BOUNDING THE SIZE OF A VERTEX-STABILISER IN A
FINITE VERTEX-TRANSITIVE GRAPH

CHERYL E. PRÄGER, PABLO SPIGA, AND GABRIEL VERRET

Abstract. In this paper we discuss a method for bounding the size of the stabiliser of a vertex in a \(G\)-vertex-transitive graph \(\Gamma\). In the main result the group \(G\) is quasiprimitive or biquasiprimitive on the vertices of \(\Gamma\), and we obtain a genuine reduction to the case where \(G\) is a nonabelian simple group.

Using normal quotient techniques developed by the first author, the main theorem applies to general \(G\)-vertex-transitive graphs which are \(G\)-locally primitive (respectively, \(G\)-locally quasiprimitive), that is, the stabiliser \(G_\alpha\) of a vertex \(\alpha\) acts primitively (respectively quasiprimitively) on the set of vertices adjacent to \(\alpha\). We discuss how our results may be used to investigate conjectures by Richard Weiss (in 1978) and the first author (in 1998) that the order of \(G_\alpha\) is bounded above by some function depending only on the valency of \(\Gamma\), when \(\Gamma\) is \(G\)-locally primitive or \(G\)-locally quasiprimitive, respectively.

To Richard Weiss for the inspiration of his beautiful conjecture

1. Introduction

In this paper we study the family \(A(d)\) defined as follows (where a graph \(\Gamma\) is \(G\)-vertex-transitive if \(G\) is a subgroup of \(\operatorname{Aut}(\Gamma)\) acting transitively on the vertex set \(V_\Gamma\) of \(\Gamma\)).

**Definition 1.** Let \(d\) be a positive integer. The family \(A(d)\) consists of the ordered pairs \((\Gamma, G)\), with \(\Gamma\) a connected \(G\)-vertex-transitive graph of valency at most \(d\).

We study the order of the stabilisers \(G_\alpha\) for pairs \((\Gamma, G) \in A(d)\) and \(\alpha \in V_\Gamma\), focusing on the case where \(G\) is quasiprimitive or biquasiprimitive in its action on \(V_\Gamma\). A permutation group \(G\) is said to be quasiprimitive if every non-identity normal subgroup of \(G\) is transitive, and biquasiprimitive if \(G\) is not quasiprimitive and every non-trivial normal subgroup of \(G\) has at most two orbits.

We briefly explain the context: the family \(A(d)\) is closed under forming normal quotients in the sense that, for \((\Gamma, G) \in A(d)\), and a normal subgroup \(N \leq G\) with at least three orbits in \(V_\Gamma\), the pair \((\Gamma_N, G_N) \in A(d)\), where \(\Gamma_N\) is the \(G\)-normal quotient of \(\Gamma\) modulo \(N\) (see Definition \ref{def:quotient}) and \(G_N\) is the group induced by \(G\) on the set of \(N\)-orbits. We regard pairs \((\Gamma, G)\) which admit no proper normal quotient reduction to smaller graphs as 'irreducible'. Thus the irreducible pairs in \(A(d)\), are those for which \(G\) is quasiprimitive or biquasiprimitive on \(V_\Gamma\).

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Address correspondence to P. Spiga, E-mail: spiga@maths.uwa.edu.au
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Before stating our main results we give two definitions.

**Definition 2.** If $G$ is a quasiprimitive or a biquasiprimitive permutation group, then the socle $\text{Soc}(G)$ of $G$ is isomorphic to a direct product $T^l$ of isomorphic simple groups (where $T$ is possibly abelian). We call $T$ the socle factor of $G$.

**Definition 3.** Let $f : \mathbb{N} \to \mathbb{N}$. Then $(\Gamma, G) \in \mathcal{A}(d)$ is said to be $f$-bounded if $|G_\alpha| \leq f(d)$ for every $\alpha \in \mathcal{V}$. Define $\hat{f}, \tilde{f} : \mathbb{N} \to \mathbb{N}$ by

\[
\hat{f}(d) = (df(d)^{df(d)^2}!)!
\]

where $d'$ is the unique element of $\mathbb{N}$ such that $(d' - 1)(d' - 2) < d \leq d'(d' - 1)$. For functions $g_1, g_2 : \mathbb{N} \to \mathbb{N}$ and setting $d_0 = d(d - 1)$, define $\tilde{g_1} \ast \tilde{g_2} : \mathbb{N} \to \mathbb{N}$ by

\[
\tilde{g_1} \ast \tilde{g_2}(d) = (d_0(g_1(d_0)g_2(d_0)))^{d_0}(\min\{g_1(d_0).g_2(d_0)\})^{d_0}!
\]

Theorems 4 and 5 are our main results for quasiprimitive groups and biquasiprimitive groups, respectively.

**Theorem 4.** Let $(\Gamma, G) \in \mathcal{A}(d)$ where $G$ is quasiprimitive on $\mathcal{V} T$ with socle factor $T$. Then either

1: $(\Gamma, G)$ is $(dd!)!$-bounded, or
2: the pair $(\Gamma, G)$ uniquely determines a pair $(\Lambda, T) \in \mathcal{A}(d)$, and if $(\Lambda, T)$ is $g$-bounded for some $g : \mathbb{N} \to \mathbb{N}$, then $(\Gamma, G)$ is $\tilde{g}$-bounded. Conversely, if $(\Gamma, G)$ is $f$-bounded for some $f : \mathbb{N} \to \mathbb{N}$, then also $(\Lambda, T)$ is $f$-bounded.

**Theorem 5.** Let $(\Gamma, G) \in \mathcal{A}(d)$ where $G$ is biquasiprimitive on $\mathcal{V} T$ with socle factor $T$. Then either

1: $(\Gamma, G)$ is $(d^2((d(d - 1))!)^2)!$-bounded, or
2: $(\Gamma, G)$ uniquely determines two (possibly isomorphic) pairs $(\Lambda_i, T) \in \mathcal{A}(d(d - 1))$, for $i = 1, 2$, and if $(\Lambda_i, T)$ is $g_i$-bounded for $i = 1, 2$, then $(\Gamma, G)$ is $\tilde{g_1} \ast \tilde{g_2}$-bounded. Conversely, if $(\Gamma, G)$ is $f$-bounded for some $f : \mathbb{N} \to \mathbb{N}$, then each of the $(\Lambda_i, T)$ is $f$-bounded.

The class of quasiprimitive permutation groups may be described (see [11]) in a fashion very similar to the description given by he O’Nan-Scott Theorem for primitive permutation groups. In [13] this description is refined and eight types of quasiprimitive groups are defined, namely HA, HS, HC, SD, CD, TW, PA and AS, such that every quasiprimitive group belongs to exactly one of these types. In proving Theorem 4 we show, in Corollaries 15 and 16 that there exists a function $h : \mathbb{N} \to \mathbb{N}$ such that, if $(\Gamma, G)$ is in $\mathcal{A}(d)$, with $G$ quasiprimitive on vertices of type HA, HS, HC, SD, CD or TW, then $(\Gamma, G)$ is $h$-bounded. Furthermore, if $G$ is of type PA with socle $T^l$, then $l$ is bounded above by a function of $d$ and of the size of the vertex-stabiliser of a $T$-vertex-transitive graph $\Lambda$ uniquely determined by $(\Gamma, G)$. Therefore Theorem 4 reduces the problem of bounding (as a function of $d$) the size of the vertex-stabiliser $G_\alpha$ to the case of nonabelian simple groups $G$. Similarly, the class of biquasiprimitive groups is described in detail in [15] and Theorem 5 applies to this more complicated family.

The novelty of these (to us, remarkable) results lies in the fact that Theorems 4 and 5 do not require any assumption on the local action, that is, on the action of $G_\alpha$ on the set $\Gamma(\alpha)$ of vertices adjacent to $\alpha$. In particular, $G_\alpha$ is not assumed to be transitive on $\Gamma(\alpha)$ in Theorems 4 and 5.
In the remainder of this introductory section we discuss how Theorems 4 and 5 can be used to study general $G$-vertex-transitive graphs. Also we show that Theorems 4 and 5 are relevant for some open problems in algebraic graph theory.

1.1. Application of Theorems 4 and 5 to studying general $G$-vertex-transitive graphs. Although Theorems 4 and 5 are stated for quasiprimitive and biquasiprimitive groups, using normal quotient techniques they can be fruitfully applied in more general situations.

**Definition 6.** Let $(\Gamma, G)$ be in $\mathcal{A}(d)$, and let $N$ be a normal subgroup of $G$ with $N$ intransitive on $V$. Let $\alpha^N$ denote the $N$-orbit containing $\alpha \in V$. Then the normal quotient $\Gamma_N$ is the graph whose vertices are the $N$-orbits on $V$, with an edge between distinct vertices $\alpha^N$ and $\beta^N$ if and only if there is an edge of $\Gamma$ between $\alpha'$ and $\beta'$, for some $\alpha' \in \alpha^N$ and some $\beta' \in \beta^N$. The normal quotient is non-trivial if $N \neq 1$.

Note that the group $G$ induces a transitive action on the vertices of the normal quotient $\Gamma_N$. Also, for adjacent $\alpha^N, \beta^N$ of $\Gamma_N$, each vertex of $\alpha^N$ is adjacent to at least one vertex of $\beta^N$ (since $N$ is transitive on both sets) and so, if $d$ is the valency of $\Gamma$ and $d'$ is the valency of $\Gamma_N$, then $d' \leq d$. Thus $(\Gamma, G) \in \mathcal{A}(d)$ implies that $(\Gamma_N, G_N) \in \mathcal{A}(d)$, where $G_N$ is the permutation group induced by $G$ on the set of $N$-orbits.

Following Wielandt [27], for a subgroup $H$ of a permutation group $G$, the 1-closure of $H$ in $G$ is the largest subgroup of $G$ with the same orbits as $H$. The subgroup $H$ is 1-closed in $G$ if $H$ equals its 1-closure in $G$. Note that if $N$ is a normal subgroup of $G$, then the 1-closure of $N$ in $G$ is normal in $G$ and equals the kernel of the action of $G$ on the $N$-orbits. Thus the induced group $G_N$ is the quotient of $G$ modulo the 1-closure of $N$ in $G$. In particular, if $N$ is a 1-closed normal subgroup of $G$, then $G_N = G/N$.

Let $(\Gamma, G) \in \mathcal{A}(d)$ and let $N$ be a normal subgroup which is 1-closed in $G$, and is maximal subject to having more than two orbits on $V$. By Definition 6, the pair $(\Gamma_N, G_N)$ lies in $\mathcal{A}(d)$ and the group $G/N$ is quasiprimitive or biquasiprimitive on $V/\Gamma_N$. Hence Theorems 4 and 5 apply to $(\Gamma_N, G/N)$. Now the stabiliser of the vertex $\alpha^N$ of $\Gamma_N$ is $G_{\alpha}N/N \cong G_{\alpha}/N_{\alpha}$ and therefore Theorems 4 and 5 provide information about bounds on $|G_{\alpha}/N_{\alpha}|$. In general, without further restrictions on $(\Gamma, G)$, it is difficult to obtain useful information about $N_{\alpha}$. In general, $|N_{\alpha}|$ is not bounded by a function of $d$ (see Example 39 which gives a family of examples in $\mathcal{A}(d)$ for which $|N_{\alpha}|$ grows exponentially with $d$). Nevertheless, there are some remarkable families of $G$-vertex-transitive graphs where fairly weak conditions on the local action lead to an upper bound on $|N_{\alpha}|$ as a function of $d$. We discuss in detail some of these families in the rest of this subsection.

For a property $\mathcal{P}$ of a group action, a pair $(\Gamma, G) \in \mathcal{A}(d)$ is said to be locally $\mathcal{P}$ if the permutation group $G_{\alpha}^{\Gamma(\alpha)}$ induced by $G_{\alpha}$ on $\Gamma(\alpha)$ has the property $\mathcal{P}$. We will consider four properties $\mathcal{P}$, the first three of which are the properties of being 2-transitive, primitive or quasiprimitive. The fourth property is semiprimitivity: a finite permutation group $L$ is semiprimitive if every normal subgroup of $L$ is either transitive or semiregular [11, 6, 9].

The following proposition was proved in [10] Lemmas 1.1, 1.4(p), 1.5 and 1.6 (see the summary in [14] Theorem 4.1), or [12] for a more general treatment). The
boundedness assertion in the last sentence follows since, as noted above, the vertex stabilisers for \((\Gamma, G)\) and \((\Gamma_N, G/N)\) are isomorphic.

**Proposition 7.** Let \(\mathcal{P}\) be one of the properties: 2-transitive, primitive or quasiprimitive. Let \((\Gamma, G) \in A(d)\) be locally \(\mathcal{P}\), and let \(N\) be a normal subgroup which is 1-closed in \(G\), and maximal subject to having more than two orbits on \(V\). Then \((\Gamma_N, G/N) \in A(d)\) is locally \(\mathcal{P}\), \(G/N\) is quasiprimitive or biquasiprimitive on \(V\), and \(N\alpha = 1\) for every \(\alpha \in V\). In particular, for any function \(f : \mathbb{N} \to \mathbb{N}\), \((\Gamma, G)\) is \(f\)-bounded if and only if \((\Gamma_N, G/N)\) is \(f\)-bounded.

Proposition 7 shows that, for pairs \((\Gamma, G)\) which are locally 2-transitive, locally primitive, or locally quasiprimitive, normal quotient reduction together with Theorems 4 and 5 can be used to obtain useful information about the vertex-stabiliser \(G\).

1.2. The Weiss and Praeger conjectures. In 1973, Gardiner [5] proved that, if \(\Gamma\) is a connected \(G\)-vertex-transitive locally primitive graph and \((u, v)\) is an arc of \(\Gamma\), then the pointwise stabiliser in \(G\) of \(u, v\), and all vertices adjacent to either \(u\) or \(v\), is a \(p\)-group, for some prime \(p\). A series of papers (see [20, 21, 22, 23] for example) followed in which various additional constraints on the local action led to upper bounds on the order of a vertex-stabiliser. This eventually led Richard Weiss [24] to conjecture in 1978 that local primitivity should imply boundedness. In our terminology his conjecture is the following.

**Weiss Conjecture.** There exists a function \(f : \mathbb{N} \to \mathbb{N}\) such that, if \((\Gamma, G) \in A(d)\) is locally primitive, then \((\Gamma, G)\) is \(f\)-bounded.

In 1998, the first author [14, Problem 7] conjectured that local primitivity can be weakened to local quasiprimitivity.

**Praeger Conjecture.** There exists a function \(f : \mathbb{N} \to \mathbb{N}\) such that, if \((\Gamma, G) \in A(d)\) is locally quasiprimitive, then \((\Gamma, G)\) is \(f\)-bounded.

Moreover, in [9] the local assumption was weakened further to semiprimitivity (defined above).

**PSV Conjecture.** There exists a function \(f : \mathbb{N} \to \mathbb{N}\) such that, if \((\Gamma, G) \in A(d)\) is locally semiprimitive, then \((\Gamma, G)\) is \(f\)-bounded.

In spirit, these conjectures are similar to the 1967 conjecture of Charles Sims [19], proved in [2], that for a \(G\)-vertex-primitive graph \(\Gamma\), the order of the stabiliser of a vertex is bounded above by some function of the valency of \(\Gamma\). Unfortunately the methods in [2], using information about maximal subgroups of nonabelian simple groups, are not transferable to attack the other conjectures, and all three remain open.

As we hinted to above, one approach towards proving the Weiss Conjecture was to prove subcases of it by placing additional constraints on the local action. After a series of papers by Trofimov [20, 21, 22, 23], this approach culminated in 1994 in a proof of the Weiss Conjecture in the case of locally 2-transitive graphs. Also
some progress on the PSV Conjecture was made in [9], by restricting further the local semiprimitive action.

An alternative approach to studying the Weiss Conjecture was initiated by the first author in [10, 14] using normal quotients to reduce to quasiprimitive and bi-quasiprimitive actions. This was taken further in [3], where an analysis of $G$-locally primitive graphs with $G$ quasiprimitive on vertices was undertaken, considering separately each of the eight types of quasiprimitive groups according to the quasiprimitive group subdivision described in [13]. For six of the eight quasiprimitive types it was proved that $|G_\alpha|$ is bounded above by an explicit function of the valency, reducing the problem of proving the Weiss conjecture for quasiprimitive group actions to the almost simple and product action types AS and PA ([3, Section 2]). The PA type was also examined in [3, Proposition 2.2] but, unfortunately, the proof contains an error which we discovered while working on the Praeger conjecture. (Example 42 gives a counter-example to a claim made in the proof of [3, Proposition 2.2].)

