(2.18), we obtain
\[ \tilde{u}_{c,j}(r) \leq g_j(r) = \left( \frac{R_j}{r} \right)^{\frac{N+1}{2}} e^{-\frac{\pi}{\epsilon} (R_j-r)}, \quad \text{for} \quad r \in (0, R_j]. \]  

When \( N = 2 \), (2.14)-(2.15) is the standard Bessel equation. Hence,
\[ \tilde{u}_{c,j}(r) = \frac{I_0(\sqrt{C_{c,j}} r/\epsilon)}{I_0(\sqrt{C_{c,j}} R_j/\epsilon)}, \quad \text{for} \quad r \in (0, R_j], \]  

where \( I_0(s) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{s \cos \theta} \, d\theta \) is the modified Bessel function of zero order (cf. [9]).

In order to deal with (2.20), we need to claim:

**Claim 1.** For any \( s > 0 \), there holds
\[ e^s \sqrt{\frac{1}{2\pi s} \left( 1 - e^{-\frac{\pi^2}{4s}} \right)} < I_0(s) < e^s \sqrt{\frac{\pi}{8s} \left( 1 - e^{-4s} \right)}. \]  

**Proof of Claim 1.** Note that \( I_0(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{s \cos \theta} \, d\theta = \frac{e^s}{2\pi} \int_{-\pi}^{\pi} e^{-2s \sin^2(\theta/2)} \, d\theta \).

Along with the elementary inequality \( \frac{\theta}{\pi} \leq |\sin \frac{\theta}{2}| \leq \frac{\theta}{\pi} \) for \( \theta \in [-\pi, \pi] \), we have
\[ \frac{e^s}{2\pi} G_1(s) \leq I_0(s) \leq \frac{e^s}{2\pi} G_2(s), \]  

where \( G_1(s) = \int_{-\pi}^{\pi} e^{-2s \theta^2} \, d\theta \) and \( G_2(s) = \int_{-\pi}^{\pi} e^{-\frac{s\theta^2}{2}} \, d\theta \).
Using the polar transformation and passing through calculation directly, we find
\[ G_1^t(s) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-\frac{\pi}{2}(x^2+y^2)}dxdy > 2\pi \int_0^\pi r e^{-\frac{\pi}{8}r^2}dr = \frac{2\pi}{\sqrt{8}} \left(1 - e^{-\frac{\pi^2}{2}}\right). \] Hence,
\[ G_1(s) > \sqrt{\frac{2\pi}{\sqrt{8}}} \left(1 - e^{-\frac{\pi^2}{2}}\right). \] (2.23)

On the other hand, it is easy to see that \[ \frac{\partial}{\partial \theta} \text{ is monotonically increasing} \]
for \( \omega \). Thus,
\[ u_{\epsilon,j}^2(r) = \frac{R_j}{2r} e^{-\frac{\pi}{\sqrt{8}}(R_j-r)}, \quad \text{for } r \in (0, R_j]. \] (2.24)
as \( 0 < \epsilon < \sqrt{8}C^*R_j \).

Recall that \( K \subset \bigcup_{j=1}^m B_{R_j}(x_j) \). Let \( \tilde{R} = \min_{1 \leq j \leq m} R_j \). Then by Lemma 2.2, for
\( 0 < \epsilon < \sqrt{8}C^*\tilde{R} \), we have\[ \max_K \left|u_\epsilon\right| \leq \max_{1 \leq j \leq m} \tilde{u}_{\epsilon,j}(R_j/2) \leq 2\frac{\sqrt{\pi^2}}{2\pi} e^{-\frac{\pi}{\sqrt{8}}(R_j-R_j)}, \] (2.25)\[ \text{where } M(K) = \frac{\sqrt{\pi^2}}{2\pi} \tilde{R}. \] As a consequence, we get (2.11) and complete the proof of Lemma 2.1(i).

It remains to prove (2.12). Since \( \Omega \) is a bounded open domain with smooth boundary \( \partial \Omega \), \( \Omega \) satisfies the uniform interior ball condition. Hence, there is a \( \tilde{R} > 0 \) such that for each \( x \in \partial \Omega \), we have \( B_{\tilde{R}}(\tilde{x}) \subset \Omega \) and \( \partial B_{\tilde{R}}(\tilde{x}) \cap \partial\Omega = \{x\} \),
where \( \tilde{x} = x - \tilde{R}v_x \) and \( v_x \) is the unit outward normal vector at \( x \). Let
\[ \Gamma_1 = \left\{ x - \frac{\tilde{R}}{2}v_x : x \in \partial \Omega \right\}, \quad \Gamma_2 = \left\{ x - \tilde{R}v_x : x \in \partial \Omega \right\}, \] (2.27)\[ \text{and } \tilde{\Omega} = \left\{ y + t\tilde{v}_y : 0 < t < \frac{\tilde{R}}{2}, y \in \Gamma_1 \right\}. \] (2.28)
Note that $K = \Omega - \overline{\Omega}$ is a compact subset of $\Omega$. Then by (2.26), there exists a positive constant $C(R, K)$ independent of $\epsilon$ such that
\[
\int_{\Omega - \overline{\Omega}} |U_\epsilon|^p dx \leq \left( \frac{2^{N+2} \pi e^{-c(R, \Omega)}}{\epsilon} \right)^p,
\]
for $0 < \epsilon < \widetilde{C}^*$. Note that $B_R(z) \subset \Omega$ for $z \in \Gamma_2$. Using the same argument as in Lemma 2.2, we may find
\[
0 \leq U_\epsilon(z + t\delta z) \leq 2^{\frac{N+2}{2}} \pi e^{-\frac{2\pi}{\epsilon} (R-t)} \quad \text{for} \quad t \in [\frac{R}{2}, R],
\]
as $0 < \epsilon < \widetilde{C}^*$. As a consequence,
\[
\int_{\overline{\Omega}} |U_\epsilon|^p dx \leq \int_{\Gamma_2} \int_{\frac{R}{2}}^R [U_\epsilon(z + t\delta z)]^p dt dz \leq \frac{\left( \frac{2^{N+2} \pi}{\epsilon} \right)^p |\Gamma_2|}{p\sqrt{C^*}} \epsilon,
\]
where $|\Gamma_2|$ is the surface area of $\Gamma_2$. Therefore, by (2.29) and (2.31), we get (2.12) and complete the proof of Lemma 2.1.

By (2.10) and Lemma 2.1, we get (2.11) and (2.12) for $0 < \epsilon < \min\{\delta(g), \widetilde{C}^* \}$. As $\epsilon \geq \min\{\delta(g), \widetilde{C}^* \}$, (2.11) and (2.12) are trivial due to $0 \leq V_\epsilon \leq 1$. Therefore, we complete the proof of Proposition 2.

3. Proof of Theorem 1.1. To prove Theorem 1.1, we need the following lemmas.

Lemma 3.1. Under the same hypotheses as in Theorem 1.1, for all $\epsilon > 0$ we have

-i) If (A1) holds true, then $\phi_\epsilon$ has to attain its minimum value on the boundary $\partial \Omega$, and $\min_{\Omega} \phi_\epsilon = \min_{\partial \Omega} \phi_\epsilon \geq \min_{\partial \Omega} \phi_{bd}$.

-ii) If (A2) holds true, then $\phi_\epsilon$ has to attain its maximum value on the boundary $\partial \Omega$, and $\max_{\Omega} \phi_\epsilon = \max_{\partial \Omega} \phi_\epsilon \leq \max_{\partial \Omega} \phi_{bd}$.

