Short Communication

A note on “Seller's optimal credit period and cycle time in a supply chain for deteriorating items with maximum lifetime”

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A B S T R A C T

In 2014, Wang et al. (2014) extended the model of Lou and Wang (2012) to incorporate the credit period dependent demand and default risk for deteriorating items with maximum lifetime. However, the rates of demand, default risk and deterioration in the model of Wang et al. (2014) are assumed to be specific functions of credit period which limits the contributions. In this note, we first generalize the theoretical results of Wang et al. (2014) under some certain conditions. Furthermore, we also present some structural results instead of a numerical analysis on variation of optimal replenishment and trade credit strategies with respect to key parameters.

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1. Introduction

Trade credit is an increasingly important payment behavior in real business transactions. Since the trade credit management needs to balance the trade-off between increased sales and the risk of granting credit, Teng and Lou (2012) first incorporated the credit-linked demand and default risks to establish an EOQ model for the retailer in a supply chain with up-stream and down-stream trade credits. In an article published in Journal of the Operational Research Society, Lou and Wang (2012) extended the model of Teng and Lou (2012) by introducing the credit-linked demand and default risks. Later, Wang, Teng, and Lou (2014) extended the model of Lou and Wang (2012) for deterioration with a maximum lifetime. Based on the model of Teng and Lou (2012), Wu, Ouyang, Cárdenas-Barrón, and Goyal (2014) subsequently built an EOQ model for deteriorating items with expiration dates under two-level trade credit financing. However, the rates of demand and default risk in the above-mentioned studies are assumed to be specific functions of the credit period. In this short note, we first generalize the theoretical results of Lou and Wang (2012) and Wang et al. (2014) under some certain conditions. Next, a sensitivity analysis with respect to key parameters are further performed in rigorous discussions.

2. The model

For easy tractability with Wang et al. (2014), we use the same notations and assumptions. Further, in order to generalize the models of Lou and Wang (2012) and Wang et al. (2014), we make assumptions on the rates of demand and default risk. Let \( n \) be the credit period offered by the retailer to its customers, the demand is given by a demand function \( D(n) \), which is assumed to be nonnegative, continuous and twice differentiable. Because the trade credit allows customers to enjoy the benefits of delayed payments, lengthening the credit period will stimulate sales. The longer the credit period, the higher the demand; hence, the demand strictly increases in the credit period, that is \( D(n) > 0 \). Meanwhile, since a longer credit period may tie up the retailer’s capital and then increase the rate of default, the rate of default risk giving the credit period \( n \) is assumed to be specific function of credit period, that is \( F(n) > 0 \). Meanwhile, since a longer credit period may tie up the retailer’s capital and then increase the rate of default, the rate of default risk giving the credit period \( n \) is assumed to be specific function of credit period, that is \( F(n) > 0 \). Hence, the total profit per unit time constructed by Wang et al. (2014) can be reviewed as follows:

\[
II(n, T) = pD(n)[1 - F(n)] - \frac{cD(n)}{T} \int_0^T e^{\delta(t)} dt - \frac{hD(n)}{T} \int_0^T \int_t^T e^{(\theta - \theta(t))} d\theta dt,
\]

(1)

where \( \delta(t) = \int_0^t \theta(u) du \) and \( \theta(t) \) is the time-varying deterioration rate at time \( t \) with \( 0 \leq \theta(t) \leq 1 \). Then the problem can be formulated as

\[
\max_{0 \leq T \leq \min(n, T)} II(n, T).
\]

Because \( p[1 - F(n)] - c \) represents the unit gross revenue after the default risk for the retailer and is strictly decreasing in \( n \), a unique \( n \) exists, denoted by \( n^* \), such that \( p[1 - F(n^*)] - c = 0 \). Furthermore, if \( n > n^* \), we can observe that

\[
\frac{dK(n, T)}{dT} = D(n)(p[1 - F(n)] - c) - D(n)F(n) < 0,
\]

which represents the gross sale revenue after the default risk decreases strictly as \( n \) increases. By using the result, we can then obtain that \( \frac{dK(n, T)}{dT} < 0 \) for \( n > n^* \), and

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so \( n \) is the upper bound for the optimal value \( n^* \). Hence, we can exclude the region \( n > n \) from further consideration, and the problem can be then rewritten as \( \max_{n \in [0,n]} I(n,T) \).

We now state our first structural result which indicates that a unique optimal credit period exists such that \( I(n,T) \) is maximized under certain possible conditions as the following theorem.