Our results shed new light on the Weiss and Praeger Conjectures. Indeed, Proposition 7 together with Theorems 4 and 5 show that the Weiss and Praeger Conjectures hold true if and only if the graphs $(\Lambda, T)$ in Theorem 4 and $(\Lambda_i, T)$ in Theorem 5 are $f$-bounded for some function $f$ depending only on their valencies. Thus Theorems 4 and 5 reduce the Weiss and Praeger Conjectures to a sequence of problems about $T$-vertex-transitive graphs, for various families of non-abelian simple groups $T$ with $|T| \to \infty$. These problems are discussed and successfully solved for many families of simple groups in [16].

Although $(\Lambda, T)$ in Theorem 4 and the $(\Lambda_i, T)$ in Theorem 5 are uniquely determined by $(\Gamma, G)$ and inherit many of the properties of $(\Gamma, G)$, we will see in Examples 40, 41 and 42 that 'local properties' are not necessarily preserved by this reduction (for instance, if $(\Gamma, G)$ is locally primitive, then $(\Lambda, T)$ is not necessarily even locally quasiprimitive).

It would therefore be very interesting to find which local properties of $(\Gamma, G)$ are inherited by the $(\Lambda_i, T)$. In fact, it may be possible to prove the Weiss and the Praeger Conjectures, using Theorems 4 and 5 by proving a stronger conjecture for $(\Gamma, T) \in \mathcal{A}(d)$, with $T$ in a family of non-abelian simple groups, in which the local action of $(\Gamma, T)$ is further relaxed. We leave this as an open problem.

**Problem 8.** Which properties of $(\Gamma, G) \in \mathcal{A}(d)$, with $G$ quasiprimitive on vertices, are inherited by $(\Lambda, T)$ in Theorem 4? Which properties of $(\Gamma, G) \in \mathcal{A}(d)$, with $G$ biquasiprimitive on vertices, are inherited by $(\Lambda_1, T), (\Lambda_2, T)$ in Theorem 5?

1.3. **Structure of the paper.** In Section 2 we give some preliminary and fundamental results that are of importance in the rest of our work. Before stating the main theorem of Section 2 in its full generality, we need a definition extending Definition 4.

**Definition 9.** Let $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ be functions. Let $\Gamma$ be a connected graph with every vertex of valency at most $d$ and $N \leq \text{Aut}(\Gamma)$. We say that $(\Gamma, N)$ is $(f_1, f_2)$-bounded if the number of orbits of $N$ on $V\Gamma$ is at most $f_1(d)$, and also $|N_\alpha| \leq f_2(d)$ for every $\alpha \in V\Gamma$.

For example, if $(\Gamma, G) \in \mathcal{A}(d)$ is $f$-bounded, then $(\Gamma, G)$ is $(1, f)$-bounded where $1$ denotes the constant function with value 1. In Section 2, we prove the following result which often leads to reductions in 'boundedness proofs'.
Theorem 10. Let \((\Gamma, N)\) be \((f_1, f_2)\)-bounded and \(G \leq \text{Aut}(\Gamma)\) with \(N \leq G\). Then there exists a function \(f_3 : \mathbb{N} \to \mathbb{N}\) such that \((\Gamma, G)\) is \((f_1, f_3)\)-bounded.

Remark 11. The proof of Theorem 10 is constructive and it shows that we can take \(f_3(d) = d^{f_1(d) - 1}(d^{f_1(d)} f_2(d))!\). Here we do not claim that such an \(f_3\) is the best possible function for Theorem 10. But it would be interesting to know whether our choice of \(f_3\) could be significantly improved.

Section 3 contains an auxiliary lemma needed in the proof of Theorem 10. Section 4 contains the proof of Theorem 4 and Section 5 contains the proof of Theorem 5. Finally, Section 6 contains the examples mentioned in the introduction.

In this paper all groups will be finite and graphs will be finite and simple.

2. Boundedness

We start this section recalling some standard definitions. Given a group \(N\) and a subset \(S\) of \(N\), we define the Cayley digraph of \(N\) over \(S\) (denoted by \(\text{Cay}(N, S)\)) as the digraph with vertex set \(N\) and arc set \{\((n, n') \in N \times N \mid n(n')^{-1} \in S\}\). Clearly, \(\text{Cay}(N, S)\) is an undirected graph if and only if \(S = S^{-1}\). Also the number of connected components of \(\text{Cay}(N, S)\) is \(|N : \langle S\rangle|\). The main aim in this section is to prove Theorem 10 for the function \(f_3(d)\) given in Remark 11.

Proof of Theorem 10. Let \((\Gamma, N)\) be a connected \((f_1, f_2)\)-bounded graph and \(N \leq G \leq \text{Aut}(\Gamma)\). Since \(N \leq G\) and \(N\) has at most \(f_1(d)\) orbits, the group \(G\) also has at most \(f_1(d)\) orbits. It remains to show that \(|G_\alpha| \leq f_3(d)\) for every \(\alpha \in \text{VT}\). Let \(O_1, \ldots, O_t\) be the orbits of \(N\) on \(\text{VT}\). We claim that \(\Gamma\) contains \(t\) vertices \(\beta_1, \ldots, \beta_t\), with \(\beta_i \in O_i\) for every \(i\), such that the subgraph induced by \(\Gamma\) on \(\{\beta_1, \ldots, \beta_t\}\) is connected. Let \(X\) be a subset of vertices of \(\Gamma\) of maximal size with the properties \(|X \cap O_i| \leq 1\) for every \(i\), and the subgraph induced by \(\Gamma\) on \(X\) is connected. If \(|X| = t\), then the claim is proved. Suppose then that \(|X| = l < t\). Without loss of generality we may assume that \(X = \{\beta_1, \ldots, \beta_l\}\). Let \(v\) be a vertex in \(O_{l+1}\). Since \(\Gamma\) is connected, there exists a path \(\beta_1 = v_1, \ldots, v_u = v\) in \(\Gamma\) from \(\beta_1\) to \(v\). Let \(i\) be minimal such that \(v_i \notin \bigcup_{j=1}^{l} O_j\). In particular, \(i \geq 2\) and \(v_{i-1} \in O_k\) for some \(k \leq l\). Since \(N\) is transitive on \(O_k\), there exists \(n \in N\) such that \(\beta_i = v^n\) is adjacent to \(\beta_1\) and \(v^n \notin \bigcup_{j=1}^{l} O_j\). Set \(X' = X \cup \{v^n\}\). By construction, \(X \subseteq X'\), the subgraph induced by \(\Gamma\) on \(X'\) is connected and \(X'\) contains at most one vertex from each \(O_i\). This contradicts the maximality of \(X\). Thus \(|X| = t\) and the claim is proved.

Fix \(\beta_1, \ldots, \beta_t\), with \(\beta_i \in O_i\) for every \(i\), such that the subgraph induced by \(\Gamma\) on \(\{\beta_1, \ldots, \beta_t\}\) is connected. Let \(S\) be the set \(S = \{n \in N \mid \text{there exists } i \text{ with } \beta^n_i \in \bigcup_{j=1}^{l} \Gamma(\beta_j)\}\) and let \(\tilde{\Gamma}\) be the Cayley digraph on \(N\) with connection set \(S\), that is, \(\tilde{\Gamma} = \text{Cay}(N, S)\). Given \(1 \leq i, j \leq t\), the number of elements \(n \in N\) with \(\beta^n_i \in \Gamma(\beta_j)\) is at most \(|\Gamma(\beta_j)||N_{\beta_i}| \leq f_2(d)\). Therefore \(|S| \leq t^2 f_2(d) \leq f_1(d)^2 f_2(d)\).

Set \(H = \bigcap_{i=1}^{t} \Gamma(\beta_i)\) and note that, by connectivity of the induced graph on \(\{\beta_1, \ldots, \beta_t\}\), for each \(i\), the index \(|G_{\beta_i} : H| \leq d(d - 1)^{t-2} < d^{t-1}\). Since \(H\) fixes a vertex in each orbit of \(N\) on \(\text{VT}\) and since \(C = \text{Sym}(\text{VT})\) permutes the \(O_i\) and the setwise stabiliser in \(C\) of each \(O_i\) induces a semiregular action on it, we obtain that \(C_H(N) = 1\). Thus the group \(H\) acts faithfully on \(N\) by conjugation.
Let $\tilde{G} = N \rtimes H$ be the semidirect product of $N$ by $H$ with respect to this action. We claim that $\tilde{G}$ acts as a group of automorphisms on $\tilde{\Gamma}$ by setting

$$\gamma^g = (\gamma n)^g,$$

for $\gamma \in V\tilde{\Gamma}$, $n \in N$ and $g \in H$. It is straightforward to check that this is a well defined action of $\tilde{G}$ on $V\tilde{\Gamma} = N$. Let $n$ be in $N$, $g$ be in $H$ and $(\gamma, \gamma')$ be an arc of $\tilde{\Gamma}$, that is, $\gamma\gamma^{-1}$ is in $S$. By definition of a Cayley digraph, $(\gamma, \gamma')^g = ((\gamma n)^g, (\gamma' n)^g) = (\gamma^g n^g, (\gamma'^g n^g)^{-1})$ is an arc of $\tilde{\Gamma}$ if and only if $\gamma^g n^g (\gamma'^g n^g)^{-1} = \gamma^g (\gamma'^g)^{-1} = (\gamma\gamma'^{-1})^g$ lies in $S$. Hence to prove the claim it remains to prove that $H$ leaves the set $S$ invariant under conjugation. Let $s \in S$ (with $\beta_i^s \in \Gamma(\beta_j)$ say) and $g \in H$. We have

$$\beta_i^s = \beta_i^{n^g} = \beta_i^g \in \Gamma(\beta_j)^g = \Gamma(\beta_j^g) = \Gamma(\beta_j),$$

and thus $\beta_i^s \in \Gamma(\beta_j)$. By definition of $S$, we have $s^g \in S$. Since $s$ is an arbitrary element of $S$, this shows that $S^g = S$ and the claim is proved.

Let $\Sigma$ be the subgraph of $\Gamma$ induced on the set $\{\beta_i^s \mid 1 \leq i \leq t, x \in \langle S \rangle\}$. We claim that $\Gamma = \Sigma$. We argue by contradiction. Let $v$ be a vertex in $V\Gamma \setminus V\Sigma$. Since $\Gamma$ is connected, there exists a path $\beta_1 = v_1, \ldots, v_u = v$ in $\Gamma$ from $\beta_1$ to $v$. Let $i$ be the minimum such that $v_i \notin V\Sigma$. In particular, $i \geq 2$ and $v_{i-1} \in V\Sigma$ and so $v_{i-1} = \beta_k^s$ for some $k \leq t$ and $x \in \langle S \rangle$. Also, the vertex $v_i$ lies in $\mathcal{O}_k$ for some $k' \leq t$, so as $N$ is transitive on $\mathcal{O}_{k'}$, we have $v_i = \beta_{i'}^n$ for some $n \in N$. Since $v_{i-1}$ is adjacent to $v_i$, we get $\beta_{i-1}^{nx} = v_{i-1}^{x^{-1}} \in \Gamma(v_{i-1}^{x^{-1}}) = \Gamma(\beta_k)$. By definition of $S$, we have $nx^{x^{-1}} \in S$. Finally, as $x \in \langle S \rangle$, we obtain $n \in \langle S \rangle$, and $v_i = \beta_{i'}^n \in V\Sigma$, a contradiction. Thus $V\Sigma = V\Gamma$ and hence $\Sigma = \Gamma$.

Since $\Gamma = \Sigma$, the group $\langle S \rangle$ acts transitively on each $N$-orbit. As $H$ fixes a vertex from each $\langle S \rangle$-orbit, we have $C_H(\langle S \rangle) = 1$. Let $\tilde{\Gamma}$ be the connected component of $\tilde{\Gamma}$ containing 1 (that is, $V\tilde{\Gamma}_1 = \langle S \rangle$) and let $L$ be the permutation group induced by $H$ on $\tilde{\Gamma}_1$. Since $\tilde{\Gamma}$ acts as a group of automorphisms on $\langle S \rangle = V\tilde{\Gamma}_1$, we obtain $L \cong H/C_H(\langle S \rangle) = H$. Thus $H$ acts faithfully on $V\tilde{\Gamma}_1$ and $H \leq \text{Aut}(\langle S \rangle)$. Also, as $H$ fixes setwise $S$, we get $|H| \leq |S|! \leq (d_{f_1}(d)^2 f_2(d))!$. Therefore $|G_{\alpha}| \leq d^{\alpha-1} |H| \leq d_{f_1}(d)^{\alpha-1} (d_{f_1}(d)^2 f_2(d))!$ for every $\alpha \in V\Gamma$ and the proof is complete.

**Remark 12.** Although Theorem 10 is a very general statement, we often use it when $N$ is vertex-transitive, in which case we can take $f_1(d) = 1$ and $f_3(d) = (d f_2(d))!$.

In the following corollary we single out the special case of Theorem 10 most useful for the rest of the paper.

**Corollary 13.** Let $(\Gamma, G)$ be in $A(d)$ and $N$ a normal subgroup of $G$. If $N$ acts regularly on $V\Gamma$, then $|G_{\alpha}| \leq d^\alpha$ for every $\alpha \in V\Gamma$.

*Proof.* If $N$ acts regularly on $V\Gamma$, then $N_\alpha = 1$ for every $\alpha \in V\Gamma$ and so $(\Gamma, N)$ is $(1, 1)$-bounded. Theorem 10 with $f_3(d)$ as in Remark 11 yields $(\Gamma, G)$ is $(1, d!)$-bounded. 

**3. Auxiliary lemma**

This section contains only Lemma 14. This very technical result will be important in the proof of Theorem 4.
Lemma 14. Let $T$ be a non-abelian simple group, $l \geq 1$, and $R$ a proper subgroup of $T$. Let $m^{(i)} = (m_{i1}^{(1)}, \ldots, m_{il}^{(1)}), \ldots, m^{(d)} = (m_{i1}^{(d)}, \ldots, m_{il}^{(d)})$ be elements of $T^l$ such that, for each $i \in \{1, \ldots, d\}$, the set of entries $\{m_{ij}^{(i)}\}_{j}$ of $m^{(i)}$ contains at most $d$ distinct elements from $T$. Let $y^{(i)}$ and $z^{(i)}$ be in $R^l$, and set $n^{(i)} := y^{(i)}m^{(i)}z^{(i)}$ for $i = 1, \ldots, d$. If $T^l = \langle n^{(1)}, \ldots, n^{(d)} \rangle R^l$, then $l \leq d^d|R|^{2d}$.

Proof. Write $y^{(i)} = (y_{i1}^{(i)}, \ldots, y_{il}^{(i)})$ and $z^{(i)} = (z_{i1}^{(i)}, \ldots, z_{il}^{(i)})$ with $y_{ij}^{(i)}, z_{ij}^{(i)} \in R$. We denote by $(n^{(i)})_j$ the $j$th coordinate of $n^{(i)}$. Set $U = \langle n^{(1)}, \ldots, n^{(d)} \rangle$, and assume that $T^l = UR^l$ and $l > d^d|R|^{2d}$.

Since the element $m^{(i)}$ has at most $d$ distinct entries, by the pigeonhole principle we obtain that $m^{(i)}$ has more than $d^{d-1}|R|^{2d}$ coordinates with the same entry. Applying this argument for each $i \in \{1, \ldots, d\}$, we obtain that there exists a set of coordinates $X \subseteq \{1, \ldots, l\}$ with $|X| > |R|^{2d}$ and such that every $m^{(i)}$ is constant on the coordinates from $X$.