Moreover, we have
\[
\sup_{\epsilon > 0} \int_{\Omega} e^{\phi_\epsilon(x)} dx \int_{\Omega} e^{-\phi_\epsilon(x)} dx \leq |\Omega|^2 \max\left\{ \frac{1}{\alpha \min_{\Omega} a \min_{\Omega} b}, \frac{\beta \max_{\Omega} a \max_{\Omega} b}{\alpha \min_{\Omega} a \min_{\Omega} b} \right\}.
\]

Proof. Note that (A1) implies
\[
\frac{\alpha \max_{x \in \Omega} a(x)}{\int_{\Omega} a(y) dy} \leq \frac{\alpha \max_{x \in \Omega} a(x)}{|\Omega| \min_{\Omega} a(x)} \leq \frac{\beta \min_{\Omega} b(x)}{|\Omega| \max_{\Omega} b(x)} \leq \frac{\beta \min_{\Omega} b(x)}{\int_{\Omega} b(y) dy},
\]
and (A2) implies
\[
\frac{\beta \max_{x \in \Omega} b(x)}{\int_{\Omega} b(y) dy} \leq \frac{\beta \max_{\Omega} b(x)}{|\Omega| \min_{\Omega} b(x)} \leq \frac{\alpha \min_{\Omega} a(x)}{|\Omega| \max_{\Omega} a(x)} \leq \frac{\alpha \min_{\Omega} a(x)}{\int_{\Omega} a(y) dy}.
\]
$\phi_\epsilon$ is a non-constant valued function trivially due to (3.2) and (3.3). Now we state the proof of Lemma 3.1(i). Assume (A1) holds true. Suppose by contradiction that
Consequently, we have

\[ \forall \text{ Case 2.} \]

If \( \phi \) shall consider two cases as follows:

When (A2) holds true, we can use the same argument to get (3.1). Therefore, we complete the proof of Lemma 3.1(ii).

It remains to prove (3.1). Assume (A1) holds true. Due to Lemma 3.1(i), we shall consider two cases as follows:

**Case 1.** If \( \phi \) attains its maximum value at an interior point \( x_M \in \Omega \), then we have

\[
0 \geq \varepsilon^2 \left( \nabla \cdot (g^2 \nabla \phi) \right)(x_M) = \frac{\alpha a(x_M)e^{\phi}(x_M)}{\int_{\Omega} a(y)e^{\phi}(y)dy} - \frac{\beta b(x_M)e^{\phi}(x_M)}{\int_{\Omega} b(y)e^{\phi}(y)dy}.
\]

This implies that for \( x \in \Omega \),

\[
e^{2\phi}(x) \leq e^{2\phi}(x_M) \leq \frac{\beta b(x_M)\int_{\Omega} a(y)e^{\phi}(y)dy}{\alpha a(x_M)\int_{\Omega} b(y)e^{\phi}(y)dy} \leq \frac{\beta \max_{\Omega} a \max_{\Omega} b \int_{\Omega} e^{\phi}(y)dy}{\alpha \min_{\Omega} a \min_{\Omega} b \int_{\Omega} e^{-\phi}(y)dy}.
\]

Integrating (3.4) over \( \Omega \) and using the Hölder’s inequality, one finds

\[
\left( \int_{\Omega} e^{\phi}(x)dx \right)^2 \leq |\Omega| \int_{\Omega} e^{2\phi}(x)dx \leq \frac{|\Omega|^2 \beta \max_{\Omega} a \max_{\Omega} b \int_{\Omega} e^{\phi}(y)dy}{\alpha \min_{\Omega} a \min_{\Omega} b \int_{\Omega} e^{-\phi}(y)dy}.
\]

Consequently,

\[
\int_{\Omega} e^{\phi}(x)dx \int_{\Omega} e^{-\phi}(x)dx \leq \frac{|\Omega|^2 \beta \max_{\Omega} a \max_{\Omega} b}{\alpha \min_{\Omega} a \min_{\Omega} b}. \tag{3.5}
\]

**Case 2.** If \( \phi \) attains its maximum value at a boundary point \( x_M \in \partial \Omega \), then we may use the same argument as Lemma 3.1(ii) to obtain \( \max_{\Omega} \phi = \max_{\partial \Omega} \phi \leq \max_{\partial \Omega} \phi_{bd} \). Along with Lemma 3.1(i), we have \( \min_{\partial \Omega} \phi_{bd} \leq \phi(x) \leq \max_{\partial \Omega} \phi_{bd} \), \( \forall x \in \Omega \), which immediately implies

\[
\int_{\Omega} e^{\phi}(x)dx \int_{\Omega} e^{-\phi}(x)dx \leq |\Omega|^2 e^{\max_{\partial \Omega} \phi_{bd} - \min_{\partial \Omega} \phi_{bd}}. \tag{3.6}
\]

Combining (3.5) and (3.6), we get (3.1). Therefore, we complete the proof of Lemma 3.1.

\[ \square \]

**Lemma 3.2.** Let \( B_\varepsilon \) be defined by (1.16). Under the same hypotheses as in Theorem 1.1, we have

- i) If (A1) holds true, then

\[
\min_{\partial \Omega} B_\varepsilon = \min_{\Omega} B_\varepsilon = \frac{A e^{\min_{\Omega} \phi}}{\int_{\Omega} e^{\phi}(y)dy} - \frac{B e^{-\min_{\Omega} \phi}}{\int_{\Omega} e^{\phi}(y)dy} < 0. \tag{3.7}
\]
On the other hand, integrating (1.16) over $\Omega$ and using (3.14), we obtain

$$
e^2 \nabla \cdot \left( g^2(x) \nabla \left( \frac{B_e}{\min_{\partial \Omega} B_e} \right) \right) \geq C(\phi_e, \nabla \phi_e) \left( \frac{B_e}{\min_{\partial \Omega} B_e} \right) \text{ in } \Omega,$$

(3.8)

and

$$\frac{B_e}{\min_{\partial \Omega} B_e} \leq 1 \quad \text{ on } \partial \Omega.$$  

(ii) If (A2) holds true, then

$$\max_{\partial \Omega} B_e = \max_{\Omega} B_e = \frac{A e^{\max_{\Omega} \phi_e}}{\int_{\Omega} e^{\phi_e(y)} dy} - \frac{B e^{-\max_{\Omega} \phi_e}}{\int_{\Omega} e^{-\phi_e(y)} dy} > 0,$$

(3.9)

and

$$\frac{B_e}{\max_{\partial \Omega} B_e} \text{ satisfies}$$

$$e^2 \nabla \cdot \left( g^2(x) \nabla \left( \frac{B_e}{\max_{\partial \Omega} B_e} \right) \right) \geq C(\phi_e, \nabla \phi_e) \left( \frac{B_e}{\max_{\partial \Omega} B_e} \right) \text{ in } \Omega,$$

(3.10)

$$\frac{B_e}{\max_{\partial \Omega} B_e} \leq 1 \quad \text{ on } \partial \Omega.$$  

Here

$$C(\phi_e, \nabla \phi_e) = \frac{2\sqrt{AB}}{|\Omega| \sqrt{\max \left\{ e^{m_{\partial \Omega} \phi_e} - \min_{\partial \Omega} \phi_e, \frac{\beta_{\max_{\Omega} A \max_{\Omega} B}}{\min_{\Omega} \eta_{\Omega}} \right\}}}$$

(3.11)

which is a positive constant independent of $\epsilon$.