**Theorem 1.** For a given \( T \), if \( D(n) \) and \( 1-F(n) \) are both log-concave in \( n \), and suppose that \( \frac{dD(n)}{dn} - \frac{dF(n)}{dn} > 0 \), then a unique \( n \in [0,n] \) exists such that \( I(n,T) \) is maximized.

**Proof.** If \( D(n) \) and \( 1-F(n) \) are both log-concave in \( n \), we have \( D'(n)D(n) - \left[ D'(n) \right]^2 \leq 0 \) and \( F'(n) > \frac{-[F(n)]^2}{T} \), respectively. Furthermore, \( \frac{dD(n)}{dn} - \frac{dF(n)}{dn} > 0 \) implies that \( D'(n)|1-F(n)| - D(n)F'(n) > 0 \). Assuming there is an interior solution, the first order condition is

\[
\frac{dI(n,T)}{dn} = p\left\{ D(n)|1-F(n)| - D(n)F'(n) \right\} - \frac{D'(n)}{T} \left\{ c_0 T e^{D(n)T} dt + \int_0^T \int_1^T e^{D(n)T} du dt \right\} = 0. \tag{2}
\]

Substituting these results into (2) into the second order derivative, it follows that

\[
\frac{d^2I(n,T)}{dn^2} = p\left[ 1 - F(n) \right] D'(n) - 2pD'(n)F'(n) + pD(n)F'(n) \leq p\left[ 1 - F(n) \right] D'(n) - 2pD'(n)F'(n) + pD(n)F'(n)
\]

\[
= \frac{pD(n)}{1-F(n)} \left\{ D(n)F'(n) - D'(n)|1-F(n)| \right\} = \frac{pD(n)}{1-F(n)} \left\{ D(n)\frac{d}{dn}\left[ D(n)\right] \right\}
\]

\[
< 0.
\]

According to Theorems 3.4.7 in Cambini and Martein (2009), since \( I(n,T) \) is strictly concave and convex at any interior critical point, \( I(n,T) \) is strictly pseudoconcave in \( n \).

Since a strictly pseudoconcave function can be either unimodal or monotone, we discuss the two situations separately below. If \( \frac{dD(n)}{dn}|_{n=0} > 0 \), then \( \frac{dI(n,T)}{dn}|_{n=0} < 0 \), \( I(n,T) \) is unimodal in \( n \). Therefore, the solution to (2) is the unique interior maximum.

Conversely, if \( \frac{dD(n)}{dn}|_{n=0} < 0 \), then \( \frac{dI(n,T)}{dn}|_{n=0} < 0 \), \( I(n,T) \) is strictly monotonic decreasing in \( n \), and it is straightforward to see that \( n^* = 0 \). Combining the two arguments stated above, we can conclude that there exists a unique global maximum for \( I(n,T) \). This completes the proof of Theorem 1.

**Remark 1.** If \( \frac{dD(n)}{dn}|_{n=0} < 0 \), it is obvious to see from (2) that \( \frac{dI(n,T)}{dn}|_{n=0} < 0 \) for a given \( T \). Therefore, \( I(n,T) \) is maximized at \( n^* = 0 \).

The intuition of the condition \( \frac{dD(n)}{dn}|_{n=0} > 0 \) in Theorem 1 is simple. Once \( \frac{dD(n)}{dn}|_{n=0} > 0 \), it represents that the marginal revenue after the default risk is less than or equal to zero. It is no benefit for the retailer to offer the customer a credit period. And thus, the retailer should shorten its credit period as much as possible.

We next state our second structural result as the following theorem, which proves that for arbitrary deterioration rate, the optimal replenishment strategy is unique.

**Theorem 2.** For a given \( n \), a unique \( T \in (0,m) \) exists such that \( I(n,T) \) is maximized.

**Proof.** Assuming there is an interior solution, the first order condition is

\[
\frac{dI(n,T)}{dT} = \frac{cD(n)}{T} \int_0^T \left[ e^{D(n)T} - e^{D(T)} \right] \frac{dT}{T} - hD(n) \left\{ \int_0^T e^{D(T)-D(n)T} \frac{dT}{T} - \int_0^T e^{D(n)T} \frac{dT}{T} \right\} = 0. \tag{3}
\]

Substituting (3) into the second order derivative and simplifying yields

\[
\frac{d^2I(n,T)}{dT^2} = - \frac{D(n)}{T} \left\{ e^{D(n)T}h(T) + h(T) \int_0^T e^{D(T)-D(n)T} \frac{dT}{T} \right\}.
\]

Since \( h(T) > 0 \) for all \( T \in (0,m) \), we then have \( \frac{d^2I(n,T)}{dT^2} < 0 \). Because it is strictly concave at any interior critical point, \( I(n,T) \) is strictly pseudoconcave in \( T \).

