Bit Error Rate Optimized Time-Domain Equalizers for DMT Systems

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Abstract—Time-domain equalizers (TEQs) have been applied extensively to shorten the channel impulse response, thus enhancing the transmission efficiency of multitone and multicarrier systems with cyclic prefix. Recent developments on TEQs have mainly been focused on minimizing the additive noise power, minimizing the ISI power or maximizing the total throughput. This paper takes a different approach by minimizing the detection error or bit error rate (BER) using the Chernoff bound. Under a fixed ISI power of the equalized channel, simulation results showed that this approach achieves better performance in terms of the bit error rate when compared to the aforementioned approaches and is robust to channel noise and channel length. Both numerical and analytical solutions have been derived.

I. INTRODUCTION

High speed communications system using Orthogonal Frequency Division Multiplexing (OFDM) has been suffering from intersymbol interference (ISI), intercarrier interference (ICI), magnitude and phase distortion. Various techniques have been developed to improve the performance of OFDM communications, in particular cyclic prefix (CP) has been used extensively to reduce the effects of ISI [1].

The CP is obtained by taking the last $p$ samples from the length $N$ block of OFDM symbols, and it is appended at the start of the symbol block. As a result, the transmitted OFDM symbol block is of length $N+p$. Furthermore, the linear convolution of the transmitted signal with the channel impulse response (CIR) is converted into a circular one. For each OFDM symbol to be independent and to avoid any ISI or ICI, the length $p$ of the CP should be at least one sample longer than that of the CIR. Hence the distortion caused by the CIR only affects the samples within the CP. Therefore the receiver can truncate the received signal sequence by discarding the CP, and retains the last $N$ samples for ISI-free decoding. The effects of the CIR on the remaining $N$ samples can be easily equalized by an array of one-tap Frequency Domain Equalizers (FEQ) following the demodulation by FFT.

Although CP improved the robustness of OFDM, it reduced the transmission efficiency by a factor of $N/(N + p)$. One way to increase the efficiency is to increase the FFT size $N$. However, this increases the complexity of the system and reduces the intercarrier spacing of the subcarriers which subsequently makes the system more susceptible to frequency offset and oscillator phase noise. Also a higher number of subcarriers will increase the Peak to Average Power Ratio (PAPR), demanding the use of linear and consequently inefficient power amplifiers. The alternative is to use a Time Domain Equalizer (TEQ) preceding the FFT demodulator at the receiver in order to constrain the length of the Effective Channel Impulse Response (EIR) to be shorter than the selected CP duration. This permits the use of a much shorter CP than could otherwise be employed and thus raises the transmission efficiency.

The TEQ is an “imperfect” equalizer compared to the zero-forcing equalizer. The TEQ has been very popular because of the following reasons: 1. CIR is not required to be minimum phase [2], 2. Robustness towards channel noise amplification, and 3. Simple to implement. The problem of designing TEQ is to find the filter coefficients $h(n)$ of the TEQ such that the transfer function resulted from the convolution of the TEQ and the CIR approximates a desired response in a specified manner. As such, the “TEQ design problem” is basically a mathematical approximation problem and can be approached strictly from a mathematician’s point of view. The optimality condition in which the approximation problem is solved determines how and where the TEQ can be used. A flurry of TEQ algorithms have been proposed with different optimal conditions. In [3], the TEQ was designed to minimize a mean-squared error (MSE) function, such that the total power of the impulse response obtained from the convolution between the TEQ and CIR outside the windowing region (residual) is minimized. The effect of channel noise in the channel shortening performance is considered in [4][5], where they proposed to maximize the shortened SNR (MSSNR).

Furthermore, to avoid trivial solutions, additional constraints are enforced in most of the TEQ design problems, which includes the Unit Energy Constraint (UEC) and Unit Tap Constraint (UTC) imposed on the Target Impulse Response (TIR) [9]. The design criteria considered in [3][4][5] emphasized on reducing the power of the residuals of the EIR without any regard for the communication efficiency of the resulting TEQ system. The communication efficiency can be optimized by maximising the bit rate of the TEQ system as considered in [6][7][8][9] where the design criteria GSNR and MBR were proposed. The communication efficiency can also be optimized by optimizing the SNR of the TEQ system which is equivalent to the minimization of the ISI and additive noise power as shown in [10] where EIGFILT was proposed. It is the purpose of this paper to propose a novel TEQ design that optimizes the communication efficiency of the TEQ system by considering the bit error rate of the detected symbols.

