Decision Support

Flexible supply contracts under price uncertainty

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Abstract

This article develops supply contracts covering environments with changing prices. We investigate characterization properties of the price processes, while considering costs and discount factors. We determine expressions of the contract’s expected low price and its second moment for a given horizon. We then employ these expected price and second moment values to identify an expected optimum time before the contract expires at which the lowest price occurs. Simulation experiments verify our analysis, and they illustrate how the optimum purchase time decreases as the drift term increases.

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1. Introduction

The purpose of this study is to investigate supply contracts subject to an environment of uncertain prices. Purchase prices fluctuate for a variety of reasons, including exchange rate movements, uncertainty of supply, lack of a futures market, information disclosure, hyperinflation conditions, technical developments, political events, environmental influences, and changing risk preferences of consumers. For example, floating exchange rates may cause a buyer to pay substantially more or less than the original contract price (Carter and Vickery, 1988), especially when the contract terms are expressed at an agreed-upon purchase price in the supplier’s home currency. Carter and Vickery (1988) report that more than 50% of surveyed firms used some form of risk-sharing agreements with their supplier. Nevertheless, “For the numerous purchasing managers of a global manufacturing concern, the presence of risk-sharing agreements still implies purchasing price uncertainty, even if there exists a contract at an agreed-upon purchase price in the buyer’s home currency” (Arcelus et al., 2002).

We consider the situation where a firm signs a contract with its supplier for the purchase of a certain amount of a material in order to satisfy its customers’ future demand. We assume that a deterministic demand $D$ needed by time $T$ is fixed. Further, we assume that the firm specifies the amount of material needed, but at
the same time, we assume that the time of purchasing the material should be flexible within the period \([0, T]\). A “time flexible” contract allows the firm to specify the purchase amount over a given period without specifying the exact time of purchase. Fixed-quantity contracts can arise in numerous settings, for example, purchasing supplies in response to contractual commitments with the buyer’s customers, or purchasing supplies to prepare for a fixed six-month production plan, etc. Examples of time-flexible contracts from industry include HP (Nagali et al., 2002) and Ben and Jerry’s entry into the Japanese market (Hagen, 1999).

Some recent literature analyzing supply contracts of a specific form include Lee and Nahmias (1993), Puerto et al. (1990), Tsay et al. (1999), Bassok and Anupindi (1997), Li and Kouvelis (1999) and Milner and Kouvelis (2005), just to name a few. Our study is closely related to Li and Kouvelis (1999).

In this article, and consistent with Dixit and Pindyck (1994), Hull (1997), Li and Kouvelis (1999) and Kamrad and Siddique (2004), we assume that the market material price per unit satisfies the usual Black-Scholes equation. The price is a process \( \{ S(t), t \geq 0 \} \), which can be expressed by the stochastic differential equation

\[
dS(t) = S(t)\{ \mu dt + \sigma dW(t) \}, \quad t \geq 0, \tag{1.1}
\]

where \( \mu \in \mathbb{R} \) denotes the usual appreciation rate and \( \sigma \in \mathbb{R}^+ \) represents the volatility rate. For purely modeling purposes, we assume that both the appreciation and volatility rates are constants. The process \( \{ W(t), t \geq 0 \} \) is a standard Brownian motion defined on the filtered probability space \((\Omega, \mathcal{F}, \mathbb{P} = \{ \mathcal{F}_t, t \geq 0 \}), \mathbb{P})\), where \( \Omega \) is a space of continuous functions such that \( W(0) = 0 \) a.s., \( \mathbb{E}[W(t)] = 0 \) and \( \mathbb{E}[W(t)^2] = t, t > 0 \). The geometric Brownian process \( \{ S(t), t \geq 0 \} \) is by now well known in financial economics and is routinely used to model prices under uncertainty (see, e.g., Karatzas and Shreve, 1988 or Øksendal, 1995).

Continuous time models built out of Brownian motion play a crucial role in modern mathematical finance. These models provide the basis of most option pricing, asset allocation and term structure theory currently being used. The examples referred to above have been routinely modeled in the literature, using as a basis model \((1.1)\). These models imply that the log-returns over intervals of length \( \delta > 0 \) are normal and independently distributed with a mean of \( (\mu - \sigma^2/2)\delta \) and a standard deviation of \( \sigma \). Unfortunately, for moderate to large values of \( \delta \) (corresponding to returns measured over five-minute to one-day intervals), returns are typically heavy tailed, exhibit volatility clustering and are skewed. For higher values of \( \delta \), a central limit theorem seems to hold and so Gaussianity becomes a less poor assumption for the log-returns (see, e.g., Campbell et al., 1997). This means that at this “macroscopic” time scale every single assumption underlying the Black-Scholes model is routinely rejected by the type of data usually seen in practice. Given the empirical facts, we strive to improve model \((1.1)\) by adding a compound Poisson process into \((1.1)\) (see e.g. Dufresne and Gerber, 1993). This extension coincides with jump diffusion processes and constitutes the family of Lévy processes, i.e., processes expressed as a linear combination of Brownian motion and a pure jump process. Thus, Lévy sample paths are more credible to fit asset prices over time than the traditional standard Brownian motions with drifts.