Furthermore, since a strictly pseudoconcave function can be either unimodal or monotone, we discuss the two situations separately below. By L’Hospital’s rule, we have

\[
\lim_{T \to 0} \frac{\int_0^T e^{D(n)T} \frac{dT}{T^2}}{2} = \lim_{T \to 0} \frac{\int_0^T \int_0^T e^{D(n)T} \frac{dT}{T^2}}{2} = \frac{1}{2} \theta(0)
\]

and

\[
\lim_{T \to 0} \int_0^T e^{D(n)T} \frac{dT}{T} - \int_0^T e^{D(n)T} \frac{dT}{T} = \lim_{T \to 0} \frac{1}{2} \int_0^T \int_0^T e^{D(n)T} \frac{dT}{T} = \frac{1}{2}.
\]

Combining these result yields

\[
\lim_{T \to 0} \frac{dI(n,T)}{dT} = \lim_{T \to 0} \frac{cD(n)h(0)}{2} = \frac{hD(n)}{2} = \infty.
\]

If \( \frac{dI(n,T)}{dT}|_{T=m} > 0 \), then \( \lim_{T \to m} \frac{dI(n,T)}{dT} > 0 \), \( I(n,T) \) is unimodal in \( T \). Therefore, the solution to (2) is the unique interior maximum. Conversely, if \( \frac{dD(n)}{dn}|_{n=0} > 0 \), then \( \lim_{T \to m} \frac{dI(n,T)}{dT} > 0 \), \( I(n,T) \) is strictly monotonic increasing in \( T \), and it is straightforward to see that \( T = m \). Combining the two arguments stated above, we can conclude that there exists a unique global maximum for \( I(n,T) \). This completes the proof of Theorem 2. □

From the analysis carried out so far, it follows that, if \( \frac{dD(n)}{dn}|_{n=0} < 0 \), we can obtain from Remark 1 that \( \max_{n \in [0,n]} I(n,T) = \max_{n \in [0,n]} I(0,T) \). Hence, the uniqueness of the optimal replenishment time and trade credit strategies is guaranteed by Theorem 2. On the other hand, if \( \frac{dD(n)}{dn}|_{n=0} > 0 \), since the optimal value of \( T \) can be uniquely determined for a given \( n \), (3) implicitly defines \( T \) as a function of \( n \). Let \( I''(n,T(n)) \) denote the maximum value function of \( I(n,T) \) for a given \( n \), then the problem becomes \( \max_{n \in [0,n]} I(n,T(n)) \). By Berge’s Theorem of the Maximum, since the maximum value function \( I''(n,T(n)) \) is continuous in the closed and bounded interval \( [0,n] \), the existence of a \( n \in [0,n] \) which maximizes \( I''(n,T(n)) \) is guaranteed by Weierstrass’ theorem. In addition, if \( \frac{dD(n)}{dn}|_{n=0} > 0 \) and \( T''(0,T(0)) > 0 \), because \( I''(n,T(n)) \) is convex, there exists at least one \( n \in (0,n) \) such that \( \frac{dD(n)}{dn}|_{n=0} = 0 \). However, since we cannot show the concavity property of \( I''(n,T(n)) \), one can solve \( I''(n,T(n)) \) many times from distinct starting values, and take the highest total profit obtained as the global maximum to avoid saddle points or local maximum.
3. Optimal replenishment time and trade credit strategies

Although we know the optimal replenishment time and trade credit strategies exist from the above discussions, we do not have any further information about the system behavior. In this section we study how the optimal replenishment time and trade credit strategies vary with respect to the key parameters by using the property of supermodular function. In the following, we first introduce the definition of supermodular function.

Definition 1. A function \( F : X \rightarrow R \) is supermodular if
\[
F(x \vee y) + F(x \wedge y) \geq F(x) + F(y), \forall x, y \in A,
\]
where
\[
x \vee y = \max\{x_1, x_2, \ldots, x_n\},
\]
and
\[
x \wedge y = \min\{x_1, x_2, \ldots, x_n\}.
\]
If \( F \) is a smooth function, \( F \) is supermodular if and only if \( \frac{\partial^2 F}{\partial x \partial y} \geq 0 \). Furthermore, if the inequality is strict, \( F \) is strictly supermodular.