The new TEQ design criterion aims at optimizing the communication efficiency of the underlying system by minimizing the bit error rate (BER) of the system while the CIR is shortened. To simplify our discussion, a Maximum Likelihood Sequence Estimation (MLSE) receiver based on Viterbi algorithm is considered, such that a closed form of the BER is readily available in terms of the $Q$-function. However, an exact computation of the BER using the $Q$-function may not be feasible. As a result, we considered a tight bound of the BER as an alternative. In particular, the Chernoff bound of the probability of error, $P_e$, is considered in this paper. The design problem is therefore formulated as an ISI constrained BER optimization problem. The ISI constraint will guarantee the channel shortening performance of the designed TEQ. The rest of the design freedom is used to optimize the BER of the resulting system. Such a design criteria makes sense because both the ISI and additive channel noise are important factors that affect the BER of the overall system. As will be shown in Section II, the trade-off between ISI and additive noise power can be imposed on the design problem as design constraint. The rest of the paper is organized as follows. The expression for the constrained optimization problem will be derived in Section III followed by the results and
discussions in Section IV. The paper is concluded in Section V.

II. ISI Constrained BER Optimal TEQ

We propose a new optimal criterion for designing TEQ. The new criterion aims at optimizing the communication efficiency of the underlying system by considering the BER of the detected symbols as the benchmark for our design. Ideally, the BER is mitigated when both the channel noise and ISI are minimized. ISI can be eliminated by zero-forcing equalizer where the equalization filter is an exact inverse of the CIR. However, such an approach will lead to channel noise amplification if there are zeros close to the unit circle in the CIR.

Different unconstrained optimization approaches [6][7][8][9][10] have been used to jointly optimize the channel noise and ISI. However, this approach will result in suboptimal reduction for each of the interferences which adversely affects the BER performance or the bit rate. This can be explained by Figure 1 which is created by varying the tradeoff factor $\alpha$ of the TEQ output from 0 to 0.9 in the EIGFILT TEQ design method, where $\alpha$ determines the relative importance of the channel noise power and ISI power of the TEQ equalized channel. The figure shows the relationship between channel noise power and ISI power. On one end, the channel noise power is maximum, while the ISI power is maximum on the other. Point A indicates the channel noise power and ISI power of the TEQ equalized signal when the TEQ is designed by joint optimization with $\alpha = 0.5$ in EIGFILT TEQ. As shown by $A$, even though the ISI can be eliminated by the CP, a relatively large amount of channel noise still exists compared to point $B$. Due to the presence of the CP, it is more advantageous for the TEQ to be designed to operate around point $B$, i.e. focusing on minimizing the channel noise, given that the CIR can be shortened to a certain degree as not to significantly impact the communication efficiency. This cannot be achieved simply by setting $\alpha$ to a value closer to 1 because the EIGFILT design inherently involves solving an unconstrained optimization problem and it limits how much design freedom one has in reducing the channel noise. Instead, a constrained optimization approach is needed.

We propose to fix the ISI power in our TEQ design problem. As a result, the TEQ power will be a design constraint in our optimization problem. Having such a constraint provides us with the design freedom we need to construct an equalizer that mainly targets at reducing the channel noise while the EIR is short enough to use a shorter CP to eliminate the ISI. Simulation results have shown that the OFDM system using the proposed TEQ achieves a low BER which outperforms that obtained by MBR [9] and EIGFILT [10]. The simulation results also demonstrated the robustness of the proposed TEQ design method by considering different channel parameters, such as the channel SNRs, CIR length, and EIR length.

III. Methodology

The ISI is caused by the heavy tail of the CIR. The purpose of the TEQ is to minimize the ISI in the time domain by shortening the channel, i.e. minimizing the heavy tail. As a result, the design criterion of the TEQ problem should be formulated as a minimization problem of the power of the tail, i.e. the power of the residual of the time domain equalized channel. Equivalently, the effectiveness of the TEQ can be measured by the power kept in the EIR after the TEQ has been applied on the CIR. Define

$$\mathbf{c} \equiv [c(0) \ c(1) \ \cdots \ c(L_c - 1)],$$

$$\mathbf{h} \equiv [h(0) \ h(1) \ \cdots \ h(L_c - 1)],$$

$$\mathbf{c}^\prime = \begin{bmatrix}
0 & c(0) & c(1) & \cdots & c(L_c - 1) & 0 & \cdots & 0 \\
0 & c(0) & c(1) & \cdots & c(L_c - 1) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & c(0) & c(1) & \cdots & c(L_c - 1) \\
\end{bmatrix},$$

where $c \in \mathbb{R}_{1 \times L_c}$, $h \in \mathbb{R}_{1 \times L_c}$ and $C \in \mathbb{R}_{L_c \times M}$ are the original CIR with length $L_c$, TEQ with length $L_c$, and the channel convolution matrix with $M = L_c + L_c - 1$ being the length of the convolution result, respectively. The EIR can be represented by vector $\mathbf{c}^\prime$ given by

$$\mathbf{c}^\prime = \mathbf{hC}.$$

The desired channel response, $\mathbf{C}_{des}$, can be obtained by truncating $\mathbf{c}^\prime$ with a window of size $L_d$.