To sketch how supply contracts under price uncertainty work, we assume that the firm will pay the supplier \( S_t > 0 \) dollars per unit when purchasing at time \( t \). The total number of units needed for the project to complete by time \( T \) is \( D \). The \( D \) units are not necessarily purchased at the same time. The time flexible contract allows them to be partitioned throughout the period \( T \). The purchasing cost of the unit is a function of the spot price at a purchased time \( t \). Thus, given a supply contract, the firm’s decision is to determine when each purchase occurs and how many units are required to be purchased each time, such that the expected net present value \((\text{NPV})\) of the purchasing cost plus the inventory holding cost is minimized. This is referred to as the discounted total cost at time \( t \). The purchasing cost of the material and the inventory holding cost are discounted at a positive function of the holding coefficient “\( h \)” and the difference \( T - t \). Obviously, when \( t = T \), \( g(h; 0) = 0 \), i.e., there is no holding cost. We let \( g(\cdot) \) be differentiable at its second argument. Thus, the discounted total cost \( \text{DTC} = \{ \text{DTC}(t), t \geq 0 \} \) per unit is expressed by

\[
dB(t) = B(t)\mu dt \quad \text{or} \quad B(t) = S_0 e^{rt}, \quad t > 0. \tag{1.2}
\]

If the firm purchases a unit at time \( t \) and uses it to satisfy the demand at time \( T \), then the purchasing cost becomes \( S_t \) and the holding cost for the same unit is \( S_t \exp(g(h; T - t - 1)) \), where \( g(h; T - t) \) is a continuous positive function of the holding coefficient “\( h \)” and the difference \( T - t \). Obviously, when \( t = T \), \( g(h; 0) = 0 \), i.e., there is no holding cost. We let \( g(\cdot) \) be differentiable at its second argument. Thus, the discounted total cost

\[
\text{DTC} = \{ \text{DTC}(t), t \geq 0 \} \text{ per unit is expressed by }
\]
Using the above formulation, the buyer should decide what instant of time to pay the supplier. It is therefore of interest to study those periods of time that the process DTC spends below certain levels. In particular, conditioning upon DTC being below its initial level, it is of concern to examine the expected duration that the process remains below that level. Ideally, it would be of use to provide table values below the initial process value with their corresponding probability values as a function of time \( t \in [0, T] \). Further, knowledge of the unconditional and conditional probabilities related to the expected minimum of the process by time \( T \) (and even for times earlier than \( T \)) would play a significant role in the firm’s decision making. To that end, we will propose extensive analytical procedures in determining the expected minimum of the process and its variance. We then utilize these quantities to obtain an optimum time, prior to the time the discounted total cost process achieves its minimum. In Section 4 we demonstrate via simulations the existence of an optimum stopping time, which varies according to the drift term. Finally, we offer concluding remarks in Section 5.

2. The discounted total cost process

We begin with some preliminary steps. Specifically, in solving Eq. (1.1), one simply uses that the price process \( S \) satisfies

\[
S(t) = S(0) \exp \left\{ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W(t) \right\}, \quad \mu \in \mathbb{R}, \ \sigma > 0 \text{ and } t > 0. \tag{2.1}
\]

Using the one-dimensional Ito’s formula, one can then have that

\[
d\text{DTC}(t) = \left( \frac{\partial \text{DTC}(t)}{\partial t} + \mu S(t) \frac{\partial \text{DTC}(t)}{\partial S(t)} + \frac{1}{2} \sigma^2 \frac{\partial^2 \text{DTC}(t)}{\partial S(t)^2} \right) dt + \sigma \frac{\partial \text{DTC}(t)}{\partial S(t)} dW(t),
\]

which leads to

\[
d\text{DTC}(t) = \text{DTC}(t) \left\{ \left( \mu - r - \frac{\sigma^2}{2} \right) dt - \frac{\partial g(h; T-t)}{\partial t} + \sigma dW(t) \right\}. \tag{2.2}
\]

The solution of (2.2) as in (2.1) can be then shown to satisfy

\[
\text{DTC}(t) = \text{DTC}(0) \exp \left\{ \left( \mu - r - \frac{\sigma^2}{2} \right) t + g(h; T-t) + \sigma W(t) \right\}, \quad \mu, r \in \mathbb{R}, \ \sigma > 0 \text{ and } t > 0. \tag{2.3}
\]

It is analytically convenient to analyze the log-DTC price instead of the actual discounted total cost process. Thus, using once more Ito’s formula, it can be shown that

\[
Y(t) := \ln \text{DTC}(t) = Y(0) + \left( \mu - r - \frac{\sigma^2}{2} \right) t + g(h; T-t) + \sigma W(t), \quad \mu \in \mathbb{R}, \ \sigma > 0 \text{ and } t > 0. \tag{2.4}
\]