Let $Y$ be the $(d \times |X|)$-array $(y_{x}^{(i)})_{1 \leq i \leq d, x \in X}$, and let $Z$ be the $(d \times |X|)$-array $(z_{x}^{(i)})_{1 \leq i \leq d, x \in X}$. The columns of $Y$ and $Z$ are elements of $R^d$. Therefore, for each of $Y$ and $Z$, there are at most $|R|^d$ possibilities for each column. As $|X| > |R|^{2d}$, by the pigeonhole principle, there exist distinct $x, x' \in X$ such that the $x$th and $x'$th columns of $Y$ are equal and the $x$th and $x'$th columns of $Z$ are equal. Hence

$$y_{x}^{(i)} = y_{x'}^{(i)}, \quad z_{x}^{(i)} = z_{x'}^{(i)} \quad \text{for every } i = 1, \ldots, d.$$ 

This yields

$$(n^{(i)})_x = y_{x}^{(i)}m_{x}^{(i)}z_{x}^{(i)} = y_{x'}^{(i)}m_{x'}^{(i)}z_{x'}^{(i)} = (n^{(i)})_{x'} \quad \text{for every } i = 1, \ldots, d.$$ 

Therefore the projection of $U$ to the group $T \times T$ (obtained from taking the $x$th and $x'$th coordinate entries in $T^l$) is contained in the diagonal subgroup $\{(t, t) \mid t \in T\}$. As $T^l = UR^l$, we obtain

$$T \times T = \{(t, t) \mid t \in T\}(R \times R).$$

Let $t$ be an element of $T \setminus R$. From \((1)\), we have $(1, t) = (a, a)(b, c)$ for some $a \in T$ and $b, c \in R$. This yields $a = b^{-1} \in R$ and $t = ac \in R$, a contradiction. This contradiction arose from the assumption $l > d^d|R|^{2d}$. Hence $l \leq d^d|R|^{2d}$ and the lemma is proved. \hfill \Box

4. Proof of Theorem \ref{thm:main}

In this section we use the subdivision into eight types (namely HA, HS, HC, SD, CD, TW, PA and AS) of the finite quasiprimitive permutation groups, and we refer the reader to \cite{praeger2013} for details. Our main tool for dealing with quasiprimitive groups is Corollary \ref{cor:upperbound}. Namely, in Corollaries \ref{cor:upperbound} and \ref{cor:upperbound2} for $(\Gamma, G) \in \mathcal{A}(d)$, we give an absolute upper bound (in terms of the valency $d$) on the size of the stabiliser of a vertex in $G$, if $G$ is quasiprimitive of type HA, HS, HC, TW, SD or CD. We note that these results were proved in \cite{praeger2013} in the case where $G$ is locally primitive.

Corollary 15. Let $(\Gamma, G)$ be in $\mathcal{A}(d)$ with $G$ a quasiprimitive group of type HA, HS, HC or TW on $VT$. Then $|G_\alpha| \leq d!$ for every $\alpha \in VT$.

Proof. As $G$ is quasiprimitive of type HA, HS, HC or TW, the group $G$ admits a regular normal subgroup $N$. From Corollary \ref{cor:upperbound} we have $|G_\alpha| \leq d!$ for every $\alpha \in VT$. \hfill \Box
Corollary 16. Let \((\Gamma, G)\) be in \(\mathcal{A}(d)\) with \(G\) a quasiprimitive group of type SD or CD on \(VT\). Then \(|G_\alpha| \leq (dd!)!\) for every \(\alpha \in VT\).

Proof. Assume that \(G\) is of type SD. Let \(N\) be the socle of \(G\). So, \(N = T_1 \times \cdots \times T_k\) for some \(k \geq 2\), and each \(T_i \cong T\) for some non-abelian simple group \(T\). By definition of type SD, the group \(M = T_1 \times \cdots \times T_{k-1}\) acts regularly on \(VT\). Hence \((\Gamma, M)\) is \((1,1)\)-bounded. Since \(M \leq N\), Theorem 10 with \(f_5(d)\) as in Remark 11 yields that \((\Gamma, N)\) is \((1, f)\)-bounded with \(f(d) = d!\). As \(N \leq G\), a similar application of Theorem 10 with \(f_5(d)\) as in Remark 11 yields that \((\Gamma, G)\) is \((1, f')\)-bounded with \(f'(d) = (df(d))! = (dd!)!\).

Assume now that \(G\) is of type CD. Then the vertex set \(VT\) admits a cartesian decomposition, that is, \(VT = \Delta^l\) for some set \(\Delta\) and for some \(l \geq 2\). Let \(N\) be the socle of \(G\). Then \(N \cong T^u\) for some non-abelian simple group \(T\) and some \(u \geq 2\). Also, \(G\) is permutation isomorphic to a subgroup of the wreath product \(H \wr \text{Sym}(l)\) (endowed with the product action), where \(H \subseteq \text{Sym}(\Delta)\) is quasiprimitive of type SD with socle \(T^u\).

As the socle of \(H\) contains a regular normal subgroup isomorphic to \(T^{u-1}\), the group \(N\) contains a normal subgroup \(M\) isomorphic to \(T^{(u-1)l}\) acting regularly on \(\Delta^l\). Hence \((\Gamma, M)\) is \((1,1)\)-bounded. Since \(M \leq N\), Theorem 10 with \(f_5(d)\) as in Remark 11 yields that \((\Gamma, N)\) is \((1, f)\)-bounded with \(f(d) = d!\). As \(N \leq G\), a similar application of Theorem 10 yields that \((\Gamma, G)\) is \((1, f')\)-bounded with \(f'(d) = (df(d))! = (dd!)!\).

\(\square\)

Remark 17. It is worth pointing out here that in Corollaries 15 and 16 there is no local assumption on \(G\). This quite remarkably shows that in a quasiprimitive group \(G\) of type HA, HS, HC, TW, SD or CD acting vertex-transitively on a connected graph \(\Gamma\), the size of the stabiliser of a vertex is bounded above by a function of the valency of \(\Gamma\) (see Theorem 4 (1)).

In the rest of this section we deal with the case that \(G\) is of type PA. We start with a definition and a lemma required in the proof of Theorem 22.

Definition 18. Let \(\Gamma\) be a \(G\)-vertex-transitive graph and \(\Sigma\) a system of imprimitivity for \(G\) in its action on \(VT\). The quotient \(\Gamma_\Sigma\) is the graph whose vertices are the blocks \(\sigma\) of \(\Sigma\), with an edge between two distinct blocks \(\sigma\) and \(\eta\) of \(\Sigma\), if and only if there is an edge of \(\Gamma\) between \(\alpha\) and \(\beta\), for some \(\alpha \in \sigma\) and some \(\beta \in \eta\).

The graph \(\Gamma_\Sigma\) is \(G\)-vertex-transitive but (despite the similarity with Definition 6) there is no upper bound on the valency of \(\Gamma_\Sigma\) in terms of the valency of \(\Gamma\).

Lemma 19. Let \((\Gamma, G)\) be in \(\mathcal{A}(d)\), \(\Sigma\) a system of imprimitivity for the action of \(G\) on \(VT\) and \(N\) a normal subgroup of \(G\) with \(N_\sigma\) transitive on \(\sigma\) for every \(\sigma\) in \(\Sigma\). Then the number of orbits of \(N_\sigma\) on \(\Gamma_\Sigma(\sigma)\) is at most \(d\), for every vertex \(\sigma\) of \(\Gamma_\Sigma\).

Proof. Fix a vertex of \(\Gamma_\Sigma\) and \(\alpha\) in \(\sigma\). Let \(\beta_1, \ldots, \beta_r\) be representatives of the orbits of \(G_\alpha\) on \(\Gamma(\alpha)\). Then \(r \leq |\Gamma(\alpha)| \leq d\). Let \(\eta_1, \ldots, \eta_r\) be in \(\Sigma\) with \(\beta_i \in \eta_i\), for \(i = 1, \ldots, r\). Now we prove two preliminary claims from which the lemma will follow immediately.

Claim 1. \(\Gamma_\Sigma(\sigma) = \bigcup_{i=1}^{r} \eta_i G_\sigma\).

By the definitions of \(\Gamma_\Sigma\) and of the \(\eta_i\), the right hand side is contained in the left hand side. Let \(\eta\) be in \(\Gamma_\Sigma(\sigma)\). By definition, there exists \(\alpha' \in \sigma\) and \(\beta \in \eta\) with \(\beta \in \Gamma(\alpha')\). Since \(G_\sigma\) is transitive on \(\sigma\), there exists \(g \in G_\sigma\) such that \(\alpha = (\alpha')^g\).
In particular, $\beta^g \in \Gamma(\alpha)$ and so there exists $h \in G_\alpha$ with $\beta_i = \beta^gh$, for some $i \in \{1, \ldots, r\}$. As $\beta_i \in \eta_i$, we get $\eta_i = \eta_i^gh$ with $gh \in G_\sigma$ and so Claim 1 follows. □

Claim 2. The number of orbits of $N_\sigma$ on $\eta_i^{G_\sigma}$ is at most $|G_\alpha : G_{\alpha,\beta_i}|$.

Clearly, $|\eta_i^{G_\sigma}| = |G_\sigma : G_{\sigma,\eta_i}|$. Also, as $N_\sigma$ is a normal subgroup of $G_\sigma$, we have that each orbit of $N_\sigma$ on $\eta_i^{G_\sigma}$ has size $|G_{\sigma,\eta_i}N_\sigma : G_{\sigma,\eta_i}|$. Therefore the number of orbits of $N_\sigma$ on $\eta_i^{G_\sigma}$ equals $|G_\sigma : G_{\sigma,\eta_i}N_\sigma|$.

Since by hypothesis $N_\sigma$ is transitive on $\sigma$, we get $G_\sigma = G_\alpha N_\sigma$. Note that, all of $G_\sigma$, $N_\sigma$ and $G_{\sigma,\eta_i}N_\sigma$ are transitive on $\sigma$ and hence $|G_\sigma : G_\alpha| = |G_{\sigma,\eta_i}N_\sigma : (G_{\sigma,\eta_i}N_\sigma)\cap G_\alpha| = |\sigma|$. Thus $|G_\sigma : G_{\sigma,\eta_i}N_\sigma| = |G_\alpha : (G_{\sigma,\eta_i}N_\sigma)\cap G_\alpha| \leq |G_\alpha : G_{\alpha,\beta_i}|$, (see Figure 1). □

\[ G_\sigma = G_\alpha N_\sigma \]

\[ G_{\sigma,\eta_i}N_\sigma \]

\[ G_\alpha \]

\[ G_{\sigma,\eta_i}N_\sigma \cap G_\alpha \]

\[ G_{\alpha,\beta_i} \]

**Figure 1.** Subgroup lattice for Lemma 19

From Claims 1 and 2, the number of orbits of $N_\sigma$ on $\Gamma_\Sigma(\sigma)$ is at most $\sum_{i=1}^r |G_\alpha : G_{\alpha,\beta_i}| = d$. □

We start our analysis of quasiprimitive groups of type PA by setting some notation (as usual we follow [11] and [13]).

**Notation 20.** Let $\Gamma$ be a connected $G$-vertex-transitive graph of valency $d$ and assume that $G$ is a quasiprimitive group of type PA. The socle $N = T_1 \times \cdots \times T_l$ of $G$ is isomorphic to $T^l$, where $T$ is a non-abelian simple group, and $l \geq 2$. Moreover, there is a $G$-invariant partition $\Sigma$ of VT such that the action of $G$ on $\Sigma$ is faithful and is permutationally isomorphic to the product action of $G$ on a set $\Delta^l$. By identifying $\Sigma$ with $\Delta^l$ we have $G \subseteq W = H wr \text{Sym}(l)$, where $H \subseteq \text{Sym}(\Delta)$ is an almost simple group with socle $T$ which is quasiprimitive on $\Delta$, $N$ is the socle of $W$, and $W$ acts on $\Sigma$ in product action. Fix $\delta$ an element of $\Delta$. We denote by $\sigma$ the element $(\delta, \ldots, \delta)$ of $\Sigma = \Delta^l$. Also, we fix $\alpha_0$ a vertex of $\Gamma$ in $\sigma$ and let $\beta_1, \ldots, \beta_r$ be representatives of the orbits of $G_{\alpha_0}$ on $\Gamma(\alpha_0)$. We have $W_\sigma = H_\delta wr \text{Sym}(l)$ and $T_\delta$ is a proper subgroup of $T$.

The group $G$ acts transitively on the set $\{T_1, \ldots, T_l\}$ of minimal normal subgroups of $N$. As $G = NG_{\alpha_0}$ and $N$ acts trivially on $\{T_1, \ldots, T_l\}$, we obtain that $G_{\alpha_0}$ acts transitively on $\{T_1, \ldots, T_l\}$. In the sequel, we use this fact repeatedly.

It is proved in [11] that $N_{\alpha_0}$ is a subdirect subgroup of $N_\sigma = T_\delta^l$, that is, $N_{\alpha_0}$ projects onto $T_\delta$ for each of the $l$ direct factors of $N_\sigma$. Furthermore, the subgroup $G_i := N_{\alpha}(T_i)$ has index $l$ in $G$ and $G_i$ induces a subgroup of $\text{Sym}(\Delta)$; this subgroup
is almost simple with socle $T$ and without loss of generality we may take $H$ to be this subgroup for each $i$. We denote by $\pi_i : G_i \to H$ the projection of $G_i$ onto $H$.

**Lemma 21.** $\pi_i(G_i \cap G_{\alpha_0}) = \pi_i(G_i \cap G_{\sigma}) = H_{\delta}$.

**Proof.** Since $G_{\alpha_0} \subseteq G_{\sigma}$, we have $\pi_i(G_i \cap G_{\alpha_0}) \subseteq \pi_i(G_i \cap G_{\sigma}) \subseteq H_{\delta}$ and hence it suffices to prove that $\pi_i(G_i \cap G_{\alpha_0}) = H_{\delta}$. Let $L = \pi_i(G_i \cap G_{\alpha_0})$. It was proved in [11] that $N_{\alpha_0}$ is a subdirect group of $N_{\sigma}$ and hence $\pi_i(N_{\alpha_0}) = \pi_i(N_{\sigma}) = T_{\delta} \subseteq L$. Since $G_{\alpha_0} \subseteq W_{\sigma} = H_{\delta}$ wr $\text{Sym}(l)$, we have $L \subseteq H_{\delta}$. As $N$ is transitive on $\text{VT}$, we have $G = N G_{\alpha_0}$. Hence, from the modular law, we get $G_i = N(G_i \cap G_{\alpha_0})$ and, applying $\pi_i$ on both sides, we have $H = T L$. Thus $H_{\delta} = H \cap (T L) = (H_{\delta} \cap T) L = T_{\delta} L = L$. □

**Theorem 22.** Let $(\Gamma, G)$ be in $\mathcal{A}(d)$ with $G$ a quasiprimitive group of type PA on $\text{VT}$ (as in Notation 20). Then $l \leq d^4 |T_{\delta}|^{2d}$. Furthermore, $(\Gamma, G)$ uniquely determines an element $(\Lambda, T)$ in $\mathcal{A}(d)$ with the stabilisers of the vertices of $\Lambda$ conjugate to $T_{\delta}$.

**Proof.** Let $\eta^i = (\delta^i_1, \ldots, \delta^i_1)$, for $1 \leq i \leq s$, be representatives of the orbits of $N_\nu$ in its action on $\Gamma_\Sigma(\sigma)$. Since $N$ is transitive on $\text{VT}$ and $\Sigma$ is a system of imprimitivity for $G$, we obtain that $N_\nu$ is transitive on $\nu$ for every $\nu \in \Sigma$. So Lemma [11] applies and we have $s \leq d$. Since $N_\nu = T_{\delta} \times \cdots \times T_{\delta}$, it follows from the definition of the $\eta^i$ that

\[
\Gamma_\Sigma(\sigma) = \bigcup_{i=1}^s (\eta^i)^{N_\nu} = \bigcup_{i=1}^s (\delta^1_j)^{T_{\delta}} \times \cdots \times (\delta^l_j)^{T_{\delta}}.
\]

Now we prove two preliminary claims from which the theorem will follow.

**Claim 1.** $\bigcup_{i=1}^s (\delta^1_j)^{H_{\delta}} = \bigcup_{i=1}^s (\delta^l_j)^{T_{\delta}}$ for every $j \in \{1, \ldots, l\}$.