Proof. Assume that (A1) holds true. Then by (1.1) and (3.2), we have $0 < A < B$ and

$$e^2 \nabla \cdot \left( g^2 \nabla \phi_e \right) (x) = \frac{\alpha}{a(x)} e^{\phi_e(x)} - \frac{\beta}{b(x)} e^{-\phi_e(x)} \leq B_e(x).$$

(3.12)

Note that

$$\nabla \cdot \left( g^2 \nabla B_e \right) = g^2 \Delta B_e + \nabla (g^2) \nabla B_e.$$

By (1.16) and (3.12), one may check that

$$e^2 \nabla \cdot \left( g^2 \nabla B_e \right) = g^2 \left[ \left( \frac{A e^{\phi_e}}{\int_{\Omega} e^{\phi_e(y)} dy} + \frac{B e^{-\phi_e}}{\int_{\Omega} e^{-\phi_e(y)} dy} \right) e^2 \Delta \phi_e + e^2 |\nabla \phi_e|^2 B_e \right]$$

$$+ \left( \frac{A e^{\phi_e}}{\int_{\Omega} e^{\phi_e(y)} dy} + \frac{B e^{-\phi_e}}{\int_{\Omega} e^{-\phi_e(y)} dy} \right) e^2 \nabla \cdot \left( g^2 \nabla \phi_e \right)$$

$$\leq \left( \frac{A e^{\phi_e}}{\int_{\Omega} e^{\phi_e(y)} dy} + \frac{B e^{-\phi_e}}{\int_{\Omega} e^{-\phi_e(y)} dy} \right) e^2 \nabla \cdot \left( g^2 \nabla \phi_e \right) + e^2 g^2 |\nabla \phi_e|^2 B_e$$

(3.13)

By Lemma 3.1(i) and the fact that $B_e(x_1) \leq B_e(x_2)$ as $\phi_e(x_1) \leq \phi_e(x_2)$ (obtained directly from (1.16)), we have

$$\min_{\partial \Omega} B_e = \min_{\Omega} B_e = \frac{A e^{\min_{\Omega} \phi_e}}{\int_{\Omega} e^{\phi_e(y)} dy} - \frac{B e^{-\min_{\Omega} \phi_e}}{\int_{\Omega} e^{-\phi_e(y)} dy}.$$  

(3.14)

On the other hand, integrating (1.16) over $\Omega$ and using (3.14), we obtain

$$\min_{\partial \Omega} B_e \leq \frac{1}{|\Omega|} \int_{\Omega} B_e(x) dx = \frac{A - B}{|\Omega|} < 0.$$  

(3.15)
Therefore, (3.7) and (3.8) immediately follow from (3.13)-(3.15). Similarly, we can prove (3.9) and (3.10).

Now we want to prove (3.11). Using the inequality of arithmetic and geometric means, we have

\[
C(\phi_\epsilon, \nabla \phi_\epsilon) \equiv \frac{A e^{\phi_\epsilon}}{\int_\Omega e^{\phi_\epsilon(y)} dy} + \frac{B e^{-\phi_\epsilon}}{\int_\Omega e^{-\phi_\epsilon(y)} dy} + \epsilon^2 |\nabla \phi_\epsilon|^2 \geq \frac{2\sqrt{AB}}{\sqrt{\int_\Omega e^{\phi_\epsilon(y)} dy} \int_\Omega e^{-\phi_\epsilon(y)} dy}.
\]

Along with (3.1), we get (3.11). Therefore, we complete the proof of Lemma 3.2.

Proof of Theorem 1.1. Note that \( B_\epsilon \) satisfies one of the structure of (3.8) and (3.10) under assumptions (A1)-(A2) (see Lemma 3.2). Thus, we shall apply the comparison theorem to give an upper bound of \( B_\epsilon \). For this motivation, we consider a linear elliptic equation

\[
\epsilon^2 \nabla \cdot \left( g^2(x) \nabla V_\epsilon \right) = \tilde{C} V_\epsilon \quad \text{in } \Omega,
\]

\[
V_\epsilon = 1 \quad \text{on } \partial \Omega,
\]

where \( \tilde{C} = \frac{2\sqrt{AB}}{|\Omega| \max \left\{ e^{\max \partial \Omega} \phi_{bd} - \min \partial \Omega \phi_{bd}, \frac{e^{\max \Omega} - e^{\min \Omega}}{2} \right\}} \). Now we give the proof of Theorem 1.1(i). Assume that (A1) holds true. Then by (3.8), (3.11) and (3.16), one may check that

\[
\epsilon^2 \nabla \cdot \left( g^2(x) \nabla \left( \frac{B_\epsilon}{\min_{\partial \Omega} B_\epsilon} - V_\epsilon \right) \right) \geq C(\phi_\epsilon, \nabla \phi_\epsilon) \left( \frac{B_\epsilon}{\min_{\partial \Omega} B_\epsilon} - \tilde{C} V_\epsilon \right) \geq C(\phi_\epsilon, \nabla \phi_\epsilon) \left( \frac{B_\epsilon}{\min_{\partial \Omega} B_\epsilon} - V_\epsilon \right) \text{ in } \Omega,
\]

and

\[
\frac{B_\epsilon}{\min_{\partial \Omega} B_\epsilon} - V_\epsilon \leq 0 \quad \text{on } \partial \Omega.
\]

Here we have used the fact \( V_\epsilon \geq 0 \) to get the second line of (3.17). Applying the standard comparison theorem to (3.17)-(3.18), we conclude

\[
\frac{B_\epsilon(x)}{\min_{\partial \Omega} B_\epsilon} \leq V_\epsilon(x), \quad \forall x \in \overline{\Omega}. \tag{3.19}
\]

Integrating (3.19) over \( \Omega \) and using (1.16) and Proposition 2 for \((\mathcal{C}, p) = (\tilde{C}, 1)\), one gets

\[
\frac{A - B}{\min_{\partial \Omega} B_\epsilon} \leq O(1) \epsilon, \tag{3.20}
\]

where \( O(1) \) denotes as a positive bounded quantity independent of \( \epsilon \). Note that (3.7) and \( 0 < A < B \). Along with (3.20), it yields

\[
\frac{O(1)(B - A)}{\epsilon} \leq - \min_{\partial \Omega} B_\epsilon < \frac{B e^{-\min \partial \Omega \phi_\epsilon}}{\int_\Omega e^{-\phi_\epsilon(y)} dy} \leq \frac{B}{|\Omega|} e^{\max \partial \Omega \phi_\epsilon - \min \partial \Omega \phi_\epsilon},
\]
which leads to
\[
\liminf_{\epsilon \downarrow 0} \left( \frac{1}{\epsilon} \right)^{-1} \left( \max \phi_\epsilon - \min \phi_\epsilon \right) \\
\geq \lim_{\epsilon \downarrow 0} \left[ 1 - \left( \frac{1}{\epsilon} \right)^{-1} \log \frac{O(1)|\Omega|(B - A)}{B} \right] = 1.
\]

Therefore, we prove (1.15).