Assuming that \( g \) is purely linear (see e.g., Li and Kouvelis, 1999), i.e., \( g(h; T-t) = h(T-t) \), Eq. (2.4) is then formed as

\[
Y(t) = Y(0) + hT + \left( \mu - r - \frac{\sigma^2}{2} \right) t + \sigma W(t) = \sigma \{ u + \theta t + W(t) \}, \tag{2.5}
\]

where \( u := \frac{Y(0) + hT}{\sigma} \) and \( \theta = \frac{\mu - r - \frac{\sigma^2}{2}}{\sigma^2} \) with \( \mu, r, h \in \mathbb{R}^+ \) and \( \sigma > 0 \).

It is known that \( \frac{\sigma^2}{2} \) represents the market price of risk. Further, transforming (2.5), the process of interest then becomes \( X(t) := Y(t)/\sigma = u + \theta t + W(t) \), with \( W(t), t > 0 \), being the standard Brownian motion shifted at \( u (u > 0) \), and with drift \( \theta \).
3. Development of the optimum rule

In this section we begin by preparing a few elementary inequalities of the distribution of the log-DTC. We establish the distribution of the minimum of the log-DTC, which is further utilized to obtain the first two moments. Finally, we demonstrate the existence of an optimum stopping time, and we provide a mechanism to compute the expected optimum time.

Using the Kolmogorov’s forward equation, it is not hard to see that the diffusion Eq. (2.5) satisfies

\[ \frac{\partial}{\partial t} f = -\theta \frac{\partial}{\partial y} f + \frac{\partial^2}{2\partial y^2} f, \]

where \( f(t, y|s, x) = \frac{\partial}{\partial s} P(X(t) \leq y|X(s) = x) \), \( \theta = \frac{e^{-\theta^2/2}}{\sqrt{2\pi}} \) and \( x \in \mathbb{R} \) and \( 0 \leq s < t \). This, in return, shows that the transition probability distribution of the process \( X(t), t \geq 0 \), satisfies the usual normal law.

In light of the above partial differential equation, we observe that the following elementary inequalities for the process \( X(t), t \geq 0 \) are satisfied.

**Lemma 1.** The process \( X = \{X(t), t \geq 0\} \) satisfies

\[
P(X(t) \leq u) \begin{cases} < \frac{1}{2} & \theta > 0 \\ = \frac{1}{2} & \theta = 0 \\ > \frac{1}{2} & \theta < 0. \end{cases}
\]

Note that as \( t \) increases, the probability that the process will stay below level \( u \) by time \( t \) tends to zero, remains equal to \( \frac{1}{2} \), or tends to one according to \( \theta \) being greater than, equal to, or smaller than zero, respectively. To understand the speed at which the probability \( P(X(t) \leq u), \theta > 0 \), tends to zero as a function of the time \( t \), the following inequality is of great help:

\[
\frac{\theta \sqrt{t}}{1 + \theta^2 t} \phi(\theta \sqrt{t}) < P(X(t) \leq u) < \frac{1}{\theta \sqrt{t}} \phi(\theta \sqrt{t}), \quad t \in [0, T].
\]

(3.1)

When \( \theta < 0 \), the inequalities in (3.1) are reversed. Along the same vein, one can also show that the process \( X \) drifts towards \( \infty \), oscillates or drifts towards \( -\infty \), as \( \theta \) is positive, zero or negative, respectively. For the case of \( \theta > 0 \), it is clear that the process \( X \) will stay below the level \( u \) (if it stays) for only a finite amount of time. Eventually, it will cross the \( u \) level and will finally drift towards infinity. Similar arguments also hold if \( \theta < 0 \) but the statements above are now reversed. It is, however, clear that the case of \( \theta < 0 \) cannot occur in most contract scenarios.

To develop the following property, let \( I(T) := \inf_{t \leq T} X(t) \) denote the minimum of the process \( X(t), t \geq 0 \) by time \( T \). Incorporating Dassios (1995), it can be shown that for \( \theta \in \mathbb{R} \) and \( T \geq 0 \), the minimum \( I(T) \) is absolutely continuous, i.e., \( P(I(T) \in dx) = f_I(x) dx \) (using the reflection principle) with probability density function given by

\[
f_I(x) = \begin{cases} \left( \frac{2}{\pi \theta^2} \right)^{1/2} \exp \left( -\frac{(x-\theta^2)^2}{2\theta^2} \right) + 2\theta \exp (2\theta(x-u)) \phi \left( \frac{x-u}{\sqrt{\theta^2}} \right), & x < u \\ 0, & x \geq u, \end{cases}
\]

(3.2)

where \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy \), and \( x \in \mathbb{R} \).