As $T_{\delta} \subseteq H_{\delta}$, the right hand side is contained in the left hand side. Fix $j$ in $\{1, \ldots, l\}$, $i$ in $\{1, \ldots, s\}$ and $h$ in $H_{\delta}$. From Lemma 21, we have $\pi_i(G_j \cap G_{\sigma}) = H_{\delta}$. Hence there exists $c = (h_1, \ldots, h_l)g \in G_j \cap G_{\sigma}$ with $h_j = h$. Now, $(\eta^i)^c \in \Gamma_\Sigma(\sigma)$ and the $j$th coordinate of $(\eta^i)^c$ is $(\delta^1_j)^h$. So, from (2), we get $(\delta^1_j)^h \in \bigcup_{i=1}^s (\delta^1_j)^{T_{\delta}}$ and Claim 1 follows. ■

**Claim 2.** $\bigcup_{i=1}^s (\delta^1_j)^{T_{\delta}} = \bigcup_{i=1}^s (\delta^l_j)^{T_{\delta}}$ for every $j \in \{1, \ldots, l\}$.

Fix $j$ in $\{1, \ldots, l\}$. Since the left and the right hand side are $T_{\delta}$-invariant, it suffices to show that $\delta^v_j \in \bigcup_{i=1}^s (\delta^1_j)^{T_{\delta}}$ for every $v \in \{1, \ldots, s\}$. (This will imply that the right hand side is contained in the left hand side, and an analogous argument proves the reverse inclusion.) Fix $v$ in $\{1, \ldots, s\}$. Recall that $G_{\sigma}$ is transitive on $\{T_1, \ldots, T_l\}$. So, there exists $c = (h_1, \ldots, h_l)g \in G_\sigma \subseteq H_{\delta}$ wr $\text{Sym}(l)$ such that $g$ conjugates $T_1$ to $T_j$, that is, $1g = j$. Then $\Gamma_\Sigma(\sigma)$ contains $\eta^v$ and hence contains

\[
(\eta^v)^c = ((\delta^v_{1g^{-1}})^{h_1g^{-1}} \ldots , (\delta^v_{lg^{-1}})^{h_lg^{-1}}),
\]

which has $j$th entry $(\delta^v_{1g^{-1}})^{h_1g^{-1}} = (\delta^v_1)^{h_1}$. From (2), we obtain that the $j$th entries of the elements of $\Gamma_\Sigma(\sigma)$ lie in $\bigcup_{i=1}^s (\delta^v_1)^{T_{\delta}}$. Therefore

\[
(\delta^v_1)^{h_1} \in \bigcup_{i=1}^s (\delta^v_1)^{T_{\delta}}.
\]
Now, as $h_i^{-1} \in H^\delta_i$, Claim 1 yields $\delta_i^T \in \cup_{s=1}^n (\delta_i^{T_s})^{T_s}$ and Claim 2 follows. ■

Claim 2 yields that the $ls$ elements $\{\delta_i\}_{1,i}$ lie in at most $s$ distinct $T^\delta_i$-orbits. In particular, since $N_\sigma = T^\delta_1$, for each $i \in \{1, \ldots, s\}$, the $N_\sigma$-orbit $(\eta_i)^{N_\sigma}$ contains an element with at most $s$ distinct entries from $\Delta$. Therefore, by replacing $\eta_i$ with a suitable element from $(\eta_i)^{N_\sigma}$ if necessary, we may assume that there are at most $s$ distinct elements of $\Delta$ among the $l$ entries of $\eta_i$.

Since $N$ is transitive on $VT$ we may choose $m_i \in N$ such that $\sigma m_i = \eta_i$. Moreover, since $\sigma = (\delta, \ldots, \delta)$, and since $\eta_i$ has at most $s$ distinct entries from $\Delta$, the element $m_i$ can be chosen so that its $l$ coordinates contain at most $s$ distinct entries from $T$.

For each $\beta \in \Gamma(a_0)$, let $n_\beta$ be an element of $N$ with $\beta = a_0^{n_\beta}$. Set $U = \{n_\beta \mid \beta \in \Gamma(a_0)\}$. Let $\Gamma$ be the subgraph of $\Gamma$ induced on the set $a_0^U$. We claim that $\Gamma = \overline{\Gamma}$. By the definitions of $\Gamma$ and $U$, we have $a_0 \in V\overline{\Gamma}$ and $\Gamma(a_0) \subseteq V\overline{\Gamma}$. Therefore, since $\overline{\Gamma}$ is $U$-vertex-transitive, every vertex of $\overline{\Gamma}$ has valency $|\Gamma(a_0)| = |\Gamma(a_0)|$. Since $\overline{\Gamma}$ is connected, this yields $\Gamma = \overline{\Gamma}$. In particular, $U$ acts transitively on $VT$ and so $N = U N_{a_0}$. As $N_{a_0}$ is a subgroup of $N_\sigma$, we have $N = U N_{a_0}$.

Fix $\beta \in \Gamma(a_0)$. Since $a_0^{n_\beta} \in \Gamma(a_0)$, we get $\sigma^{n_\beta} \in \Gamma_\Sigma(\sigma)$ and hence $\sigma^{n_\beta} \in (\eta_i)^{N_\sigma}$ for some $i \in \{1, \ldots, s\}$. In particular, there exists $z_\beta \in N_\sigma$ such that $\sigma z_\beta = \eta_i^\beta$.

Since by the definition of $m_i$, we have $\sigma m_i = \eta_i^\beta$, we obtain $\sigma = \sigma^{n_\beta} z_\beta m_i^{-1}$, that is to say, $y_\beta := n_\beta z_\beta m_i^{-1} \in N_\sigma$. Therefore, for every $\beta \in \Gamma(a_0)$, there exists $i_\beta \in \{1, \ldots, s\}$ and $y_\beta, z_\beta \in N_\sigma$ such that

$$n_\beta = y_\beta m_{i_\beta} (z_\beta)^{-1}.$$  

Now Lemma [14] applied to $m_1, \ldots, m_s \in N = T^l$ gives $l \leq d^4|T^\delta|^{2d}$.

Set $M_1 = T_1 \times \cdots \times T_{i-1} \times T_{i+1} \times \cdots \times T_l$ for $i \in \{1, \ldots, l\}$. Since $M_i$ is a maximal normal subgroup of $N$, we obtain that either $M_i$ is transitive on $VT$ or $M_i$ is 1-closed in $N$. As $N_\alpha$ is a subdirect subgroup of $N_\sigma$, we have $N_\alpha M_i = N_\sigma M_i = T_\delta \times M_i < N$ and so $M_i$ is 1-closed in $N$. Therefore, by Definition [9] the pair $(M_i, N/M_i)$ lies in $A(d)$ for each $i \in \{1, \ldots, l\}$. Furthermore, since $G$ acts transitively on $\{M_1, \ldots, M_l\}$, we obtain that $G \cong N/M_i$ for every $i, j \in \{1, \ldots, l\}$. This shows that $(\Gamma, G)$ uniquely determines the element $(\Gamma_{M_i}, N/M_i)$ of $A(d)$ up to isomorphism. Finally, the stabiliser in $N/M_1$ of the vertex $\alpha M_i$ of $\Gamma_{M_i}$ is $N_\alpha M_1/M_1 = (T_\delta \times M_1)/M_1 \cong T_\delta$ and theorem is proved. □

Now we are ready to prove Theorem [4].

**Proof of Theorem [4]**

If $G$ is of type HA, HS, HC, TW, SD or CD, then by Corollaries [15] and [16] the graph $(\Gamma, G)$ is $(dd!)!$-bounded.

Assume that $G$ is of type AS, and set $\Lambda = \Gamma$ and $T = \text{Soc}(G)$. Clearly, $(\Lambda, T)$ is uniquely determined by $(\Gamma, G)$. Also, as $T$ is transitive on $VT$, from Theorem [14] with $f_3(d)$ as in Remark [14] if $(\Lambda, T)$ is $g$-bounded, then $(\Gamma, G)$ is $f$-bounded where $f(d) = (dg(d))!$. In particular, as $f(d) \leq \tilde{g}(d)$, we get that $(\Gamma, G)$ is $\tilde{g}$-bounded. Conversely, if $(\Gamma, G)$ is $f$-bounded, then $(\Lambda, T)$ is also $f$-bounded.

Finally assume that $G$ is of type PA and let $(\Lambda, T)$ be as in Theorem [22]. We use Notation [20]. From Theorem [22] we have $l \leq d^4|T^\delta|^{2d}$ with $\lambda \in VA$. Assume that $(\Gamma, G)$ is $f$-bounded for some $f : \mathbb{N} \to \mathbb{N}$. As $N_\alpha$ is a subdirect subgroup of $N_\sigma \cong T^\delta_1$ and $|T^\delta| = |T^\lambda|$, we have that $|T^\lambda| \leq |N_\alpha| \leq f(d)$ and so $(\Lambda, T)$ is
f-bounded. Finally, assume that \((A,T)\) is \(g\)-bounded for some \(g : \mathbb{N} \to \mathbb{N}\). Now, 
\[|N_\alpha| \leq |N_\sigma| = |T_\lambda|^l \leq g(d)^l \leq g(d)^d g(d)^2d\]
and so \((\Gamma,N)\) is \(f'\)-bounded with 
\[f'(d) = g(d)^d g(d)^2d\]. Finally, Theorem 10 with \(f_3(d)\) as in Remark 11 yields that 
\((\Gamma,G)\) is \(f\)-bounded for 
\[f(d) = (dg(d)^d g(d)^2d)! = \hat{g}(d)\]. □

5. Proof of Theorem 5

Recall that a permutation group \(G\) is biquasiprimitive if every non-trivial normal subgroup of \(G\) has at most two orbits and \(G\) does have a normal subgroup with exactly two orbits. In [15, Theorem 1.1], the structure of a biquasiprimitive group is described in detail and here we use [15] as a reference. We set some notation for the rest of the paper, which follows [15].

**Notation 23.** For a finite biquasiprimitive permutation group \(G\) on \(\Omega\), there is at least one non-trivial intransitive normal subgroup \(N\) (since \(G\) is not quasiprimitive) and \(N\) must therefore have two orbits, say \(\Delta, \Delta'\). Each element of \(G\) either fixes setwise these two orbits or interchanges them. Thus the elements of \(G\) that fix \(\Delta, \Delta'\) setwise form a subgroup \(G^+\) of index 2, and \(G^+\) induces a transitive permutation group \(H\) on \(\Delta\). By the embedding theorem for permutation groups, \(G\) is conjugate in \(\text{Sym}(\Omega)\) to a subgroup of the wreath product \(H \wr \text{Sym}(2) = (H \times H) \rtimes \text{Sym}(2)\). The set \(\Omega\) may be identified with \(\Delta \times \{1, 2\}\) such that, for \((y_1, y_2)\) in the base group \(H \times H\), and \((1, 2) \in \text{Sym}(2)\),

\[(\delta, i)^{(y_1, y_2)} = \left(\delta^{y_1}, i\right)\quad \text{and} \quad (\delta, i)^{(1,2)} = (\delta, i^{(1,2)})\]

for all \((\delta, i) \in \Omega\). Theorem 1.1 in [15] (which we report below) defines various distinct possibilities for \(\text{Soc}(G)\).

Let \(M\) be a group. For each \(\varphi \in \text{Aut}(M)\), we denote by \(\text{Diag}_\varphi(M \times M)\) the full diagonal subgroup \(\{(x, x^\varphi) \mid x \in M\}\) of \(M \times M\). We write \(\iota_x : y \mapsto x^{-1}yx\) for the inner automorphism induced by the element \(x \in M\).

Before stating Theorem 24 we remark that if \(\Gamma\) is a \(G\)-vertex-biquasiprimitive graph, then \(\Gamma\) is not necessarily bipartite with bipartition \(\{\Delta \times \{1\}, \Delta \times \{2\}\}\). Indeed, the hypothesis of \(\Gamma\) being vertex-transitive does not imply that every edge of \(\Gamma\) joins vertices from distinct \(G^+\)-orbits (but this is the case if \(\Gamma\) is connected and \(G\)-arc-transitive).

**Theorem 24 ([15, Theorem 1.1 and Theorem 1.2]).** Let \(G\) be a biquasiprimitive group on \(\Omega\), and \(H\) the permutation group induced by \(G^+\) on \(\Delta \times \{1\}\) (as in Notation 23). Assume that \(G_{(\alpha,1)}\) is intransitive on \(\Delta \times \{2\}\) for \(\alpha \in \Delta\). Replacing \(G\) by a conjugate in \(\text{Sym}(\Omega)\) if necessary, \(G \leq H \wr \text{Sym}(2), G \setminus G^+\) contains an element \(g = (x, 1)(1, 2)\) for some \(x \in H\), \(G^+ = \text{Diag}_\varphi(H \times H)\) where \(\varphi \in \text{Aut}(H)\) and \(\varphi^2 = \iota_x\), and one of the following holds.

\((A)\): \(H\) is quasiprimitive on \(\Delta\).

\((B)\): \(H\) is not quasiprimitive on \(\Delta\); there exists an intransitive minimal normal subgroup \(R\) of \(H\) such that \(R \neq R^\varphi\), \(M = R \times R^\varphi\) is a transitive normal subgroup of \(H\), and \(N = \text{Diag}_\varphi(M \times M)\) is a minimal normal subgroup of \(G\); and one of:

(i) \(\text{Soc}(G) = N\).

(ii) \(\text{Soc}(G) = N \times \overline{N}\), where \(N, \overline{N}\) are isomorphic non-abelian minimal normal subgroups of \(G\), and \(\overline{N} = \text{Diag}_\varphi(M \times M)\); \(M, \overline{M}\) are isomorphic.
regular normal subgroups of \( H \), \( \text{Soc}(H) = M \times \overline{M} \), and \( \overline{M} = \overline{R} \times \overline{R}^c \) for an intransitive minimal normal subgroup \( \overline{R} \) of \( H \).

**Remark 25.** We warn the reader that [15, Theorem 1] describes the structure of any finite biquasiprimitive permutation group (that is, \( G_{(\alpha,1)} \) is not necessarily assumed to be intransitive on \( \Delta \times \{2\} \) for \( \alpha \in \Delta \)). The refinement of [15, Theorem 1] in [15, Theorem 2] is concerned with biquasiprimitive groups acting transitively on the vertices of a bipartite graph \( \Gamma \). The statement of [15, Theorem 1] requires \( G \) to be arc-transitive on \( \Gamma \), but the proof uses simply that \( G_{(\alpha,1)} \) is intransitive on \( \Delta \times \{2\} \) for \( \alpha \in \Delta \). Therefore a proof of Theorem 24 combines [15, Theorem 1] and Theorem 1.2 together with this remark.

In the rest of this section we use the subdivision into types given in Theorem 24 and we prove Theorem 3. We start our analysis with the easiest example.

**Lemma 26.** Let \((\Gamma,G) \in \mathcal{A}(d) \) with \( G \) biquasiprimitive on \( VT \) and \( G_{(\alpha,1)} \) transitive on \( \Delta \times \{2\} \) for \( \alpha \in \Delta \). Then \( |G_{(\alpha,i)}| \leq d! (d-1)! \) for every \( \alpha \in \Delta \) and \( i = 1,2 \).

**Proof.** Fix \( \alpha \) in \( \Delta \). As \( \Gamma \) is connected, \((\alpha,1)\) has a neighbour, \((\alpha',2)\), say, in \( \Delta \times \{2\} \). Since \( G_{(\alpha,1)} \) is transitive on \( \Delta \times \{2\} \) and \( G \leq \text{Aut}(\Gamma) \), we obtain \( \Gamma((\alpha,1)) \supseteq \Delta \times \{2\} \). Therefore \( G_{(\alpha,1)} \) is intransitive on \( \Delta \times \{2\} \) for \( \alpha \in \Delta \). Hence, without risk of ambiguity, we may assume that \( G_{(\alpha,1)} \) is intransitive on \( \Delta \times \{2\} \) and so Theorem 23 applies.

**Notation 27.** Fix \((\Gamma,G) \in \mathcal{A}(d) \) with \( G \) biquasiprimitive on \( VT \) and \( G_{(\delta,1)} \) intransitive on \( \Delta \times \{2\} \) for \( \delta \in \Delta \) (as in Notation 23). We denote by \( \Gamma_{\Delta,i}^{\delta,1} \) (respectively, \( \Gamma_{\Delta,2}^{\delta,1} \)) the graph whose vertices are the elements of \( \Delta \), with an edge between two distinct elements \( \delta_1 \) and \( \delta_2 \) of \( \Delta \) if and only if the distance of \((\delta_1,1)\) from \((\delta_2,1)\) (respectively, \((\delta_1,2)\) from \((\delta_2,2)\)) in \( \Gamma \) is at most 2.