To complete the proof of Theorem 1.1(i), it remains to prove that as \(0 < \epsilon \ll 1\), \(\phi_\epsilon\) does not attain its maximum value on the boundary \(\partial \Omega\). Suppose by contradiction that there exists a sequence \(\epsilon_j \to 0\) as \(j \to \infty\) such that the maximum value of \(\phi_{\epsilon_j}\) locates at \(x_{\epsilon_j} \in \partial \Omega\). Then we have \(\frac{\partial \phi_{\epsilon_j}}{\partial \nu}(x_{\epsilon_j}) \geq 0\). Along with the boundary condition (1.2), we obtain \(\max_{\Omega} \phi_{\epsilon_j} = \phi_{\epsilon_j}(x_{\epsilon_j}) = \phi_{bd}(x_{\epsilon_j}) - \eta_{\epsilon_j} \frac{\partial \phi_{\epsilon_j}}{\partial \nu}(x_{\epsilon_j}) \leq \max_{\partial \Omega} \phi_{bd}\). On the other hand, by Lemma 3.1(i), we have \(\min_{\Omega} \phi_{\epsilon_j} \geq \min_{\partial \Omega} \phi_{bd}\). Thus, \(\max_{\Omega} \phi_{\epsilon_j} - \min_{\Omega} \phi_{\epsilon_j} \leq \max_{\partial \Omega} \phi_{bd} - \min_{\partial \Omega} \phi_{bd}\) which is independent of \(\epsilon_j\). As a consequence, by (1.15) we find
\[
0 = \lim_{\epsilon_j \downarrow 0} \left( \frac{1}{\epsilon_j} \right)^{-1} \left( \max \phi_{bd} - \min \phi_{bd} \right) \\
\geq \limsup_{\epsilon_j \downarrow 0} \left( \frac{1}{\epsilon_j} \right)^{-1} \left( \max \phi_{\epsilon_j} - \min \phi_{\epsilon_j} \right) \geq 1,
\]
which gives a contradiction. Consequently, as \(0 < \epsilon \ll 1\), \(\phi_\epsilon\) has to attain its maximum value at an interior point of \(\Omega\). This completes the proof of Theorem 1.1(i).

When (A2) holds true, we can use the same argument to prove (1.15) and Theorem 1.1(ii). Therefore, we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.2. Under condition (GN), equation (1.1) becomes
\[
\epsilon^2 \nabla \cdot (g^2(x)\nabla \phi_\epsilon) = \alpha \frac{a(x)}{M_\epsilon} \left( e^{\phi_\epsilon} - e^{-\phi_\epsilon} \right) \quad \text{in} \quad \Omega. \tag{4.1}
\]

Following the same argument as Lemma 3.1, we immediately get the following result:

**Lemma 4.1.** Let \(\phi_\epsilon\) be the solution of equation (4.1) with the boundary condition (1.2). Then for all \(\epsilon > 0\), \(\phi_\epsilon\) attains both its minimum value and maximum value on the boundary \(\partial \Omega\), and \(\min_{\partial \Omega} \phi_{bd} \leq \phi_\epsilon(x) \leq \max_{\partial \Omega} \phi_{bd}\) for \(x \in \Omega\).

The proof of Lemma 4.1 is similar to that of Lemma 3.1, so we omit the details here.

4.1. Proof of Theorem 1.2(i). To estimate \(\phi_\epsilon\), we rewrite (4.1) and the boundary condition (1.2) as
\[
\epsilon^2 \nabla \cdot (g^2(x)\nabla \psi_\epsilon) = \frac{\alpha a(x)}{M_\epsilon} \left( e^{\psi_\epsilon} - e^{-\psi_\epsilon} \right) \quad \text{in} \quad \Omega, \tag{4.2}
\]
\[
\psi_\epsilon + \eta_\epsilon \frac{\partial \psi_\epsilon}{\partial \nu} = \phi_{bd} + \mu_\epsilon \quad \text{on} \quad \partial \Omega. \tag{4.3}
\]
where
\[ \psi_\epsilon = \phi_\epsilon + \mu_\epsilon, \quad (4.4) \]
\[ \mathcal{M}_\epsilon = \left( \int_{\Omega} a(y)e^{\phi_\epsilon(y)}dy \int_{\Omega} a(y)e^{-\phi_\epsilon(y)}dy \right)^{\frac{1}{2}}, \quad (4.5) \]
\[ \mu_\epsilon = \frac{1}{2} \log \frac{\int_{\Omega} a(y)e^{-\phi_\epsilon(y)}dy}{\int_{\Omega} a(y)e^{\phi_\epsilon(y)}dy}. \quad (4.6) \]

Multiplying (4.2) by \( \psi_\epsilon \), one may check that
\[ \epsilon^2 \nabla \cdot \left( g^2(x) \nabla \left( \psi_\epsilon^2 \right) \right) \geq 2 \epsilon^2 \nabla \cdot \left( g^2(x) \nabla \psi_\epsilon \right) \psi_\epsilon \]
\[ = \frac{2\alpha a(x)}{\mathcal{M}_\epsilon} \left( e^{\psi_\epsilon} - e^{-\psi_\epsilon} \right) \psi_\epsilon \]
\[ \geq \frac{4\alpha a(x)}{\tilde{M}_\epsilon} \psi_\epsilon^2 \geq \tilde{C} \psi_\epsilon^2 \quad \text{in} \ \Omega, \quad (4.7) \]
where \( \tilde{C} = \frac{4\alpha}{|\Omega| e^{\max_{\Omega} a}} \max_{\partial \Omega} a \max_{\Omega} a \) is a positive constant independent of \( \epsilon \). Here we have used Lemma 4.1 and (4.5)-(4.6) to assert the second line of (4.7).

On the other hand, by Lemma 4.1 and (4.4)-(4.6), there exists a positive constant \( \hat{M} \) independent of \( \epsilon \) such that
\[ \psi_\epsilon^2 \leq \hat{M} \quad \text{on} \ \partial \Omega. \quad (4.8) \]

Due to the structure of (4.7) and (4.8), we consider the following problem:
\[ \epsilon^2 \nabla \cdot \left( g^2(x) \nabla \tilde{\psi}_\epsilon \right) = \tilde{C} \tilde{\psi}_\epsilon \quad \text{in} \ \Omega, \quad \tilde{\psi}_\epsilon = \hat{M} \quad \text{on} \ \partial \Omega. \quad (4.9) \]

Applying the comparison theorem to (4.7)-(4.8) and (4.9) and using Proposition 2 for \( \mathcal{C} = \tilde{C} \), we find \( \max_K |\tilde{\psi}_\epsilon| \leq \max_K \tilde{\psi}_\epsilon \leq \tilde{M} C(K,g) e^{-\frac{M(K,g)}{\epsilon}} \) as \( 0 < \epsilon \ll 1 \), for any compact subset \( K \) of \( \Omega \). As a consequence, for \( 0 < \epsilon \ll 1 \) we have
\[ \max_{x_1, x_2 \in K} |\phi_\epsilon(x_1) - \phi_\epsilon(x_2)| \leq 2 \max_K |\psi_\epsilon| \leq 2 \tilde{M} C(K,g) e^{-\frac{M(K,g)}{\epsilon}}. \quad (4.10) \]

Therefore, by Lemma 4.1 and (4.10), we complete the proof of Theorem 1.2(i).

**Remark 3.** By (4.4), (4.7)-(4.9) and Proposition 2(ii), we have
\[ \sup_{0 < \epsilon \ll 1} \frac{1}{\epsilon} \int_{\Omega} |\phi_\epsilon + \mu_\epsilon| dx < \infty. \quad (4.11) \]

We will verify the limiting value of \( \mu_\epsilon \) as \( \epsilon \) goes to zero, under a suitable condition of \( \eta_\epsilon \) with respect to \( \epsilon \).