Substituting \( \theta = 0 \) into (3.2), one can then demonstrate that

\[
P(I(T) \in dx) = \begin{cases} \left( \frac{2}{\pi} \right)^{1/2} \exp \left( -\frac{(x-\theta^2)^2}{2\theta^2} \right) dx, & x < u \\ 0, & x \geq u. \end{cases}
\]

(3.3)

The density of log-DTC in (3.3) represents the case where the process oscillates around its initial level.

In view of (3.2), Theorem 1 below establishes computational expressions for the first two moments of the random variable \( I(T) \).
Theorem 1. The first two moments of \( I(T) \), the standardized minimum of log-DTC process by time \( T \geq 0 \), are given by

\[
E[I(T)] = \left( u - \frac{1}{2\vartheta} \right) \Phi \left( \sqrt{\frac{T}{2\pi}} - \frac{1}{2} \left( u - \frac{1}{2\vartheta} \right) \right) \exp \left( -\frac{u^2}{2\vartheta^2} + 2(u + \vartheta) \Phi \left( -\sqrt{\frac{T}{2\pi}} \right) \right),
\]

\[
E[I^2(T)] = \left[ u^2 - \frac{7u}{2\vartheta} + \frac{1}{2\vartheta^2} \right] \Phi \left( \sqrt{\frac{T}{2\pi}} \right) + \left\{ \frac{T^{3/2}}{2} - 2(u + \vartheta)^2 \right\} \Phi \left( -\sqrt{\frac{T}{2\pi}} \right) + \left\{ \frac{5T^{3/2}}{2\sqrt{2\pi}} - \frac{T}{2} + 2 \left( u + \frac{1}{\vartheta} \right) \sqrt{\frac{T}{2\pi}} - \frac{1}{2} \left( u - \frac{1}{\vartheta} \right) \left( u - \frac{1}{2\vartheta} \right) + 1 \right\} \exp \left( -\frac{u^2}{2\vartheta^2} \right),
\]

where \( u = Y(0)^{1/\sigma} + \frac{k\pi}{\sigma} \) and \( \theta = \frac{\mu - k - \sigma^2/2}{\sigma} \in \mathbb{R} \).

Furthermore, as \( T \to \infty \), then

\[
\lim_{T \to \infty} E[I(T)] = \left( u - \frac{1}{2\vartheta} \right)
\]

and for \( u < (100)^{-1} \),

\[
\lim_{T \to \infty} \text{Var}(I(T)) = -\frac{5u}{2\vartheta} + \frac{1}{4\vartheta^2}.
\]

Proof. See Appendix. □

The following result presents a stochastic integral representation of the infimum of the log-DTC. It will play a significant role in identifying an optimum time before the horizon \( T \), at which the loss function becomes minimum. Specifically, if \( \mathcal{M} \) denotes a family of Markov times with respect to \( \sigma \)-algebra \( \mathcal{F} \), then \( \tau^* \in \mathcal{M} \) is chosen such that

\[
E[(I(T) - X(\tau^*))^2] = \inf_{\tau \in \mathcal{M}} E[(I(T) - X(\tau))^2].
\] (3.4)

This part of the work was influenced by Graversen et al. (2001), Graversen et al. (2007) and Shiryaev and Yor (2004), who studied the existence of optimum times when the deviation considered in (3.4) is just the \( \max_{0 \leq t \leq T} W(t) - W(\tau) \), where \( W(\cdot) \) here is only a standard Brownian motion. Even though there are a few similar ideas between those studies and ours, the investigation here assumes initial values and, of course, assumes Brownian motions with drifts. To that aspect the calculations become more complex and the use of Theorem 1 is essential. Further, the aim here is to achieve the ultimate minimum (not the maximum) of the log-DCT, which was by no means approached earlier.

Proposition 1. Let \( X(t) = u + \theta t + W(t) = u + W(t), \ t \geq 0 \) denote the log-DTC process. The ultimate minimum of \( X(t) \) and \( W(t), \ t \geq 0 \) are \( I(T) = \inf_{0 \leq t \leq T} X(t) \) and \( I_0(T) = \inf_{0 \leq u \leq T} W(u), \ T \geq 0 \), respectively. Let \( F_{t-s}(x) = P(I_0(t-s) \leq x) \) and \( \mathcal{F}_s = \sigma(X(s) : u \leq s) \). Then,

\[
I(T) = E[I(T)] + \int_0^T F_{T-s}(I(s) - X(s)) \, dW(s).
\]