Since \( \Gamma \) is connected of valency \( d \), the graph \( \Gamma_{\Delta,i}^{\delta,1} \) is connected of valency at most \( d(d-1) \) for each \( i \in \{1,2\} \). The subgroup \( G^{+} = \text{Diag}_d(H \times H) \) of \( G \) that fixes setwise \( \Delta \times \{1\} \) and \( \Delta \times \{2\} \) acts transitively on \( \Delta \times \{i\} \) and the action of \( G^{+} \) on \( \Delta \times \{i\} \) is equivalent to the action of \( H \) on \( \Delta \) and preserves the edge set of \( \Gamma_{\Delta,2}^{\delta,1} \) for each \( i \in \{1,2\} \). Therefore \( \Gamma_{\Delta,i}^{\delta,1} \) is an \( H \)-vertex-transitive graph of valency at most \( d(d-1) \), that is, \( (\Gamma_{\Delta,i}^{\delta,1}, H) \in \mathcal{A}(d(d-1)) \) for each \( i \in \{1,2\} \). As \( g = (x,1)(1,2) \in G \) and \( (\Delta \times \{1\})^g = \Delta \times \{2\} \), we have \( \Gamma_{\Delta,i}^{\delta,1} \cong \Gamma_{\Delta,2}^{\delta,1} \). Hence, without risk of ambiguity, we write simply \( \Gamma_{\Delta,i}^{\delta,1} \) for \( \Gamma_{\Delta,i}^{\delta,1} \).

The subgroup \( G^{+} \) is the unique subgroup of \( G \) of index 2 except for the case where \( |VT| = 4 \) and \( G = Z_2 \times Z_2 \) (see [15, Remarks 1.1 (1)]), in which case Theorem 3 is obvious. Therefore, in the remainder of the section, we assume that \( G^{+} \) is the unique subgroup of index 2 in \( G \) and hence \( \Delta \) is uniquely determined by \( G \).

Summing up, the element \((\Gamma,G) \in \mathcal{A}(d) \) uniquely determines the element \((\Gamma_{\Delta,i}^{\delta,1}, H) \in \mathcal{A}(d(d-1)) \) with \( G_{(\alpha,i)} \cong H_{\alpha} \) for every \( \alpha \in \Delta \) and for every \( i \in \{1,2\} \).

**Lemma 28.** Let \((\Gamma,G) \in \mathcal{A}(d) \) and \((\Gamma_{\Delta,i}^{\delta,1}, H) \in \mathcal{A}(d(d-1)) \) as in Notation 27. If \((\Gamma_{\Delta,i}^{\delta,1}, H) \) is \( f \)-bounded, then \((\Gamma,G) \) is \( f' \)-bounded where \( f'(d) = f(d(d-1)) \), and if \((\Gamma,G) \) is \( f \)-bounded, then \((\Gamma_{\Delta,i}^{\delta,1}, H) \) is \( f \)-bounded where \( f \) is as in Definition 1.

**Proof.** We have \( |G_{(\alpha,i)}| = |H_{\alpha}| \) for every \( \alpha \in \Delta \) and \( i \in \{1,2\} \). As \((\Gamma,G) \in \mathcal{A}(d) \) and \((\Gamma_{\Delta,i}^{\delta,1}, H) \in \mathcal{A}(d(d-1)) \), from Definition 1 we have to show that \( |G_{(\alpha,i)}| \) is
bounded above by a function of \( d \) if and only if \( |H_\alpha| \) is bounded above by a function of \( d(d-1) \). If \((\Gamma^\Delta, H)\) is \( f \)-bounded, then \( |H_\alpha| \leq f(d') \) where \( d' = d(d-1) \) and so \( |G_{(\alpha,i)}| \leq f'(d) \) where \( f'(d) = f(d(d-1)) \). Conversely, if \((\Gamma, G)\) is \( f \)-bounded, then \( |G_{(\alpha,i)}| \leq f(d) \) and so \( |H_\alpha| \leq f(d) = \tilde{f}(d(d-1)) \).\( \square \)

**Theorem 29.** Let \((\Gamma, G)\) be in \( \mathcal{A}(d) \) with \( G \) biquasiprimitive of type (A) or (B) (ii) on \( \text{VT} \) (notation as in Theorem 27). Then Theorem 3 holds for \((\Gamma, G)\).

**Proof.** Assume that \( G \) is of type (A), that is, \( H \) is quasiprimitive on \( \Delta \). In particular, Theorem 4 applies to \((\Gamma^\Delta, H)\). If Part (1) of Theorem 4 holds for \((\Gamma^\Delta, H)\), then from Notation 26 and Lemma 25 we obtain that \((\Gamma, G)\) is \( f \)-bounded with \( f(d) = (d(d-1)(d(d-1))! \) and Part (1) of Theorem 4 holds for \((\Gamma, G)\). Assume that Part (2) of Theorem 4 holds for \((\Gamma^\Delta, H)\) and let \((\Lambda, T)\) be as in Theorem 4(2). We show that Part (2) of Theorem 5 holds by taking \( \Lambda_1 = \Lambda_2 = \Lambda \). If \((\Lambda, T)\) is \( g_i \)-bounded, then by Theorem 4 for each \( i \in \{1,2\} \), we have that \((\Gamma^\Delta, H)\) is \( \hat{g}_i \)-bounded and hence from Notation 27 and Lemma 25 \((\Gamma, G)\) is \( f \)-bounded with \( f(d) = \hat{g}_i(d_0) \) and \( d_0 = d(d-1) \). In particular, \((\Gamma, G)\) is \( \hat{g}_1 \ast \hat{g}_2 \)-bounded. Conversely, assume that \((\Gamma, G)\) is \( f \)-bounded. By Notation 24 \( |H_\alpha| = |G_{(\alpha,1)}| \leq f(d) \) and \((\Gamma^\Delta, H) \in \mathcal{A}(d(d-1)) \). So \((\Gamma^\Delta, H)\) is \( \hat{f} \)-bounded. Now Theorem 4(2) yields that \((\Lambda, T)\) is \( f \)-bounded.

Assume now that \( G \) is of type (B) (ii). From Theorem 24 \( M \) is a regular normal subgroup of \( H \). Therefore from Corollary 13 applied to \( M, H \) and \( \Gamma^\Delta \), we have \( |H_\alpha| \leq (d(d-1))! \) for every \( \alpha \in \Delta \). Hence \((\Gamma, f)\) is \( f \)-bounded with \( f(d) = (d(d-1))! \) and Theorem 5(1) holds for \((\Gamma, G)\).\( \square \)

**Remark 30.** It is worth pointing out here that Theorem 29 together with Remark 17 show that there is a function \( f : \mathbb{N} \to \mathbb{N} \) such that, if \( G \) is of type (A) (with \( H \) a quasiprimitive group of type HA, HS, HC, TW, SD or CD) or if \( G \) is of type (B) (ii), then \((\Gamma, G)\) is \( f \)-bounded. Furthermore a suitable function \( f \) can be explicitly determined from Corollaries 13 and 16 and Theorem 29.

Throughout the remainder this section we assume Notations 26 and 27 and we fix \((\Gamma, G)\) in \( \mathcal{A}(d) \) with \( G \) biquasiprimitive of type (B) (i) on \( \text{VT} \) (notation as in Theorem 23). Write \( S = R^2 \) and recall that \( R \) and \( S \) are intransitive minimal normal subgroups of \( H \). Furthermore \( \text{Soc}(H) = M = R \times S \) is transitive on \( \Delta \).

**Lemma 31.** If \( R \) is abelian, then \((\Gamma, G)\) is \( f \)-bounded with \( f(d) = (d(d-1))! \).

**Proof.** If \( R \) is abelian, then so is \( S \) and \( M \). In particular, \( M \) is an abelian normal transitive subgroup of \( H \). Therefore from Corollary 13 applied to \( M, H \) and \( \Gamma^\Delta \), we have \( |H_\alpha| \leq (d(d-1))! \). Hence \((\Gamma, G)\) is \( f \)-bounded with \( f(d) = (d(d-1))! \).\( \square \)

**Notation 32.** Given Lemma 31 from now on we may assume that \( R \) is the direct product of \( l \) isomorphic non-abelian simple groups, each isomorphic to \( T \) say. Write \( R = T_{R,1} \times \cdots \times T_{R,l} \) and \( S = T_{S,1} \times \cdots \times T_{S,l} \) with \( T_{R,j} \cong T_{S,j} \cong T \) for every \( j \in \{1, \ldots, l\} \). Since \( R \) and \( S \) are minimal normal subgroups of \( H \), the group \( H \) permutes transitively the sets \( \{T_{R,j}\}_j \) and \( \{T_{S,j}\}_j \). Let \( \alpha \in \Delta \) and denote by \( \pi_{R,j,\alpha} : M_\alpha \to T_{R,j} \) the natural projection on the \( j \)th coordinate of \( R \). Similarly, denote by \( \pi_{S,j,\alpha} : M_\alpha \to T_{S,j} \) the natural projection on the \( j \)th coordinate of \( S \).

**Lemma 33.** If \( \pi_{R,j,\alpha} \) or \( \pi_{S,j,\alpha} \) is surjective for some \( j \in \{1, \ldots, l\} \) and for some \( \alpha \in \Delta \), then \((\Gamma, G)\) is \( f \)-bounded with \( f(d) = (d^2((d(d-1))!)^2)! \).
Claim 1. Let $L$ be normal in $H$. If $L_{\alpha} \neq 1$, then $L_{\alpha}^{\Gamma_{\alpha}(\alpha)} \neq 1$. Since $H$ is transitive on $\Delta$, $L$ is normal in $H$ and $L_{\alpha} \neq 1$, we have $L_{\alpha'} \neq 1$ for every $\alpha' \in \Delta$. Fix $\alpha \in \Delta$. As $\Gamma_{\alpha}$ is connected, there exists a vertex $\alpha'$ of $\Gamma_{\alpha}$ such that $L_{\alpha}$ fixes $\alpha'$ and $L_{\alpha}^{\Gamma_{\alpha}(\alpha')} \neq 1$. Replacing $\alpha'$ if necessary, we may assume that $\alpha'$ is chosen so that its distance $r$ from $\alpha$ is minimal. As $L_{\alpha} \leq L_{\alpha'}$ and $\alpha$, $\alpha'$ are conjugate under $H$, we obtain $L_{\alpha} = L_{\alpha'}$. By minimality of $r$, this yields $r = 0$, $\alpha = \alpha'$ and $L_{\alpha}^{\Gamma_{\alpha}(\alpha)} = L_{\alpha}^{\Gamma_{\alpha}(\alpha')} \neq 1$, and the claim is proved.

Claim 2. If $L_{\alpha} = R_{\alpha} \times S_{\alpha}$, $R_{\alpha} \neq 1$ and $R_{\alpha}$ is a minimal normal subgroup of $H_{\alpha}$, then $(\Gamma_{\alpha}, G)$ is $f$-bounded with $f(d) = (d^2((d(d-1)))^2)!$.

As $R_{\alpha} \neq 1$, from Claim 1 we have that $R_{\alpha}$ acts non-trivially on $\Gamma_{\alpha}(\alpha)$. As $R_{\alpha}$ is a minimal normal subgroup of $H_{\alpha}$, we obtain that $R_{\alpha}$ acts faithfully on $\Gamma_{\alpha}(\alpha)$ and hence $|R_{\alpha}| = |\Gamma_{\alpha}(\alpha)|! \leq (d(d-1))!$ for every $\alpha \in \Delta$.

Write $R = \{(r, r^x) \mid r \in R \subseteq G^+ \}$ and $S = \{(s, s^x) \mid s \in S \subseteq G^+ \}$. Since $R^x = S$, $\varphi^x = \epsilon_x$, we have

$$(r, r^x)^g = (r, r^x)^{(x, 1)(1, 2)} = (r^x, r^x)^{(1, 2)} = (r^x, r^x) = ((r^x), (r^x)^x)$$

and $R^g = S$. Therefore $(\hat{R}(\alpha, 1))^g = \hat{S}(\alpha, 1)^g = \hat{S}(\alpha, 2)^g$ and so $|\hat{S}(\alpha, 2)| = |\hat{R}(\alpha, 1)| = |R_{\alpha}| \leq (d(d-1))!$ for every $\alpha \in \Delta$. Let $(\beta, 2)$ be in $\Gamma((\alpha, 1))$. As $S(\alpha, 2) \subseteq S(\beta, 2)$ and $|\hat{S}(\alpha, 1) : \hat{S}(\alpha, 1), (\beta, 2) \leq d$, we obtain $|S_{\alpha}| = |\hat{S}(\alpha, 1)| \leq (d(d(d-1)))!$. Therefore $|M_{\alpha}| = |R_{\alpha} \times S_{\alpha}| \leq d((d(d-1)))!$. From Theorem 12 with $f_3(d)$ as in Remark 14 applied to $H$, $M$ and $\Gamma_{\alpha}$, we get $|H_{\alpha}| \leq (d^2((d(d-1))))!$. Hence $(\Gamma, G)$ is $f$-bounded with $f(d) = (d^2((d(d-1))))!$.

Claim 3. If $\pi_{R, j, \alpha}$ is surjective for some $j \in \{1, \ldots, l\}$ and for some $\alpha \in \Delta$, then $\pi_{R, j, \alpha}$ is surjective for every $j \in \{1, \ldots, l\}$ and for every $\alpha \in \Delta$. A similar claim holds replacing $R$ by $S$.

Assume that $\pi_{R, j, \alpha}$ is surjective for some $j$ and for some $\alpha$. Since $M$ is transitive on $\Delta$ and $M$ acts trivially on the set $\{T_{R,j}\}$, the projection $\pi_{R, j, \alpha}$ is surjective for every $\beta \in \Delta$. Since $H = H_{\alpha}M$, the group $H_{\alpha}$ acts transitively on $\{T_{R,j}\}$. As $M_{\alpha}$ is normal in $H_{\alpha}$, we obtain that $\pi_{R, j, \alpha}$ is surjective for every $j \in \{1, \ldots, l\}$.

Now we continue the proof of the lemma. Replacing $R$ by $S$ if necessary, we may assume that $\pi_{R', j', \alpha'}$ is surjective for some $j'$ and for some $\alpha'$. From Claim 3, $\pi_{R, j, \alpha}$ is surjective for every $j$ and for every $\alpha$. We divide the proof in various cases, depending on whether the projection $\pi_{S, j', \alpha'}$ is also surjective.

Assume that $\pi_{S, j', \alpha'}$ is surjective for some $j'$ and for some $\alpha'$. Then, from Claim 3, the mapping $\pi_{S, j, \alpha}$ is surjective for every $j$ and for every $\alpha$. This yields that $M_{\alpha}$ is a subdirect subgroup of $M \cong T^{2l}$, and hence by Scott’s Lemma (see [18], Lemma, page 328), $M_{\alpha}$ is isomorphic to a direct product of $s \geq 1$ copies of $T$. Namely, $M_{\alpha} = D_1 \times \cdots \times D_s$ where $D_i \cong T$ for each $i$. (Specifically, each $D_i$ is a diagonal subgroup of some subproduct of $T^{2l} \cong M$.)

Suppose first that $H_{\alpha}$ acts transitively on the set $\{D_i\}_i$ by conjugation. Then $M_{\alpha}$ is a minimal normal subgroup of $H_{\alpha}$. This implies that either $M_{\alpha}$ acts faithfully or trivially on $\Gamma_{\alpha}(\alpha)$. Since $M_{\alpha} \neq 1$, Claim 1 gives that $M_{\alpha}$ acts faithfully on $\Gamma_{\alpha}(\alpha)$ and so $|M_{\alpha}| \leq (d(d-1))!$. From Theorem 10 with $f_3(d)$ as in Remark 14 applied to $H$, $M$ and $\Gamma_{\alpha}$, we get $|H_{\alpha}| \leq (d((d(d-1))))!$. Hence $(\Gamma, G)$ is $f$-bounded with $f(d) = (d((d(d-1))))!$. 