### 4.2. Proof of Theorem 1.2(ii)

To prove Theorem 1.2(ii), we need the following Pohozaev’s identity for solutions \( \phi_\epsilon \) of equation (4.1) with the boundary condition (1.2):
Lemma 4.2. (Pohozaev's identity) Let \( \phi_e \) be the solution of equation \((4.1)\) with the boundary condition \((1.2)\). Under the same hypotheses as in Theorem 1.2, we have

\[
e^2 \int_{\Omega} \left[ \left( \frac{N}{2} - 1 \right) g^2 + (x \cdot \nabla g) \right] |\nabla \phi_e|^2 \, dx
+ e^2 \int_{\partial \Omega} g^2 \left\{ \frac{1}{2} \left( \frac{\partial \phi_e}{\partial \nu} \right)^2 - \chi^2(\bar{\tau}) \right\} x \cdot \bar{\nu} + \left( \frac{\partial \phi_e}{\partial \nu} \chi(\bar{\tau}) \right) x \cdot \bar{\tau} \right\} \, dS
= \alpha \left( -2N + - \int_{\Omega} (x \cdot \nabla a)e^{\phi_e} \, dx + \int_{\partial \Omega} a e^{\phi_e} x \cdot \bar{\nu} dS \right.
+ \left. - \int_{\Omega} (x \cdot \nabla a)e^{-\phi_e} \, dx + \int_{\partial \Omega} a e^{-\phi_e} x \cdot \bar{\nu} dS \right),
\]

where \( \bar{\tau} \equiv \bar{\tau}(x) \) and \( \chi(\bar{\tau}) \equiv \chi(\bar{\tau}(x)) \in \mathbb{R} \) for \( x \in \partial \Omega \) are defined as follows:

(D1) If \( \nabla \phi_e \parallel \bar{\nu} \) at \( x \in \partial \Omega \), we define \( \bar{\tau} = \bar{\nu} \) and \( \chi(\bar{\tau}) = 0 \);

(D2) If \( \nabla \phi_e \parallel \bar{\nu} \) at \( x \in \partial \Omega \), then \( \bar{\tau} \) which lies on the plane determined by vectors \( \nabla \phi_e \) and \( \bar{\nu} \) is a unit tangent vector to the boundary \( \partial \Omega \) at \( x \), and \( \chi(\bar{\tau}) = \frac{\bar{\nu}}{|\nabla \phi_e|}. \)

Proof. Multiplying \((4.1)\) by \( x \cdot \nabla \phi_e \) and integrating the result over \( \Omega \), we derive an identity

\[
P_I = P_g,
\]

where

\[
P_I \equiv e^2 \int_{\Omega} \nabla \cdot \left( g^2 \nabla \phi_e \right) (x \cdot \nabla \phi_e) \, dx,
\]

\[
P_g \equiv \alpha \int_{\Omega} a \left( \int_{\partial \Omega} a(y)e^{\phi_e(y)} dy - \int_{\partial \Omega} a(y)e^{-\phi_e(y)} dy \right) x \cdot \nabla \phi_e \, dx.
\]

Now we compute the left-hand and right-hand sides of \((4.13)\), respectively. The term on the left is

\[
P_I = -e^2 \int_{\Omega} g^2 \nabla \phi_e \cdot (x \cdot \nabla \phi_e) \, dx + e^2 \int_{\partial \Omega} (g^2 \nabla \phi_e) (x \cdot \nabla \phi_e) \cdot \bar{\nu} \, dS
= -e^2 \int_{\Omega} \sum_{k=1}^{N} g^2 \partial_{x_k} \phi_e \cdot \partial_{x_k} (x \cdot \nabla \phi_e) \, dx + e^2 \int_{\partial \Omega} g^2 (x \cdot \nabla \phi_e) \frac{\partial \phi_e}{\partial \nu} \, dS
\]

\[
(P_{IV})
\]

Using integration by parts and passing through calculation directly, one may check that

\[
P_g = e^2 \int_{\Omega} g^2 \left( \sum_{k=1}^{N} x_k \partial_{x_k} \phi_e \partial_{x_k} x_k \phi_e + |\nabla \phi_e|^2 \right) \, dx
= e^2 \int_{\Omega} g^2 \left[ \frac{x}{2} \cdot \nabla (|\nabla \phi_e|^2) + |\nabla \phi_e|^2 \right] \, dx
= e^2 \int_{\Omega} \left( \frac{1}{2} \nabla \cdot (g^2 x) + g^2 \right) |\nabla \phi_e|^2 \, dx + \frac{e^2}{2} \int_{\partial \Omega} g^2 |\nabla \phi_e|^2 x \cdot \bar{\nu} \, dS
= -e^2 \int_{\Omega} \left[ \left( \frac{N}{2} - 1 \right) g^2 + (x \cdot \nabla g) g \right] |\nabla \phi_e|^2 \, dx + \frac{e^2}{2} \int_{\partial \Omega} g^2 |\nabla \phi_e|^2 x \cdot \bar{\nu} \, dS.
\]
Due consideration of the Robin boundary condition (1.2), we shall decompose $\nabla \phi_\epsilon$ into

$$\nabla \phi_\epsilon = \left( \frac{\partial \phi_\epsilon}{\partial \nu} \right) \vec{v} + \chi(\vec{r}) \vec{r} \quad \text{on} \quad \partial \Omega,$$

(4.16)

where $\chi(\vec{r})$ and $\vec{r}$ are defined by (D1) and (D2). Putting (4.16) into $P_\Sigma$ and $P_N$ and using $\vec{v} \cdot \vec{r} = 0$, we conclude

$$P_\Sigma = \epsilon^2 \int \left( \frac{N}{2} - 1 \right) g^2 + (x \cdot \nabla g)g \right) |\nabla \phi_\epsilon|^2 \, dx$$

$$- \frac{\epsilon^2}{2} \int_{\partial \Omega} g^2 \left[ \left( \frac{\partial \phi_\epsilon}{\partial \nu} \right)^2 + \chi^2(\vec{r}) \right] x \cdot \vec{v} \, dS$$

$$+ \epsilon^2 \int_{\partial \Omega} g^2 \left[ \left( \frac{\partial \phi_\epsilon}{\partial \nu} \right)^2 x \cdot \vec{v} + \left( \frac{\partial \phi_\epsilon}{\partial \nu} \chi(\vec{r}) \right) x \cdot \vec{r} \right] dS \quad \text{(4.17)}$$

$$= \epsilon^2 \int \left( \frac{N}{2} - 1 \right) g^2 + (x \cdot \nabla g)g \right) |\nabla \phi_\epsilon|^2 \, dx$$

$$+ \epsilon^2 \int_{\partial \Omega} g^2 \left[ \frac{1}{2} \left( \frac{\partial \phi_\epsilon}{\partial \nu} \right)^2 - \chi^2(\vec{r}) \right] x \cdot \vec{v} + \left( \frac{\partial \phi_\epsilon}{\partial \nu} \chi(\vec{r}) \right) x \cdot \vec{r} \right] dS.$$
Proof. Multiplying (4.1) by $\phi_\epsilon$, integrating the expression over $\Omega$ and using the integration by parts, we obtain

$$- \epsilon^2 \int_{\Omega} g^2 |\nabla \phi_\epsilon|^2 \, dx + \epsilon^2 \int_{\partial \Omega} g^2 \phi_\epsilon \frac{\partial \phi_\epsilon}{\partial \nu} \, dS$$

$$= \alpha \int_{\Omega} a \phi_\epsilon \left( \frac{e^{\phi_\epsilon}}{\int_{\Omega} a(y) e^{\phi_\epsilon(y)} \, dy} - \frac{e^{-\phi_\epsilon}}{\int_{\Omega} a(y) e^{-\phi_\epsilon(y)} \, dy} \right) \, dx. \tag{4.19}$$