Proof. Using the time homogeneity property, it follows that

\[
E[I(T) | \mathcal{F}_t] = I(t) + E[\inf_{t \leq s \leq T} (X(s) - X(t)) | \mathcal{F}_t] = I(t) + E[\inf_{t \leq s \leq T} (X(s) - X(t)) - (I(t) - X(t)) | \mathcal{F}_t],
\]

where \( x^* = (-x) \vee 0 \). It is known that \( E[(X - c)^+] = \int_{-\infty}^- P(X < z) \, dz \). Upon substituting the last identity into (3.5), we obtain that

\[
E[I(T) | \mathcal{F}_t] = I(t) + \int_{-\infty}^{I(t)-X(t)} F_{T-t}(z) \, dz = f(t, X(t), I(t)).
\] (3.6)
Applying Itô’s formula to the right-hand side of (3.6) and using the fact that the left-hand side defines a continuous martingale, we have

\[
E[I(T)|\mathcal{F}_t] = E[I(T)] + \int_0^t \frac{\partial f}{\partial x}(s, X(s), I(s)) \, dW(s) = E[I(T)] + \int_0^t F_{T-s}(I(s) - X(s)) \, dW(s).
\]  

(3.7)

This is a nontrivial continuous martingale and does not have paths of bounded variation.

Setting \( t = T \) and then equalizing the left-hand sides of (3.6) and (3.7) the desired result easily follows. \( \square \)

Using Dassios (1995), Eq. (3.2), or Shiryae et al. (1993), it can be seen that

\[
F_{I}(x) = \phi \left( -\frac{x + \theta t}{\sqrt{t}} \right) - e^{2\theta t} \phi \left( \frac{x + \theta t}{\sqrt{t}} \right).
\]  

(3.8)

Eq. (3.8) will be used for Proposition 1 in order to evaluate the cumulative distribution \( F_{T-\cdot}(\cdot) \).

The drive of the next result is to find \( \{\mathcal{F}_t\} \) – stopping time, \( \tau^* \leq T \), such that \( X(\tau^*) \) is the closest to \( I(T) = \inf_{0 \leq t \leq T} X(t) \) in some sense. Clearly, \( X(t) \) describes the evolution of the log-DTC process on the interval \( 0 \leq t \leq T \). The financial motivation of such a problem is to observe the log-DTC and then pay off the contract at its lowest price. The next theorem suggests that such a time does exist. The determination of evaluating the optimum time is very complex and tedious. It is, however, available for the case when the underlying process is only a standard Brownian motion and when the process attains its maximum (see, e.g., Urusov, 2005). Below, we provide various steps of how this can be achieved for the case of Brownian motion with a drift with initial value and when the process attains its minimum.

In light of Proposition 1, the main theorem of this article is then formulated as follows.

**Theorem 2.** Let \( M \) denote a family of Markov times with respect to \( \sigma \) – algebra \( \mathcal{F} \). There exists a stopping time \( \tau^* \in M, \tau^* \leq T \), such that

\[
E[(I(T) - X(\tau^*))^2] = \inf_{\tau \in M} E[(I(T) - X(\tau))^2].
\]

The expression of \( E[(I(T) - X(\tau))^2] \) in terms of \( u, \theta, \tau \in M, \) and \( I(T) \) is given by

\[
E[(I(T) - X(\tau))^2] = E \left[ \int_0^\tau \left\{ \frac{\theta^2}{2} s + 20(u - E[I(T)]) + 1 - 2F_{T-s}(I(s) - X(s)) \right\} \, ds \right] + uE[I(T)] + u^2 + E[I^2(T)].
\]

**Proof.** Note that for any \( \tau \in M \), we have

\[
E[(I(T) - X(\tau))^2] = E[I^2(T)] + E[X^2(\tau)] - 2E[I(T)X(\tau)].
\]  

(3.9)

The terms of interest in Eq. (3.9) are, of course, the second and the third term. The first term is given in Theorem 1. To evaluate the third term, we utilize Proposition 1. Specifically, we substitute \( I(T) \) by its integral representation, as follows

\[
E[I(T)X(\tau)] = E[I(T)]E[X(\tau)] + E \left[ X(\tau) \int_0^T F_{T-s}(I(s) - X(s)) \, dW(s) \right] = E[I(T)]E[X(\tau)]
\]

\[
+ E \left[ X(\tau) \left\{ \int_0^T F_{T-s}(I(s) - X(s)) \, dW(s) + \int_t^T F_{T-s}(I(s) - X(s)) \, dW(s) \right\} \right].
\]  

(3.10)

Note that \( \int_0^T F_{T-s}(I(s) - X(s)) \, dW(s) \) and \( X(\tau) \) are independent, thus the second product in the expectation (3.10) vanishes. Since \( X(\tau) = u + \theta \tau + \int_0^\tau dW(s) \), one can then use Itô’s isometry property to finally obtain

\[
E[I(T)X(\tau)] = E[I(T)](u + \theta E[\tau]) + E \left[ \int_0^\tau F_{T-s}(I(s) - X(s)) \, ds \right].
\]  

(3.11)
To evaluate the second term in (3.9), we note that
\[
E[X^2(\tau)] = E[(u + \theta \tau + W(\tau))^2] = u^2 + (2u\theta + 1)E[\tau^2] + \theta^2 E[\tau^2].
\] (3.12)

Combining (3.9), (3.11) and (3.12), we obtain
\[
E[(I(T) - X(\tau))^2] = \theta^2 E[\tau^2] + \{20(u - E[I(T)]) + 1\}E[\tau] - 2E\left[\int_0^T F_{T-s}(I(s) - X(s)) \, ds\right] + uE[I(T)]
\] + \left. u^2 + E[I^2(T)] \right].
\] (3.13)

This completes the proof of Theorem 2. □

To complete our investigation, as Eq. (3.13) remains complex, we seek to obtain a more explicit expression. To achieve a more thorough understanding of (3.4), one can condition the loss function and notice that
\[
E[(I(T) - X(\tau))^2] = E[E[(I(T) - X(\tau))^2] | \tau].
\]

It is thus appropriate to study the inside expectation first. In light of this, the following corollary is in order.