Assume now that $H_\alpha$ is intransitive on $\{D_i\}_i$. As $H_\alpha$ permutes transitively the sets $\{T_{R,j}\}_j$ and $\{T_{S,j}\}_j$, the group $H_\alpha$ has two orbits on $\{D_i\}_i$ and each $D_i$ is contained in $R$ or in $S$. Relabelling the indices of the subgroups $D_i$ if necessary, we may assume that $\{D_1, \ldots, D_k\}$ and $\{D_{k+1}, \ldots, D_s\}$ are the two orbits of $H_\alpha$ on $\{D_i\}_i$, with $D_i \subseteq R \cap M_\alpha$ for $i \in \{1, \ldots, k\}$ and $D_i \subseteq S \cap M_\alpha$ for $i \in \{k+1, \ldots, s\}$. Therefore

$$R_\alpha \times S_\alpha \subseteq M_\alpha = (D_1 \times \cdots \times D_k) \times (D_{k+1} \times \cdots \times D_s) \subseteq (R \cap M_\alpha) \times (S \cap M_\alpha) = R_\alpha \times S_\alpha.$$ 

Hence $M_\alpha = R_\alpha \times S_\alpha$, $R_\alpha, S_\alpha \neq 1$ and $R_\alpha, S_\alpha$ are minimal normal subgroups of $H_\alpha$. In particular, from Claim 2 the lemma is proved.

Finally we may assume that, for every $j \in \{1, \ldots, l\}$, $\pi_{S,j,\alpha}$ is not surjective. Let $\pi_{R,\alpha} : M_\alpha \to R$ and $\pi_\alpha : M_\alpha \to S$ be the projections of $M_\alpha$ on $R$ and $S$. Let $K_R$ and $K_S$ be the kernels of $\pi_{R,\alpha}$ and $\pi_\alpha$ respectively. As $\pi_{R,\alpha}$ is surjective for every $j$, the group $M_\alpha/K_R$ is isomorphic to a direct product of $s \geq 1$ copies of $T$, that is, every composition factor of $M_\alpha/K_R$ is isomorphic to $T$. As, for all $j$, $\pi_{S,j,\alpha}$ is not surjective, the group $M_\alpha/K_S$ has no composition factor isomorphic to $T$. Since $M_\alpha/(K_RK_S)$ is a homomorph image of $M_\alpha/K_R$ and of $M_\alpha/K_S$, we obtain that $M_\alpha/(K_RK_S) = 1$ and $M_\alpha = K_RK_S = K_R \times K_S$. It follows that $K_S = M_\alpha \cap R = R_\alpha$ and $K_R = M_\alpha \cap S = S_\alpha$. Therefore $M_\alpha = R_\alpha \times S_\alpha$.

Furthermore, as $R_\alpha \cong M_\alpha/K_R \cong T^s$ and $H_\alpha$ acts transitively on $\{T_{R,j}\}_j$, we see that $R_\alpha$ is a minimal normal subgroup of $H_\alpha$. In particular, from Claim 2 the lemma is proved.

**Notation 34.** From Lemma 33, we may now assume that, for every $j \in \{1, \ldots, l\}$ and for every $\alpha \in \Delta$, $\pi_{R,\alpha}$ and $\pi_{S,j,\alpha}$ are not surjective. Let $\alpha \in \Delta$. Let $R_{U,j} = \pi_{R,\alpha}(M_\alpha)$, $U_{S,j} = \pi_{S,j,\alpha}(M_\alpha)$ and define $U_R = U_{R,1} \times \cdots \times U_{R,l}$, $U_S = U_{S,1} \times \cdots \times U_{S,l}$. By construction, $M_\alpha$ projects surjectively on each of the $2l$ direct factors of $U_R \times U_S$, that is, $M_\alpha$ is a subdirect subgroup of $\prod_j U_{R,j} \times \prod_j U_{S,j}$. Let $\Sigma_R$ be the set of right cosets of $U_R$ in $R$, which we denote by $\Sigma_R = R/U_R$. Similarly, let $\Sigma_S = S/U_S$ be the set of right cosets of $U_S$ in $S$. Since $U_R = \prod_j U_{R,j}$ and $U_S = \prod_j U_{S,j}$, the $R$-action on $\Sigma_R$ and the $S$-action on $\Sigma_S$ admit cartesian decompositions, namely $\Sigma_R = \prod_{j=1}^l (T_{R,j}/U_{R,j})$ and $\Sigma_S = \prod_{j=1}^l (T_{S,j}/U_{S,j})$. By construction, $M = R \times S$ acts transitively and faithfully with product action on $\Sigma = \Sigma_R \times \Sigma_S$.

We claim that $H_\alpha$ normalises $U_R$ and $U_S$. In fact, since $H_\alpha$ normalises $M_\alpha$ and acts transitively on $\{T_{R,j}\}_j$ and on $\{T_{S,j}\}_j$, we get that $H_\alpha$ acts transitively on $\{U_{R,j}\}_j$ and on $\{U_{S,j}\}_j$. In particular, $H_\alpha$ normalises $\prod_{j=1}^l U_{R,j} = U_R$ and $\prod_{j=1}^l U_{S,j} = U_S$ proving the claim. By transitivity, $|T_{R,j} : U_{R,j}|$ does not depend on $j$, and $|T_{S,j} : U_{S,j}|$ does not depend on $j$, that is, $\Sigma_R$ and $\Sigma_S$ are homogeneous cartesian decompositions (a cartesian decomposition $\Lambda_1 \times \cdots \times \Lambda_l$ is said to be homogeneous if $|\Lambda_i| = |\Lambda_j|$ for every $i, j \in \{1, \ldots, l\}$). Furthermore, since $H_\alpha$ normalises $U_R$, we have that $U_R H_\alpha$ is a subgroup of $H$ and hence $H$ preserves the cartesian decomposition $\Sigma_R$. Similarly, $H$ preserves the cartesian decomposition $\Sigma_S$. Since $R$ and $S$ are the only minimal normal subgroups of $H$ and $M$ acts faithfully on $\Sigma = \Sigma_R \times \Sigma_S$, the group $H$ acts faithfully with the natural product action on $\Sigma = \Sigma_R \times \Sigma_S$.

Let $H_R$ and $H_S$ be the permutation groups induced by $H$ on $\Sigma_R$ and on $\Sigma_S$ respectively. So $H \leq H_R \times H_S \leq \text{Sym}(\Sigma_R) \times \text{Sym}(\Sigma_S) \subseteq \text{Sym}(\Sigma)$. Since $R$ is the
only minimal normal subgroup of $H_R$ and $R$ is transitive on $\Sigma_R$, we obtain that $\text{Soc}(H_R) = R$ and $H_R$ is a quasiprimitive group on $\Sigma_R$. Since $\text{Soc}(H_R)$ is the unique minimal normal subgroup of $H_R$ and $\pi_{R,j}(M_\alpha) = U_{R,1} < T$, the quasiprimitive group $H_R$ is of type TW, AS or PA. The group $H_R$ is of type TW if $U_R = 1$ and $l > 1$ (in particular, $R$ acts regularly on $\Sigma_R$), of type AS if $l = 1$, and of type PA if $U_R > 1$ and $l > 1$. Similarly, $H_S$ is a quasiprimitive group of type TW, AS or PA on $\Sigma_S$.

We recall the following result (for a proof see [11, proof of Theorem 1 Case 2(b)]).

**Theorem 35.** Let $G$ be a permutation group on a finite set $\Omega$ preserving a homogeneous cartesian decomposition $\Lambda_1 \times \cdots \times \Lambda_l$ of $\Omega$. Then there is a permutational isomorphism that maps $G$ to a subgroup of $\text{Sym}(\Lambda_1) \wr \text{Sym}(l)$ in its natural product action on $\Lambda_1^l$ and that maps $\Omega$ to the natural cartesian decomposition $\Lambda_1^l$.

To state the next result we need a standard definition. Assume that $G$ is a subgroup of $\text{Sym}(\Lambda) \wr \text{Sym}(l)$ in its natural product action on $\Lambda^l$. Recall that each element of $G$ is of the form $f\sigma$, where $f \in \text{Sym}(\Lambda^l)$ and $\sigma \in \text{Sym}(l)$. We define the $j$th component of $G$ as the permutation group induced by $\{f\sigma \in G \mid j\sigma = j\}$ on the $j$th coordinate of $\Lambda$.

**Theorem 36.** Suppose that $G \leq \text{Sym}(\Lambda) \wr \text{Sym}(l)$ is transitive in its product action on $\Lambda^l$. Then there exists an element $x$ in the base group $\text{Sym}(\Lambda)^l$ and a transitive subgroup $K$ of $\text{Sym}(\Lambda)$ such that the $j$th component of $x^{-1}Gx$ is $K$ for every $j \in \{1, \ldots, l\}$. In particular, $G^x \leq K \wr \text{Sym}(l)$.

(A proof of Theorems 35 and 36 can also be found in [17].) Our next step is to replace, if necessary, the group $H$ by a suitable conjugate to obtain a simpler form for the action of $H$ on $\Sigma$ (see Notation 52).

**Notation 37.** Assume Notation 23 and 54. Applying Theorem 35 and 36 to $H_R$ and $H_S$ separately, we obtain that, up to replacing $H_R$ and $\Sigma_R$, $H_S$ and $\Sigma_S$, by suitable conjugates, we may assume that $\Sigma_R = \Lambda_1^l$, $\Sigma_S = \Lambda_S^l$ (for some sets $\Lambda_R$ and $\Lambda_S$), $H_R \subseteq K_R \wr \text{Sym}(l)$ and $H_S \subseteq K_S \wr \text{Sym}(l)$ in their natural product actions on $\Lambda_1^l$ and $\Lambda_S^l$, where $K_R$ is a transitive subgroup of $\text{Sym}(\Lambda_R)$ and $K_S$ is a transitive subgroup of $\text{Sym}(\Lambda_S)$. Furthermore, since for each $j \in \{1, \ldots, l\}$ the group $T_{R,j}$ is a normal transitive subgroup of the $j$th component of $H_R$ in its action on $\Lambda_R$, we may assume that $K_R$ is almost simple with socle $T = T_{R,j}$. Similarly, we may assume that $K_S$ is almost simple with socle $T$. Let $\pi_{R,j} : H_{R,j} = N_H(T_{R,j}) \to K_R$ and $\pi_{S,j} : H_{S,j} = N_H(T_{S,j}) \to K_S$ be the natural projections. From Theorem 36 we may assume that $\pi_{R,j}$ and $\pi_{S,j}$ are surjective for each $j$. Summing up, there is an $H$-invariant partition $\Sigma$ of $\Delta$ such that

\[ H \subseteq H_R \times H_S \subseteq (K_R \wr \text{Sym}(l)) \times (K_S \wr \text{Sym}(l)) \]

and the faithful action of $H$ on $\Sigma$ is the natural product action on $\Lambda_1^l \times \Lambda_S^l$. In particular, the elements of $H$ can be written as $h = (k_{R,1}, \ldots, k_{R,l}, k_{S,1}, \ldots, k_{S,l})s_{RS}$ with $k_{R,j} \in K_R$ and $k_{S,j} \in K_S$ for each $j$, $s_R$ a permutation of the $l$ labels $\{R, j\}_j$ and $s_S$ a permutation of the $l$ labels $\{S, j\}_j$.

Fix $\lambda_R$ an element of $\Lambda_R$ and $\lambda_S$ an element of $\Lambda_S$. We denote by $\sigma_R$ the element $(\lambda_R, \ldots, \lambda_R)$ of $\Sigma_R = \Lambda_R^l$ and by $\sigma_S$ the element $(\lambda_S, \ldots, \lambda_S)$ of $\Sigma_S = \Lambda_S^l$. Also, we fix $\alpha_0$ a vertex of $V^{T\Delta} = \Delta$ with $\alpha_0$ lying in the block $(\sigma_R, \sigma_S)$ of $\Sigma$. 


Theorem 38. Assume Notations\(^{27,23,34,37}\). Then \(l \) is less than or equal to \((d(d - 1))^d(d - 1)\). Furthermore, \((\Gamma, G)\) uniquely determines two elements \((\Lambda_R, T), (\Lambda_S, T)\) in \(\mathcal{A}(d(d - 1))\) with the stabilisers of the vertices of \(\Lambda_R\) conjugate to \(T_{\Lambda_R}\) and with the stabilisers of the vertices of \(\Lambda_S\) conjugate to \(T_{\Lambda_S}\).

Proof. Let \(\Gamma\) be the quotient graph of \(\Gamma^\Delta\) corresponding to the partition \(\Sigma = \Lambda^l_R \times \Lambda^l_S\) of \(V T^\Delta = \Delta\) and let \(\Gamma^\Sigma((\sigma_R, \sigma_S))\) denote the set of neighbours of \((\sigma_R, \sigma_S)\) in \(\Gamma^\Sigma\). Let \((\eta^i_R, \eta^i_S) = (\lambda^i_R, 1, \ldots, \lambda^i_R, 1, \ldots, \lambda^i_S, 1, \ldots, \lambda^i_S, 1)\), for \(1 \leq i \leq s\), be representatives of the orbits of \(M_{(\sigma_R, \sigma_S)}\) in the action on \(\Gamma^\Sigma((\sigma_R, \sigma_S))\). Since \(M\) is transitive on \(\Delta\) and \(\Sigma = \Sigma_R \times \Sigma_S\) is a system of imprimitivity for \(H\) acting on \(\Delta\), we obtain that \(M_{(\nu_R, \nu_S)}\) is transitive on \((\nu_R, \nu_S)\) for every \((\nu_R, \nu_S) \in \Sigma\). So Lemma\(^{19}\) applies to \(H, M\) and \(\Sigma\), and we have \(s \leq d(d - 1)\). Since \(M_{(\sigma_R, \sigma_S)} = R_{\sigma_R} \times S_{\sigma_S} = (T_{\Lambda_R})^s \times (T_{\Lambda_S})^s\), we get

\[
\Gamma^\Sigma((\sigma_R, \sigma_S)) = \bigcup_{i=1}^{s} (\eta^i_R, \eta^i_S)^{M_{(\sigma_R, \sigma_S)}} \bigcup_{i=1}^{s} (\eta^i_R)^{R_{\sigma_R}} \times (\eta^i_S)^{S_{\sigma_S}}
\]

\[
= \bigcup_{i=1}^{s} (\lambda^i_R, 1)^{T_{\Lambda_R}} \times \ldots \times (\lambda^i_S, 1)^{T_{\Lambda_S}}.
\]

Now we prove five claims from which the theorem will follow.

Claim 1. \(\bigcup_{i=1}^{s} (\lambda^i_R, 1)^{K_{\Lambda_R}} = \bigcup_{i=1}^{s} (\lambda^i_R, 1)^{T_{\Lambda_R}}\) for every \(j \in \{1, \ldots, l\}\). As \(T_{\Lambda_R} \subseteq K_{\Lambda_R}\), the right hand side is contained in the left hand side. Fix \(i \) in \(\{1, \ldots, s\}\) and \(k \) in \(K_{\Lambda_R}\). From Lemma\(^{21}\) we have that \(\pi_{R,j}(H_{R,j} \cap H_{(\sigma_R, \sigma_S)}) = K_{\Lambda_R}\) for each \(j \in \{1, \ldots, l\}\). Hence there exists \(h = (k^1_{R,1}, \ldots, k^l_{R,1}) s \in H_{R,j} \cap H_{(\sigma_R, \sigma_S)}\) with \(k^i_{R,j} = k\). We obtain that \((\eta^i_R, \eta^i_S) h \in \Gamma^\Sigma((\sigma_R, \sigma_S))\). Since \((R, j)s = (R, j),\) the \((R, j)\)th coordinate of \((\eta^i_R, \eta^i_S) h\) is

\[
(\lambda^i_{R,j})^{k^i_{R,j} -1} = (\lambda^i_{R,j})^{k^i_{R,j}} = (\lambda^i_{R,j})^k.
\]

So, from \(^3\), we have \((\lambda^i_{R,j})^k \in \bigcup_{i=1}^{s} (\lambda^i_R, 1)^{T_{\Lambda_R}}\) and Claim 1 follows.

Claim 1. \(\bigcup_{i=1}^{s} (\lambda^i_S, 1)^{K_{\Lambda_S}} = \bigcup_{i=1}^{s} (\lambda^i_S, 1)^{T_{\Lambda_S}}\) for every \(j \in \{1, \ldots, l\}\). The proof of this claim is similar to the proof of Claim 1.