We now state our third structural result as the following proposition.

Proposition 1. Suppose that the conditions in Theorem 1 are met, then:

1. The optimal value of \( T \) decreases as \( n \) increases.
2. The optimal value of \( n \) increases, but the optimal value of \( T \) decreases as \( p \) increases.
3. The optimal value of \( n \) decreases, but the optimal value of \( T \) increases as \( o \) increases.
4. For a given \( T \), the optimal value of \( n \) decreases as \( c \) or \( h \) increases. Likewise, for a given \( n \), the optimal value of \( T \) decreases as \( c \) or \( h \) increases.

Proof. (1) Taking the mixed derivative of \( II(n, T) \) with respect to \( n \) and \( T \) we get
\[
\frac{\partial^2 II(n, T)}{\partial n \partial T} = \frac{cD(n) \int_0^T [e^{(T-n)}(T-n)] dt}{T^2} + hD(n) \left\{ \int_0^T e^{(T-n)}(T-n) du - T \int_0^T e^{(T-n)}(T-n) du \right\}.
\]
Since \( D(n) > 0 \) and \( \frac{\partial^2 F}{\partial x \partial y} = \theta(z) > 0 \) for all \( z \in (0, m) \), it is straightforward to see that \( \int_0^T e^{(T-n)}(T-n) du < T \int_0^T e^{(T-n)}(T-n) du \), which implies that \( \frac{\partial^2 II(n, T)}{\partial n \partial T} < 0 \). According to Lemma 1 in Amir et al. (2005), \( \frac{\partial^2 II(n, T)}{\partial n \partial T} < 0 \) implies that \( II(n, T) \) is supermodular in \((n, -T)\). Hence the optimal value of \( T \) is a decreasing function of \( n \).

(2) Taking the mixed derivative of \( II(n, T) \) with respect to \( n \) and \( p \) gives \( \frac{\partial^2 II(n, T)}{\partial n \partial p} = 0 \) and \( \frac{\partial^2 II(n, T)}{\partial n \partial T} = D'(n) [1 - F(n)] - D(n)F(n) > 0 \). Combining these with the fact that \( \frac{\partial^2 II(n, T)}{\partial n \partial T} < 0 \), we obtain that \( II(n, T) \) is supermodular in \((n, -T, p)\). Therefore, the optimal value of \( n \) increases, but the optimal value of \( T \) decreases as \( p \) increases.

(3) Taking the mixed derivative of \( II(n, T) \) with respect to \( n \) and \( o \) yields \( \frac{\partial^2 II(n, T)}{\partial n \partial o} = \frac{1}{T} > 0 \) and \( \frac{\partial^2 II(n, T)}{\partial n \partial o} = 0 \). Combining these with the fact that \( \frac{\partial^2 II(n, T)}{\partial n \partial o} < 0 \), we then obtain that \( II(n, T) \) is supermodular in \((-n, -T, o)\). Hence the optimal value of \( n \) decreases, but the optimal value of \( T \) increases as \( o \) increases.

(4) Taking the mixed derivative of \( II(n, T) \) with respect to \( n \) and \( T \) and \( c \) and \( h \) yields
\[
\frac{\partial^2 II(n, T)}{\partial n \partial c} = \frac{D(n) \int_0^T e^{(T-n)} dt - T e^{(T-n)}}{T^2} < 0,
\]
\[
\frac{\partial^2 II(n, T)}{\partial n \partial h} = -\frac{D(n) \int_0^T e^{(T-n)} dt}{T} < 0,
\]
\[
\frac{\partial^2 II(n, T)}{\partial T \partial c} = -\frac{D(n) \int_0^T e^{(T-n)} dt - T \int_0^T e^{(T-n)} dt}{T^2} < 0,
\]
and
\[
\frac{\partial^2 II(n, T)}{\partial T \partial h} = -\frac{D(n) \int_0^T e^{(T-n)} dt}{T} < 0,
\]
respectively. Using the definition of supermodular function, we obtain that \( II(n, T) \) is supermodular in \((-n, c, h) \) and \((-T, c, h)\), respectively. Therefore, for a given \( T \), the optimal value of \( n \) decreases as \( c \) or \( h \) increases. Likewise, for a given \( n \), the optimal value of \( T \) decreases as \( c \) or \( h \) increases. \( \square \)

Finally, the rates of default risk and deterioration defined in the model of Wang et al. (2014) are used to study the impacts of default risk and deterioration on the optimal replenishment time and trade credit strategies. We then state the structural results on the variation of optimal replenishment time and trade credit strategies with respect to \( b \) and \( m \) as the following proposition.