**Corollary 1.** Let \( \tau^* \in \mathcal{M} \) be as in Theorem 2. Then the optimum time can be computed from the following:
\[
E[(I(T) - X(\tau^*))^2] = \inf_{\tau \in \mathcal{M}} E\left[\int_0^T (\sigma^2 + 20(u - E[I(T)]) + 1 - 2 \int_{-\infty}^{0} F_{T-s}(y) \, dF_s(y)) \, ds\right] + uE[I(T)] + u^2
\] + \left. E[I^2(T)] \right].
\]

**Proof.** From Theorem 2, it is clear that \( E[(I(T) - X(\tau))^2] \) will change only through the first expectation in the right-hand side. In particular, the change will occur through the term
\[
V(\theta) = E\left[\int_0^T F_{T-s}(I(s) - X(s)) \, ds\right].
\]

It is known (Bertoin, 1996, p156) that the reflected process \( \{I(t) - X(t); 0 \leq t \leq T\} \) is a Markov process in the filtration \( \mathcal{F}_s = \sigma(W(u); u \leq t) \) and its semi-group has the Feller property. That is, we need only to consider stopping times, which are hitting times for \( I(\cdot) - X(\cdot) \).

Using the homogeneity property for the Brownian motions, it can be seen that for any \( x < 0 \)
\[
P(I(t) - X(t) \leq x) = P(W(s) - W(t) = x)
\] for some \( s < t \) \( W(t) = y \)
\[
= P(W(t-s) - \theta(s-t) \geq -x)
\] for some \( s < t \) \( W(t) = y \)
\[
= P(W(s) - \theta s \geq -x)
\] for some \( s < t \) \( W(t) = y \).
\] (3.14)

In view of the well-known formula of Siegmund (1986), we have that
\[
P(W(s) - \theta s \geq -x)
\] for some \( s < t \) \( W(t) = y \) = \( \exp(2\theta(xt - x - y)/t) \), \( x < 0 \).
\] (3.15)

In conjunction with (3.13) and since \( V(\theta) = E[V(\theta) | \tau] \), the conditioning upon \( \tau \), \( V(\theta | \tau) \) can be then expressed as follows
\[
V(\theta | \tau) = \int_0^\tau E[F_{T-s}(I(s) - X(s)) \, ds]
\]
\[
= \int_0^\tau \left[ \phi \left(\frac{-I(s) + X(s) + \theta(T-s)}{\sqrt{T-s}}\right) - e^{2\phi(I(s) - X(s))} \phi \left(\frac{I(s) - X(s) + \theta(T-s)}{\sqrt{T-s}}\right) \right] \, ds
\]
\[
= \int_0^\tau \left[ \int_{-\infty}^0 \left\{ \phi \left(\frac{-y + \theta(T-s)}{\sqrt{T-s}}\right) - e^{2\phi y} \phi \left(\frac{y + \theta(T-s)}{\sqrt{T-s}}\right) \right\} \, dP(I(s) - X(s) \leq y) \right] \, ds. \] (3.16)
Using (3.15) and (3.16), it follows that
\[
P(I(t) - X(t) \leq x) = 1 - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( \frac{-(y+\theta t)^2}{2\sigma^2} \right) \Phi \left( \frac{y+\theta t}{\sqrt{X(t)}} \right) dy
\]
and its density is given by (3.2), for \( u = 0 \).

Substituting (3.17) into (3.16), the result follows immediately. \( \square \)

4. Simulation analysis of stopping times

The best stopping time for a given contract is highly dependent upon the drift term \( \theta \). When \( \theta \) is positive and large enough, the units should be purchased at the beginning of the time horizon; conversely, when \( \theta \) is negative and large enough in absolute value, the units should be purchased at the end of the time horizon. However, when the drift term approaches 0, the best purchase time to minimize expected cost will not be at the beginning or the end.