Claim 2. \(\bigcup_{i=1}^{s} (\lambda^i_R, 1)^{T_{\Lambda_R}} = \bigcup_{i=1}^{s} (\lambda^i_R, 1)^{T_{\Lambda_R}}\) for every \(j \in \{1, \ldots, l\}\). Fix \(j \in \{1, \ldots, l\}\). Since the left hand side and the right hand side are \(T_{\Lambda_R}\)-invariant, it suffices to show that \(\lambda^v_{R,j} \in \bigcup_{i=1}^{s} (\lambda^i_R, 1)^{T_{\Lambda_R}}\) for every \(v \in \{1, \ldots, s\}\). (This would prove the left hand side is contained in the right hand side and the same argument proves the reverse inclusion.) Fix \(v \in \{1, \ldots, s\}\). Recall that \(H_{(\sigma_R, \sigma_S)}\) is transitive on \(\{T_{R,j}\}_j\) and hence so is \(H_{(\sigma_R, \sigma_S)}\). Therefore there exists

\[
h = (k_{R,1}, \ldots, k_{S,1}) s \in H_{(\sigma_R, \sigma_S)}\] with \((R, 1)s^{-1} = (R, j)\).

We obtain that \((\eta^v_R, \eta^v_S) h \in \Gamma^\Sigma((\sigma_R, \sigma_S))\). Thus

\[
(\eta^v_R, \eta^v_S) h = (\lambda^v_{R,1}, \ldots, \lambda^v_{S,1})^{(k_{R,1}, \ldots, k_{S,1}) s} = ((\lambda^v_{R,1})^{k_{R,1} - 1}, \ldots, ((\lambda^v_{S,1})^{k_{S,1} - 1})),
\]
Recalling that \((R, 1) s^{-1} = (R, j)\), we see that the \((R, 1)\)th entry of \((\eta_R^i, \eta_S^j)^h\) is \((\lambda_{R,j}^i)^{k_{R,j}}\). Therefore from \(\boxed{}\) we obtain that

\[
(\lambda_{R,j}^i)^{k_{R,j}} \in \bigcup_{i=1}^s (\lambda_{R,1}^i)^{T_{\lambda_R}^i}.
\]

As \(k_{R,j} \in K_{\lambda_R}\), Claim 1\(R\) yields \(\lambda_{R,j}^i v \in \bigcup_{i=1}^s (\lambda_{R,1}^i)^{T_{\lambda_R}^i}\) and Claim 2\(R\) follows. ■

Claim 2\(R\). \(\bigcup_{i=1}^s (\lambda_{S,j}^i)^{T_{\lambda_S}^i} = \bigcup_{i=1}^s (\lambda_{S,1}^i)^{T_{\lambda_S}^i}\) for every \(j \in \{1, \ldots, l\}\).

The proof of this claim is similar to the proof of Claim 2\(R\). ■

Claim 2\(R\) yields that the \(l\) elements \(\{\lambda_{R,j}^i\}_{(R,j)}\) are in at most \(s\) distinct \(T_{\lambda_R}\)-orbits. In particular, for each \(i \in \{1, \ldots, s\}\), the \(R_{\sigma_R}\)-orbit \((\eta_R^i)^{R_{\sigma_R}}\) contains an element with at most \(s\) distinct entries from \(\Lambda_R^i\). Therefore, replacing \(\eta_R^i\) with a suitable element from \((\eta_R^i)^{R_{\sigma_R}}\) if necessary, we may assume that there are at most \(s\) distinct elements of \(\Lambda_R\) among the entries of \(\eta_R^i\). A similar argument applies for \(\eta_S^i\) and we may assume that there are at most \(s\) distinct elements of \(\Lambda_S\) among the entries of \(\eta_S^i\).

Since \(M = R \times S\) is transitive on \(V^T\Delta\), we may choose \(r_i \in R\) and \(s_i \in S\) such that \((\sigma_R, \sigma_S)^{(r_i, s_i)} = (\eta_R^i, \eta_S^i)\). Note that as \(\eta_R^i\) has at most \(s\) distinct entries from \(\Lambda_R^i\), \(\eta_S^i\) has at most \(s\) distinct entries from \(\Lambda_S^i\) and all the entries of \(\sigma_R\) and of \(\sigma_S\) are equal, the elements \(r_i \in R\) and \(s_i \in S\) can be chosen so that their \(t\) coordinates contain at most \(s\) distinct entries from \(T\).

For each \(\beta \in \Gamma^\Delta(\alpha_0)\), let \(r_\beta\) be an element in \(R\) and \(s_\beta\) an element in \(S\) with \(\beta = \alpha_0^{r_\beta s_\beta}\).

Claim 3. \(T_R^i \times T_S^j = \langle r_\beta \mid \beta \in \Gamma^\Delta(\alpha_0) \rangle T_{\lambda_R}^i\) and \(T_S^j = \langle s_\beta \mid \beta \in \Gamma^\Delta(\alpha_0) \rangle T_{\lambda_S}^j\).

Set \(U = \langle r_\beta s_\beta \mid \beta \in \Gamma^\Delta(\alpha_0) \rangle\) and let \(\Gamma\) be the subgraph of \(\Gamma^\Delta\) induced on the set \(\alpha_0^U\). We show that \(\Gamma^\Delta = \Gamma\). By the definitions of \(U\) and \(\Gamma\), we have \(\alpha_0 \in V^T\Delta\) and \(\Gamma(\alpha_0) \subseteq V \Gamma\). Therefore, since \(\Gamma\) is \(U\)-vertex-transitive, every vertex of \(\Gamma\) has valency \(\Gamma(\alpha_0) = |\Gamma^\Delta(\alpha_0)|\). Since \(\Gamma^\Delta\) is connected, this yields \(\Gamma^\Delta = \Gamma\). In particular, \(U\) acts transitively on \(V^T\Delta\) and so \(M = U M_{\alpha_0}\). As \(M_{\alpha_0}\) is a subgroup of \(M_{(\sigma_R, \sigma_S)}\), we have \(M = U M_{(\sigma_R, \sigma_S)}\) and

\[
T_R^i \times T_S^j = M = \langle r_\beta s_\beta \mid \beta \in \Gamma^\Delta(\alpha_0) \rangle M_{(\sigma_R, \sigma_S)} \\
\subseteq \langle \langle r_\beta \mid \beta \in \Gamma^\Delta(\alpha_0) \rangle T_{\lambda_R}^i \rangle \times \langle \langle s_\beta \mid \beta \in \Gamma^\Delta(\alpha_0) \rangle T_{\lambda_S}^j \rangle.
\]

The claim follows. ■

Fix \(\beta \in \Gamma^\Delta(\alpha_0)\). Since \(\beta = \alpha_0^{r_\beta s_\beta} \in \Gamma^\Delta(\alpha_0)\), we get \((\sigma_R, \sigma_S)^{r_\beta s_\beta} \in \Gamma^\Delta(\sigma_R, \sigma_S)\) and hence \((\sigma_R, \sigma_S)^{r_\beta s_\beta} \in (\eta_R^i, \eta_S^j)^{M_{(\sigma_R, \sigma_S)}}\) for some \(i_\beta \in \{1, \ldots, s\}\). In particular, as \(M_{(\sigma_R, \sigma_S)} = R_{\sigma_R} \times S_{\sigma_S}\), there exists \(z_{R,\beta} \in R_{\sigma_R}\), \(z_{S,\beta} \in S_{\sigma_S}\) such that

\[
(\sigma_R, \sigma_S)^{r_\beta s_\beta z_{R,\beta} z_{S,\beta}} = (\eta_R^i, \eta_S^j).
\]

Since by the definitions of the \(r_i\) and \(s_i\) we have \((\sigma_R, \sigma_S)^{r_i s_i} = (\eta_R^i, \eta_S^j)\), we obtain

\[
(\sigma_R, \sigma_S)^{r_\beta z_{R,\beta} r_{i_\beta}^{-1} s_{S,\beta} s_{i_\beta}^{-1}} = (\sigma_R, \sigma_S),
\]

that is to say, \(y_{R,\beta} = r_\beta z_{R,\beta} r_{i_\beta}^{-1} \in R_{\sigma_R}\) and \(y_{S,\beta} = s_{\beta} z_{S,\beta} s_{i_\beta}^{-1} \in S_{\sigma_S}\). Therefore, for every \(\beta \in \Gamma^\Delta(\alpha_0)\), there exists \(i_\beta \in \{1, \ldots, s\}\), \(y_{R,\beta}, z_{R,\beta} \in R_{\sigma_R}\) and \(y_{S,\beta}, z_{S,\beta} \in S_{\sigma_S}\) such that

\[
y_{R,\beta} = y_{R,\beta} r_{i_\beta} (z_{R,\beta})^{-1} \quad \text{and} \quad s_{\beta} = y_{S,\beta} s_{i_\beta} (z_{S,\beta})^{-1}.
\]
Now Lemma [14] and Claim 3 applied to \( \{r_\beta\}_\beta \) and \( \{s_\beta\}_\beta \) imply that \( l \) is less than or equal to \( d(d-1)^{d(d-1)\min\{|T_{R,0}|,|T_{\lambda_0}|\}^{2d(d-1)}}. \)

Set \( M_{R,j} = T_{R,1} \times \cdots \times T_{R,j-1} \times T_{R,j+1} \times \cdots \times T_{R,l} \times S \) and \( M_{S,j} = R \times T_{S,1} \times \cdots \times T_{S,j-1} \times T_{S,j+1} \times \cdots \times T_{S,l} \) for \( j \in \{1,\ldots,l\} \). Since \( M_{R,j} \) is a maximal normal subgroup of \( M \), we obtain that either \( M_{R,j} \) is transitive on \( V \Gamma \Delta \) or \( M_{R,j} \) is 1-closed in \( M \). As \( M_{a_0} \) is a subdirect subgroup of \( M(\sigma_R,\sigma_S) \), we have \( M_{a_0}M_{R,j} = M(\sigma_R,\sigma_S)M_{R,j} = T_{\lambda_0} \times M_{R,j} < M \) and so \( M_{R,j} \) is 1-closed in \( M \). Similarly, \( M_{a_0}M_{S,j} = T_{\lambda_0} \times M_{S,j} \) and \( M_{S,j} \) is 1-closed in \( M \). Therefore, by Definition [9] the pairs \((\Gamma_R^\Delta,M/M_{R,j})\) and \((\Gamma_S^\Delta,M/M_{S,j})\) lie in \( A(d(d-1)) \) for each \( j \in \{1,\ldots,l\} \). Furthermore, since \( H \) acts transitively on \( \{M_{R,j}\}_j \) and \( \{M_{S,j}\}_j \), we obtain that \( \Gamma_R^\Delta \) and \( \Gamma_S^\Delta \), for every \( i,j \in \{1,\ldots,l\} \). This shows that \((\Gamma,G)\) uniquely determines the elements \((\Gamma_R^\Delta,M/M_{R,1})\) and \((\Gamma_S^\Delta,M/M_{S,1})\) of \( A(d(d-1)) \). Finally, the stabiliser in \( M/M_{R,1} \) of the vertex \( \alpha_0^M_{R,1} \) of \( \Gamma_R^\Delta \) is \( M_{a_0}M_{R,1}/M_{R,1} \cong T_{\lambda_R} \). A similar argument holds for \( S \) and the theorem is proved. \( \square \)

Now we are ready to prove Theorem [5].

**Proof of Theorem 5.** If \( G(\alpha,1) \) is transitive on \( \Delta \times \{2\} \), then from Lemma 29 the graph \((\Gamma,G)\) is \( f \)-bounded for \( f(d) = d!d(d-1)! \) and Part (1) of Theorem 5 holds for \((\Gamma,G)\). So assume that this is not the case. Then \( G \) satisfies Part (A) or (B) of Theorem 23. If \( G \) is of type (A) or of type (B) (ii), then the result follows from Theorem 29. Assume that \( G \) of type (B) (i). We use Notations 27, 32, 34 and 37. If \( R \) is abelian or if \( \pi_{R,j,\alpha} \text{ or } \pi_{S,j,\alpha} \) are surjective for some \( j \) and for some \( \alpha \), then from Lemmas 31 and 33 the graph \((\Gamma,G)\) is \( f \)-bounded for \( f \) as in Part (1) of Theorem 5. Finally, assume that \( R \) is non-abelian and \( \pi_{R,j,\alpha} \text{ or } \pi_{S,j,\alpha} \) are not surjective. Let \((\Lambda_R,T) \text{ and } (\Lambda_S,T) \) be as in Theorem 38. We have \( l \leq d_0^{d_0} \min\{|T_{\lambda_R}|,|T_{\lambda_S}|\}^{2d_0} \) with \( d_0 = d(d-1)! \) and with \( \lambda_R \in VA_R, \lambda_S \in VA_S \). Assume that \((\Lambda_R,T) \) is \( g_R \)-bounded and \((\Lambda_S,T) \) is \( g_S \)-bounded for some \( g_R \) and \( g_S \). Then, \( |M(\alpha,\beta)| \leq |M(\sigma_R,\sigma_S)| = (|T_{\lambda_R}|^{|T_{\lambda_S}|})^{d_0} \leq (g_R(d_0)g_S(d_0))^{d_0} \) with \( d_0 = d(d-1)! \). So \((\Gamma^\Delta,M) \) is \( f' \)-bounded for \( f'(d_0) = (g_R(d_0)g_S(d_0))^{d_0} \). Theorem 10 with \( f_3(d) \) as in Remark 11 yields that \((\Gamma^\Delta,H) \) is \( f \)-bounded for \( f(d_0) = (d_0!f'(d_0))! \). From Notation 27

\[ |G(\alpha,1)| = |H_{a_0}| \leq f(d_0) = (d_0!f'(d_0))! \]

\[ = (d_0!g_R(d_0)g_S(d_0))^{d_0} \leq g_{R^*g_{S^*}}(d) \]

and hence \((\Gamma,G)\) is \( g_R^*g_S^* \)-bounded. Conversely, assume that \((\Gamma,G)\) is \( f \)-bounded for some \( f \). By Lemma 28 \((\Gamma^\Delta,H) \) is \( f \)-bounded. As \( M_{a_0} \) is a subdirect subgroup of \( M(\sigma_R,\sigma_S) \cong T_{\lambda_R} \times T_{\lambda_S} \), we have \( |T_{\lambda_R}| \leq |M_{a_0}| \leq |H_{a_0}| \) and similarly \( |T_{\lambda_S}| \leq |H_{a_0}| \). So \((\Lambda_R,T) \) and \((\Lambda_S,T) \) are \( f \)-bounded. \( \square \)

6. **Examples**

**Example 39.** There are many natural examples of \((\Gamma,G) \in A(d)\) admitting a 1-closed subgroup \( N \) where \((\Gamma_N,G/N) \) is \( f \)-bounded and \( N_{a_0} \) is not bounded by a function of \( d \). For instance, let \( X \) and \( Y \) be connected vertex-transitive graphs of valency \( d_X \) and \( d_Y \), respectively. We recall that the **lexicographic product** \( X[Y] \) of \( X \) and \( Y \) is the graph with vertex set \( VX \times VY \) where \((x,y) \) is adjacent to \((x',y') \) if and only if \( x,x' \) are adjacent in \( X \) or \( x = x' \) and \( y,y' \) are adjacent in \( Y \). Note
that \(X[Y]\) is connected of valency \(d = d_Y + d_X |VY|\). Clearly, the wreath product \(G = Aut(Y) \wr Aut(X)\) acts vertex-transitively on \(X[Y]\). If \(N = Aut(Y)^{V_X}\) is the base group of \(G\), then \(G/N \cong Aut(X)\), the normal quotient \(X[Y]/X\) is isomorphic to \(X\) and \(N_{(x,y)} = Aut(Y)_y \times Aut(Y)^{V_X}\{x\}\). In particular, if \(|VX|\) is not bounded by a function of \(d\), then \(N_{(x,y)}\) is not bounded by a function of \(d\). Also, if \((X, Aut(X))\) is \(f\)-bounded, then \((X[Y]/X, G/N)\) is \(f\)-bounded.

As an explicit example take \(X = C_n\) the cycle of length \(n\) and \(Y = K_2\) the complete graph on two vertices. We have \((\Gamma, G) \in \mathcal{A}(4)\) and \((\Gamma_N, G/N)\) is 2-bounded because \(\Gamma_N \cong X\) and \(Aut(X)\) is the dihedral group of order \(2n\). Furthermore, \(|N_{(x,y)}| = 2^{n-1}\) and hence \(N_{(x,y)}\) can be exponential in the number of vertices of \(\Gamma\) with \(G/N\) having stabiliser \(C_2\).