For the right-hand side of (4.19), we have

Claim 2. If

$$\int_{\Omega} a \phi_\epsilon \left( \frac{e^{\phi_\epsilon}}{\int_{\Omega} a(y) e^{\phi_\epsilon(y)} \, dy} - \frac{e^{-\phi_\epsilon}}{\int_{\Omega} a(y) e^{-\phi_\epsilon(y)} \, dy} \right) \, dx \geq 0.$$  

Proof of Claim 2. By simple calculations, one may check

$$\begin{align*}
&\int_{\Omega} a \phi_\epsilon \left( \frac{e^{\phi_\epsilon}}{\int_{\Omega} a(y) e^{\phi_\epsilon(y)} \, dy} - \frac{e^{-\phi_\epsilon}}{\int_{\Omega} a(y) e^{-\phi_\epsilon(y)} \, dy} \right) \\
&= \alpha \left( t_1(\phi_\epsilon) - t_2(\phi_\epsilon) + \log \frac{\int_{\Omega} a(y) e^{\phi_\epsilon(y)} \, dy}{\int_{\Omega} a(y) e^{-\phi_\epsilon(y)} \, dy} \right) \left( e^{t_1(\phi_\epsilon)} - e^{t_2(\phi_\epsilon)} \right) \\
&\geq \alpha \left( e^{t_1(\phi_\epsilon)} - e^{t_2(\phi_\epsilon)} \right) \log \frac{\int_{\Omega} a(y) e^{\phi_\epsilon(y)} \, dy}{\int_{\Omega} a(y) e^{-\phi_\epsilon(y)} \, dy}.
\end{align*} \tag{4.20}$$

where $t_1(\phi_\epsilon) = \phi_\epsilon - \log \int_{\Omega} a(y) e^{\phi_\epsilon(y)} \, dy$ and $t_2(\phi_\epsilon) = -\phi_\epsilon - \log \int_{\Omega} a(y) e^{-\phi_\epsilon(y)} \, dy$.

Note that $\int_{\Omega} ae^{t_1(\phi_\epsilon)} \, dx = \int_{\Omega} ae^{t_2(\phi_\epsilon)} \, dx = 1$. Thus, by (4.20), we obtain

$$\int_{\Omega} a \phi_\epsilon \left( \frac{e^{\phi_\epsilon}}{\int_{\Omega} a(y) e^{\phi_\epsilon(y)} \, dy} - \frac{e^{-\phi_\epsilon}}{\int_{\Omega} a(y) e^{-\phi_\epsilon(y)} \, dy} \right) \, dx \geq \left( \log \frac{\int_{\Omega} a(y) e^{\phi_\epsilon(y)} \, dy}{\int_{\Omega} a(y) e^{-\phi_\epsilon(y)} \, dy} \right) \int_{\Omega} a \left( e^{t_1(\phi_\epsilon)} - e^{t_2(\phi_\epsilon)} \right) \, dx = 0.$$

This completes the proof of Claim 2.

By (4.19), the boundary condition (1.2) and Claim 2, we have

$$\epsilon \left( \min_\Omega g^2 \right) \int_{\Omega} |\nabla \phi_\epsilon|^2 \, dx \leq \epsilon \int_{\Omega} g^2 |\nabla \phi_\epsilon|^2 \, dx \leq \epsilon \int_{\partial \Omega} g^2 \phi_\epsilon \frac{\partial \phi_\epsilon}{\partial \nu} \, dS = \frac{\epsilon}{\eta_\epsilon} \int_{\partial \Omega} g^2 \phi_\epsilon (\phi_{bd} - \phi_\epsilon) \, dS.$$  

Since $\lim_{\epsilon \to 0} \frac{\eta_\epsilon}{\epsilon} = \infty$, from the above inequality and Lemma 4.1 we immediately get

$$\lim_{\epsilon \to 0} \epsilon \int_{\Omega} |\nabla \phi_\epsilon|^2 \, dx = 0.$$  

This proves Lemma 4.3(i).
Now we want to prove (ii). By the Cauchy-Schwarz inequality, for any $\kappa > 0$ we have
\[
\frac{1}{2} \left( \frac{\partial \phi_e}{\partial \nu} \right)^2 - \chi^2(\vec{r}) \leq \frac{1}{2} \left( \frac{\partial \phi_e}{\partial \nu} \right)^2 \chi^2(\vec{r}) x \cdot \vec{v} + \left( \frac{\partial \phi_e}{\partial \nu} \right) \chi(\vec{r}) x \cdot \vec{r}
\]
(4.21)
Here we have used $|x \cdot \vec{r}| \leq |x|$ to get the second line of (4.21). Note that $\Omega$ is a strictly star-shaped domain with respect to $0 \in \Omega$. Hence, there exists $\sigma > 0$ such that
\[
x \cdot \vec{v} \geq \sigma \quad \text{on} \quad \partial \Omega.
\]
(4.22)
Setting $\kappa = \frac{\sigma}{2 \max_{\partial \Omega} |x|}$ in (4.21) and using (1.2), we obtain
\[
\frac{1}{2} \left( \frac{\partial \phi_e}{\partial \nu} \right)^2 \leq \frac{1}{2} \left( \max_{\partial \Omega} |x| + \frac{\max_{\partial \Omega} |x|^2}{\sigma} \right) \frac{\partial \phi_e}{\partial \nu}^2 \chi(\vec{r}) x \cdot \vec{r}
\]
(4.23)
\[
\leq \frac{1}{2} \left( \max_{\partial \Omega} |x| + \frac{\max_{\partial \Omega} |x|^2}{\sigma} \right) (\phi_{bd} - \phi_e)^2.
\]
Since $\lim_{\epsilon \downarrow 0} \frac{\eta}{\epsilon} = \infty$, Lemma 4.3(ii) immediately follows from (4.23) and Lemma 4.1.

It remains to prove Lemma 4.3(iii). Fixed a point $x_0 \in \Omega$, by (4.10) we have
\[
\lim_{\epsilon \downarrow 0} \|\phi_e(x) - \phi_e(x_0)\| = 0, \quad \text{uniformly in} \quad K,
\]
(4.24)
for any compact subset $K$ of $\Omega$. Along with Lemma 4.1, we can use the Lebesgue’s dominated convergence theorem to get
\[
\lim_{\epsilon \downarrow 0} \frac{\int_{\Omega} (x \cdot \nabla a) e^{\phi_e} \, dx}{\int_{\Omega} a e^{\phi_e} \, dx} = \lim_{\epsilon \downarrow 0} \frac{\int_{\Omega} (x \cdot \nabla a) e^{\phi_e(x) - \phi_e(x_0)} \, dx}{\int_{\Omega} a e^{\phi_e(x) - \phi_e(x_0)} \, dx}
\]
(4.25)
\[
= \frac{\int_{\Omega} (x \cdot \nabla a) \, dx}{\int_{\Omega} a \, dx} = -N + \frac{\int_{\partial \Omega} ax \cdot \vec{v} \, dS}{\int_{\Omega} a \, dx}.
\]
Here we have used integration by parts in the last equality of (4.25). Similarly,
\[
\frac{\int_{\Omega} (x \cdot \nabla a) e^{-\phi_e} \, dx}{\int_{\Omega} a e^{-\phi_e} \, dx} = -N + \frac{\int_{\partial \Omega} ax \cdot \vec{v} \, dS}{\int_{\Omega} a \, dx}.
\]
(4.26)
This proves Lemma 4.3(iii).