$$\mathbf{C}_{des} = \mathbf{hC} \mathbf{W}_\Delta,$$

where the window $\mathbf{W}_\Delta$ is given by

$$\mathbf{W}_\Delta \equiv \begin{bmatrix}
0_\Delta & 0 & 0 \\
0 & I_{L_d} & 0 \\
0 & 0 & 0_{M-L_d-\Delta}
\end{bmatrix}.$$

The variable $\Delta$ is the delay of the TEQ equalized channel. In the following discussion, we assume that a priori information is given to determine $\Delta$, e.g. such information can be determined by the system delay tolerance of the receiver, such that $\Delta$ is considered to be a system parameter, instead of an optimization parameter for TEQ system performance enhancement. Similarly, the residual of the TEQ is given by

$$\mathbf{C}_{res} = \mathbf{hC} \mathbf{W}_\Delta,$$

where the window $\mathbf{W}_\Delta$ is defined as

$$\mathbf{W}_\Delta \equiv \mathbf{I}_M - \mathbf{W}_\Delta.$$

The probability of error obtained by communicating through the EIR with a given modulation method is given by [11].

$$P_{e,\text{eff}} = \frac{1}{N} \sum_{k=0}^{N-1} Q(\sqrt{m \cdot SNIR_k}),$$

(1)
where $k$ is the subchannel index, and $N$ is the total number of subchannels. $m$ is the modulation constant and $Q$ is defined as the area underneath a standardized Gaussian tail and is related to the complementary error function \( \text{erfc}(x) \) by \( Q(x) = \frac{1}{2} \text{erfc} \left( \frac{x}{\sqrt{2}} \right) \). For simplicity, the modulation constant, $m$, is assumed to be 1 in the following discussions. The signal to noise interference ratio of the $k$-th subchannel, SNIR$_k$, is given by [10]

$$
\text{SNIR}_k = \frac{\sigma^2_k |C_{\text{des}}(e^{j\omega_k})|^2}{\sigma^2_k |C_{\text{des}}(e^{j\omega_k})|^2 + S_{qq}(e^{j\omega_k})}, \quad \omega_k = \frac{2\pi k}{N_{\text{DFT}}},
$$

(2)

where $C_{\text{des}}(e^{j\omega_k})$, $C_{\text{res}}(e^{j\omega_k})$, and $S_{qq}(e^{j\omega_k})$ are the Fourier Transform of $c_{\text{des},k}(n)$, $c_{\text{res},k}(n)$, and $s_{qq,k}(n)$ for each subchannel. In particular, $S_{qq}(e^{j\omega_k})$ is the spectrum density of the additive noise $\eta_k(n)$ after it passes through the equalizer $h(n)$. Noted that the $Q$-function in (1) is bounded by the Chernoff bound,

$$
Q(\sqrt{\text{SNIR}_k}) \leq \exp \left( - \frac{\text{SNIR}_k}{2} \right).
$$

(3)

Since the Chernoff bound for each subchannel is less than the Chernoff bound of the entire channel, the index $k$ can be ignored. Noted that the Fourier Transform of $c_{\text{des},k}(n)$, the desired response of the entire channel, in vector form is given by $C_{\text{des}} = hC W_M$, where $W_M$ is the $M \times M$ DFT matrix. As a result,

$$
|C_{\text{des}}(e^{j\omega})|^2 = hC W_M W_M^\dagger C W_M^\dagger,
$$

(4)

where we make use of the fact that $W_M W_M^\dagger = W$. Similarly,

$$
|C_{\text{res}}(e^{j\omega})|^2 = M hC W M^\dagger C^\dagger h^\dagger.
$$

(5)

According to (7), $S_{qq}(e^{j\omega})$ can be assumed to be constant within each subchannel $k$, especially when the FFT size increases. In that case, $S_{qq}(e^{j\omega})$ equals

$$
S_{qq}(e^{j\omega}) = R_{qq}(0) = h R_{qq} h^\dagger.
$$

(6)