In Examples \([40, 41, 42]\) we use the notation of Theorem \([4]\). In each of the examples \(G\) is a quasiprimitive group of type PA with socle \(T^2\). We denote by \(D_n\) the dihedral group of order \(n\).

**Example 40.** In this example we give an infinite family of \((\Gamma, G)\) vertex-primitive and locally quasiprimitive with \((\Lambda, H)\) not quasiprimitive.

Let \(q\) be a prime power \(q = p^r \geq 4\) and \(n \geq 3\) with \(Gcd(q^2 - 1, n) = 1\). Let \(T\) be the simple group \(\text{PSL}(n, q^2) = \text{SL}(n, q^2)\) and \(H = T \rtimes \langle \tau \rangle\) where \(F\) is the field automorphism of order 2 of \(\mathbb{F}_{q^2}\) and \(\tau\) is the graph automorphism, that is, \(x^\tau = (x^{-1})^{tr}\). Let \(K\) be the group \(\text{C}_H(F) = (\text{SL}(n, q) \rtimes \langle \tau \rangle) \rtimes \langle F \rangle\). From \([7]\), we see that \(K\) is maximal in \(H\). Let \(\Delta\) be the set of right cosets of \(K\) in \(H\) and denote by \(\delta_0\) the coset \(K\) of \(\Delta\). So, \(H\) acts primitively on \(\Delta\). Let \(\lambda\) be an element of order \(q + 1\) in \(\mathbb{F}_{q^2}\) and

\[
x = \begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda^{-1} & 0 \\
0 & 0 & I_{n-2}
\end{pmatrix}.
\]

Denote by \(\delta_1\) the coset \(Kx\) in \(\Delta\). We claim that \(H_{\delta_0}\) acts faithfully on the suborbit \(\delta_1^{H_{\delta_0}}\) of \(H\). By our choice of \(x\), the element \(F\) does not fix \(\delta_1\). Also, it is easy to find elements of \(\text{SL}(n, q)\) not fixing \(\delta_1\). Since \(\text{SL}(n, q)\) and \(\langle F \rangle\) are the only minimal normal subgroups of \(H_{\delta_0}\), our claim is proved. Now we claim that \(\text{SL}(n, q)\) acts transitively on \(\delta_1^{H_{\delta_0}}\). It is easy to check that the element

\[
Fy, \quad \text{with} \quad y = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & I_{n-2}
\end{pmatrix},
\]

of \(H_{\delta_0}\) fixes \(\delta_1\). Similarly, \(\tau y\) fixes \(\delta_1\). Since \(H_{\delta_0} = \text{SL}(n, q) \langle Fy, \tau y \rangle\), we get that \(\text{SL}(n, q)\) is transitive on \(\delta_1^{H_{\delta_0}}\). Finally, a direct computation shows that \((\delta_0, \delta_1)^{H_{\delta_0}} = (\delta_1, \delta_0)\). Let \(\Lambda\) be the \(H\)-orbital graph containing the arc \((\delta_0, \delta_1)\). Since \(H\) is primitive, \(\Lambda\) is connected.

We have shown that \(\Lambda\) is an undirected \(H\)-arc-transitive graph, that \(H\) acts primitively on \(\text{VA}\) and that \(H_{\delta_0}\) acts faithfully on \(\Lambda(\delta_0)\). Also, \(\text{SL}(n, q)\) acts transitively on \(\Lambda(\delta_0)\) and \(\langle F \rangle\) acts intransitively and semiregularly on \(\Lambda(\delta_0)\). In particular \((\Lambda, H)\) is not locally quasiprimitive.

Let \(W\) be the wreath product \(H \rtimes \text{Sym}(2)\) endowed with the product action on \(\Omega = \Delta^2\). Write \(W = (H \times H) \rtimes \langle \pi \rangle\), where \(\pi^2 = 1\) and \((h, 1)^\pi = (1, h)\) for \(h \in H\). Let \(T\) be the socle of \(H\) and \(N = T^2\) the socle of \(W\). Consider \(G = N \langle (F, \tau), (\tau, F), \pi \rangle\). Note that each of \((F, \tau), (\tau, F), \pi\) has order 2 and \((F, \tau)^\pi = (\tau, F)\), so \(G/N \cong D_8\).
The projection of $N_G(T \times 1) = (T \times T)/\langle (F, \tau), (\tau, F) \rangle$ onto the first coordinate is the whole of $H$. As $G$ contains $\langle N, \pi \rangle$ and as $H$ is primitive on $\Delta$, we obtain that $G$ acts primitively on $\Omega$.

Let $\Gamma$ be the $W$-orbital graph corresponding to the suborbit $\delta_1^{H_{\delta_0}} \times \delta_1^{H_{\delta_0}}$ of $W_{(\delta_0, \delta_0)}$. Since $G$ is primitive, $\Gamma$ is connected and since $\Lambda$ is undirected, so is $\Gamma$.

As $T_{\delta_0} = SL(n, q)$ is transitive on $\delta_1^{H_{\delta_0}}$, the group $N_{(\delta_0, \delta_0)}$ acts transitively on $\Gamma((\delta_0, \delta_0))$. Therefore the graph $\Gamma$ is $G$-arc-transitive.

We claim that $G_{(\delta_0, \delta_0)}$ is quasiprimitive on $\Gamma((\delta_0, \delta_0))$. We have

$$G_{(\delta_0, \delta_0)} = (T_{\delta_0} \times T_{\delta_0})/\langle (F, \tau), (\tau, F), \pi \rangle.$$ 

Let $X$ be a normal non-trivial subgroup of $G_{(\delta_0, \delta_0)}$. As $T_{\delta_0}$ is simple and $\pi \in G_{(\delta_0, \delta_0)}$, the group $N_{(\delta_0, \delta_0)} = T_{\delta_0} \times T_{\delta_0}$ is a minimal normal subgroup of $G_{(\delta_0, \delta_0)}$. If $X \cap N_{(\delta_0, \delta_0)} \neq 1$, then by minimality $N_{(\delta_0, \delta_0)} \subseteq X$ and $X$ is transitive on $\Gamma((\delta_0, \delta_0))$.

If $X \cap N_{(\delta_0, \delta_0)} = 1$, then $X$ centralises $N_{(\delta_0, \delta_0)}$. The centraliser of $N_{(\delta_0, \delta_0)}$ in $W_{(\delta_0, \delta_0)}$ has order 4 and is generated by $(F, 1), (1, F)$. Since $\langle (1, F), (F, 1) \rangle \cap G = 1$, no non-trivial element of $G_{(\delta_0, \delta_0)}$ centralises $N_{(\delta_0, \delta_0)}$. Hence $X = 1$, a contradiction. This proves that $G_{(\delta_0, \delta_0)}$ is quasiprimitive on $\Gamma((\delta_0, \delta_0))$.

**Example 41.** In this example we give $(\Gamma, G)$ locally semiprimitive with $\langle \Lambda, T \rangle$ not semiprimitive. Let $T$ be the simple group $SL(3, 9)$ and let $T = x = (F, \tau) \in G$ be the Frobenius automorphism of $T$ and $\pi$ is the automorphism of $T$ defined by $x^\pi = x^{-1}$. Let $C$ be a Singer cycle of $T$. Since $|C| = (9^3 - 1)/(9 - 1) = 7 \cdot 13$, we have $C = \langle x, y \rangle$ where $x$ has order 7 and $y$ has order 13. The normaliser of $C$ in $T$ is $C \rtimes z$ for some $z$ of order 3. From [4], we get that the normaliser $N$ in $H = C \rtimes z$. From [4], we get that $N$ is the unique proper subgroup of $H$ containing $K$ and $N \cap T = C \times z$ is the unique proper subgroup of $T$ containing $C$. In particular, $N$ is a maximal subgroup of $H$ and $N \cap T$ is a maximal subgroup of $T$.

Let $\Delta$ be the set of right cosets of $K$ in $H$ and denote by $\delta_0$ the coset $Kx$ in $\Delta$. So, $H$ acts quasiprimitively on $\Delta$. Let $x$ be an involution of $T$ such that $x^\pi = xF = x$ and denote by $\delta_1$ the coset $Kx$ in $\Delta$. From [4] we see that $H_{\delta_0}$ acts faithfully on the suborbit $\delta_1^{H_{\delta_0}}$ and $(H_{\delta_0}, \delta_1) = (F, \tau)$. In particular, $T_{\delta_0}$ acts regularly on $\delta_1^{H_{\delta_0}}$.

Since $x^2 = 1$, we get $(\delta_0, \delta_1) = (\delta_1, \delta_0) = (\delta_1, \delta_0)$. Let $\Lambda$ be the $H$-orbital graph containing the arc $(\delta_0, \delta_1)$. Since $N$ is the unique proper subgroup of $H$ containing $K$ and since $x / N$ (because $|N : K| = 3$), the graph $\Lambda$ is connected. As $T_{\delta_0}$ is transitive on $\Lambda(\delta_0)$, the group $T$ acts arc-transitively on $\Lambda$. Furthermore $\langle x, F \rangle$ and $\langle y, \tau \rangle$ are normal intransitive and non semiregular subgroups of $H_{\delta_1}$, so $(\Lambda, H)$ is not locally semiprimitive.

Let $W$ be the wreath product $H \wr Sym(2)$ endowed with the product action on $\Omega = \Delta^2$. Write $W = (H \times H) \rtimes \{\pi\}$, where $\pi^2 = 1$ and $(h, 1)^\pi = (1, h)$ for $h \in H$. Let $T$ be the socle of $H$ and $N = T^T$ the socle of $W$. Consider $G = N/\langle (\tau, F), (F, \tau), \pi \rangle$. Note that each of $(\tau, F), (F, \tau)$, $\pi$ has order 2 and $(\tau, F)^\pi = (F, \tau)$, so $G/N \cong D_4$. The projection of $N_G(T \times 1) = N/\langle (\tau, F), (F, \tau) \rangle$ onto the first coordinate is the whole of $H$. As $G$ contains $\langle N, \pi \rangle$ and as $H$ is quasiprimitive on $\Delta$, we obtain that $G$ acts quasiprimitively on $\Omega$. 


Let $\Gamma$ be the $W$-orbital graph corresponding to the suborbit $\delta_1^{H_{l_0}} \times \delta_1^{H_{l_0}}$ of $W_{(l_0, l_0)}$. We claim that $\Gamma$ is connected, that is, $G = \langle G_{(l_0, l_0)}, (x, x) \rangle$. We have

$$G_{(l_0, l_0)}N_{(l_0, l_0)}\langle (\tau_1, F_2), (F_1, \tau_2), \pi \rangle.$$  

Since the only proper subgroup of $T$ containing $T_{l_0} = C$ is $C \rtimes \langle z \rangle$ and $x \notin C \rtimes \langle z \rangle$, we get $T = \langle T_{l_0}, x \rangle$. Hence $N = \langle N_{(l_0, l_0)}, (x, x) \rangle$. Therefore $G = \langle G_{(l_0, l_0)}, (x, x) \rangle$.

Since $\Lambda$ is undirected, so is $\Gamma$. As $\alpha$ is a non-trivial normal subgroup of the dihedral group $G$ of degree 9, we claim that $G_{(l_0, l_0)}$ acts regularly on $\Gamma((l_0, l_0))$. In particular, the graph $\Gamma$ is $G$-arc-transitive.

We claim that $G_{(l_0, l_0)}$ is semiregular on $\Gamma((l_0, l_0))$. Let $L$ be a normal non-trivial subgroup of $G_{(l_0, l_0)}$. We have to prove that $L$ is either transitive or semiregular. Assume that $L$ is not semiregular. So without loss of generality we may assume that some non-identity element $l$ of $L$ fixes $\beta = (\delta_1, \delta_1)$. As $N_{(l_0, l_0)}$ acts regularly on $\Gamma((l_0, l_0))$, we have $l \in \langle (\tau, F), (F, \tau), \pi \rangle$. The group $LN_{(l_0, l_0)}/N_{(l_0, l_0)}$ is a non-trivial normal subgroup of $\Gamma((l_0, l_0))$. Hence $LN_{(l_0, l_0)}/N_{(l_0, l_0)}$ contains the center of $G_{(l_0, l_0)}/N_{(l_0, l_0)}$, that is, $(\tau F, F \tau)N_{(l_0, l_0)} \in LN_{(l_0, l_0)}N_{(l_0, l_0)}$. Hence we may assume that $l = (\tau F, F \tau)$. Now, $L$ contains the element

$$l(x, 1)l^{-1} = (x^{-1}, 1)(\tau F, F \tau)(x, 1)(F \tau, \tau F) = (x^{-1} \tau F F \tau F, 1) = (\tau x^{-1} F x F \tau, 1) = (\tau F x^2 F \tau, 1) = (\tau x^{-2} F, 1) = (x^{-2}, 1).$$

Therefore $L$ contains $(x, 1)$. A similar computation shows that $L$ contains $(1, x)$, $(y, 1)$ and $(1, y)$. Also $L$ contains $N_{(l_0, l_0)}$. In particular, $L$ is transitive on $\Gamma((l_0, l_0))$.

**Example 42.** This remarkable example is described in detail in [16, Example 16] and we recall here some significant properties related to the work in this paper. We refer to [16] for the proofs of our claims.

Let $H = \text{Sym}(10)$, $x = (1, 2, 3)(4, 5, 6)(7, 8, 9)$, $y = (1, 4, 7)(2, 5, 8)(3, 6, 9)$, $z = (2, 3)(5, 6)(8, 9)$, $t = (4, 7)(5, 8)(6, 9)$ and $\iota = (1, 10)$. Write $K = \langle x, y, z, t, \iota \rangle$. Clearly, $K = \langle x, z \rangle \times \langle y, t \rangle \cong \text{Sym}(3)^2$. Let $\Delta$ be the $H$-set $H/K$ and $\Lambda$ be the orbital graph $(K, K\iota)^H$. The graph $\Lambda$ is connected, $H$-arc-transitive, vertex-quasiprimitive and the local action is the natural product action of $\text{Sym}(3) \times \text{Sym}(3)$ of degree 9, which is not quasiprimitive.

Let $W$ be the wreath product $W := \text{wr Sym}(2) = (H \times H) \rtimes \langle \pi \rangle$ where $\pi^2 = 1$ and $(h_1, h_2)^\pi = (h_2, h_1)$ for $h_1, h_2 \in H$. Let $T$ be the socle of $H$ and $N = T^2$ the socle of $W$. Consider $G = N \rtimes \langle \pi, (l, t) \rangle$ and the subgroup $L = \langle (x, y), (y, x), (z, t), (l, z), \pi \rangle$ of $G$. The projection of $N_{C}(T \times 1) = N \langle (l, t) \rangle$ onto the first coordinate of $H^2$ is the whole of $H$; furthermore, $|L| = 72$ and $L$ is isomorphic to $\text{Sym}(3) \rtimes \text{Sym}(2)$. Let $\Omega$ be the $G$-set $G/L$. The group $G$ is quasiprimitive of type PA with socle $N$ in its action on $\Omega$.

Denote by $\alpha$ the element $L$ of $\Omega$, by $\beta$ the element $L(lzt, l)$ of $\Omega$ and by $\Gamma$ the $G$-orbital graph $(\alpha, \beta)^G$. The graph $\Gamma$ is connected, $G$-arc-transitive, vertex-quasiprimitive and the $G$-local action is the natural primitive action of $\text{Sym}(3) \rtimes \text{Sym}(2)$ of degree 9. Finally, since the projection of $L \cap (H \times H)$ on the first coordinate is exactly the group $K$, the graph uniquely determined by $\langle (\Gamma, G) \rangle$ in Theorem [4] is $\Lambda$. Therefore in this example we have $\langle (\Gamma, G) \rangle$ locally primitive with $(\Lambda, T)$ arc-transitive, but not even locally quasiprimitive.
References


[27] H. Wielandt, Permutation groups through invariant relations and invariant functions, Ohio State University, Columbus, Ohio, 1969. Reprinted in: Wielandt, Helmut, Mathematische

Cheryl E. Praeger, School of Mathematics and Statistics, The University of Western Australia, Crawley, WA 6009, Australia
E-mail address: praeger@maths.uwa.edu.au

Pablo Spiga, School of Mathematics and Statistics, The University of Western Australia, Crawley, WA 6009, Australia
E-mail address: spiga@maths.uwa.edu.au

Gabriel Verret, Institute of Mathematics, Physics, and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia
E-mail address: gabriel.verret@fmf.uni-lj.si