We conducted simulations to illustrate the optimum purchase time under various values of \( \theta \). In particular, we generated 2000 price processes of length \( T = 100 \) periods. We let \( \sigma = 1 \) and \( u = 0 \) (note that the results are

Fig. 1. Simulation results plotting the squared loss \( [I(T) - X(t)]^2 \), averaged over 2000 trials for each of nine values of the drift term \( \theta \).
independent of the value of \( u \) chosen. For purchasing at each stopping time period \( \tau \) between 0 and 100, we calculated the squared loss of the difference between the process \( X(\tau) \) and \( I(T) \) (the minimum of the process \( X(t) \) over the full time horizon). Thus, each graph in Fig. 1 represents a plot of 101 points.

Fig. 1 presents simulation results for \( \theta \) values of 0.20, 0.03, 0.02, 0.01, 0.00, −0.01, −0.02, −0.03, and −0.20. The corresponding expected optimum stopping times based on the simulation results were \( \tau = 0, 11, 30, 40, 50, 60, 70, 88, \) and 100, respectively.

5. Concluding remarks

Today’s increased globalization has opened up many more possibilities for procuring goods from around the world. However, the increased options generate even more exposure to purchase price fluctuations, particularly with regard to issues such as exchange rate movements, political turmoil, and supply and demand shifts in emerging markets like China. In response, supply contracts have begun to include a time flexibility component, allowing the buyer to choose the time of purchase. With a risky option of purchase timing in hand, firms need assistance to help determine when, in fact, the best time to purchase might be. This paper provides a solution for that decision.

In the formulation developed in this paper we regard log-DTC as a state of a game at time \( T \), where each realization, \( \omega \in \Omega \), corresponds to one sample of the game. For each time period prior to the end of the horizon \( T \), the buyer has the option of stopping the game and accepting the current cost or continuing the game in the hope that purchasing later will reduce the cost further. The problem is of course that we do not know in what state the game will be in the future; we can only estimate the probability distribution of the ‘future’. Among all possible stopping times \( \tau < T \) in the above formulation, we have demonstrated a procedure for obtaining an optimum time \( \tau^* < T \) such that log-DTC(\( \cdot \)) gives us the best result ‘in a long run’, i.e., the expected loss \( E[(I(T) - X(\tau))^2] \) becomes minimum.

Appendix

Proof of Theorem 1. Calling upon (3.2), we redefine the density of the random variable \( I(T) \) as \( f_I(x) = (I_1(x) + 20I_2(x))I(x < \theta) \), where \( I(x \in A) \) denotes the indicator function, which becomes one if the event occurs and zero otherwise.

Consequently, \( E[I(T)] = \int_{-\infty}^{\theta} xI_1(x) \, dx + 20 \int_{-\infty}^{\theta} xI_2(x) \, dx \equiv I_1 + 20I_2 \). We thus compute the two parts separately. The first can be expressed as

\[
I_1 = 2 \left( \frac{T}{2\pi} \right)^{1/2} \exp(-\theta^2T/2) + (u + \theta T) \int_{-\infty}^{\theta} \varphi(u) \, dx
\]

\[
= 2 \left( \frac{T}{2\pi} \right)^{1/2} \exp(-\theta^2T/2) + (u + \theta T) \Phi(-\theta\sqrt{T}), \quad \theta \in \mathbb{R}, \tag{A.1}
\]

where \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \varphi(y) \, dy \) and \( \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \ x \in \mathbb{R} \).

To determine the second term, we first set \( y = x - \theta \). Further, based upon \( 2\theta y \exp(2\theta y) \, dy = d(y \exp(2\theta y)) - \exp(2\theta y) \, dy \) and an integration by parts, it follows that

\[
20I_2 = \int_{-\infty}^{0} \Phi\left( \frac{y + \theta T}{\sqrt{T}} \right) d(y \exp(2\theta y)) + 20 \left( u - \frac{1}{2\theta} \right) \left( \int_{-\infty}^{0} \exp(2\theta y) \Phi\left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy \right)
\]

\[
= - \int_{-\infty}^{0} y \exp(2\theta y) \frac{1}{\sqrt{2\pi T}} e^{-(y+\theta T)^2/2T} \, dy + 20 \left( u - \frac{1}{2\theta} \right) \left( \int_{-\infty}^{0} \exp(2\theta y) \Phi\left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy \right)
\]

\[
= - \left( \sqrt{T} + \frac{1}{2} \left( u - \frac{1}{2\theta} \right) \right) \Phi(\theta\sqrt{T}). \tag{A.2}
\]

Using (A.1) and (A.2), the expected value of \( I(T) \), \( T > 0 \), immediately follows.
Arguing the same way as in the determination of the first moment, we have that $E[I^2(T)] = \int_{-\infty}^{\infty} x^2 I_1(x) \, dx + 20 \int_{-\infty}^{\infty} x^2 I_2(x) \, dx \equiv I_1 + 20I_2$. The first term $I_1$ can be expressed as

$$I_1 = T \frac{\sqrt{\frac{\pi}{2}}}{\sqrt{\pi}T} \int_{-\infty}^{\infty} v^2e^{-v^2/2} \, dv + 4(u + \theta T) \frac{\sqrt{\pi}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ve^{-v^2/2} \, dv - 2(u + \theta T)^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-v^2/2} \, dv$$

$$= T \frac{\sqrt{\frac{\pi}{2}}}{\sqrt{\pi}T} \int_{0}^{\infty} v^{1/2} e^{-v} \, dv + 4(u + \theta T) \frac{\sqrt{\pi}}{\sqrt{2\pi}} e^{-\theta^2T/2} - 2(u + \theta T)^2 \Phi(-\theta\sqrt{T}). \quad (A.3)$$