Proposition 2. If \( D(n) = ke^{bm} F(n) = 1 - e^{-bn} \) and \( \theta(t) = \frac{1}{1 + m + t} \) with \( a > b \), then

1. The optimal value of \( n \) decreases, but \( T \) increases as \( b \) increases.
2. For a given \( T \), the optimal value of \( n \) increases as \( m \) increases. Likewise, for a given \( n \), the optimal value of \( T \) increases as \( m \) increases.

Proof. (1) If \( a > b \), \( \frac{d\ln[D(n)]}{dt} = (a-b)Ke^{(a-b)n} > 0 \). Because \( \frac{d^2 \ln[D(n)]}{dt^2} = 0 \) and \( \frac{d\ln[D(n)]}{dt} = 0 \), \( D(n) \) and \( 1 - F(n) \) are both log-concave in \( n \). Therefore, based on Proposition 1.1, we can conclude that \( \frac{\partial^2 II(n, T)}{\partial n \partial T} < 0 \). Further, since \( \frac{\partial^2 II(n, T)}{\partial n \partial m} = 0 \), \( II(n, T) \) is supermodular in \((-n, T, b)\) and \( \theta(t) \) implies that optimal value of \( n \) decreases, but \( T \) increases as \( b \) increases.

(2) Taking the mixed derivative of \( II(n, T) \) with respect to \( n \) and \( m \) yields
\[
\frac{\partial^2 II(n, T)}{\partial n \partial m} = \frac{ak^e}{4T} \left\{ 4c \left[ \frac{T}{1 + m + T} \ln \left( 1 - \frac{T}{1 + m + T} \right) \right] + h \left[ \frac{2(2 + 2m - T)T}{1 + m - T} + 4(1 + m) \ln \left( 1 - \frac{T}{1 + m + T} \right) \right] \right\},
\]
and
\[
\frac{\partial^2 II(n, T)}{\partial T \partial m} = \frac{ak^e}{4T^2} \left\{ -4c \left[ \frac{(1 + m - 2T)(T)}{(1 + m - T)^2} + \ln \left( 1 - \frac{T}{1 + m} \right) \right] + 2h(1 + m) \left[ \frac{T(2 - 2m + 3T)}{(1 + m + T)^2} - 2 \ln \left( 1 - \frac{T}{1 + m} \right) \right] \right\}.
\]
For convenience, we let
\[
V_1 = 4c \left[ \frac{T}{1 + m - T} + \ln \left( 1 - \frac{T}{1 + m} \right) \right] \\
+ \frac{2(2 + 2m - T)T}{1 + m - T} + 4(1 + m) \ln \left( 1 - \frac{T}{1 + m} \right)
\]
and
\[
V_2 = -4c \left[ \frac{(1 + m - 2T)T}{1 + m - T} + \ln \left( 1 - \frac{T}{1 + m} \right) \right] \\
+ 2h(1 + m) \left[ \frac{T(-2 - 2m + 3T)}{(1 + m - T)^2} - 2 \ln \left( 1 - \frac{T}{1 + m} \right) \right].
\]

For all \( T \in (0, m) \), since \( \frac{\partial V_1}{\partial T} < 0 \) and \( \lim_{T \to 0} V_1 = 0 \), it is straightforward to see that \( V_1 > 0 \), which also implies that \( \frac{\partial^2 V_1}{\partial T^2} > 0 \). Using similar arguments, we can show that \( \frac{\partial V_2}{\partial T} > 0 \) and \( \lim_{T \to 0} V_2 = 0 \); thus, it is obvious to see that \( V_2 > 0 \). Therefore, we have \( \frac{\partial^2 V_2}{\partial T^2} > 0 \). Using the definition of supermodular function, we can easily obtain that \( II(n, T) \) is supermodular in \((n, m)\) and \((T, m)\), respectively. Therefore, for a given \( T \), the optimal value of \( n \) increases as \( m \) increases. Likewise, for a given \( n \), the optimal value of \( T \) increases as \( m \) increases. □

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