Therefore, we complete the proof of Lemma 4.3. \qed

Now we return to prove Theorem 1.2(ii). By (4.12) and Lemma 4.3, we conclude
\[
\limsup_{\epsilon \downarrow 0} \left( \frac{\int_{\partial \Omega} ax \cdot \vec{v} \, dS}{\int_{\Omega} a e^{\phi_e} \, dx} + \frac{\int_{\Omega} a e^{-\phi_e} \, dx}{\int_{\Omega} a e^{\phi_e} \, dx} - \frac{2 \int_{\partial \Omega} ax \cdot \vec{v} \, dS}{\int_{\Omega} a \, dx} \right) \leq 0.
\]
(4.27)
On the other hand, by using the argument similar to (4.25), we have
\[
\limsup_{\epsilon \downarrow 0} \int_{\partial \Omega} \epsilon a e^{\phi_{\epsilon} (x)} x \cdot \vec{v} dS = \frac{1}{\int_{\Omega} a \, dx} \limsup_{\epsilon \downarrow 0} \int_{\partial \Omega} ae^{\phi (x)} \phi (x_0) x \cdot \vec{v} dS,
\]
(4.28)
\[
\limsup_{\epsilon \downarrow 0} \int_{\partial \Omega} \epsilon a e^{\phi (x)} x \cdot \vec{v} dS = \frac{1}{\int_{\Omega} a \, dx} \limsup_{\epsilon \downarrow 0} \int_{\partial \Omega} ae^{-\phi (x)} \phi (x_0) x \cdot \vec{v} dS.
\]
(4.29)
Note that (4.22) and \( \min_{\Omega} a > 0 \). Thus, (4.27)-(4.29) immediately implies
\[
\lim_{\epsilon \downarrow 0} \int_{\partial \Omega} \left( e^{\frac{1}{2} (\phi (x) - \phi (x_0))} - e^{-\frac{1}{2} (\phi (x) - \phi (x_0))} \right)^2 dS = 0.
\]
(4.30)
In particular, we have
\[
\lim_{\epsilon \downarrow 0} \int_{\partial \Omega} (\phi (x) - \phi (x_0))^2 dS = 0.
\]
(4.31)
Here we have used an elementary inequality \((e^a - e^{-a})^2 \geq 4a^2\) to get (4.31). On the other hand, by integrating (4.1) over \( \Omega \) and using (1.2), we obtain
\[
\int_{\partial \Omega} g^2 \phi_x dS = \int_{\partial \Omega} g^2 \phi_{bd} dS.
\]
(4.32)
Applying the Hölder’s inequality to the integral term of (4.31) and using (4.32), we then get
\[
\lim_{\epsilon \downarrow 0} \int_{\partial \Omega} g^2 (\phi_{bd} - \phi (x_0)) \epsilon dS = \lim_{\epsilon \downarrow 0} \int_{\partial \Omega} g^2 (\phi (x) - \phi (x_0)) \epsilon dS = 0.
\]
As a consequence, \( \lim_{\epsilon \downarrow 0} \phi (x_0) = \int_{\partial \Omega} g^2 \phi_{bd} dS \). Along with (4.24) and (4.31), we get (1.20) and complete the proof of Theorem 1.2(ii).

5. Conclusion remarks and further works. Motivated by works [11, 22, 24, 25, 29] on the ion transport in electrolyte solutions with constant-valued dielectric coefficient, we study the electrostatic model (1.1)-(1.2) with variable dielectric coefficients. In this paper, we consider a natural charge neutrality constraint (GN) for (1.1)-(1.2) and show the effects of the dielectric coefficient and the applied potentials at the charged surface on the behavior of the electrostatic potential (cf. Theorem 1.2(ii)). It is interesting to note that although Theorem 1.2 is a seemingly simple result, it requires some ingredients such as the comparison principle of PDE, the uniform estimate for the linear elliptic equation with small parameter \( \epsilon \) (cf. Proposition 2) and the Pohozaev’s identity (cf. Lemma 4.2) to answer this basic question.

On the other hand, in contrast to the phenomenon under the situation (GN) where the solution \( \phi \) of (1.1)-(1.2) always remains bounded for all \( \epsilon > 0 \), we investigate the asymptotic blow-up solutions under conditions (A1) and (A2). In particular, we establish a lower bound for the maximum difference between the boundary and interior values of the electrostatic potential \( \phi \) in the whole domain \( \Omega \) as \( \epsilon \) goes to zero (cf. Theorem 1.1). The method dealing with the blow-up behavior of solutions \( \phi \) in high-dimensional domains is totally different from that in one-dimensional domains (cf. [22, 23]). We would like to point out that in a high-dimensional general domain, the exact first order term of the asymptotic blow-up solution with respect to \( \epsilon \) seems a quite challenge problem which is one of our ongoing projects.
We remark that under the replacement of the operator \( \epsilon^2 \nabla \cdot (g^2(x) \nabla) \) by a strictly elliptic operator \( \epsilon^2 \left( \sum_{i,j=1}^{N} a_{ij}(x) \partial_{x_i} x_j + \sum_{i=1}^{N} b_i(x) \partial_{x_i} + c(x) \right) \) where \( a_{ij}'s, b_i's \) and \( c \) are smooth, we can use Lemma 6.1 of [13] together with the same argument of Proposition 2 to obtain the same estimates as in Proposition 2. This is a quite natural extension.

Some interesting problems also remain to be addressed on a more theoretical setting. For this, we point out an essential difference between equation (1.1) and the standard CCPB equation (1.11) from a different perspective. Let \( Q \) be a smooth function in \( \Omega \). We assume for the moment that \( a(x) \) and \( b(x) \) satisfy \( b(x) = \frac{1}{\sqrt{\epsilon(x)}} > 0 \) and \( \epsilon^2 \Delta (\log a) = Q \) in \( \Omega \). (Here we do not need to consider the boundary conditions.) Then (1.1) with \( g(x) \equiv 1 \) can be transformed into the following CCPB equation for monovalent ion species in electrolyte solutions with a permanent charge density \( Q \):

\[
\epsilon^2 \Delta \tilde{\phi}_\epsilon = \frac{\alpha e^{\tilde{\phi}_\epsilon}}{\int_{\Omega} e^{\tilde{\phi}_\epsilon(y)} dy} - \frac{\beta e^{-\tilde{\phi}_\epsilon}}{\int_{\Omega} e^{-\tilde{\phi}_\epsilon(y)} dy} + Q \quad \text{in} \quad \Omega,
\]

where \( \tilde{\phi}_\epsilon = \phi_\epsilon + \log \epsilon \). In particular, if \( N = 2, \Omega = B_1 \equiv \{ x \in \mathbb{R}^2 : |x| < 1 \}, \) and \( Q(x) \equiv Q\lambda(x) = \frac{\lambda^2}{\pi(\epsilon(x)^2 + \lambda^2)} \) (\( \lambda > 0 \) is independent of \( \epsilon \)), then as \( \lambda \) goes to zero (\( \epsilon \) is fixed), \( Q \) becomes singular at the origin and formally tends to \( \delta_0 \), where \( \delta_0 \) denotes a standard Dirac delta function at the origin. Consequently, as \( \lambda \) tends to zero, (5.1) with \( Q = Q\lambda \) may formally approach

\[
\epsilon^2 \Delta \tilde{\phi}_\epsilon = \frac{\alpha e^{\tilde{\phi}_\epsilon}}{\int_{\Omega} e^{\tilde{\phi}_\epsilon(y)} dy} - \frac{\beta e^{-\tilde{\phi}_\epsilon}}{\int_{\Omega} e^{-\tilde{\phi}_\epsilon(y)} dy} + \delta_0 \quad \text{in} \quad B_1.
\]

It is expected that solutions \( \tilde{\phi}_\epsilon \) of (5.2) and \( \phi_\epsilon \) have quite different behaviors. Here we refer the reader to [6, 10, 12, 28] for other models which are similar to (5.1) and (5.2) with \( \alpha, \beta > 0 \). The study of asymptotic behaviors (as \( \epsilon \downarrow 0 \)) for solutions \( \tilde{\phi}_\epsilon \) of (5.2) will be our project in the near future.