Since $\Gamma(a + 1, x) = a \Gamma(a, x) + x^a e^{-x}$ and $\Gamma(1/2, x^2/2) = \sqrt{2\pi}(1 - \Phi(x))$, Eq. (A.3) can be expressed as

$$I_1 = \left( \frac{T \sqrt{\frac{\pi}{2}}}{2} - 2(u + \theta T)^2 \right) \Phi(-\theta\sqrt{T}) + \frac{4u + 50T\sqrt{T}}{\sqrt{2\pi}} e^{-\theta^2T/2}. \quad (A.4)$$

To obtain the second term, we again set $y = x - u$ and then it follows that

$$20I_2 = 20 \int_{-\infty}^{\infty} y^2 \exp(20y) \Phi \left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy + 2u \left\{ 20 \int_{-\infty}^{0} y \exp(20y) \Phi \left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy \right\}$$

$$+ u^2 \left\{ 20 \int_{-\infty}^{0} \exp(20y) \Phi \left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy \right\}. \quad (A.5)$$

Since $20 \int_{-\infty}^{0} y \exp(20y) \Phi \left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy = 20I_2 - u20 \int_{-\infty}^{0} \exp(20y) \Phi \left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy$, Eq. (A.5) can be replaced by

$$20I_2 = 20 \int_{-\infty}^{0} y^2 \exp(20y) \Phi \left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy + 2u20I_2 - u^220 \int_{-\infty}^{0} \exp(20y) \Phi \left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy \quad (A.6)$$

Using the fact that $20y^2 \exp(20y) \, dy = d(y^2 \exp(20y)) - 2y \exp(20y) \, dy$, the first term in (A.5) can be further simplified by

$$20 \int_{-\infty}^{0} y^2 \exp(20y) \Phi \left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy = \int_{-\infty}^{0} \Phi \left( \frac{y + \theta T}{\sqrt{T}} \right) d(y^2 \exp(20y)) - 2 \int_{-\infty}^{0} y \exp(20y) \Phi \left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy$$

$$= - \frac{1}{\sqrt{2\pi}T} \int_{-\infty}^{0} y^2 \exp(20y) e^{-\theta^2T/2} \, dy - I_2$$

$$- u \int_{-\infty}^{0} \exp(20y) \Phi \left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy. \quad (A.7)$$

Substituting (A.6) into (A.7), we obtain

$$20I_2 = - \frac{1}{\sqrt{2\pi}T} \int_{-\infty}^{0} y^2 \exp(20y) e^{-\theta^2T/2} \, dy + 2 \left( u - \frac{1}{\theta} \right) (20I_2)$$

$$- u \left( u + \frac{1}{20} \right) \left\{ 20 \int_{-\infty}^{0} \exp(20y) \Phi \left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy \right\}. \quad (A.8)$$

Note that the first term $I_{21}$ in (A.9) can be written as

$$I_{21} = \frac{1}{\sqrt{2\pi}T} \int_{-\infty}^{0} y^2 \exp(20y) e^{-\theta^2T/2} \, dy = - \frac{T \sqrt{21 \Gamma(3/2)}}{\sqrt{2\pi}T} e^{-\theta^2T/2} = - \frac{T}{2} e^{-\theta^2T/2}. \quad (A.9)$$

Further, it is easy to see that

$$20 \int_{-\infty}^{0} \exp(20y) \Phi \left( \frac{y + \theta T}{\sqrt{T}} \right) \, dy = \Phi \left( \theta \sqrt{T} \right) - \frac{1}{2} e^{-\theta^2T/2}. \quad (A.10)$$
In light of (A.2), (A.9) and (A.10), it then follows that
\[
2\tilde{I}_2 = \left\{ 2\left( u - \frac{1}{\theta} \right) \left( u - \frac{1}{2\theta} \right) - u\left( u + \frac{1}{2\theta} \right) \right\} \Phi(\theta\sqrt{T}) \\
- \left\{ \frac{T}{2} + 2\left( u - \frac{1}{\theta} \right) \frac{\sqrt{T}}{\sqrt{2\pi}} + \frac{1}{2} \left\{ \left( u - \frac{1}{\theta} \right) \left( u - \frac{1}{2\theta} \right) + 1 \right\} \right\} e^{-\theta^2T/2}. \tag{A.11}
\]
Combining (A.4) and (A.11), the computation of the second moment is now completed.

The second part of the theorem easily follows by just letting \( T \to \infty. \)

References