Combining (2), (4), (5), and (6), the SNIR for the entire channel is

$$
\text{SNIR} = \frac{\sigma^2 hC W M^\dagger C h^\dagger}{\sigma^2 M hC W M^\dagger C h^\dagger + h R_{qq} h^\dagger}.
$$

(7)

As a result, an equivalent formulation to minimize the probability of detection error, $P_t$, is to minimize the Chernoff bound expression obtained by putting (7) into (3). The TEQ design problem is therefore formulated as a constrained optimization

$$
\min_h \exp \left( - \frac{\sigma^2 hC W M^\dagger C h^\dagger}{2(\sigma^2 M hC W M^\dagger C h^\dagger + h R_{qq} h^\dagger)} \right)
$$

(8)

s.t. $hC W M^\dagger C h^\dagger = \mu \sigma \eta + \mu E_e$,

where $E_e$ is the energy of the original channel, and $\mu$ is a constant that controls the percentage of the original channel energy we wish to preserve in the EIR. This, in effect, decides how much of the original channel will be kept after its tail is windowed out since most of the energy, determined by $\mu$, will be concentrated within the windowed portion of the original channel. Substituting the constraint into the objective function, and making use of the fact that $W_M^\dagger \equiv 1_M - W$, we get

$$
\min_h \exp \left( - \frac{\sigma^2 \mu ME_e}{2[-\sigma^2 \mu ME_e + \sigma^2 h(M C C^\dagger + \sigma^2 R_{qq}) h^\dagger]} \right)
$$

(9)

s.t. $hC W M^\dagger C h^\dagger = \mu E_e$.

The numerator and the first term of the denominator in the above expression do not depend on $h$. As a result, they can be eliminated from the formulation without affecting the optimization results. Also notice that minimizing (9) is equivalent to maximize the argument inside $\exp$ without the minus sign. Thus, the simplified optimization formulation is given as

$$
\min_h h P h^\dagger \text{ s.t. } h Q h^\dagger = \mu E_e, \quad (10)
$$

where $P \equiv M C C^\dagger + \sigma^2 R_{qq}$ and $Q \equiv C W M^\dagger C h^\dagger$. $P$ and $Q$ are both Hermitian and positive definite matrices. Since both the objective and constraint in (10) are quadratic, therefore, it is impossible to obtain a close form solution for (10). Nonlinear optimization techniques such as quasi-Newton can be used to solve (10).

The application of nonlinear optimization to (10) is straightforward and easy to program. The simulations results presented in Section IV makes use of the constrained nonlinear optimization function $\text{fmincon}$ in MATLAB to optimize (10), and good results are obtained.

### IV. RESULTS

This section shows the simulation results of the symbol error rate (SER) obtained with the TEQ being used in the receiver of a Digital Subscriber Line (DSL) system, which is a DMT based communication system. The DSL models are obtained from the DMT TEQ Toolbox developed by [13]. We will first describe the design parameters used for the 3 TEQs that we are comparing, namely, our constrained Chernoff design, EIGFILT [10] and MBR [9], at different additive noise levels. Robustness of our design against different channel lengths will also be shown. The constrained nonlinear optimization function $\text{fmincon}$ from MATLAB was used for the proposed design method with nonlinear optimization.

#### A. Design Parameters

Table I provides the design parameters used for our simulation. For simplicity, the min-ISI design from [9] was used instead of the real MBR. According to [9], virtually identical results are obtained by both algorithms. We shall follow the same naming convention as we have done throughout and called it MBR.

<table>
<thead>
<tr>
<th>TABLE I</th>
</tr>
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<tbody>
<tr>
<td><strong>TEQ DESIGN PARAMETERS</strong></td>
</tr>
<tr>
<td><strong>Chernoff</strong></td>
</tr>
<tr>
<td>Input signal power, $\sigma_e^2$ (dBm)</td>
</tr>
<tr>
<td>AWGN power, $\sigma_n^2$ (dBm)</td>
</tr>
<tr>
<td>Equalizer’s length, $L_e$ (samples)</td>
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<tr>
<td>Desired EIR length, $L_q$ (samples)</td>
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<tr>
<td>Delay, $\Delta$ (samples)</td>
</tr>
<tr>
<td>Carrier Service Loop (CSA), $L_c$ = 512</td>
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<tr>
<td>DFT size</td>
</tr>
<tr>
<td>Sampling frequency, $f_s$ (MHz)</td>
</tr>
<tr>
<td>$\mu$</td>
</tr>
<tr>
<td>$\alpha$</td>
</tr>
</tbody>
</table>

During the design, both the effective channel, $c_{\text{eff}} = h C$, and the original channel, $c$, were normalized by their corresponding maximum value before being windowed by either $W$ or $W'$, e.g. $c_{\text{norm}} = c / \max(c)$. In other words, $c_{\text{eff}} = (hC)_{\text{norm}} W$ and $E_e = c_{\text{norm}}^\dagger c_{\text{norm}}^\dagger$.