**Appendix.** In this section, we shall state some properties of the energy functional (1.14) as follows.

**Lemma 5.1.** Under the same hypotheses as in Proposition 1, we have

- i) \( \inf_{u \in \mathbb{H}} E[u] \) is finite.
- ii) \( E[u] \) is strictly convex on \( \mathbb{H} \), i.e., for any \( u_1, u_2 \in \mathbb{H} \) with \( u_1 \neq u_2 \) and \( t \in (0, 1) \), there holds

\[
E[tu_1 + (1-t)u_2] < tE[u_1] + (1-t)E[u_2].
\]

**Proof.** For the proof, we shall focus merely on the case of \( \eta_\epsilon > 0 \). By Jensen’s inequality, we have

\[
\log \int_{\Omega} a(x) e^{u(x)} dx \geq \log |\Omega| + \frac{1}{|\Omega|} \int_{\Omega} (\log a(x) + u(x)) dx
\]

\[
\geq \log \left( \frac{|\Omega|}{\min a} \right) - \frac{1}{|\Omega|} \int_{\Omega} |u| dx.
\]
Similarly,

\[
\log \int_\Omega b(x)e^{-u(x)}dx \geq \log \left( \frac{\Omega |\min b}{\pi} \right) - \frac{1}{|\Omega|} \int_\Omega |u|dx. \tag{5.5}
\]

By (1.14), (5.4) and (5.5), we obtain

\[
E[u] \geq \frac{1}{2} \frac{\epsilon^2}{\eta_7} \int_{\Omega} |\nabla u|^2 dx - \frac{\alpha + \beta}{|\Omega|} \int_{\Omega} |u|^2 dx + \frac{\epsilon^2}{2\eta_7} \int_{\partial \Omega} (u - \phi_{bd})^2 dS
\]

\[
+ \alpha \log \left( \frac{\Omega |\min a}{\pi} \right) + \beta \log \left( \frac{\Omega |\min b}{\pi} \right)
\]

\[
\geq C \min \left\{ \frac{1}{2} \frac{\epsilon^2}{\eta_7} g^2, \frac{\epsilon^2}{4\eta_7} \right\} \int_{\Omega} u^2 dx - \frac{\alpha + \beta}{2\gamma c} \left( \frac{|\Omega|}{2\gamma c} + \frac{\gamma c}{2} \int_{\Omega} u^2 dx \right)
\]

\[
- \frac{\epsilon^2}{2\eta_7} \int_{\partial \Omega} \phi_{bd}^2 dS + \alpha \log \left( \frac{\Omega |\min a}{\pi} \right) + \beta \log \left( \frac{\Omega |\min b}{\pi} \right). \tag{5.6}
\]

Here we have used the Friedrichs' inequality (cf. Theorem 6.1 of [20])

\[
C \int_{\Omega} u^2 dx \leq \int_{\Omega} |\nabla u|^2 dx + \int_{\partial \Omega} u^2 dS \tag{5.7}
\]

and \(|u| \leq \frac{1}{\epsilon_7} + \frac{\gamma c}{2\eta_7} u^2 \) for \( \gamma c > 0 \) (by the Cauchy-Schwarz inequality) to get (5.6), where \( C \) is a positive constant depending only on the space dimension \( N \) and the measures of \( \Omega \) and \( \partial \Omega \). Setting \( \frac{\alpha + \beta}{|\Omega|} = C \min \left\{ \frac{1}{2} \frac{\epsilon^2}{\eta_7} g^2, \frac{\epsilon^2}{4\eta_7} \right\} \), we immediately obtain \( \inf_{\in H} E[u] > -\infty \) and complete the proof of Lemma 5.1 (i).

By (1.14), one may check that

\[
E[tu_1 + (1 - t)u_2] - (tE[u_1] + (1 - t)E[u_2])
\]

\[
= -t(1 - t) \left( \int_{\Omega} |\nabla (u_1 - u_2)|^2 dx + \int_{\partial \Omega} (u_1 - u_2)^2 dS \right) \tag{5.8}
\]

\[
+ \log \left( \frac{\int_{\Omega} a(x)e^{tu_1 + (1 - t)u_2} dx}{\left( \int_{\Omega} a(x)e^{u_1} dx \right)^t \left( \int_{\Omega} a(x)e^{u_2} dx \right)^{1-t}} \right)
\]

\[
+ \log \left( \frac{\int_{\Omega} b(x)e^{-tu_1 - (1 - t)u_2} dx}{\left( \int_{\Omega} b(x)e^{-u_1} dx \right)^t \left( \int_{\Omega} b(x)e^{-u_2} dx \right)^{1-t}} \right).
\]

On the other hand, by using the H"older's inequality, we get

\[
\int_{\Omega} a(x)e^{tu_1 + (1 - t)u_2} dx \leq \left( \int_{\Omega} a(x)e^{u_1} dx \right)^t \left( \int_{\Omega} a(x)e^{u_2} dx \right)^{1-t}, \tag{5.9}
\]

\[
\int_{\Omega} b(x)e^{-tu_1 - (1 - t)u_2} dx \leq \left( \int_{\Omega} b(x)e^{-u_1} dx \right)^t \left( \int_{\Omega} b(x)e^{-u_2} dx \right)^{1-t}, \tag{5.10}
\]

for \( t \in (0, 1) \). In particular, each equality of (5.9) and (5.10) holds if and only if \( u_1 - u_2 = \) constant almost everywhere in \( \Omega \). As a consequence, by (5.7) with \( u = u_1 - u_2 \) and (5.8)-(5.10), we get (5.3) for \( t \in (0, 1) \) and \( u_1 \neq u_2 \) in \( \Omega \). Therefore, we complete the proof of Lemma 5.1. \( \square \)

Due to Lemma 5.1, we can follow the same arguments of Theorem 1.1 of [21] to prove Proposition 1. Finally, when \( \eta_7 > 0 \), a stability property in \( H^1 \)-norm for solutions of equation (1.1)-(1.2) with respect to \( \phi_{bd} \) is stated as follows:
Lemma 5.2. Let $\epsilon > 0$ and $\eta_\epsilon > 0$ be fixed, and let $\phi_j^\epsilon \in C^\infty(\Omega)$ be the unique solution of equation (1.1)-(1.2) corresponding to $\phi_{bd} = \phi_j^0$, $j = 1, 2$. Then for each $\rho > 0$, there is a $\varsigma > 0$ depending on $\eta_\epsilon > 0$, $\rho$ and $\Omega$ such that $\|\phi_1^\epsilon - \phi_2^\epsilon\|_{L^2(\partial\Omega)} < \varsigma$ implies $\|\phi_1 - \phi_2\|_{H^1(\Omega)} < \rho$.

Lemma 5.2 can be easily obtained by the same arguments of Theorem 1.2 of [21], so we omit the proof here.

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E-mail address: chlee@mail.nhcue.edu.tw