According to our constraint in (10), the amount of original channel energy inside the TEQ window in our Chernoff design is strictly controlled by $\mu$. If $\mu = 0.1$, then it contains 10% of the CIR energy. If $\mu = 0.9$, then it contains 90% of the energy. The EIGFILT have shown to contain 10.67% $\sim$ 10.72% of the original channel energy inside the TEQ window as $\alpha$ varies from 1.0 $\sim$ 0.1, where the channel energy inside the TEQ equalized channel increases as $\alpha$ decreases.
The MBR design contains 47.95% of the original channel energy inside the TEQ window.

B. Simulation and results

A randomly generated 16-QAM signal and a CP length of 48 samples was used for all the simulations. An one-tap FEQ was used for each subcarrier at the receiver after removal of the CP to compensate for any magnitude and phase distortion caused by the TEQ equalized channel. Figure 2 shows the normalized impulse response of the Chernoff TEQ and the CIR of CSA 1 obtained by nonlinear optimization technique.

1) Comparison with EIGFILT and MBR: Figure 3 and 4 show the SER result for the 3 different TEQs on CSA loop 1 under low SNR (1-10 dB) and high SNR (10 dB - 40 dB) cases, respectively. Table II contains the naming convention used during the simulation.

From the two figures, they showed that the heqchern1o TEQ consistently outperform the other designs. This is expected because the heqchern1o TEQ only constrained 10% of the ISI energy, therefore, it has more design freedom to counteract the effects of channel noise. This supports our argument in Section II. Comparing the heqchern1o TEQ to the closest EIGFILT TEQ result, they have a difference in the SER in the range of 0.0016 to 0.01 as the SNR increases from 1 dB to 40 dB. The difference in the SER between the Chernoff and MBR design seems to remain relatively constant at about 0.11 as the SNR increases from 1 dB to 40 dB.

<table>
<thead>
<tr>
<th>Name</th>
<th>Type</th>
<th>$\mu$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>heqchern1o</td>
<td>Chernoff(nl opt)</td>
<td>0.1</td>
<td>-</td>
</tr>
<tr>
<td>heqchern9o</td>
<td>Chernoff(nl opt)</td>
<td>0.9</td>
<td>-</td>
</tr>
<tr>
<td>heqeig1-16</td>
<td>EIGFILT</td>
<td>-</td>
<td>0.1</td>
</tr>
<tr>
<td>heqeig5-16</td>
<td>EIGFILT</td>
<td>-</td>
<td>0.5</td>
</tr>
<tr>
<td>heqeig10-16</td>
<td>EIGFILT</td>
<td>-</td>
<td>1.0</td>
</tr>
<tr>
<td>heqmbr24</td>
<td>MBR</td>
<td>-</td>
<td>-</td>
</tr>
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2) Robustness Against Different Channel Lengths: Here we show the robustness of our design by comparing the SER for different channel length using CSA loop 1 and setting $\mu = 0.1$. The shortened channels were created by multiplying the original channel response with a Hamming window of size 384, 256, 128, and 64 samples; with the window’s peak centered at about the peak of the CIR. Figure 5 shows the result of the SER with various length, while other parameters remain the same.
V. CONCLUSION

We have designed an ISI constrained BER optimal TEQ which minimized the detection error when it is used in conjunction with the cyclic prefix and a zero-forcing equalizer. Using the detection error as a benchmark, our design was shown to outperform both the EIGFILT and MBR TEQs. It was also shown to be robust against different channel lengths.

This kind of performance gain was achieved by constraining the amount of ISI power in the TEQ equalized channel. The rest of the design freedom is used to construct the TEQ that minimizes the additive noise, which in fact is the major contributor of the detection error. Since the ISI can be minimized or eliminated by cyclic prefix, and the additive noise cannot, we argued that it is reasonable to place more emphasis on mitigating the latter. However, the performance gain will level off as more ISI power is allowed at the output of the TEQ. The Chernoff bound of the error probability density function was then used as our optimization objective. This resulted in an optimization problem involving a quadratic objective and quadratic constraint that requires using nonlinear optimization, where good results are obtained.

REFERENCES


